ON INTEGRAL FUNCTIONS OF INTEGRAL OR ZERO ORDER

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Let $F(z)$ be an integral function of finite order ρ. We write $F(z)=z^k e^{g(z)}f(z)$ where $g(z)$ is a polynomial of degree $q\leqq\rho$ and

$$f(z) = \prod_1^\infty\left\{\left(1-\frac{z}{a_n}\right)\exp\left(\frac{z}{a_n}+\cdots+\frac{1}{p}\left(\frac{z}{a_n}\right)^p\right)\right\}$$

is the canonical product of order ρ_1 and genus p. Let $M(r, F)=\max_{|z|=r}|F(z)|$ and $n(r, F-a)=n(r, a)$ be the number of zeros of $F(z)-a$ in $|z|=r$. In an earlier paper[1] I proved the following result.

THEOREM 1. *If $F(z)$ be of integral order ρ and if the genus of the canonical product $f(z)$ be $p=\rho$, then*

$$\liminf_{r=\infty}\frac{\log M(r, F)}{n(r, F)\phi(r)}=0 \tag{1}$$

where $\phi(x)$ is any positive continuous increasing function of the real variable x such that

$$\int_a^\infty\frac{dx}{x\phi(x)} \tag{2}$$

is convergent.

In this note I prove a similar result for the canonical products of order ρ and genus $p=\rho-1$, and discuss whether the result can be extended to integral functions which are not canonical products. The main result is the following.

THEOREM 2. *If $f(z)$ is a canonical product of integral order ρ and genus $p=\rho-1$ then*

$$\liminf_{r=\infty}\frac{\log M(r, f)}{n(r, f)\Phi(r)}=0 \tag{3}$$

where $\Phi(x)$ is any positive increasing function such that

$$\int_a^\infty\frac{dx}{x\Phi(x)} \tag{4}$$

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[1] *A Theorem on integral functions of integral order*, Journal of the London Mathematical Society, vol. 15 (1940), pp. 23–31. I shall refer to this paper as (1).

is convergent and

$$(4.1) \qquad \Phi(x)/x^{\alpha}$$

is monotonic for all large x, *say* $x \geqq \Delta > 0$; α *a constant such that* $0 < \alpha < 1$.

LEMMA 1. *For all* $r \geqq r_0(A, \beta)$

$$J = \int_{\Delta_1}^{r} \frac{dx}{x^{\beta}\Phi(x)} < \frac{Ar^{1-\beta}}{\log r},$$

where A *and* Δ_1 *are positive constants, and* β *is a constant such that* $0 < \beta < 1$.

PROOF. From the convergence of the integral in (4), we have $\log x < \Phi(x)$ for all $x \geqq \Delta_2$. Hence for $r \geqq r_0$

$$J = \int_{\Delta_1}^{r^{1/2}} + \int_{r^{1/2}}^{r} \leqq \frac{1}{\Phi(\Delta_1)} \frac{r^{(1-\beta)/2}}{(1-\beta)} + \frac{2}{(1-\beta)} \frac{r^{1-\beta}}{\log r} < \frac{Ar^{1-\beta}}{\log r}.$$

LEMMA 2. *Suppose that the real functions* $\psi(x)$ *and* $\theta(x)$ *satisfy the following conditions*:

(1) $\psi(x)$ *is continuous in* (δ, ∞) *where* $\delta > 0$, *except for isolated points where* $\psi(x)$ *has ordinary left-hand discontinuities.*

(2) $\psi(x)$ *is non-increasing as* $x \geqq \delta$ *increases in any interval between two consecutive discontinuities.*

(3) $\theta(x)$ *is a positive continuous increasing function for* $x \geqq \delta$.

$$(4) \qquad \limsup_{x=\infty} \psi(x) = \infty, \qquad \limsup_{x=\infty} \frac{\psi(x)}{\theta(x)} = 0.$$

Then we can find a sequence $\{x_n\}$ *of values of* x *tending to* ∞ *such that the two inequalities*

$$\psi(x) \leqq \psi(x_n), \qquad x_1 \leqq x < x_n,$$

$$\frac{\psi(x)}{\theta(x)} \leqq \frac{\psi(x_n)}{\theta(x_n)}, \qquad x > x_n,$$

are satisfied simultaneously.

The x_n *are points of discontinuity so that* $\psi(x_n) = \psi(x_n + 0)$ *and* x_1 *is the first point of discontinuity in* (δ, ∞).

