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AMALGAMS OF L^p AND l^q

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1. Introduction. The *amalgam* of L^p and l^q on the real line is the space (L^p, l^q) consisting of functions which are locally in L^p and have l^q behavior at infinity in the sense that the L^p-norms over the intervals $[n, n+1]$ form an l^q-sequence. For $1 \leqslant p, q < \infty$, the norm

$$(1.1) \qquad \|f\|_{p,q} = \left\{ \sum_{n=-\infty}^{\infty} \left[\int_n^{n+1} |f(x)|^p \, dx \right]^{q/p} \right\}^{1/q}$$

makes (L^p, l^q) into a Banach space.

The idea of considering the amalgam (L^p, l^q), as opposed to the Lebesgue space $L^p = (L^p, l^p)$, is a natural one because it allows us to separate the global behavior from the local behavior of a function. This idea goes back to 1926 and Norbert Wiener who considered the special cases (L^1, l^2) and (L^2, l^∞) in **[W1]** and (L^∞, l^1) and (L^1, l^∞) in **[W2]**. Other special cases have appeared sporadically since then, but the first systematic study of these spaces was undertaken in 1975 by F. Holland **[H1]**.

After giving an account of the basic theory of amalgams on groups in §2, we show in the following sections how amalgams have arisen in various areas of analysis: almost periodic functions **[W1]**, Tauberian theorems **[W2]**, extending the domain of the Fourier transform **[Sz1]**, Fourier multipliers **[EHR]**, integral operators **[BiS]**, product-convolution operators **[BuS]**, positive definite functions **[Coop]**, Fourier transforms of unbounded measures **[H2]**, lacunarity **[Fou2]**, the lower majorant property for $H^p(R^n)$ **[BaS]**, approximation theory **[JR]**, algebras and modules **[LVW]**, and the range of the Fourier transform **[Ke]**. The common theme is that, in many situations, an amalgam space (L^p, l^q) turns out to be exactly the right space that is needed to solve a problem or formulate a theory.

In view of these occurrences, "the amalgam spaces ... appear to be an idea whose time has come" **[GdL]**. We hope that the present article will help to make these spaces more widely known in the mathematical community.

2. Amalgams on groups. If G is a locally compact abelian group, we use the structure theorem to write $G = R^a \times G_1$, where a is a nonnegative integer and G_1 is a group with a compact open subgroup H. We let $I = [0,1)^a \times H$ and

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$I_\alpha = \alpha + I$, where $\alpha = (n_1, \ldots, n_a, t)$, the n_i's are integers, and the t's form a transversal of G_1 in H, i.e., $G_1 = \bigcup_t (t + H)$ is a coset decomposition of G_1. (Thus, each I_α is a Cartesian product of a unit cube in R^a, sitting at a lattice point, and a coset of H.) Then, in terms of the disjoint union $G = \bigcup_{\alpha \in J} I_\alpha$, we define

$$(2.1)\qquad \begin{aligned} \|f\|_{p,q} &= \left[\sum_{\alpha \in J} \|f\|^q_{L^p(I_\alpha)}\right]^{1/q} \qquad (0 < p \leqslant \infty, 0 < q < \infty), \\ \|f\|_{p,\infty} &= \sup_{\alpha \in J} \|f\|_{L^p(I_\alpha)} \qquad (0 < p \leqslant \infty). \end{aligned}$$

The *amalgam* of L^p and l^q on G is then

$$(L^p, l^q)(G) = \{f \colon \|f\|_{p,q} < \infty\}.$$

Notice that when $G = R$, we have $I_\alpha = [\alpha, \alpha + 1)$ and (2.1) becomes (1.1). (The reader should keep this example in mind throughout the paper.) Notice also that if G is compact, then $(L^p, l^q)(G) = L^p(G)$. If G is discrete, then $(L^p, l^q)(G) = l^q(G)$.

This definition of $(L^p, l^q)(G)$ is the one given in Stewart **[Stw2]** and is useful in making explicit computations. Bertrandias, Datry, and Dupuis **[BDD]** gave an equivalent definition which has the advantage of not using the structure theorem. Given an open precompact neighborhood E of 0, and the family $\mathscr{P}$ of all tilings $\{E_i\}$ of G by disjoint translates of E, they used the norm

$$\sup_{\{E_i\} \in \mathscr{P}} \left[\sum_i \|f\|^q_{L^p(E_i)}\right]^{1/q}$$

and the notation $l^q(L^p)$ instead of (L^p, l^q). (Different choices of E give equivalent norms.) They also made use of the equivalent translation-invariant norm given by

$$(2.2)\qquad \left[\int_G \|f\|^q_{L^p(x+E)}\, dx\right]^{1/q}.$$

Busby and Smith **[BuS]** defined amalgams on locally compact groups which are not necessarily abelian. Their definition is similar to (2.1) and is given in terms of a "uniform partition" of the group. See **[GdL]** for a comparison of these three definitions of amalgams.

We begin our study of these spaces by listing the following inclusion relations.

$$(2.3)\qquad \text{If } q_1 \leqslant q_2, \quad \text{then} \quad (L^p, l^{q_1}) \subset (L^p, l^{q_2}).$$

$$(2.4)\qquad \text{If } p_1 \leqslant p_2, \quad \text{then} \quad (L^{p_2}, l^q) \subset (L^{p_1}, l^q).$$

If we combine these relations with the fact that $(L^p, l^p)(G) = L^p(G)$, we obtain the following:

$$\text{If } q \leqslant p, \quad \text{then} \quad (L^p, l^q) \subset L^p \cap L^q.$$

$$\text{If } p \leqslant q, \quad \text{then} \quad (L^p \cup L^q) \subset (L^p, l^q).$$

There are several facts about amalgams that are not surprising in view of what we know about L^p-spaces. For instance, it is not hard to show that $(L^p, l^q)(G)$ is a Banach space if $p, q \geqslant 1$, and is an F-space if either p or q is less than 1. Also, Holder's inequality extends as expected: If $f \in (L^p, l^q)$ and $g \in (L^{p'}, l^{q'})$, where $p, q \geqslant 1$ and $1/p' = 1 - 1/p$, then $fg \in L^1$ and

$$\|fg\|_1 \leqslant \|f\|_{p,q}\|g\|_{p',q'}. \tag{2.5}$$

Likewise, it is not hard to guess what the dual space of (L^p, l^q) is.

THEOREM 2.6. *If* $1 \leqslant p, q < \infty$, *then* $(L^p, l^q)^* = (L^{p'}, l^{q'})$.

Theorem 2.6 can be found in [**H1, BDD**, and **BuS**], but it also follows from a general fact that was known earlier. If $\{E_n\}$ is a sequence of Banach spaces, $l^q(E_n)$ denotes the space of sequences $x = (x_n)$, where $x_n \in E_n$ and $\|x\| = [\Sigma \|x_n\|^q]^{1/q} < \infty$. (See, for instance, Day [**Da**].) It is known that $l^q(E_n)^* = l^{q'}(E_n^*)$ when $1 \leqslant q < \infty$ [**K**, p. 359]. If we take $E_n = L^p([n, n+1))$, or more generally $E_\alpha = L^p(I_\alpha)$, then we have $l^q(E_n) = (L^p, l^q)$ and Theorem 2.6 follows.

