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SOME PROBLEMS IN POTENTIAL THEORY AND THE NOTION OF HARMONIC ENTROPY

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ABSTRACT. Blaschke regions are studied for certain classes of subharmonic functions in connection with the notion of harmonic entropy. A complete description of Riesz measures for some of these classes is obtained. A new analytic inequality is established.

1. Definitions, notations and two basic problems. $k(r)$ $(0 \leq r < 1)$ will always denote a continuous nonnegative function such that $k(|z|)$ is subharmonic in the open unit disc $\mathbf{D}$ (or, equivalently, such that $k(r)$ and $rk'(r)$ are nondecreasing).

DEFINITION 1. Let $\mathcal{M} \subset \mathbf{D}$ be a given set, and let $\mathcal{H}_{\langle k \rangle}(\mathcal{M})$ be the set of all nonnegative harmonic functions $u(z)$ in $\mathbf{D}$ such that $u(z) \geq k(|z|)$ on $\mathcal{M}$. The following quantity will be called the *harmonic k-entropy* of $\mathcal{M}$:

$$\mathcal{E}(\mathcal{M}; k) = \min\{u(0)\colon u \in \mathcal{H}_{\langle k \rangle}(\mathcal{M})\}.^2 \tag{1.1}$$

If $\mathcal{H}_{\langle k \rangle}(\mathcal{M})$ is empty, we set $\mathcal{E}(\mathcal{M}; k) = +\infty$.

DEFINITION 2. $\mathcal{SH}^{\langle k \rangle}$ will denote the class of subharmonic functions $u(z)$ in $\mathbf{D}$ such that

$$u(z) \leq C_u k(|z|) \quad (z \in \mathbf{D}), \tag{1.2}$$

where C_u is some constant (depending on u).

DEFINITION 3. $\mathcal{A}^{\langle k \rangle}$ will denote the class of analytic functions $f(z)$ in $\mathbf{D}$ such that $\log|f(z)| \in \mathcal{SH}^{\langle k \rangle}$.

DEFINITION 4. A region $G \subset \mathbf{D}$ is called a k-Blaschke region if either of two equivalent[3] conditions holds:

(a) for every $u \in \mathcal{SH}^{\langle k \rangle}$

$$b(G; d\mu) = \int_G (1 - |z|)\, d\mu(z) < \infty, \tag{1.3}$$

where $d\mu = \Delta u$ is the Riesz measure (i.e. generalized Laplacian) of u;

(b) for every $f \in \mathcal{A}^{\langle k \rangle}$

$$\sum_{z_\nu \in G} (1 - |z_\nu|) < \infty, \tag{1.3'}$$

where $\{z_\nu\}$ is the zero set of f.

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[2]The use of that term, borrowed from Information Theory, is suggested by this interpretation: if $u(z)$ is conceived as a "signal" of strength $u(0)$ and $k(|z|)$ as the "noise", then $\mathcal{E}(\mathcal{M}; k)$ is the strength of the weakest signal that overcomes the noise on $\mathcal{M}$.

[3]The equivalence of (a) and (b) is easily proved.

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DEFINITION 5. (1) S_ς is the open Stolz angle whose closure is the convex hull of the disk $|z| < 1/\sqrt{2}$ and the point $\varsigma \in \partial\mathbf{D}$.

(2) For a given closed set $F \subset \partial\mathbf{D}$, G_F is the union of $S_\varsigma(\varsigma \in F)$.

(3) $\mathcal{L}$ is the class of regions $G = \{z = re^{i\theta}: 0 \leq r < f(\theta) \leq 1\}$, where $f(\theta)$ is a 2π-periodic function satisfying the Lipschitz condition $|f(\theta_1) - f(\theta_2)| \leq |\theta_1 - \theta_2|$. It is easily seen that every G_F is an $\mathcal{L}$-region.

All results announced[4] below are assoicated with the following two basic problems.

(A) Given $k(r)$, characterize regions of finite k-entropy and find estimates of that quantity.

(B) Given $k(r)$, characterize k-Blaschke regions and find effective estimates of the integral (1.3) and the sum (1.3′).

The main motivation for (B) is to ultimately obtain a complete description of zero sets for $\mathcal{A}^{\langle k\rangle}$–an objective that we are able to realize only for the case of "slowly increasing" $k(r)$. Since the problem of $\mathcal{A}^{\langle k\rangle}$-zero sets is essentially a potential-theoretic one, there seems to be no good reason for studying only the special Riesz measures $d\mu = \Delta \log|f(z)|$ determined by the zeros of an $f \in \mathcal{A}^{\langle k\rangle}$, rather than the general Riesz measures for $S\mathcal{H}^{\langle k\rangle}$. In emphasizing the potential-theoretic, rather than complex-analytic, aspect, we also aim at similar multidimensional problems; in fact, some interesting results [**4**] for the unit ball in $\mathbf{R}^m$ have recently been obtained in this circle of ideas (see also §3 below).