The proof is similar to that of Lemma 2 of my paper referred to above, and is based on the following lemma of Pólya.[2]

If

$$l_1, l_2, l_3, \cdots, \qquad l_m > 0,$$

$$s_1, s_2, s_3, \cdots, \qquad s_1 > 0;\ s_{m+1} > s_m;\ m = 1, 2, 3, \cdots,$$

[2] Mathematische Annalen, vol. 88 (1923), p. 170.

are two sequences of positive numbers, of which the second is monotonic and increasing, such that

$$\lim_{m=\infty} l_m = 0, \qquad \limsup_{m=\infty} l_m s_m = \infty,$$

then we can find an infinite sequence $\{n\}$ *of the indices* n *such that the two sets of inequalities*

$$l_n > l_\nu, \qquad \nu > n,$$

$$l_n s_n > l_\mu s_\mu, \qquad \mu < n,$$

are satisfied simultaneously.

To prove Theorem 2 we first consider the case when

$$\limsup_{r=\infty} \frac{n(r, f)\Phi(r)}{r^{p+1}} > 0. \tag{5}$$

We have

$$r^{p+1} < An(r)\Phi(r) \tag{5.1}$$

for an infinity of values $r = R_n$ tending to ∞ and so

$$\frac{\log M(R_n)}{n(R_n)\Phi(R_n)} < \frac{A \log M(R_n)}{R_n^{p+1}} \to 0 \qquad \text{as } n \to \infty.$$

Hence

$$\liminf_{r=\infty} \frac{\log M(r)}{n(r)\Phi(r)} = 0.$$

Suppose secondly that

$$\lim_{r=\infty} \frac{n(r)\Phi(r)}{r^{p+1}} = 0. \tag{5.2}$$

Here $\Phi(x)/x^\alpha$ must be monotonic decreasing, for if not $\Phi(x) \geqq Ax^\alpha$, and so

$$\limsup_{r=\infty} \frac{n(r)\Phi(r)}{r^{p+1}} \geqq A \limsup_{r=\infty} \frac{n(r)}{r^{p+1-\alpha}} = \infty,$$

contradicting hypothesis (5.2); so $\Phi(x)/x^\alpha$ is monotonic decreasing for $x \geqq \Delta$. We apply Lemma 2 putting

$$\psi(x) = \frac{n(x)\Phi(x)}{x^{p+1-\beta}} = n(x) \frac{\Phi(x)}{x^\alpha} \frac{1}{x^{p+1-\alpha-\beta}},$$

and choosing $\theta(x) = x^\beta$, β a constant such that $0 < \beta < 1 - \alpha$, $\delta = \Delta$. The conditions of Lemma 2 are satisfied, and hence, putting $x_n = R$ we obtain

$$\frac{n(x)\Phi(x)}{x^{p+1-\beta}} \leqq \frac{n(R)\Phi(R)}{R^{p+1-\beta}}, \quad \text{for } \Delta \leqq x_1 \leqq x \leqq R,$$

$$\frac{n(x)\Phi(x)}{x^{p+1}} \leqq \frac{n(R)\Phi(R)}{R^{p+1}}, \quad \text{for } x > R.$$

Thus for $R > x_1$,

$$\begin{aligned}
\log M(R, f) &< AI(R, f) \\
&= A \int_0^\infty \frac{n(x, f)}{x^{p+1}} \frac{R^{p+1}}{(x+R)} dx \\
&\leqq A \left\{ A_1 R^p \int_{x_1}^R \frac{n(x)}{x^{p+1}} dx + R^{p+1} \int_R^\infty \frac{n(x)}{x^{p+2}} dx \right\} \\
&\leqq A \left\{ A_1 R^p \frac{n(R)\Phi(R)}{R^{p+1-\beta}} \int_{x_1}^R \frac{dx}{x^\beta \Phi(x)} + n(R)\Phi(R) \int_R^\infty \frac{dx}{x\Phi(x)} \right\} \\
&\leqq A \left\{ \frac{A_2 n(R)\Phi(R)}{\log R} + o(n(R)\Phi(R)) \right\}.
\end{aligned}$$

Hence

$$\liminf_{r=\infty} \frac{\log M(r, f)}{n(r, f)\Phi(r)} \leqq A \liminf_{r=\infty} \frac{I(r, f)}{n(r, f)\Phi(r)} = 0 \tag{6}$$

and this completes the proof of the theorem.