Again, convolution behaves as we would expect; Young's inequality carries over to amalgams as follows. If $f \in (L^{p_1}, l^{p_2})$ and $g \in (L^{q_1}, l^{q_2})$, where $1/p_i + 1/q_i \geqslant 1$, then $f * g \in (L^{r_1}, l^{r_2})$, where $1/r_i = 1/p_i + 1/q_i - 1$. Furthermore, there is a constant C (depending on the dimension of the factor R^a and on which subgroup H is chosen in G_1) such that

$$\|f * g\|_{r_1,r_2} \leqslant C\|f\|_{p_1,p_2}\|g\|_{q_1,q_2}. \tag{2.7}$$

(See [**BDD** and **BuS**].)

It is not easy, perhaps, to guess what the analogue of the Hausdorff-Young Theorem will be. For L^p-spaces, this theorem says that if $f \in L^p(G)$, $1 \leqslant p \leqslant 2$, then the Fourier transform, $\hat{f}$, belongs to $L^{p'}(\hat{G})$, where $\hat{G}$ is the dual group of G. For amalgams, notice that the conjugate indices, q' and p', in Theorem 2.8, appear in the opposite order to that of p and q.

THEOREM 2.8. *If* $f \in (L^p, l^q)(G)$, *where* $1 \leqslant p, q \leqslant 2$, *then* $\hat{f} \in (L^{q'}, l^{p'})(\hat{G})$ *and there is a constant* C (*depending on* p, q, *the decompositions of* G *and* $\hat{G}$, *but not on* f) *such that* $\|\hat{f}\|_{q',p'} \leqslant C\|f\|_{p,q}$.

Theorem 2.8 was first proved by Holland [**H1**] for $G = R$, although certain anticipations of this result can be seen in [**Lin**, Lemma 1] and [**Sz1**] and cases of it have been rediscovered at least three times [**A1, BaS, SS**]. For groups, it was proved by Bertrandias and Dupuis [**BD**]. See also Stewart [**Stw2**], Fournier [**Fou1**], Feichtinger [**Fei4**], and Bürger [**Bur2**].

In view of (2.3) and (2.4), the largest of the spaces (L^p, l^q) for $1 \leqslant p, q \leqslant 2$ is (L^1, l^2). In a reasonable sense, this is the largest space of functions to which we can extend the Fourier transform (see §6).

Amalgams are related to the mixed-norm spaces of Benedek and Panzone [**BP**]. If X and Y are σ-finite measure spaces, these authors defined $L^{(p,q)}(X \times Y)$ to be the space of functions g such that

$$\|g\|_{L^{(p,q)}} = \left[\int_Y \left[\int_X |g(x, y)|^p \, dx\right]^{q/p} dy\right]^{1/q} < \infty,$$

with the usual modification if p or q is ∞. Taking $G = R$, $X = [0,1]$, $Y = Z$, and $g(x, n) = f(x + n)$, we see that

$$\|g\|_{L^{(p,q)}} = \left[\sum_{n=-\infty}^{\infty} \left[\int_0^1 |f(x+n)|^p \, dx \right]^{q/p} \right]^{1/q} = \|f\|_{p,q}.$$

This shows that the amalgam $(L^p, l^q)(R)$ is isomorphic to the special mixed-norm space $L^{(p,q)}([0,1] \times Z)$. In general, we could take $X = I$ and $Y = J$. (Another connection is given in **[BDD**, Proposition IX] and **[Fei4]**. See also **[BuS**, Proposition 3.11].)

In spite of this connection, however, facts concerning mixed-norm spaces do not necessarily imply analogous facts about amalgams. For instance, Benedek and Panzone proved a version of the Hausdorff-Young Theorem for mixed-norm spaces on R^n, and their theorem can be extended to groups, but Theorem 2.8 does not follow from this extension (except for some special groups).

Finally we note that the amalgam $(L^1, l^q)(G)$ is embedded in a space of measures $M_q(G)$ which has also attracted attention. We define $M_q(G)$ to be the space of measures μ such that

$$\|\mu\|_{M_q} = \left[\sum_{\alpha \in J} [|\mu|(I_\alpha)]^q \right]^{1/q} < \infty.$$

These spaces have been studied by Các **[Cac]**, Liu et al. **[LVW]**, Holland **[H1, H2]**, Bertrandias et al. **[BDD]**, Stewart **[Stw2]**, and others. The special case $q = \infty$ is

$$M_\infty = \left\{ \mu : \sup_{\alpha \in J} |\mu|(I_\alpha) < \infty \right\},$$

which is the space of translation-bounded measures as studied by Lin **[Lin]**, Argabright and Gil de Lamadrid **[AG1]**, Berg and Forst **[BF]**, and Thornett **[Th]**.

These spaces also occur as dual spaces. If (C, l^q) denotes the space of continuous functions in (L^∞, l^q), where $1 \leqslant q < \infty$, then $(C, l^q)^* = M_{q'}$ **[Cac, H1, Stw2, BD]**. An extension of Theorem 2.8 is that if $\mu \in M_q(G)$, where $1 \leqslant q \leqslant 2$, then $\hat{\mu}$ is a function in $(L^{q'}, l^\infty)(G)$ and there is a constant C such that $\|\hat{\mu}\|_{q',\infty} \leqslant C\|\mu\|_{M_q}$.

3. Almost periodic functions. We recall that in 1924 Bohr **[Bo]** had defined an almost periodic function to be a continuous function f on R with the property that for every $\varepsilon > 0$ there is a number $L = L(\varepsilon) > 0$ such that every interval of length L contains a number τ with

$$|f(x+\tau) - f(x)| \leqslant \varepsilon \quad \text{for every real } x. \tag{3.1}$$

In 1926 both Stepanoff **[Stp]** and Wiener **[W1]** independently extended Bohr's theory of almost periodic functions by replacing (3.1) with

$$\int_\xi^{\xi+1} |f(x+\tau) - f(x)|^2 \, dx \leqslant \varepsilon \quad \text{for every real } \xi. \tag{3.2}$$

Wiener called any measurable function f satisfying (3.2) *pseudoperiodic*. He also called f *nearly bounded* if $\int_\xi^{\xi+h} |f(x)|^2\,dx$ is bounded in ξ, for fixed h. Of course, in the notation of §2, the class of nearly bounded functions is (L^2, l^∞) and (3.2) can be written as $\|f_\tau - f\|_{2,\infty}^2 \leqslant \varepsilon$, where $f_\tau(x) = f(x+\tau)$.

Wiener showed that the limit in (L^2, l^∞) of pseudoperiodic functions is again pseudoperiodic. Both Stepanoff and Wiener proved the existence of the mean value

$$M\{f(x)\} = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(x)\,dx$$

of a pseudoperiodic function f and generalized Bohr's Fundamental Theorem as follows: If F is pseudoperiodic, then there is only a finite or countable set of values $\lambda = \lambda_n$ for which $M\{f(x)e^{-i\lambda x}\} \neq 0$. If $a_n = M\{f(x)e^{-i\lambda_n x}\}$, then

$$M\left\{|f(x)|^2\right\} = \sum |a_n|^2$$

and

$$\lim_{N\to\infty} M\left\{\left|f(x) - \sum_1^N a_n e^{i\lambda_n x}\right|^2\right\} = 0.$$

In the course of his investigations of pseudoperiodic functions, Wiener also introduced ideas related to the amalgam (L^1, l^2) and the space of measures M_2. He showed that if F is a function of bounded variation over any finite interval, F assumes only a countable number of values, $F(0) = 0$, and

$$\sum_{-\infty}^{\infty}\left[\int_n^{n+1} |dF(x)|\right]^2 < \infty,$$

then $\int_{-T}^{T} e^{-i\lambda x}\,dF(x)$ converges in L^2 as $T \to \infty$ to a pseudoperiodic function.

