A NOTE ON THE ELEMENTARY DIVISOR THEORY IN NON-COMMUTATIVE DOMAINS*

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Two main features of the classical elementary divisor theory are the reduction of a matrix with integral coefficients to a normal form, which is a diagonal form with certain properties, and the uniqueness of such a normal form. The former of these two was extended to non-commutative domains by B. L. van der Waerden† and J. H. M. Wedderburn,‡ and a further contribution in this line was made by N. Jacobson.§ Moreover, O. Teichmüller|| showed recently that the so-called euclidean division process is unnecessary for the purpose and the weaker assumption that the domain is a principal ideal domain is sufficient. As for the second problem, namely the uniqueness problem, as it seems to me, little has been done in the non-commutative case except to show that the directly indecomposable components of the diagonal elements as a whole are, in virtue of the Krull-Remak-Schmidt theorem, unique up to similarity. In the present short note¶ we shall, generalizing a result in a joint note of K. Asano and the author,** see that the diagonal elements of a Jacobson-Teichmüller normal form themselves are determined uniquely up to similarity, although this uniqueness theorem is not so satisfactory and is essentially not so far from the uniqueness of the indecomposable components.

Let I be a (not necessarily commutative) domain of integrity†† in

* Presented to the Society, February 26, 1938. The writer is grateful to the referee whose comments enabled him to make some necessary changes.

† B. L. van der Waerden, *Moderne Algebra*, vol. 2, 1st edition, 1932.

‡ J. H. M. Wedderburn, *Non-commutative domains of integrity*, Journal für die reine und angewandte Mathematik, vol. 167 (1932).

§ N. Jacobson, *Pseudolinear transformations*, Annals of Mathematics, vol. 38 (1937).

|| O. Teichmüller, *Der Elementarteilersatz für nichtkommutative Ringe*, Sitzungsberichte der Preussischen Akademie der Wissenschaften, 1937.

¶ The abstract of the note has appeared in this Bulletin, abstract 44-3-118, but the present note consists of the content of the second half of the original one; the first half was omitted since the author found later that that part had been proved in Teichmüller, loc. cit. The note came out originally from an oral discussion with K. Asano and M. Takahasi.

** K. Asano and T. Nakayama, *Über halblineare Transformationen*, Mathematische Annalen, vol. 115 (1937).

†† That is, an associative ring with unit element 1 and without divisors of zero.

which all left and right ideals are principal; the left and right principal ideals generated by an element a we denote by $(a)_l$ and $(a)_r$, respectively. We have then the maximum condition for both left and right ideals, and we have also the so-called weakened minimum condition for both left and right ideals, that is, every descending chain of left or right ideals containing a fixed non-zero element is finite. Every element not a unit or the zero can be expressed as a product of a finite number of irreducible elements, and the number of the factors in such an expression is, in virtue of the Jordan-Hölder theorem, uniquely determined by the element.* We call this number the length of the element, as well as that of the left and right ideals generated by the element. The length of a unit we define to be 0. Furthermore, if a and b are both different from 0, then $(a)_r \cap (b)_r \neq 0$ and $(a)_l \cap (b)_l \neq 0$ (existence of the least common multiples); for the proof see Teichmüller, loc. cit. (Refer also to Asano and Nakayama, loc. cit., p. 89.)

The following elementary divisor theorem for I was proved in Teichmüller, loc. cit.: *Let M be a (m, n)-matrix with elements from I. Then there exist invertible square matrices U and V (of order m and n, respectively) such that $M_0 = UMV$ has the form*

$$M_0 = \begin{pmatrix} a_1 & & & & 0 \\ & a_2 & & & \\ & & \ddots & & \\ & & & a_t & \\ 0 & & & & 0 \end{pmatrix}, \qquad a_i \neq 0,$$

where $Ia_{i+1}I \subseteqq (a_i)_l \cap (a_i)_r$ $(=Ia_i \cap a_iI)$ for all $i=1, 2, \cdots, t-1$. (This M_0 we shall call a *Jacobson-Teichmüller normal form* for M.)

Let us call, after Jacobson, an element a *bounded* if there is a two-sided ideal divisible† by a and different from 0. Furthermore, we call the (set-theoretically) greatest two-sided ideal divisible by a the *bound* of a in case a is bounded. Then the above condition for a_i's, which Teichmüller expressed by saying that a_{i+1} is totally divisible by a_i, can be expressed also in the following manner: All a_i except a_t are bounded, and the bound of a_i divides a_{i+1}, $(i=1, 2, \cdots, t-1)$.

Two right (left) ideals $\mathfrak{r}_1$ and $\mathfrak{r}_2$ ($\mathfrak{l}_1$ and $\mathfrak{l}_2$) are called similar if the I-right (left)-moduli $I/\mathfrak{r}_1$ and $I/\mathfrak{r}_2$ ($I/\mathfrak{l}_1$ and $I/\mathfrak{l}_2$) are isomorphic. Two

* Not only the number of the factors but the factors themselves are determined uniquely up to similarity.

