CONFORMAL TRANSFORMATIONS IN RIEMANNIAN AND HERMITIAN SPACES

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The purpose of the present note is to show that the results recently announced by S. I. Goldberg [1] in this Bulletin are valid also in slightly more general forms.

1. Consider a conformal Killing vector v^h in an n-dimensional Riemannian space. Then the Lie derivative of the fundamental tensor g_{ji} and that of Christoffel symbols with respect to v^h are respectively given by

$$\mathfrak{L}_v g_{ji} = \nabla_j v_i + \nabla_i v_j = 2\phi g_{ji} \tag{1.1}$$

and

$$\mathfrak{L}_v \left\{ {h \atop j\ i} \right\} = \nabla_j \nabla_i v^h + K_{kji}{}^h v^k = A_j^h \phi_i + A_i^h \phi_j - \phi^h g_{ji}, \tag{1.2}$$

where ∇_j is the symbol of covariant differentiation, $K_{kji}{}^h$ the curvature tensor, A_j^h the unit tensor and $\phi_i = \nabla_i \phi$, ϕ^h being its contravariant components.

For a skew-symmetric tensor $w_{i_p i_{p-1} \cdots i_1}$, we have in general [5]

$$\begin{aligned} &\mathfrak{L}_v \nabla_j w_{i_p \cdots i_1} - \nabla_j \mathfrak{L}_v w_{i_p \cdots i_1} \\ &= -\left(\mathfrak{L}_v \left\{ {t \atop j\ i_p} \right\}\right) w_{t i_{p-1} \cdots i_1} - \cdots - \left(\mathfrak{L}_v \left\{ {t \atop j\ i_1} \right\}\right) w_{i_p \cdots i_2 t}. \end{aligned} \tag{1.3}$$

Taking the skew-symmetric part with respect to $j, i_p \cdots i_1$, we find

$$\mathfrak{L}_v \nabla_{[j} w_{i_p \cdots i_1]} = \nabla_{[j} \mathfrak{L}_v w_{i_p \cdots i_1]}, \tag{1.4}$$

from which

THEOREM 1.1. *The Lie derivative of a closed skew-symmetric tensor is closed.*

Transvecting (1.3) with g^{ji_p} and taking account of (1.1) and (1.2), we get

$$\begin{aligned} &\mathfrak{L}_v g^{ji} \nabla_j w_{i i_{p-1} \cdots i_1} + 2\phi g^{ji} \nabla_j w_{i i_{p-1} \cdots i_1} - g^{ji} \nabla_j \mathfrak{L}_v w_{i i_{p-1} \cdots i_1} \\ &\qquad = (n - 2p)\phi^t w_{t i_{p-1} \cdots i_1}, \end{aligned} \tag{1.5}$$

from which

THEOREM 1.2. *The Lie derivative of a coclosed skew-symmetric tensor of order p with respect to a conformal Killing vector is coclosed if and only if $p=n/2$, n being even, or $\nabla^t(\phi w_{ti_{p-1}\cdots i_1})=0$, that is, $\phi w_{i_p\cdots i_1}$ is also coclosed, where ϕ is the function appearing in $\mathfrak{L}_v g_{ji}=2\phi g_{ji}$.*

Combining Theorems 1.1 and 1.2 we have

THEOREM 1.3. *The Lie derivative of a harmonic tensor w of order p in an n-dimensional Riemannian space with respect to a conformal Killing vector is also harmonic if and only if $p=n/2$, n being even, or ϕw is coclosed.*

The most specific statement resulting is as follows, see [**4**; **5**; **6**].

THEOREM 1.4. *The Lie derivative of a harmonic tensor w of order p in an n-dimensional compact orientable Riemannian space with respect to a conformal Killing vector is zero if and only if $p=n/2$, n being even, or ϕw is coclosed where ϕ is a function appearing in $\mathfrak{L}_v g_{ji}=2\phi g_{ji}$* [**1**].