Understanding the structure of $\mathcal{A}^{\langle k\rangle}$-zero sets is also an essential first step towards a satisfactory factorization theory for $\mathcal{A}^{\langle k\rangle}$; see [**1**], where the case $k(r) = |\log(1-r)|$ is treated.

As to (A), this problem is instrumental in solving (B). For slowly increasing $k(r)$ the k-entropy of an $\mathcal{L}$-region G can be estimated in terms of the following integral

$$I(G;k) = \int_0^{2\pi} k[f(\theta)]\,d\theta, \tag{1.4}$$

where $k(1) = k(1^-)$ $(= \infty$, except for the trivial case of a bounded $k(r))$.

In the particular case $k(r) = (1-r)^{-\alpha}$ $(0 < \alpha < 1)$ our problems lead to a new elementary inequality (3.3).

2. Slowly increasing $k(r)$. In this section an additional condition is imposed on $k(r)$ (C is some constant):

$$k(1-x^2) \leq Ck(1-x) \qquad (0 < x < \tfrac{1}{2}). \tag{2.1}$$

THEOREM 1. (i) *A* $G \in \mathcal{L}$ *is a k-Blaschke region if and only if* $I(G;k) < \infty$.

(ii) *There is a constant* $\lambda > 1$ *depending only on* $k(r)$ *with the property that for every* $G \in \mathcal{L}$ *there is a* $G' \in \mathcal{L}$, $G' \supset G$, *such that* $I(G',k) < \lambda I(G;k)$ *and*

$$\lambda^{-1} I(G;k) \leq \mathcal{E}(G',k) < \lambda I(G;k). \tag{2.2}$$

[4]Detailed proofs will be published elsewhere.

THEOREM 2. *The necessary and sufficient condition for a nonnegative Borel measure* $d\mu$ *in* $\mathbf{D}$ *to be the Riesz measure of a function* $u \in S\mathcal{H}^{\langle k \rangle}$ *is*

$$b(G_F; d\mu) \leq CI(G_F; k) \tag{2.3}$$

for all finite sets $F \subset \partial\mathbf{D}$ *(*C *is some constant). In this case* (2.3) *holds also for all* $G \in \mathcal{L}$*, but perhaps with a greater constant* C.

3. Some other results. (1) A Stolz angle is a Blaschke region for $S\mathcal{H}^{\langle k \rangle}$ if and only if

$$\int_0^1 [k(r)(1-r)^{-1}]^{1/2}\, dr < \infty. \tag{3.1}$$

(See [2].) A similar result for the unit ball in $\mathbf{R}^m$ (with $1/m$ substituted for $1/2$ in (3.1)) has recently been obtained by Krzysztof Samotij (written communication).

(2) Consider the region $G = \{z \in \mathbf{D}\colon M(1-|z|^2)|1-z|^{-2} > k(|z|)\}$, where M is large enough to ensure that $G \supset S_1$. Then (3.1) implies $I(G; K) < \infty$ and $\mathcal{E}(G; k) < \infty$.

(3) A recent result by C. N. Linden [3] shows that, under some extra conditions on the regularity of growth of k, (3.1) implies that the above region G is a Blaschke region for $S\mathcal{H}^{\langle k \rangle}$. Similar results describing some "tangential" Blaschke regions for the ball in $\mathbf{R}^m$ are given in [4].

(4) In attempting to extend the results of §2 to wider classes of subharmonic functions, it is natural to consider the particular case $k(r) = (1-r)^{-\alpha}$, where α is fixed, $0 < \alpha < 1$. In this case the assertion (i) of Theorem 1 still holds, provided the function $r = f(\theta)$, which describes the boundary of G, has a finite number of maxima and minima. The proof of this depends on

THEOREM 3. *There is a constant* C_α *such that for arbitrary real* $x_0 < x_1 < \cdots < x_n$ *satisfying*

$$x_1 - x_0 \leq x_2 - x_1 \leq \cdots \leq x_n - x_{n-1}, \tag{3.2}$$

and for arbitrary nonnegative $\{m_i\}_0^n$*, the following inequality holds:*

$$\begin{aligned} &\int_{x_0}^{x_n} \left\{ \sum_0^n m_i (x - x_i)^{-2} \right\}^{\alpha/(\alpha+1)} dx \\ &\leq C_\alpha \left(\sum_0^n m_i\right)^{\alpha/(\alpha+1)} \left\{ \sum_1^n (x_i - x_{i-1})^{1-\alpha} \right\}^{1/(\alpha+1)}. \end{aligned} \tag{3.3}$$

Because of the restriction (3.2), which cannot be dropped altogether, our results for this case fall short of a complete description of Riesz measures.

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