COROLLARY. *If* $F(z) = z^k e^{g(z)} f(z)$ *is of integral order* ρ *and genus* $g = \rho - 1$ *then*

$$\liminf_{r=\infty} \frac{\log M(r, F)}{n(r, f)\Phi(r)} = 0. \tag{7}$$

We have $g = \rho - 1 = \max(p, q)$. It is easily seen that $p = \rho - 1$, $q \leqq \rho - 1$ and

$$\log M(r, F) < A\left\{ r^{\rho-1} + \log r \right\} + \log M(r, f). \tag{8}$$

If

$$\limsup_{r=\infty} \frac{n(r, f)\Phi(r)}{r^{p+1}} > 0$$

then R_n being defined in (5.1),

$$\frac{\log M(R_n, F)}{n(R_n, F)\Phi(R_n)} < \frac{A\{R_n^{\rho-1} + \log R_n\}}{n(R_n, F)\Phi(R_n)} + \frac{\log M(R_n, f)}{n(R_n, f)\Phi(R_n)}.$$

Hence

$$\liminf_{r=\infty} \frac{\log M(r, F)}{n(r, F)\Phi(r)} = 0.$$

If now

$$\lim_{r=\infty} \frac{n(r, f)\Phi(r)}{r^{p+1}} = 0$$

then for all large r

$$\log M(r, F) < A\{r^{\rho-1} + \log r\} + AI(r, f) < A_3 I(r, f)$$

and hence from (6) the required result follows.

The condition (4) on $\Phi(x)$ is sufficient but not necessary[3] for (3) and (7) to hold. The condition (4.1) is also not necessary for we can take $\Phi(x)$ to be any function

$$(l_1 x)(l_2 x) \cdots (l_{k-1} x)(l_k x)^{1+\eta}, \qquad \eta > 0,$$

of the logarithmic comparison scale, and hence any function for which

$$\liminf_{x=\infty} \frac{\Phi(x)}{(l_1 x)(l_2 x) \cdots (l_{k-1} x)(l_k x)^{1+\eta}} \geqq A > 0.$$

We can take $\Phi(x)$ to be any positive L function[4] which satisfies (4) but we cannot take $\Phi(x)$ (or $\phi(x)$ in Theorem 1) to be $(l_1 x)(l_2 x) \cdots (l_k x)$.

Consider for instance

$$f_1(z) = \prod_N^\infty \left\{1 - \frac{z}{a_n}\right\}, \qquad f_2(z) = \prod_N^\infty \left\{\left(1 - \frac{z}{a_n'}\right) \exp\left(\frac{z}{a_n'}\right)\right\},$$

where

$$a_n = -\, n(l_1 n) \cdots (l_k n)(l_{k+1} n)^2,$$
$$a_n' = n(l_1 n) \cdots (l_k n)(l_{k+1} n).$$

The functions $f_1(z)$ and $f_2(z)$ are canonical products of order 1. The genus of $f_1(z)$ is 0, and of $f_2(z)$ is 1. For each of them we have

$$\lim_{r=\infty} \frac{\log M(r)}{n(r)(l_1 r) \cdots (l_k r)} = \infty.$$

[3] Cf. p. 4 of (1).

[4] For definition see G. H. Hardy, *Orders of Infinity*, 1924, p. 17.

In what follows we shall take $\phi(x)$ to be a positive L function satisfying the condition (2).

Suppose now $F(z)$ is of integral order ρ. There are four possibilities:

(1) $\rho_1 < \rho,\ p \leqq \rho_1,\ q = \rho,$ (2) $\rho_1 = p = \rho,\ q \leqq \rho,$

(3) $\rho_1 = q = \rho,\ p = \rho - 1,$ (4) $\rho_1 = \rho,\ q < \rho,\ p = \rho - 1.$

Combining the above results we have in cases (2) and (4)

$$\liminf_{r=\infty} \frac{\log M(r, F)}{n(r, F)\phi(r)} = 0. \tag{9}$$

In cases (1) and (3), (9) does not hold.[5] For functions of fractional order and zero order[6] it certainly holds. In particular (9) *is true for any canonical product of finite order; it also holds for functions of maximum or minimum type, order* ρ.

It is known that if $F(z)$ is of integral order ρ, then[7]

$$\liminf_{r=\infty} \frac{\log M(r, F)}{n(r, F - a)} < \infty \tag{10}$$

for every a, except possibly a single exceptional value of a. Since $F(z)$ and $F(z) - a$ belong to the same type, we deduce from (9) that *if $F(z)$ is of maximum or minimum type, order ρ, where ρ is an integer, then*

$$\liminf_{r=\infty} \frac{\log M(r, F)}{n(r, F - a)\phi(r)} = 0 \tag{11}$$

for every a. If $F(z)$ is of mean type then (11) need not hold for one exceptional value of a. For example, ze^z and

$$e^z \prod_2^\infty \left\{1 + \frac{z}{n(\log n)^2}\right\}$$

are both functions of mean type, order 1. For each of these two functions

$$\lim_{r=\infty} \frac{\log M(r, F)}{n(r, F - 0)(\log r)^{3/2}} = \infty.$$

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[5] (1), p. 29.

[6] (1), pp. 29–30.

[7] G. Valiron, *Lectures on the General Theory of Integral Functions*, 1923, p. 86.