Besicovitch **[Bes]** considered a generalization of Wiener's pseudoperiodic functions by replacing (3.1) or (3.2) with the condition that $\|f_\tau - f\|_{p,\infty} \leqslant \varepsilon$, where $1 \leqslant p < \infty$. The resulting S^p-almost periodic functions are denoted by S^pAP. (Stepanoff had considered both $p = 1$ and $p = 2$.) We can then say that $f \in S^p$AP if and only if the set of translates of f has compact closure in (L^p, l^∞). Also, S^pAP is the closure in (L^p, l^∞) of the set of trigonometric polynomials.

Argabright and Gil de Lamadrid **[AG2]** have constructed a very general theory of almost periodic measures. Given a translation invariant topological vector space $\mathscr{M}$ of measures on a group G, they call $\mu \in \mathscr{M}$ an almost periodic measure if the set of translates of μ has compact closure in $\mathscr{M}$. When $\mathscr{M}$ is what they call a suitable G-module, they obtain generalizations of much of the classical theory of almost periodic functions. In particular, when $\mathscr{M} = L^\infty(G)$ their theory specializes to that of Bohr, von Neumann, and Eberlein, whereas if $\mathscr{M} = (L^p, l^\infty)(G)$, then their theory recovers much of the work of Wiener, Stepanoff, and Besicovitch, but in the context of groups. They also consider $\mathscr{M} = (L^p, l^q)$ and $\mathscr{M} = M_q$.

See Feichtinger **[Fei5]** for another generalization of almost periodic functions which also uses amalgams.

4. Algebras and Tauberian theorems. In his famous paper on Tauberian theorems **[W2]** Wiener considered the amalgam (L^∞, l^1) and introduced the space (C, l^1), which consists of the continuous functions in (L^∞, l^1), and the space M_∞ of translation-bounded measures. In Theorem IX he states an analogue of his better-known Tauberian theorem for L^1 (Theorem VIII):

Let F be a function of bounded variation over every finite interval and let $\int_y^{y+1} |dF(x)|$ be bounded in y. Let $K_1 \in (C, l^1)$ and suppose that $\hat{K}_1$ is never zero. Let

$$\lim_{x \to \infty} \int_{-\infty}^{\infty} K_1(\xi - x)\, dF(\xi) = A \int_{-\infty}^{\infty} K_1(\xi)\, d\xi.$$

Then, if K_2 is any function in (C, l^1), we have

$$\lim_{x \to \infty} \int_{-\infty}^{\infty} K_2(\xi - x)\, dF(\xi) = A \int_{-\infty}^{\infty} K_2(\xi)\, d\xi.$$

The space (C, l^1) turns out to be a convolution algebra which has been named *Wiener's algebra* and has been studied by several authors. For instance, Goldberg **[Go]** showed that the dual space of (C, l^1) is M_∞. See also **[LVW, Fei1, Bur1]**.

Segal **[Seg,** Theorem 3.1**]** isolated those properties of (C, l^1) which were enough to enable him to prove a Tauberian theorem like the one above. He considered a dense subspace B of $L^1(R)$ with norm $\|\cdot\|$ and satisfying the following properties: (a) $f \in B \Rightarrow f_a \in B$ and $\|f_a\| = \|f\|$; (b) the mapping $a \to f_a$ from R to B is continuous; (c) $\|f_n\| \to 0 \Rightarrow \|f_n\|_1 \to 0$. Segal's Theorem 3.1 gives Wiener's Theorems VIII or IX when $B = L^1$ or (C, l^1). Although the norm

$$\|f\|_{\infty,1} = \sum_{n=-\infty}^{\infty} \max_{n \leqslant x \leqslant n+1} |f(x)|$$

does not satisfy property (a), Segal used the equivalent translation-invariant norm on (C, l^1) given by

$$\sup_{-\infty < y < \infty} \sum_{n=-\infty}^{\infty} \max_{0 \leqslant x \leqslant 1} |f(x + y + n)|.$$

Reiter **[Rei1, Rei2]** called subalgebras of $L^1(G)$ satisfying properties (a), (b), and (c) *Segal algebras* and showed that they have the same ideal theory as that of $L^1(G)$.

It is easy to see that the amalgam (L^p, l^1) is a Banach algebra under convolution for $p \geqslant 1$. In fact, using Young's inequality for amalgams (2.7), we have

$$\|f * g\|_{p,1} \leqslant C\|f\|_1 \|g\|_{p,1} \leqslant C\|f\|_{p,1}\|g\|_{p,1}.$$

If we use the equivalent translation-invariant norm given by (2.2), then (L^p, l^1) becomes a Segal algebra when $1 \leqslant p < \infty$. (Recall that $(L^p, l^1) \subset L^1$.) To complete the picture, we state the following results which are proved in **[SW]**: If $p \geqslant 1$, $q > 1$, then $A = (L^p, l^q)(G)$ is a convolution algebra if and only if G is compact; if $0 < p < 1$, $0 < q \leqslant 1$, then A is a convolution algebra if and only if G is discrete; if $0 < p < 1$, $q > 1$, then A is a convolution algebra if and only if G is finite; if $p \geqslant 1$, $q \leqslant 1$, then A is always a convolution algebra.

Finally, we mention Feichtinger **[Fei3]** who studied Banach convolution algebras and modules which resemble the Wiener algebra, but which are more general than amalgams.

5. The Orlicz-Paley-Sidon phenomenon. If $f \in C(T)$, where T denotes the circle group, then $\hat{f} \in l^2(Z)$ because $C(T) \subset L^2(T)$; surprisingly, this is all that can be said, in general, about the size of the Fourier transform of a continuous function on T. This fact was discovered simultaneously by Sidon **[Sid]**, Orlicz **[Or]**, and Paley **[P]**. In 1932 each of the three authors proved, for functions w on Z, that if

$$\sum_{-\infty}^{\infty} |\hat{f}(n)w(n)| < \infty \quad \text{for all } f \in C(T), \tag{5.1}$$

then

$$\sum_{-\infty}^{\infty} |w(n)|^2 < \infty. \tag{5.2}$$

The converse implication that (5.2) $\Rightarrow$ (5.1) follows immediately from the inclusion $C(T)^\wedge \subset l^2(Z)$ and the Cauchy-Schwarz inequality.

An analogue of the Orlicz-Paley-Sidon Theorem for the real line did not appear until 1977 when Edwards, Hewitt, and Ritter **[EHR]** identified the space of multipliers from $C_c(G)^\wedge$ to $L^1(\hat{G})$ as follows. ($C_c(G)$ denotes the space of continuous functions with compact support.)

THEOREM 5.3. *Let w be a measurable function on G. Then w has the property that*

$$\int_{\hat{G}} |\hat{f}w| < \infty \quad \textit{for all } f \in C_c(G) \tag{5.4}$$

if and only if

$$w \in (L^1, l^2)(\hat{G}). \tag{5.5}$$

To give an indication of the usefulness of the Hausdorff-Young theorem for amalgams, we mention that the proof of the implication (5.5) $\Rightarrow$ (5.4), given in **[EHR]**, is a lengthy proof using the theory of entire functions of exponential type, whereas, using Theorem 2.8, we observe that if $f \in C_c(G)$, then $f \in (L^2, l^1)$, so $\hat{f} \in (L^\infty, l^2)$. Since $w \in (L^1, l^2)$, Hölder's inequality (2.5) gives $\hat{f}w \in L^1(\hat{G})$. Other proofs of the converse implication (5.4) $\Rightarrow$ (5.5) have appeared in **[EH]**, **[Fou1]**, and **[Ber1]**. In the next section we use the theorem to determine the extended domain of the Fourier transform. In this section we analyse the theorem further and mention some related results.

