THE HOMOLOGY OF CYCLIC PRODUCTS

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1. **Introduction.** The problem of computing the homology of cyclic products has been considered by a number of authors. The rational homology of all such products was found by Richardson [3]. The remaining results are all concerned with p-fold cyclic products where p is a prime [2; 4; 6; 7]. In [2], the cup products and the Steenrod operations are determined when the coefficient group is Z_p. In [6] and [7], the betti numbers and torsion coefficients are determined but the method used does not give induced maps and cup products.

I will outline here a method for computing the homology, with arbitrary coefficients, of n-fold cyclic products *for all* n, not merely for the case where n is prime. The construction used is natural up to homotopy and so permits the determination of induced maps and cup products.

2. **Outline of the method.** Let X be a semisimplicial complex. Let X^n be the n-fold Cartesian product of X with itself. A cyclic group π of order n acts on X^n by permuting the factors cyclically. The n-fold cyclic product CP^nX is defined to be the quotient complex X^n/π.

For any simplex x of X, there is a unique nondegenerate simplex y of X such that x is obtained by applying degeneracy operators to y. Define the reduced dimension of x by $d_r(x) = \dim y$. Define a π-stable filtration on X^n by letting $\Phi_k X^n$ be the subcomplex consisting of those $(x_1, \cdots, x_n)$ with $\sum d_r(x_i) \leqq k$. The Eilenberg-Zilber map $\nabla: \otimes^n C(X) \to C(X^n)$ has the property that ∇ maps all k-chains into $\Phi_k C(X^n)$. Here $\otimes^n C(X)$ means $C(X) \otimes \cdots \otimes C(X)$.

LEMMA 2.1. *Let π' be the subgroup of π of order r. Let e be a nondegenerate simplex of $\Phi_k X^n$ which is fixed under π'. Then* $\dim e \leqq k/r$.

We now consider the category $\mathfrak{F}$ of filtered chain complexes having a Z-base permuted by π with the property stated in Lemma 2.1. We also consider a category $\mathcal{P}$ of chain complexes having a Z-base permuted up to sign by π in the same way as the obvious base for $\otimes^n C(X)$. Define a $\mathcal{P}\mathfrak{F}$-map from an object P of $\mathcal{P}$ to an object F of $\mathfrak{F}$ to be a π-equivariant chain map which maps P_k into $\Phi_k F$ for all k.

[1] This work was done while the author was a National Science Foundation fellow.

THEOREM 2.1. *Every object P in $\mathcal{P}$ has a $\mathcal{P}\mathfrak{F}$-map $u: P\rightarrow F(P)$ which is universal up to homotopy.*

In other words, u is universal in the usual sense [**1**; **5**] if we pass to the quotient categories obtained by identifying homotopic maps.

The Eilenberg-Zilber map ∇ is a $\mathcal{P}\mathfrak{F}$-map. Consequently, it factors as $\nabla = gu$ where $g: F(\otimes^n C(X))\rightarrow C(X^n)$ is unique up to homotopy.

THEOREM 2.2. *The induced map*

$$(g/\pi)_*: H(F(\otimes^n C(X))/\pi) \rightarrow H(CP^n X)$$

is an isomorphism.

THEOREM 2.3. *If M is any Z-free chain complex, $H(F(\otimes^n M)/\pi)$ depends only on $H(M)$.*

To find $H(CP^n X)$, we now choose a small chain complex M such that $H(M)\approx H(X)$ and compute $H(F(\otimes^n M)/\pi)$. The complex $F(\otimes^n M)$ is given by an explicit construction from which $H(F(\otimes^n M)/\pi)$ can be computed explicitly.

The above outline is slightly oversimplified. We must actually construct a different F for each primary component of n and piece together the resulting complexes.

3. **Formulas.** A general formula for the integral homology of $CP^n X$ would apparently be very complicated. To get this homology, it is best to regard the above method as an algorithm for computing it. If, however, we are concerned with homology with coefficients in a field, it is quite easy to give an explicit formula for the cohomology ring of $CP^n X$.

Let K be a field of characteristic p. For each power p^k of p which divides n, define $\pi(p^k)$ to be the subgroup of π of index p^k. Let $H = H^*(X; K)$. If A is any graded module, let $[A]^k$ be the k-dimensional part of A. For all integers j, k and all $p^r | n$, define G^j_{k,n,p^r} to be $[(\otimes^{n/p^r} H)/\pi(p^r)]^k$ if $p^r k - j$ is even and define it to be $[(\otimes^{n/p^r} H)^{\pi(p^r)}]^k$ if $p^r k - j$ is odd. We regard G^j_{k,n,p^r} as having dimension j, not k. For all j, define

$$J^j = \sum_{j-2\geqq k > j/p} G^j_{k,n,p} + \sum_{r\geqq 2}\ \sum_{j-2\geqq k\geqq (j-1)/p} G^j_{k,n,p^r}.$$

THEOREM 3.1. *Assume H is finitely generated in each dimension. Then there are natural isomorphisms*

$$H^j(CP^n X; K) \approx [(\otimes^n H)/\pi]^j + J^j \qquad \text{for } j > 0,$$
$$H^0(CP^n X; K) \approx (\otimes^n H^0)^\pi.$$

THEOREM 3.2. *The cup products in* $H^*(CP^nX; K)$ *are given as follows.*

(1) *Let a and b have dimension* >0. *Then,*

(i) *If a or b is in* J^i, *then* $ab=0$.

(ii) *Let* $a, b \in (\otimes^n H)/\pi$ *be represented by* $a', b' \in \otimes^n H$. *Then ab is represented by* $(\sigma a')b'$ *and also by* $a'(\sigma b')$ *where* $\sigma = \sum_{t\in\pi} t$.

(2) *Let* $a \in (\otimes^n H^0)^\pi$. *Let b belong to any one of the summands in the formula for* $H^*(CP^nX; K)$, *say one of the form* $(\otimes^{n/p^r}H)/\pi(p^r)$ *or* $(\otimes^{n/p^r}H)^{\pi(p^r)}$. *Then* $ab = d^*(a)\cdot b$ *where the second product is the usual one in* $\otimes^{n/p^r}H$ *and* $d^*: \otimes^n H^0 \to \otimes^{n/p^r}H^0$ *is induced by the diagonal map* $d: X^{n/p^r} \to X^n$.

REMARK. This map d^* is completely determined by the ring structure of H^0.

The quotient map $f: X^n \to CP^nX$ induces a map f^* of cohomology.

THEOREM 3.3. *For all* j, $f^*(J^j)=0$. *If* $a \in (\otimes^n H)/\pi$ *is represented by* $a' \in \otimes^n H$, *then* $f^*(a) = \sigma a'$ *where* σ *is as in Theorem* 3.2. *In dimension* 0, f^* *is simply the inclusion* $(\otimes^n H^0)^\pi \to \otimes^n H^0$.

COROLLARY. *If* $p \mid n$, *the diagonal map* $d: X \to CP^nX$ *induces a trivial map of cohomology, i.e.* $d^*: H^j(CP^nX; K) \to H^j(X; K)$ *is zero for* $j>0$.

PROOF. Factor d as $X \to^{\Delta} X^n \to^{f} CP^nX$. If π is made to act trivially on X, Δ is π-equivariant. If $u \in H^j(CP^nX)$, Theorem 3.3 shows $f^*(u) = \sigma v$ for some $v \in H^j(X^n)$. Therefore, $d^*(u) = \Delta^*(\sigma v) = \sigma\Delta^*(v) = n\Delta^*(v) = 0$ since $p \mid n$.

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