A NEW UNIVERSAL WARING THEOREM FOR EIGHTH POWERS

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1. *Introduction.* Hardy and Littlewood* in their proof of Waring's theorem obtained a constant $C=C(s, k)$ beyond which every number is a sum of s integral kth powers $\geqq 0$. Recently Dickson perfected an algebraic method by which he was able to show that every positive integer $\leqq C$ is a sum of s integral kth powers $\geqq 0$. Thus we are now able to obtain universal Waring theorems for relatively small values of s.

We shall consider in this paper the problem of meeting the Hardy and Littlewood constant by Dickson's method and establishing a new universal Waring theorem for eighth powers. The earlier result for eighth powers was 575, obtained by Dickson.†

2. *Proof of the Principal Theorem.* We write

$$(1)\qquad a = 2^8, \qquad b = 3^8, \qquad c = 4^8, \qquad d = 5^8, \qquad e = 6^8.$$

The right side of

$$m = n + Aa + Bb + \cdots + Qq, \qquad (n, A, B, \cdots, Q \text{ integral}),$$

is a *resolution* of m of *weight* $w(m)=n+A+B+\cdots+Q$. When $n, A, B, \cdots, Q \geqq 0$ the resolution is a *decomposition*.

By division we obtain

$$(2)\quad b = 161 + 25a, \; c = -74 + 10b, \; d = 56 + 15a + 9b + 5c,$$

$$(3)\qquad e = 21 + 22a + 7b + c + 4d.$$

Consider an integer M, such that $2d+e \leqq M \leqq 3d+e$. We can express the integer $P=M-2d-e$ uniquely in the form $R+N$, where

$$(4)\qquad 0 \leqq R < a = 256, \qquad N = Aa + Bb + Cc,$$

$$(5)\quad C = [P/c],\ddagger \; B = [(P - Cc)/b], \; A = [(P - Bb - Cc)/a].$$

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* A simplified proof can be found in Landau, *Vorlesungen über Zahlentheorie*, vol. 1, 1927, pp. 235–360.

† This Bulletin, vol. 39 (1933), p. 713.

‡ $[x]$ denotes the largest integer $\leqq x$.

Since $N<d$, we obtain by (4) and (5) the inequalities $Cc<d$, $Bb<c$, and $Aa<b$. Hence

$$(6) \qquad 0 \leqq A < 26, \qquad 0 \leqq B < 10, \qquad 0 \leqq C < 6.$$

Since

$$(7) \qquad M = R + Aa + Bb + Cc + 2d + e,$$

then

$$(8) \qquad \begin{aligned} w(M) &= R + A + B + C + 3 \\ &\leqq 255 + 25 + 9 + 5 + 3 = 297. \end{aligned}$$

Since (7) defines a decomposition of M, we can state the following lemma.

LEMMA 1. *Every integer M, such that $2d+e \leqq M \leqq 3d+e$, is a sum of 297 eighth powers.*

Let us now consider the problem of obtaining a smaller value for $w(M)$. Table I contains a list of certain equations of the form

$$(9) \qquad r = A'a + B'b + C'c + D'd + E'e.$$

Such an equation defines a resolution of r of weight w. We shall refer to an equation of Table I by citing its r value whenever we may do so without ambiguity. For example, equation 31, $31 = -10a+6c-d$, which defines a resolution of 31 of weight -5, is the first equation listed in Table I. We can readily verify these equations by (1).

We write (7) in the form $M_1 = A_1a+B_1b+C_1c+2d+e+r+r'$, A_1, B_1, C_1 fixed and $R=r+r'$, where $0 \leqq r' < r_f - r$ (the subscript f is used to denote that r_f is the equation immediately following r in Table I and possessing the property $r_f > r$). Eliminating r between this equation and (9), we obtain

$$(10) \qquad \begin{aligned} M_1 = {} & (A_1 + A')a + (B_1 + B')b + (C_1 + C')c \\ & + (2 + D')d + (1 + E')e + r'. \end{aligned}$$

We construct a tablette $A=A_1$, $B=B_1$, $C=C_1$ by listing the r and w values of resolutions of Table I whose coefficients satisfy the inequalities $A_1+A' \geqq 0$, $B_1+B' \geqq 0$, $C_1+C' \geqq 0$ (it should be noted that for such resolutions, (10) gives decompositions of M_1 since all the resolutions in Table I satisfy the inequalities