The *Köthe dual* of a space X of measurable functions on $\hat{G}$ is the space consisting of all functions g for which $\int |fg| < \infty$ for all functions f in X. Theorem 5.3 may be summarized by saying that the space $(L^\infty, l^2)(\hat{G})$ and its subspace $C_c(G)^\wedge$ have the same Köthe dual. So, at least in this dual sense, the most that can be said about the size of $\hat{f}$, for a general function f in $C_c(G)$, is that $\hat{f} \in (L^\infty, l^2)(\hat{G})$.

Given a compact subset K of G, let C_K denote the space of all continuous functions on G that vanish off K. If w is in the Köthe dual of $C_K^\wedge$, let T_w be the linear operator from C_K (with the supremum norm) to $L^1(\hat{G})$, which maps f to $\hat{f}w$. It is easy to verify that the graph of T_w is closed, and hence, T_w is bounded; denote the norm of this operator by $\|T_w\|_K$. It follows **[EHR]** from Theorem 5.3 that if K has nonempty interior, then

$$(5.6) \qquad \|w\|_{1,2} \leqslant C(K)\|T_w\|_K.$$

Moreover, some proofs **[Fou1]** of the implication (5.4) $\Rightarrow$ (5.5) proceed via this inequality.

Inequality (5.6) leads to a more direct comparison between the spaces $C_K^\wedge$ and $(L^\infty, l^2)(\hat{G})$. Using the machinery **[Rie]** that represents spaces of Fourier multipliers as duals of certain function spaces, one can prove

THEOREM 5.7. *Let K be a compact subset of G with nonempty interior. Then, for each $v \in (L^\infty, l^2)(\hat{G})$ and each $\varepsilon > 0$, there is a sequence (f_n) of functions in C_K with the properties that*

$$(5.8) \qquad \sum_{n=1}^{\infty} |\hat{f}_n(\gamma)| \geqslant |v(\gamma)|$$

for almost all γ in $\hat{G}$, and

$$(5.9) \qquad \sum_{n=1}^{\infty} \|f_n\|_\infty \leqslant (1+\varepsilon)C(K)\|v\|_{\infty,2}.$$

In other words, every function in $(L^\infty, l^2)(\hat{G})$ can be essentially majorized by a sum of absolute values of transforms of functions in C_K *with control on norms*. For the case of the circle group, this observation was made in **[Cav]** and **[Kh]**, and it was later proved **[DKK]** that *only one function is needed*. That is, for each function v in $l^2(Z)$ there is a function f in $C(T)$ with $|\hat{f}| \geqslant |v|$. It is not known whether the analogous statement holds for noncompact groups.

CONJECTURE 5.10. For each function v in $(L^\infty, l^2)(\hat{G})$ there is a function f in $C_c(G)$ such that $|v| \leqslant |\hat{f}|$ almost everywhere.

Theorem 5.3 identifies the pointwise multipliers from $C_c(G)^\wedge$ to $L^1(\hat{G})$. One can also consider multipliers from $C_c(G)^\wedge$ to $L^p(\hat{G})$, where $p \neq 1$. Special cases of this problem were considered by Steckin **[Stc]** ($G = T$, $0 < p \leqslant 2$) and Edwards **[Ed]** (G compact abelian, $1 < p \leqslant 2$), and it is easy to transfer the method of Edwards to the present context. Fix $p \geqslant 1$ and $1 \leqslant q \leqslant 2$ and suppose that $w\hat{f} \in (L^p, l^q)(\hat{G})$ for all $f \in C_K$, where K is compact with nonempty interior. Then $v(w\hat{f}) \in L^1(\hat{G})$ for all $v \in (L^{p'}, l^{q'})(\hat{G})$ and all such f. By (5.6), $vw \in (L^1, l^2)$ for all such v. It then follows, by a converse of Hölder's inequality, that $w \in (L^r, l^s)(\hat{G})$, where $1/r + 1/p' = 1$ and $1/s + 1/q' = 1/2$; that is, $w \in (L^p, l^{2q/(2-q)})(\hat{G})$. Moreover, if w satisfies the latter condition, then $w\hat{f} \in (L^p, l^q)(\hat{G})$ for all $f \in (L^2, l^1)(G)$.

This has a connection with a problem of Muckenhoupt about weighted L^p-inequalities for Fourier transforms. It is shown by Aguilera and Harboure **[AH]** that if $1 \leqslant p \leqslant 2$, then $w \in (L^1, l^{2/(2-p)})$ is a necessary condition on a nonnegative function w, in order that

$$\int_{-\infty}^{\infty} |\hat{f}|^p w \leqslant C \int_{-\infty}^{\infty} |f|^p \tag{5.11}$$

for all f in $L^p(R)$. From the analysis in the preceding paragraph, this can be seen as follows. If (5.11) holds, then

$$w^{1/p} \cdot (C_{[0,1]})^{\wedge} \subset L^p = (L^p, l^p)$$

and so $w^{1/p} \in (L^p, l^{2p/(2-p)})$, that is, $w \in (L^1, l^{2/(2-p)})$.

6. The extended domain of the Fourier transform. The largest amalgam satisfying the hypotheses of Theorem 2.8 is (L^1, l^2). Our purpose in this section is to explain the fact that (L^1, l^2) is the largest solid space of functions that is mapped into a space of functions by the Fourier transform. We consider two ways to extend the definition of the Fourier transform, first by continuity with respect to suitable topologies on function spaces and second by duality as in the theory of tempered distributions. See **[Sz2, BD, Du, Ber2]** for other treatments of this topic.

Consider a topological vector space $\mathscr{F}$ of measurable functions on G. Call $\mathscr{F}$ *solid* if it has a basis of neighbourhoods V of 0 such that if $f \in V$ and g is measurable with $|g| \leqslant |f|$, then $g \in V$ also. Suppose that $\mathscr{F}$ is solid and that the Fourier transform, defined initially on $\mathscr{F} \cap L_c^\infty$, has an extension that is a continuous linear transformation from $\mathscr{F}$ into a space $\hat{\mathscr{F}}$ of measurable functions on $\hat{G}$. Suppose, finally, that convergence in $\hat{\mathscr{F}}$ implies local convergence in measure.

THEOREM 6.1. *$\mathscr{F} \subset (L^1, l^2)(G)$ and the injection $\mathscr{F} \to (L^1, l^2)(G)$ is continuous.*

PROOF. Let K be a block (one of the I_α's) in the definition of amalgams on $\hat{G}$. The hypotheses above imply that there is a solid neighbourhood V of 0 in $\mathscr{F}$ such that the set

$$S_g = \{ y \in K : |\hat{g}(y)| > 1 \}$$

has measure $< \varepsilon$ for all $g \in V \cap L_c^\infty$. For any such g, and all $f \in C_K(\hat{G})$,

$$\left| \int_G g\hat{f} \right| = \left| \int_{\hat{G}} \hat{g} f \right| \leqslant \int_{S_g} |\hat{g} f| + \int_{K \setminus S_g} |\hat{g} f|$$
$$\leqslant \|\hat{g}\|_{2,\infty} \|f\|_\infty \sqrt{\varepsilon} + |K| \, \|f\|_\infty$$

because of our assumptions about the measure of S_g and the size of $\hat{g}$ off S_g. Applying Theorem 2.8 to g yields that

$$\left| \int_G g\hat{f} \right| \leqslant \left(\sqrt{\varepsilon}\, C \|g\|_{1,2} + |K| \right) \|f\|_\infty$$

for all such f and g. Let $\phi = \overline{\operatorname{sgn}(g\hat{f})}$. Then $\phi g \in V \cap L_c^\infty$, so

$$\int_G |g\hat{f}| = \int_G (\phi g)\hat{f} \leqslant \left(C\sqrt{\varepsilon}\,\|g\|_{1,2} + |K|\right)\|f\|_\infty$$

for all such f. Thus, the operator T_g: $C_K(\hat{G}) \to L^1(G)$ defined by $f \to g\hat{f}$ has norm at most $C\sqrt{\varepsilon}\,\|g\|_{1,2} + |K|$. By (5.6), with the roles of G and $\hat{G}$ interchanged,

$$\|g\|_{1,2} \leqslant C'(K)\|T_g\| \leqslant C'(K)\left[C\sqrt{\varepsilon}\,\|g\|_{1,2} + |K|\right].$$

