

## ON SYMMETRIC BERNOULLI CONVOLUTIONS

BY AUREL WINTNER

In a previous note\* a class of sequences  $\{a_n\}$  has been determined such that

$$(1) \quad \prod_{n=1}^{\infty} \cos(a_n t) = O(\exp(-At^\gamma)) \text{ as } t \rightarrow +\infty,$$

$$a_n > 0, \quad \sum_{n=1}^{\infty} a_n^2 < +\infty,$$

where  $A > 0$  and  $\gamma > 0$ . The role of this estimate was that of assuring a high degree of smoothness for the distribution function having the infinite product as Fourier-Stieltjes transform. In fact, this distribution function is an entire function if  $\gamma > 1$ , it is regular analytic if  $\gamma = 1$  and it has derivatives of arbitrarily high order whenever there exists a  $\gamma (> 0)$ . For the sequences  $\{a_n\}$  in question and also for some more general sequences a simple proof of (1) will be given by means of an appraisal indicated in the reference cited. This appraisal has the advantage of being valid for every  $\{a_n\}$  and it gives results also if (1) is not satisfied. It implies for instance the fact that the infinite product  $\rightarrow 0$  as  $t \rightarrow +\infty$  in cases of the type  $a_n = c^{-n^\delta}$ , where  $c > 1$  and  $0 < \delta < 1$ .

The  $n$ th factor of (1) does not come near to  $\pm 1$  if  $1 \leq a_n t \leq 2$ , since this limitation implies  $|\cos(a_n t)| < 2/3$ . Thus

$$\left| \prod_{n=1}^{\infty} \cos(a_n t) \right| < (2/3)^{K(t)},$$

if  $1 \leq a_n t \leq 2$  is satisfied for  $K = K(t)$  values of  $n$ . Now

$$K(t) \geq N(2t) - N(t),$$

if  $a_n \geq 1/t$  holds for exactly  $N(t)$  values of  $n$ .

Thus if  $a_n = n^{-\alpha}$ , then  $N(t) = [t^{1/\alpha}]$ , hence  $K(t) > Ct^{1/\alpha}$  for

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\* A. Wintner, *On analytic convolutions of Bernoulli distributions*, American Journal of Mathematics, vol. 56 (1934), pp. 659-663.

some  $C > 0$  so that (1) is satisfied by  $\gamma = 1/\alpha$ . Similarly, if  $N(t) \sim t^\beta S(t)^{\pm 1}$ , where  $\beta > 0$  and  $S(t)$  is, as in the reference cited, a "slow" function, then (1) holds for every  $\gamma < \beta$ . If  $1/a_n$  is the  $n$ th prime number, then (1) holds for every  $\gamma < 1$  since  $K(t) > Ct^\gamma$  in virtue of the elementary inequalities of Tchebycheff. Thus it is not necessary to use the prime number theorem  $\beta = 1$ ,  $S(t)^{\pm 1} = 1/\log t$ , applied in the reference cited. Correspondingly, the present method enables us to prove (1) also for sequences for which  $N(t)$  is not  $\sim t^\beta S(t)^{\pm 1}$ .

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## PROOF OF THREE PROPOSITIONS OF PALEY

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1. *Introduction.* In a letter to Fejér,\* Paley stated three interesting propositions concerning Fourier series of bounded and continuous functions whose Fourier coefficients satisfy the conditions  $na_n, nb_n \geq -K$ ,  $K \geq 0$ . The letter of Paley contains only a very brief sketch of the proof. After knowing Paley's results the author succeeded in developing complete proofs with various improvements of the estimates, and even in extending them to a wider class of Fourier series of almost periodic functions. These extensions will be treated elsewhere. In the present note we prove the following three theorems.

It will be assumed throughout that  $f(x)$  is real-valued and periodic, of period  $2\pi$ , and Lebesgue integrable over  $(-\pi, \pi)$ . Let

$$(1) \quad f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be its Fourier series expansion and let

$$(2) \quad s_n = s_n(x) = a_0 + \sum_{\nu=1}^n (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

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\* This letter is reproduced in a note by Fejér, *On a theorem of Paley*, this Bulletin, vol. 40 (1934), pp. 469-475, especially pp. 474-475. It was communicated to the author by Professor Fejér in September, 1933.