TABLE I*
LIST OF EQUATIONS

r	w	a	b	c	d	e	r	w	a	b	c	d	e
31	−5	−10	0	6	−1	0	140	9	−3	8	1	4	−1
41	23	2	7	13	2	−1	148	18	0	20	−2	0	0
62	−10	−20	0	12	−2	0	157	15	6	−1	12	−2	0
66	0	−3	−2	2	4	−1	158	46	12	27	5	3	−1
70	48	31	8	11	−2	0	169	34	26	9	−1	0	0
72	18	−8	7	19	1	−1	171	4	−13	8	7	3	−1
81	58	1	47	9	2	−1	189	41	2	27	11	2	−1
84	37	12	17	6	3	−1	192	20	13	−3	8	3	−1
95	25	26	−1	0	0	0	200	29	16	9	5	−1	0
97	−5	−13	−2	8	3	−1	210	8	−20	20	10	−2	0
105	4	−10	10	5	−1	0	214	18	−3	18	0	4	−1
105	53	38	6	7	3	−1	223	15	3	−3	14	2	−1
115	32	2	17	12	2	−1	231	24	6	9	11	−2	0
126	20	16	−1	6	−1	0	245	13	−13	18	6	3	−1

$2+D'\geqq 0$, $1+E'\geqq 0$). From the r and w columns we form a new column, the W column, where $W=(r_f-1)-r+w$. We obtain our tablette in its final form by deleting all equations r_f for which $w_f>W$. We denote the greatest W value in this tablette by $G(A_1, B_1, C_1)$. For example, upon constructing the tablette $A=0$, $B=0$, $C=0$ we get

$A=0$,	$B=0$,	$C=0$	$A=0$	$B=0$	$C=0$
r	w	W	r	w	W
0	0	40	115	32	74
41	23	51	158	46	76
70	48	58	189	41	51
81	58	60	200	29	59
84	37	57	231	24	48
105	53	62		$G=76$	

By constructing eleven such tablettes, we obtain

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(11)\quad
\begin{aligned}
&G(0,0,0)=76, && G(3,0,0)=62, && G(15,4,0)=41,\\
&G(0,1,0)=60, && G(5,4,0)=47, && G(20,0,0)=39,\\
&G(0,0,1)=74, && G(14,0,0)=50, && G(10,0,0)\ \ 57.\\
&G(0,0,2)=64, && G(20,4,0)=37,
\end{aligned}
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* The method of obtaining these equations is analogous to that explained in Dickson's paper on ninth powers in this Bulletin, vol. 40 (1934), pp. 487–493.

Because of the exclusive condition $w_f > W$, the tablettes have the property (P), $G(A_2, B_2, C_2) \leqq G(A_1, B_1, C_1)$ if $A_1 \leqq A_2$, $B_1 \leqq B_2$, $C_1 \leqq C_2$.

Let us now consider $M_1 = R + A_1 a + B_1 b + C_1 c + 2d + e$. Since $w(R) \leqq G(A_1, B_1, C_1)$,

$$w(M_1) \leqq A_1 + B_1 + C_1 + 3 + G(A_1, B_1, C_1).$$

Dropping the subscripts we seek the maximum value H of the function $A + B + C + 3 + G(A, B, C)$ in the range (6). From (11) and (P) it is evident that H 81. This result may be expressed as follows.

LEMMA 2. *Every integer M, such that* $2{,}460{,}866 \leqq M \leqq 2{,}851{,}491$, *is a sum of* 81 *eighth powers.*

From this interval we ascend to the Hardy and Littlewood constant by employing two theorems of Dickson, Theorems 10 and 12 in this Bulletin.* Using Theorem 10 and a table of eighth powers it was found that 102 eighth powers suffice for the enlarged interval from $q = 2d + e$ to $L_0 = 2{,}235{,}617 \cdot 10^9$. Ap plying Theorem 12, we obtain

$$\log L_t = (8/7)^t (\log L_0 + h) - h, \qquad h = -8 \log 8,$$

$$h = \bar{8}.775280, \ \log(\log L_0 + h) = 0.909806, \ \log(8/7) = 0.057992.$$

We take $t = 464$; then $\log\log L_t = 27.818$. Hence 566 eighth powers suffice from q to L_t.

James showed that $\log_e C = 20 \cdot 8^3 2^\eta$, where $C = C(s)$ is the Hardy and Littlewood constant, s is the number of eighth powers, and

$$\eta = \frac{20.1s - 162}{s - 426}.$$

For $s = 566$ we obtain $\log\log C = 27.762$. Hence every integer $> q$ is a sum of 566 eighth powers. It is evident from (8) that 294 eighth powers suffice from 0 to d. Consequently 300 eighth powers suffice from 0 to $7d$. Since $q < 7d$, we have the following theorem.

THEOREM. *Every positive integer is a sum of* 566 *integral eighth powers.*

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* Loc. cit., vol. 39 (1933), pp. 710–711.