Choosing ε so that $C'(K)C\sqrt{\varepsilon} = 1/2$, we have $\|g\|_{1,2} \leqslant 2C'(K)|K|$ for all $g \in V \cap L_c^\infty$. For any $h \in V$, it is the case that

$$\|h\|_{1,2} = \sup\left\{\|g\|_{1,2}\colon g \in V \cap L_c^\infty,\ |g| \leqslant |h|\right\}.$$

Hence, $\|h\|_{1,2} \leqslant 2C'(K)|K|$ for all $h \in V$; since V is a neighbourhood of 0 in $\mathscr{F}$, the inclusion $\mathscr{F} \subset (L^1, l^2)$ holds and the injection is continuous.

Theorem 6.1 was first proved, for R, by Aronszajn and Szeptycki **[AS]**. Later Szeptycki **[Sz1]** completed this result by showing that $(L^1, l^2)(R)^\wedge$ is a space of functions; it is implicit in his proof that $(L^1, l^2)(R)^\wedge \subset (L^2, l^\infty)(\hat{R})$.

Another way to extend the domain of the Fourier transform is to use duality as in the theory of tempered distributions. For locally compact abelian groups, various spaces of test functions have been used **[Br, Gau, Cow]**. For our purposes it is particularly convenient to work with a test space that is an amalgam. Recall that for a subset E of G one can define the algebra $A(E)$ of restrictions to E of functions in the Fourier algebra $A(G)$; the space $A(E)$ is normed by letting

$$\|f\|_{A(E)} = \inf\left\{\|\hat{F}\|_1\colon F \in A(G),\ F|E = f\right\}.$$

Given a (continuous) function f on G, let

$$\|f\|_{(A,l^1)} = \sup_y \sum_{\alpha \in J} \|f_y\|_{A(I_\alpha)}$$

and let $(A, l^1)(G)$ be the space of functions f for which this norm is finite. This space was introduced by Feichtinger **[Fei2]** and Bertrandias **[Ber2]**; both authors showed that

$$(6.2) \qquad (A, l^1)(G)^\wedge = (A, l^1)(\hat{G}).$$

With this fact in hand, it is easy to define the Fourier transform of any (bounded) linear functional on $(A, l^1)(G)$. Given such a functional, Φ say, we let Φ be the functional on $(A, l^1)(\hat{G})$ defined by $\hat{\Phi}(\phi) = \Phi(\hat{\phi})$.

Inequality 5.6 can be used as above to prove the statement below. Given a measurable function f, let the *band determined by* f consist of all measurable functions g such that $|g| \leqslant |f|$ locally almost everywhere.

THEOREM 6.3 **[Du]**. *Let f be a locally integrable function on G with the property that every function g in the band determined by f defines a bounded linear functional on $(A, l^1)(G)$. Suppose further that $\hat{g}$ is a locally integrable function for all such functions g. Then $f \in (L^1, l^2)(G)$.*

A similar argument shows that $M_2(G)$ is the largest solid space of measures that is mapped into a space of measures by the Fourier transform.

Feichtinger and Bertrandias call elements of the dual space of (A, l^1) *translation-bounded quasimeasures*. The amalgams (L^p, l^q) with $1 \leqslant p, q \leqslant \infty$ are all included in the dual space of (A, l^1) and the sets $(L^p, l^q)(G)$ can therefore be defined as spaces of such quasimeasures. One can then define restriction norms and spaces $L^p(G)^\wedge|E$ as was done above for the Fourier algebra. Formula 6.2 and the Hausdorff-Young theorem are both special cases of the following more general inclusion which is due to Feichtinger **[Fei4]** and Bertrandias **[Ber2]**.

If $1 \leqslant q \leqslant p \leqslant \infty$, then $(L^p(\hat{G})^\wedge, l^q)(G)^\wedge \subset (L^q(G)^\wedge, l^p)(\hat{G})$.

7. Product-convolution operators. Mixed norms appear frequently in discussions of integral operators between L^p spaces. For instance **[Ka, Ga]**, an operator K, formally given by

$$(Kf)(x) = \int_Y k(x, y) f(y)\, dy$$

with a *positive* kernel k, extends to a bounded operator from $L^2(Y)$ to $L^2(X)$ if and only if k can be factored as a product $k_1 k_2$, where the factors satisfy the conditions

$$\operatorname*{ess\,sup}_x \int_Y [k_1(x, y)]^2\, dy < \infty \quad \text{and} \quad \operatorname*{ess\,sup}_y \int_X [k_2(x, y)]^2\, dx < \infty.$$

Similarly **[CFR]**, membership of the kernel in certain amalgams implies that the corresponding integral operator is bounded between certain amalgams.

Our goal here is to discuss instances where amalgams arise in essential ways in the study of certain classes of operators on $L^2(G)$. Recall that the *singular values* of a compact operator T, from a Hilbert space H_1 to a Hilbert space H_2, are the eigenvalues μ_n of the operator $|T| \equiv (T^*T)^{1/2}$, enumerated in decreasing order, with multiplicity. One then defines $\|T\|_p$ to be the l^p-norm of the sequence $(\mu_n)_{n=1}^\infty$, and the space c_p to be the set of such operators T for which $\|T\|_p < \infty$. The space c_∞ is just the set of all compact operators from H_1 to H_2, and $\|T\|_\infty$ is just the usual operator norm $\|T\|_{\text{op}}$; it is conventional to declare that $\|T\|_\infty = \|T\|_{\text{op}}$ even if T is not compact. We refer to **[Sim]** for more details about these norms and spaces of operators.

Given measurable functions g on $\hat{G}$ and f on G, let $T_{g,f}$ be the operator on $L^2(G)$ given formally by $\phi \to f(g\hat{\phi})^\vee$. In **[Sim]**, Simon gives several examples to illustrate his claim that "some of the most celebrated estimates in analysis" assert that certain operators $T_{g,f}$ are bounded. It is also useful to consider the operator $V_{g,f}$ from $L^2(\hat{G})$ to $L^2(G)$ given by $\psi \to f(g\psi)^\vee$ and the operator $S_{h,f}$ on $L^2(G)$ given by $\phi \to f(h * \phi)$, where $\hat{h} = g$. These operators belong to the same classes c_p, with the same norms, as $T_{g,f}$.

THEOREM 7.1. *Suppose that neither g nor f is 0 locally almost everywhere. Then the following statements hold*:

(i) *If $\|T_{g,f}\|_\infty < \infty$, then $g \in (L^2, l^\infty)$ and $f \in (L^2, l^\infty)$.*

(ii) *$T_{g,f} \in c_2$ if and only if $g \in L^2$ and $f \in L^2$.*

(iii) *$T_{g,f} \in c_1$ if and only if $g \in (L^2, l^1)$ and $f \in (L^2, l^1)$.*

Assertion (ii) follows easily from the fact that an integral operator belongs to the Hilbert-Schmidt class c_2 if and only if its kernel is square-summable. For $G = R^n$, assertion (iii) is due partly to Birman-Solomjak **[BiS]** and partly to Simon **[Sim]**; their proofs extend easily to more general groups. Assertion (i) is new, although there is precedent for it in **[BuS]**.