2. In an almost complex space, a contravariant almost analytic vector is defined as a vector v^h which satisfies

$$\mathfrak{L}_v F_i^{\ h} = v^t\partial_t F_i^{\ h} - F_i^{\ t}\partial_t v^h + F_t^{\ h}\partial_i v^t = 0. \tag{2.1}$$

In an almost Hermitian space, (2.1) may be written as

$$\mathfrak{L}_v F_i^{\ h} = v^t\nabla_t F_i^{\ h} - F_i^{\ t}\nabla_t v^h + F_t^{\ h}\nabla_i v^t = 0, \tag{2.2}$$

from which, by a straightforward calculation,

$$\nabla^i\nabla_i v^h + K_i^{\ h} v^i - F_i^{\ h}(\mathfrak{L}_v F^i) - \frac{1}{2} F_{ji}^{\ \ h}(\mathfrak{L}_v F^{ji}) = 0, \tag{2.3}$$

where $K_i^{\ h}$ is the Ricci tensor and

$$F^i = \nabla^j F_j^{\ i},$$

$$F_{jih} = \nabla_j F_{ih} + \nabla_i F_{hj} + \nabla_h F_{ji}.$$

If we put

$$S^{ji} = g^{jt}(\mathfrak{L}_v F_t^{\ i}),$$

and suppose that the space is compact, we have

$$\int \left[\left\{ \nabla^i \nabla_i v^h + K_i{}^h v^i - F_i{}^h (\mathfrak{L}_v F^i) - \frac{1}{2} F_{ji}{}^h (\mathfrak{L}_v F^{ji}) \right\} v_h + \frac{1}{2} S^{ji} S_{ji} \right] d\sigma = 0, \tag{2.4}$$

$d\sigma$ being volume element of the space.

From (2.3) and (2.4) we have

THEOREM 2.1. *A necessary and sufficient condition for a vector v^h in a compact almost Hermitian space to be contravariant analytic is* (2.3).

Suppose that a conformal Killing vector v^h satisfies

$$F_i{}^h (\mathfrak{L}_v F^i) + \frac{1}{2} F_{ji}{}^h (\mathfrak{L}_v F^{ji}) = 0.$$

Substituting

$$\nabla^i \nabla_i v^h + K_i{}^h v^i = - \frac{n-2}{n} \nabla^h (\nabla_i v^i)$$

obtained from (1.2) into (2.4), we find

$$\int \left[\frac{n-2}{n} (\nabla_i v^i)^2 + \frac{1}{2} S^{ji} S_{ji} \right] d\sigma = 0, \tag{2.5}$$

from which, for $n > 2$,

$$\nabla_i v^i = 0, \qquad S_{ji} = 0$$

and consequently v^h is a Killing vector [**4**; **6**] and at the same time a contravariant almost analytic vector, and for $n = 2$, we have $S_{ji} = 0$. Thus we have

THEOREM 2.2. *If a conformal Killing vector v^h in an n-dimensional compact almost Hermitian space satisfies*

$$F_i{}^h (\mathfrak{L}_v F^i) + \frac{1}{2} F_{ji}{}^h (\mathfrak{L}_v F^{ji}) = 0, \tag{2.6}$$

then, for $n > 2$, it defines an automorphism of the space, that is, the infinitesimal transformation v^h does not change both the metric and the almost complex structure of the space, and for $n = 2$, it is contravariant almost analytic.

An almost Hermitian space in which $F_i = 0$ is satisfied is called an almost semi-Kählerian space. In such a space, we have

$$F_{jih}F^{ji} = 2F_tF_h^{\ t} = 0.$$

Thus from Theorem 2.2, we have

THEOREM 2.3. *If a conformal Killing vector v^h in an n (>2) dimensional compact almost semi-Kählerian space satisfies*

$$F_{jih}(\mathfrak{L}_v F^{ji}) = 0 \quad or \quad (\mathfrak{L}_v F_{jih})F^{ji} = 0, \tag{2.7}$$

then v^h defines an automorphism in the space.

An almost Hermitian space in which $F_{jih}=0$ is satisfied is called an almost Kählerian space. In such a space, we have

$$F_h = -\frac{1}{2} F_{jit}F^{ji}F_h^{\ t} = 0,$$

that is, F_{ji} is harmonic. Thus from Theorem 2.3, we have

THEOREM 2.4. *A conformal Killing vector v^h in an n (>2) dimensional compact almost Kählerian space defines an automorphism of the space (cf.* [**1**; **2**; **3**]).

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