† We say that a two-sided ideal $\mathfrak{t}$ is divisible by a if $\mathfrak{t} \subseteqq (a)_l \cap (a)_r$. This will be the case as soon as either $\mathfrak{t} \subseteqq (a)_l$ or $\mathfrak{t} \subseteqq (a)_r$, as one can see easily.

elements a and b are said to be r-similar* (l-similar) if $(a)_r$ and $(b)_r$ ($(a)_l$ and $(b)_l$) are similar. But then r-similarity and l-similarity are coexistent,† and we can therefore simply speak of similarity of two elements.

THEOREM. *The number t of the non-zero diagonal elements $a_1, a_2, \cdots, a_t$ in a Jacobson-Teichmüller normal form is uniquely determined by the matrix M, and furthermore the elements $a_1, a_2, \cdots, a_t$ are determined by M uniquely up to similarity.*

Before we go to the proof of the theorem, let us notice that in so far as we deal with bounded elements only, the situation is quite similar to the usual formal arithmetic in a maximal order of a rational algebra. For example, two-sided ideals (not zero) form a commutative multiplicative system; if $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are two two-sided ideals (not zero), then $\mathfrak{t}_1 \subseteq \mathfrak{t}_2$ if and only if there is a (uniquely determined) third two-sided ideal $\mathfrak{t}_3$ such that $\mathfrak{t}_1 = \mathfrak{t}_2\mathfrak{t}_3$ $(=\mathfrak{t}_3\mathfrak{t}_2)$; every two-sided ideal (not zero) may be expressed in a unique way as a product, as well as a direct cross cut, of powers of different (two-sided) prime ideals; a prime ideal is a direct cross cut of mutually similar irreducible right (left) ideals, and the number of such components, which is obviously independent of the special choice of the decomposition, is called the *capacity* of the prime ideal; if $\mathfrak{p}$ is a prime ideal with the capacity κ, then $\mathfrak{p}^\alpha$ is a direct cross cut of κ mutually similar (directly) indecomposable right (left) ideals of the length α; if $\mathfrak{p}^\alpha = \mathfrak{r}_1 \cap \mathfrak{r}_2 \cap \cdots \cap \mathfrak{r}_\kappa$ is a decomposition of $\mathfrak{p}^\alpha$ into indecomposable right ideals in the above sense, then $I, \mathfrak{r}_i \cup \mathfrak{p}, \mathfrak{r}_i \cup \mathfrak{p}^2, \cdots, \mathfrak{r}_i \cup \mathfrak{p}^{\alpha-1}, \mathfrak{r}_i$ exhaust all the right ideal divisors of $\mathfrak{r}_i$; every indecomposable right ideal generated by a bounded element divides a suitable power of a suitable prime ideal, say $\mathfrak{p}$, and if it has the length α, then it divides $\mathfrak{p}^\alpha$ but not $\mathfrak{p}^{\alpha-1}$, and moreover it is similar to the above $\mathfrak{r}_i$; every right ideal dividing a suitable power of a prime ideal of capacity κ is a direct cross cut of κ indecomposable right ideals ($I=(1)_r$ being allowed as components); every right ideal generated by a bounded element can be expressed uniquely as a direct cross cut of right ideals which divide the powers of different prime ideals. (The individual component in such decomposition is called the component of the given right ideal with respect to the corresponding prime ideal.)

* In Ore-Jacobson's sense r-similarity corresponds to left-similarity. The element a is r-similar to b if and only if there is an element c with $(ca)_r = (b)_r \cap (c)_r$, $(b)_r \cup (c)_r = I$.

† For this fact, although in our case it can be seen easily, see H. Fitting, *Über den Zusammenhang zwischen dem Begriff der Gleichartigkeit zweier Ideale und dem Äquivalenzbegriff der Elementarteilertheorie*, Mathematische Annalen, vol. 112 (1936).

All these statements can be verified as in the arithmetic in algebras,* and almost all of them are proved essentially in Jacobson, loc. cit. (Jacobson's case is a special case of ours, but the essential points are almost the same.) Simply expressed, the residue class ring of I with respect to a two-sided ideal (not zero) is one-rowed (einreihig) in the sense of G. Koethe.†