We illustrate the proof of assertion (i) by showing that if $|g| > 0$ on some set of positive measure, and if the operator $V_{g,f}$ is bounded, then $f \in (L^2, l^\infty)$. First note that the boundedness of $V_{g,f}$ implies that $V_{k,f}$ is also bounded for all measurable functions k with $|k| \leqslant |g|$; moreover, $\|V_{k,f}\|_\infty \leqslant \|V_{g,f}\|_\infty$. There must be a number $\varepsilon > 0$ and a compact set K having positive measure such that $|g| \geqslant \varepsilon$ on K. Let $k = 1_K$, the indicator function of the set K; then $\|V_{k,f}\|_\infty \leqslant \|V_{g,f}\|_\infty/\varepsilon$. In particular, $\|f\check{\psi}\|_2 \leqslant C\|\psi\|_2$ for all $\psi \in L^2(K)$, where any such function is extended to be 0 off K. Equivalently, $\|\phi f\check{\psi}\|_1 \leqslant C\|\psi\|_2$ for all $\psi \in L^2(K)$ and all $\phi \in L^2(G)$. But a variant **[Fou1**, p. 127] of Theorem 5.3 asserts that if $\|v\check{\psi}\|_1 \leqslant C'\|\psi\|_\infty$ for all $\psi \in L^\infty(K)$, then $v \in (L^1, l^2)$. Hence $\phi f \in (L^1, l^2)$ for all ϕ in $L^2(G)$, and it follows that $f \in (L^2, l^\infty)$, as claimed.

The converse to assertion (i) is false; for instance, $\|T_{1,f}\|_\infty < \infty$ if and only if $f \in L^\infty$. It is true, however, that if one of the functions g or f lies in (L^2, l^∞), then the operator $T_{g,f}$ is bounded for many choices of the other function. Suppose, for definiteness, that $f \in (L^2, l^\infty)(G)$ and let $g \in (L^\infty, l^2)(\hat{G})$. Then $\|g\hat{\phi}\|_{2,1} \leqslant \|g\|_{\infty,2}\|\phi\|_2$ for all $\phi \in L^2(G)$, and

$$\left\|(g\hat{\phi})^\vee\right\|_{\infty,2} \leqslant C\|g\|_{\infty,2}\|\phi\|_2$$

by Theorem 2.8. Therefore,

$$\left\|f(g\hat{\phi})^\vee\right\|_2 \leqslant C\|f\|_{2,\infty}\|g\|_{\infty,2}\|\phi\|_2.$$

Hence, $\|T_{g,f}\|_\infty \leqslant C\|f\|_{2,\infty}\|g\|_{\infty,2}$ and, more generally,

$$\left\|T_{g,f}\right\|_\infty \leqslant C\min\left\{\|f\|_{p,\infty}\|g\|_{\infty,p},\ \|f\|_{\infty,p}\|g\|_{p,\infty}\right\}$$

for all p in the interval $[2, \infty]$. When $p = \infty$, this reduces to the elementary estimate $\|T_{g,f}\|_{\mathrm{op}} \leqslant \|g\|_\infty\|f\|_\infty$.

These observations, and assertions (ii) and (iii) of the theorem, provide a variety of sufficient conditions on the sizes of g and f for $T_{g,f}$ to belong to various classes c_p. As in **[Sim]**, complex interpolation then yields a large family of such conditions, involving membership of g and f in certain amalgams, that imply that $T_{g,f}$ belongs to c_p for intermediate values of p.

Busby and Smith **[BuS]** considered the product-convolution operators $S_{h,f}$ and they showed that if $\|S_{h,f}\|_{\mathrm{op}}$ is finite but nonzero, then $f \in (L^2, l^\infty)$ and $h \in (L^1, l^2)$. We comment briefly on the relation between this work and part (i) of Theorem 7.1. Busby and Smith regard the convolution $h * \phi$ as undefined unless the integral defining $|h| * |\phi|$ converges absolutely, locally almost everywhere. If this requirement is dropped, then there are functions h, such as the kernel of the Hilbert transform, that are *not* locally integrable, but for which $h * \phi$ exists in the principal-value sense for all ϕ in $L^2(G)$, and $\|h * \phi\|_2 \leqslant C\|\phi\|_2$. These functions h lie outside (L^1, l^2). On the other hand, Busby and Smith show that if $|h| * |\phi|$ is locally almost everywhere finite for all ϕ in

$(L^\infty, l^p)(G)$, then $h \in (L^1, l^{p'})$. Part (i) of Theorem 7.1 follows from this. As noted earlier, Busby and Smith consider a more general class of groups that includes all locally compact abelian groups.

8. Lacunarity. In this section we give a brief account of another instance where amalgams provide the right setting for generalizations of classical results, and this generalization yields new information in the classical setting.

Given a closed subset E of $\hat{G}$ and a function $f \in (L^1, l^\infty)(G)$, call f an *E-function* if $\hat{f}$ vanishes outside E; in general, $\hat{f}$ will be a translation-bounded quasimeasure rather than a function and we say that $\hat{f} = 0$ off E if $\langle \psi, \hat{f} \rangle = 0$ for all $\psi \in (A, l^1)(\hat{G})$ with support in the interior of $\hat{G} \setminus E$. Given $p > 1$, call E a $\Lambda(p)$ set if every E-function in $L^1(G)$ also belongs to $L^p(G)$. The theory of $\Lambda(p)$ sets in discrete abelian groups is well established **[LR, Haj]**.

We now state an instance of a theorem which appears in **[Fou2]**. Let $G = R^N$, let E be a subset of the integer lattice Z^N in R^N, and let $F = E + [0,1]^N$. Fix $p > 1$ and regard E as a subset of the dual of T^N.

THEOREM 8.1. *The following statements are equivalent.*

(i) *E is a $\Lambda(p)$ set in Z^N.*

(ii) *F is a $\Lambda(p)$ set in R^N.*

(iii) *For every q in the interval $[1, \infty]$ it is the case that every F-function in $(L^1, l^q)(R^N)$ belongs to $(L^p, l^q)(R^N)$.*

Statement (iii) has at least two virtues. First, it provides a reasonable analogue to the fact that E-functions in the space $L^1(T^N)$ must also belong to the *smaller* space $L^p(T^N)$. Statement (ii) only asserts that F-functions in $L^1(R^N)$ belong to $L^p(R^N)$ which is *not* a smaller space than $L^1(R^N)$; on the other hand, $(L^p, l^q) \subsetneqq (L^1, l^q)$. So amalgams provide a good setting for the study of $\Lambda(p)$ sets in $\hat{R}^N$. The second virtue is that the proof of (i) $\Rightarrow$ (ii) goes by way of statement (iii). In fact, one first proves that statement (i) implies that

(ii)$'$ every F-function in $(L^1, l^1)(R^N)$ belongs to $(L^p, l^\infty)(R^N)$ and one then shows that (ii)$'$ $\Rightarrow$ (iii); finally, (iii) $\Rightarrow$ (ii) because every F-function in $L^1 = (L^1, l^1)$ belongs, by (iii), to (L^p, l^1) which is a subspace of L^p.

Theorem 8.1 yields interesting conclusions about $\Lambda(p)$ sets in Z^N, a setting where it might seem pointless to consider amalgams. Let $\mathbf{v}$ be a vector in R^N whose components are all irrational and strictly greater than 1. Given $E \subset Z^N$, form a set $\mathbf{v}E$ as follows. First pass from E to the set $\mathbf{v}E$ in R^N by componentwise multiplication. Then replace each point in $\mathbf{v}E$ by the nearest point to it in the lattice Z^N, and denote the resulting set by $\mathbf{v}E$. As in **[Fou2]**, it follows from Theorem 8.1 that $\mathbf{v}E$ is a $\Lambda(p)$ set if and only if E is.

9. Approximation theory. Let S be a linear space of functions from R to C, and let $y = (y_n)_{n \in Z}$ be a complex-valued sequence. The *cardinal interpolation problem* CIP(y; S) is to find functions $f \in S$ such that $f(n) = y_n$ for all $n \in Z$. Schoenberg **[Sch]** considered this problem for the case where S is the amalgam (L^∞, l^1). In fact, more generally, he considered the space

$$L^m_{[1]} = \left\{ f \colon f^{(m-1)} \text{ is absolutely continuous and } \|f^{(m)}\|_{\infty,1} < \infty \right\}$$

together with the sequence space

$$l_1^m = \{ y\colon \|\Delta^m y\|_1 < \infty \},$$

where Δ is a difference operator. Schoenberg showed that CIP(y; $L^m_{[1]}$) has solutions if and only if $y \in l_1^m$. Furthermore, he used spline functions to construct an optimal solution, i.e., a solution such that $\|f^{(m)}\|_{\infty,1}$ is minimized.

Jakimovski and Russell **[JR]** investigated the more general *interpolation problem* IP(y; S, x) which is to find $f \in S$ with $f(x_n) = y_n$ for all $n \in Z$, where $x = (x_n)$ is a fixed strictly increasing sequence with $x_n \to -\infty$ as $n \to -\infty$, and $x_n \to \infty$ as $n \to \infty$. In particular, if E is a normed sequence space, they take S to be the space $EL_{p,x}$ consisting of functions f with

$$\|f\|_{EL_{p,x}} = \left\| \left(\|f\|_{L^p(x_n, x_{n+1})} \right)_{n \in Z} \right\|_E < \infty.$$

Under certain conditions on E, they prove a theorem (similar to the above result of Schoenberg) on the existence of an optimal solution to the problem IP(y; $EL_{p,x}$, x).

If we take $E = l^q$ and $x_n = n$, then $EL_{p,x}$ becomes the amalgam (L^p, l^q). But if we take $E = l^q$ and let $x = (x_n)$ be an arbitrary sequence satisfying the above conditions, then we obtain the "stretched amalgam"

$$(L^p, l^q)_x = \left\{ f\colon \|f\|_{p,q,x} = \left[\sum_{n=-\infty}^{\infty} \left[\int_{x_n}^{x_{n+1}} |f(x)|^p \, dx \right]^{q/p} \right]^{1/q} < \infty \right\}.$$

Some of the theory of amalgams, including (2.5) and (2.6), goes through without change for stretched amalgams. But anything involving translation is difficult to control and we know of no analogue for (2.7) and (2.8). Moreover, although the inclusion (2.3) holds for all stretched amalgams, (2.4) does not.

10. Some properties of $H^1(R^N)^\wedge$. Amalgams arise in the intrinsic characterization of the set of functions w for which

$$\text{(10.1)} \qquad \int_{R^N} |\hat{f}w| < \infty \quad \text{for all } f \in H^1(R^N),$$

and in showing that the space $H^1(R^N)$ has the lower majorant property.

Although $H^1(R^N)$ was originally defined in a different way, it can be characterized **[La, Wln]** using the notion of atoms as explained below. By a *prototypical atom* we mean a measurable function A on R^N with the following properties:

(i) $A \equiv 0$ off the set $[0,1]^N$,
(ii) $\|A\|_\infty \leqslant 1$,
(iii) $\int_{R^N} A = 0$.

By an *atom* we mean any function a say, obtained from a prototypical atom by translation, dilation, and multiplication by scalars so that L^1-norms are preserved. Thus,

$$a(\mathbf{x}) = \lambda^N A((\mathbf{x} - \mathbf{c})/\lambda),$$

where $\mathbf{c} \in R^N$ and λ is some positive number. Note that $\|a\|_1 = \|A\|_1 \leqslant 1$ here. Finally, we say that f belongs to $H^1(R^N)$ if there are sequences $(c_n)_1^\infty$ and

$(a_n)_1^\infty$ of scalars and atoms such that $f = \Sigma c_n a_n$ and $\Sigma_{n=1}^\infty |c_n| < \infty$. See [TW] for much more about such characterizations.

This characterization of $H^1(R^N)$ reduces the problem of finding all functions w with property (10.1) to that of finding all w such that

$$(10.2) \qquad \int |\hat{a}w| < c \quad \text{for all atoms } a.$$

If A satisfies conditions (i) and (ii), then $\|A\|_{2,1} \leqslant 1$ and $\|\hat{A}\|_{\infty,2} \leqslant C$, by Theorem 2.8. Hence $\int |\hat{A}w| \leqslant C\|w\|_{1,2}$. Prototypical atoms have the further property that $\hat{A}(0) = 0$. On the other hand, (10.2) is supposed to hold for all functions a obtained from prototypical atoms by dilation and rescaling. A further analysis using these two facts shows that condition (10.4) implies (10.2).

THEOREM 10.3 (FEFFERMAN, ALEXANDROV, SLEDD-STEGENGA). *A function w satisfies*

$$\int |\hat{f}w| < \infty \quad \textit{for all } f \in H^1(R^N)$$

if and only if its dilates given by $w_t(\mathbf{x}) = t^N w(\mathbf{x}/t)$, where t is any positive number, have the property that

$$(10.4) \qquad \sup_t \sum_{\mathbf{n}\neq \mathbf{0}} \left(\int_{\mathbf{n}+[-1/2,1/2]^N} |w_t| \right)^2 < \infty.$$

To rephrase (10.4) in terms of amalgams, let W_t coincide with the function w_t except on the cube $[-\frac{1}{2}, \frac{1}{2})^N$ where $W_t \equiv 0$. Condition (10.4) asserts that $\sup_t \|W_t\|_{1,2} < \infty$. To show that this condition is necessary for the validity of (10.1), the authors of [A1] and [SS] explicitly solve the following majorization problem:

Given $v \in (L^\infty, l^2)(R^N)$ with $v \equiv 0$ on $[-\frac{1}{2}, \frac{1}{2}]^N$, construct a function $f \in H^1(R^N)$ such that

(i) $\hat{f} \geqslant |v|$ almost everywhere, and

(ii) $\|f\|_{H^1} \leqslant C\|v\|_{\infty,2}$.

Since every such function f can be represented as an l^1-sum of atoms, this problem resembles the one addressed in Theorem 5.7.

A similar analysis [Al, BaS] shows that $H^1(R^N)$ has the *lower majorant property*; that is, for each $F \in H^1(R^N)$ there exists $f \in H^1(R^N)$ such that

$$\hat{f} \geqslant |\hat{F}| \quad \text{and yet} \quad \|f\|_{H^1} \leqslant C\|F\|_{H^1}.$$

The spaces $H^p(R^N)$ with $0 < p < 1$ also have this property [Al, BaS] and there are analogues [Al, SS] of Theorem 10.3 for these spaces. There are similar characterizations [Al, SS] of spaces of measures μ for which $\int |\hat{f}|\, d\mu < \infty$ for all $f \in H^p(R^N)$, where again $0 < p \leqslant 1$.