PROOF OF THE THEOREM. Let us consider as usual the module $\mathfrak{N}$ of all linear forms $x_1c_1+x_2c_2+ \cdots +x_mc_m$ of m independent indeterminates x_i with coefficients from I, and define $\mathfrak{M}$ as the submodule of $\mathfrak{N}$ generated by n elements y_i with $(y_1, \cdots, y_n)=(x_1, \cdots, x_m)M$. Then t is the maximal number of linearly independent elements in $\mathfrak{M}$. Let $\mathfrak{N}_0$ be the submodule of $\mathfrak{N}$ consisting of all elements u such that $uc \in \mathfrak{M}$ for a suitable non-zero element c in I. Then $\mathfrak{N}_0/\mathfrak{M}$ is a direct sum of t cyclic submoduli with generators whose annihilators are $(a_1)_r$, $(a_2)_r, \cdots, (a_t)_r$, respectively. Now, let $\mathfrak{a}$ be the intersection of all principal right ideals generated by bounded left factors of a_t. Let us first prove that $(a_1)_r, \cdots, (a_{t-1})_r$, $\mathfrak{a}$ are determined by M uniquely up to similarity.‡ For that purpose, take a non-zero two-sided ideal $\mathfrak{t}$ divisible by $\mathfrak{a}$, and hence also by $(a_1)_r, \cdots, (a_{t-1})_r$. It is obvious that $(a_t)_r \cup \mathfrak{t}=\mathfrak{a}$. Consider then $\mathfrak{N}_0\mathfrak{t}$ and $\mathfrak{N}_0\mathfrak{t} \cup \mathfrak{M}$. The module $\mathfrak{N}_0/\mathfrak{N}_0\mathfrak{t} \cup \mathfrak{M}$ is a direct sum of cyclic submoduli with generators whose annihilators are $(a_1)_r, \cdots, (a_{t-1})_r$, $\mathfrak{a}$, respectively. Take any prime ideal $\mathfrak{p}$, and let κ be its capacity. Let us consider the $\mathfrak{p}$-components of $(a_1)_r, \cdots, (a_{t-1})_r$, $\mathfrak{a}$. Let the $\mathfrak{p}$-component of $(a_1)_r$ be a direct cross cut of κ indecomposable right ideals of the lengths (greater than or equal to zero) $l_{11}\leqq l_{12}\leqq \cdots \leqq l_{1\kappa}$, and let the $\mathfrak{p}$-component of $(a_2)_r$ be a direct cross cut of κ indecomposable right ideals of the lengths $l_{21}\leqq l_{22}\leqq \cdots \leqq l_{2\kappa}$, and so on. From our condition on $a_1, a_2, \cdots, a_t$ it is easy to see that then $l_{11}\leqq \cdots \leqq l_{1\kappa}\leqq l_{21}\leqq \cdots \leqq l_{2\kappa}\leqq l_{31}\leqq \cdots$. Now, if we have a second Jacobson-Teichmüller normal form of M with the diagonal elements $a_1', a_2', \cdots, a_t'$, then we can take $\mathfrak{t}$ so that $\mathfrak{t}$ is divisible also by all bounded left factors of a_t'. Then the Krull-Remak-Schmidt theorem concerning direct decompositions of a group (with chain conditions) applied to $\mathfrak{N}_0/\mathfrak{N}_0\mathfrak{t} \cup \mathfrak{M}$ shows that $l_{11}, \cdots, l_{1\kappa}, l_{21}, \cdots, l_{t\kappa}$ are, including their order, independent of the special choice of the normal form. Since $\mathfrak{p}$ is any prime ideal,

* See, for example, M. Deuring, *Algebren*, Ergebnisse der Mathematik, vol. 4, 1935; cf. particularly chap. 6, §3.

† G. Koethe, *Verallgemeinerte Abelsche Gruppen mit hyperkomplexem Operatorenring*, Mathematische Zeitschrift, vol. 39 (1934).

‡ For the following cf. Asano and Nakayama, loc. cit., Theorem 9.

$(a_1)_r, \cdots, (a_{t-1})_r$ (and $\mathfrak{a}$) are determined uniquely up to similarity. Then the Krull-Remak-Schmidt theorem, applied now to $\mathfrak{N}_0/\mathfrak{M}$, shows that $(a_t)_r$ is also determined uniquely up to similarity.

REMARK. The above uniqueness theorem is unsatisfactory, since two diagonal forms (of the same type) with diagonal elements $a_1, a_2, \cdots$ and $a_1', a_2', \cdots$ are in general not equivalent (associate) even if $a_1, a_2, \cdots$ and $a_1', a_2', \cdots$ are similar in pairs. But if, moreover, a_1 (therefore also a_1') is a unit, then they are equivalent.*

THE INSTITUTE FOR ADVANCED STUDY

ON A MIXED BOUNDARY CONDITION FOR HARMONIC FUNCTIONS

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In two recent notes in this Bulletin† (referred to below as I, II) I considered the boundary conditions

$$\frac{\partial u}{\partial n} + au = 0, \qquad a = \text{const.}, \tag{1}$$

for harmonic functions, investigating in particular the "reflection" of singularities across a plane at which (1) obtains and indicating several applications of the results.

Dr. A. Weinstein has kindly called my attention to an application of (1) that I have overlooked, namely, to the problem of gravity surface waves of liquids. Under the assumption of small irrotational motion, the velocity potential ϕ satisfies along the free boundary the condition‡

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial n} = 0. \tag{2}$$

For simple harmonic motions with time entering as $e^{i\sigma t}$, this reduces to (1) with

$$a = -\sigma^2/g.\S \tag{3}$$

Again, equation (1) may be applied, for two-dimensional motions, to the flow function ψ which is the conjugate harmonic to ϕ by assuming

* Fitting, loc. cit.

† I, this Bulletin, vol. 43 (1936), p. 873; II, this Bulletin, vol. 44 (1938), p. 443.

‡ Lamb, *Hydrodynamics*, Cambridge, 1924, p. 342.

§ Lamb, loc. cit., p. 342.