In comparing condition (10.4) with conditions for membership in amalgams it is useful to note that $\lim_{t\to\infty} \|w_t\|_{1,2} = \|w\|_1$. This fact can be used to show that there are functions w in $(L^1, l^2)(R^N)$ that do *not* satisfy condition (10.4). On the other hand, there are functions w that satisfy condition (10.4) but are not locally integrable at 0 and, in particular, do not belong to the amalgam $(L^1, l^2)(R^N)$.

11. Dyadic amalgams and the range of the Fourier transform. We know that (L^1, l^2) is the *largest* solid space of functions that is mapped into a space of functions by the Fourier transform. Amalgams also arise in the study of $S_p(\hat{G})$, the *smallest* solid space of functions that includes $L^p(G)^\wedge$; here p is a fixed index in the interval $1 < p < 2$.

No intrinsic characterization of the space $S_p(\hat{G})$ is known. It was conjectured by Bichteller **[Sz3]** that $S_p(\hat{G}) = L^{p'}(\hat{G})$; note that $S_p(\hat{G}) \subset L^{p'}(\hat{G})$ by the Hausdorff-Young theorem. In fact, however, this inclusion is strict whenever $1 < p < 2$ and $\hat{G}$ is infinite. One way to see this **[M]** is to use the theory of Lorentz spaces $L(p, q)$ as presented in **[Hu]**. Another way, when G is noncompact, is to use amalgams.

To simplify the presentation we suppose for the rest of this section that $G = R$. (See **[EG]** and **[Fou2]** for the general case.) Then any infinite $\Lambda(p')$ subset E of Z has the property that

$$\sum_{n \in E} \int_n^{n+1} |\hat{f}|^2 < \infty \tag{11.1}$$

for all $f \in L^p(R)$. (See §8.) Even more can be said if the thin set E has a special form. For each integer n, let I_n be the union of the intervals $[2^n, 2^{n+1})$ and $(-2^{n+1}, -2^n]$. Then

$$\sum_{n=-\infty}^{\infty} \left\{ \left[\int_{I_n} |\hat{f}|^{p'} \right]^{1/p'} \right\}^2 < \infty \tag{11.2}$$

for all functions $f \in L^p(R)$, $1 < p \leqslant 2$. In other words, $\hat{f}$ lies in the *dyadic amalgam* $(L^{p'}, l^2)_{I_n}$ based on the sets I_n instead of $[n, n+1]$. This is the most important special case of the stretched amalgams introduced in §9.

The result given by (11.2) is due to J. W. Wells **[Wil**, p. 824**]**. Kellogg **[Ke]** had proved the corresponding result for $G = T$ based on Hedlund's **[Hed]** use of dyadic amalgams in identifying multipliers of H^p spaces. Williams **[Wil]** proved an analogue of (11.2) for connected groups. See also the papers by Herz **[Her]** and Johnson **[Jo]**.

Condition (11.2) does *not* characterize the set $S_p(\hat{R})$. By the M. Riesz theorem, the function $\hat{f} \cdot 1_{I_n}$ is the transform of a function f_n in $L^p(R)$. By Littlewood-Paley theory **[Stn]**,

$$\int_{-\infty}^{\infty} \left[\left\{ \sum_{n=-\infty}^{\infty} |f_n(x)|^2 \right\}^{1/2} \right]^p dx < \infty. \tag{11.3}$$

It follows from the integral form of Minkowski's inequality that

$$\sum_{n=-\infty}^{\infty} \|f_n\|_p^2 < \infty. \tag{11.4}$$

By Hunt's Hausdorff-Young theorem **[Hu]** it follows that

$$\sum_{n=-\infty}^{\infty} \left[\|\hat{f}_n\|_{L(p',p)} \right]^2 < \infty. \tag{11.5}$$

This is a stronger restriction than (11.2). (See **[RS]** and **[Sl]** for similar observations when $G = T$.) Even condition (11.5), however, does not characterize the space $S_p(\hat{R})$ because there are functions that satisfy (11.5) but do not satisfy (11.1) for general $\Lambda(p')$ sets E.

The strongest restriction that arises in this context of dyadic decompositions of R is inequality (11.3); something is lost in each step of the implication (11.3) $\Rightarrow$ (11.4) $\Rightarrow$ (11.5) $\Rightarrow$ (11.2). The idea of splitting a given function f into pieces f_n, whose transforms are supported by dyadic intervals, has been modified and developed considerably during the last fifteen years **[Tr]**. The sharpest conclusions in this approach involve norms computed on R rather than on $\hat{R}$ where the splitting occurs; the resulting spaces are (usually) not amalgams of L^p-spaces.

12. Other occurrences of amalgams. In this section we give brief mention to several other situations in which amalgams have arisen. A common principle which unites some of the occurrences, both here and in the preceding sections, is that amalgams are extremal Banach spaces to which one can extend conclusions that are provable for objects with compact support.

(i) A function f is called positive definite for a class of functions F if the integral

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y)\phi(x)\phi(y)\,dx\,dy$$

exists and is nonnegative for every $\phi \in F$. Cooper **[Coop]** made essential use of the amalgam (L^2, l^1) in proving that any function which is positive definite for C_c (the continuous functions with compact support) is the Fourier transform of a positive measure (possibly unbounded) in the sense of Cesàro summability a.e. (see also **[Stw1]**).

(ii) The Schoenberg-Eberlein criterion **[Eb]** says that f is the Fourier transform of a bounded measure if and only if there is a constant M such that $|\int_G f\phi| \leqslant M\|\hat{\phi}\|_\infty$ for all $\phi \in L^1(G)$. Using the theory of amalgams discussed in §2, Holland **[H2]** and Stewart **[Stw2]** have extended this criterion to unbounded measures by showing, when $1 \leqslant q \leqslant 2$, that f is the Fourier transform of a measure in $M_q(\hat{G})$ if and only if there is a constant M such that $|\int_G f\phi| \leqslant M\|\hat{\phi}\|_{\infty,q'}$ whenever $\phi \in C_c(G)$.

(iii) The class M_∞ of translation-bounded measures arises in the study of second-order stationary random measures **[Th]** and also in potential theory. In particular, Berg and Forst **[BF**, Proposition 13.10**]** have shown that if a family $(\mu_t)_{t>0}$ of bounded positive measures forms a transient convolution semigroup, then the potential kernel $\int_0^\infty \mu_t\,dt$ is a translation-bounded measure.

(iv) Finally we mention the far-reaching generalization of amalgams proposed by Feichtinger **[Fei3]**. He takes Banach spaces B and C satisfying certain conditions and defines a *space of Wiener's type* $W(B, C)$ to consist of objects which are, roughly speaking, locally in B and globally in C. To be slightly more precise, we start with any compact subset E of a group such that E has nonempty interior. With each object f which is locally in B, we associate the function $F_f(x) = \|f\|_{B(x+E)}$. If $F_f \in C$, then we say $f \in W(B, C)$ and write

$$\|f\|_{W(B,C)} = \|F_f\|_C. \tag{12.1}$$

This norm makes $W(B, C)$ into a Banach space, and by comparing (12.1) with (2.2) we see that $W(L^p, L^q) = (L^p, l^q)$. For other examples, Feichtinger takes C to be a weighted L^q-space and B to be L^p, or a Besov or Sobolev space, or the space of functions of bounded variation. Of particular interest is the Wiener space $S_0 = W(A(G), L^1)$, where $A(G)$ is the Fourier algebra L^1. See §6 for an example of how the properties of S_0 are used.

This generalization can be extended [FG] to the case of nonuniform decompositions of homogeneous type. Dyadic amalgams and the Fourier transforms of Besov spaces are special cases.

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