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Componentwise equivariant estimation of order restricted location and scale parameters in bivariate models: A unified study

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Abstract. The problem of estimating location (scale) parameters θ_1 and θ_2 of two distributions when the ordering between them is known apriori (say, $\theta_1 \leq \theta_2$) has been extensively studied in the literature. Many of these studies are centered around deriving estimators that dominate the best location (scale) equivariant estimators, for the unrestricted case, by exploiting the prior information that $\theta_1 \leq \theta_2$. Several of these studies consider specific distributions such that the associated random variables are statistically independent. In this paper, we consider a general bivariate model and a general loss function, and unify various results proved in the literature. We also consider applications of these results to a bivariate normal and a Cheriyan and Ramabhadran's bivariate gamma model. A simulation study is also considered to compare the risk performances of various estimators under bivariate normal and Cheriyan and Ramabhadran's bivariate gamma models.

1 Introduction

Let $\boldsymbol{X} = (X_1, X_2)$ be a random vector having a joint probability density function (pdf) belonging to location (scale) family

$$f_{\boldsymbol{\theta}}(x_1, x_2) = f(x_1 - \theta_1, x_2 - \theta_2), \quad (x_1, x_2) \in \Re^2, \tag{1.1}$$

$$\left(f_{\boldsymbol{\theta}}(x_1, x_2) = \frac{1}{\theta_1\theta_2} f\left(\frac{x_1}{\theta_1}, \frac{x_2}{\theta_2}\right), \quad (x_1, x_2) \in \Re^2, \right) \tag{1.2}$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \Theta = \Re^2(\Re^2_{++})$ is the vector of unknown location (scale) parameters and $f(\cdot,\cdot)$ is a specified pdf on $\Re^2$; here $\Re^2 = (-\infty, \infty) \times (-\infty, \infty)$ and $\Re^2_{++} = (0, \infty) \times (0, \infty)$. Generally, $\boldsymbol{X} = (X_1, X_2)$ would be a minimal-sufficient statistic based on a bivariate random sample or two independent random samples, as the case may be. In many real life situations, ordering between the parameters θ_1 and θ_2 may be known apriori (say, $\theta_1 \leq \theta_2$) and it may be of interest to estimate θ_1 and θ_2 (see, for example, Barlow et al. (1972), Robertson, Wright and Dykstra (1988), Kumar and Sharma (1988), Kubokawa and Saleh (1994), Hwang and Peddada (1994) and references cited therein).

Let $\Theta_0 = \{\boldsymbol{\theta} \in \Theta : \theta_1 \leq \theta_2\}$ be the restricted parameter space. There is an extensive literature on estimation of θ_1 and θ_2 (simultaneously, as well as, componentwise) when it is known apriori that $\boldsymbol{\theta} \in \Theta_0$. A natural question that arises in these problems is whether the best location (scale) equivariant estimator(s) (BLEE (BSEE)) for the unrestricted case (i.e., when the prior information $\boldsymbol{\theta} \in \Theta_0$ is not available) can be improved by exploiting the prior information that $\boldsymbol{\theta} \in \Theta_0$. Many researchers have studied this and related aspects of the problem. However, several of these studies are focussed to specific distributions, having independent marginals, and specific loss functions. Some of the contributions in this direction are

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due to Cohen and Sackrowitz (1970), Brewster and Zidek (1974), Lee (1981), Kumar and Sharma (1988), Kumar and Sharma (1989), Kelly (1989), Kushary and Cohen (1989), Gupta and Singh (1992), Pal and Kushary (1992), Misra and Singh (1994), Vijayasree, Misra and Singh (1995), Misra and Dhariyal (1995), Misra, Dhariyal and Kundu (2002), Misra, Iyer and Singh (2004), Oono and Shinozaki (2006), Chang and Shinozaki (2015), Petropoulos (2010), Petropoulos (2017), Bobotas (2019a), Bobotas (2019b) and Patra, Kumar and Petropoulos (2021). For a few contributions to this problem under general setting (general probability model and/or general loss function) readers may refer to Blumenthal and Cohen (1968), Sackrowitz (1970), Hwang and Peddada (1994), Kubokawa and Saleh (1994) and Iliopoulos (2000). For a detailed account of contributions in this area of research one may refer to the research monograph by van Eeden (2006).

Kubokawa and Saleh (1994) considered the location (scale) model (1.1) ((1.2)) with

$$f(z_1, z_2) = f_1(z_1) f_2(z_2), \quad -\infty < z_i < \infty, i = 1, 2,$$

where f_1 and f_2 are specified pdfs on the real line $\Re$. They dealt with estimation of the smaller location (scale) parameter θ_1 when it is known apriori that $\boldsymbol{\theta} \in \Theta_0$. They assumed that

$$f_i \in P_L = \left\{ g : \frac{g(y + c_1)}{g(y + c_2)} \text{ is non-decreasing in } y \in \Re, \text{for every } c_1 < c_2 \right\}, \quad i = 1, 2, \quad (1.3)$$

$$\left(f_i \in P_S = \left\{ g : \frac{g(c_1 y)}{g(c_2 y)} \text{ is non-decreasing in } y \in \Re_{++}, \text{for every } 0 < c_1 < c_2 \right\}, \quad i = 1, 2, \right) \quad (1.4)$$

and considered a quite general bowl-shaped loss function

$$L_1(\boldsymbol{\theta}, a) = W(a - \theta_1), \quad \boldsymbol{\theta} \in \Theta_0, a \in \Re, \qquad \left(L_1(\boldsymbol{\theta}, a) = W\left(\frac{a}{\theta_1}\right), \quad \boldsymbol{\theta} \in \Theta_0, a \in \Re_{++} \right),$$

where $\Re_{++} = (0, \infty)$ and, $W : \Re \to [0, \infty)$ is such that $W(0) = 0$ ($W(1) = 0$), $W(t)$ is strictly decreasing for $t < 0$ ($t < 1$) and strictly increasing for $t \geq 0$ ($t \geq 1$). Under the above set-up, they derived conditions that ensure improvements over the best location (scale) equivariant estimator of θ_1. They found explicit expressions of the dominating estimators. In fact, Kubokawa and Saleh (1994) dealt with estimation of the smallest location (scale) parameter of k (≥ 2) independent location (scale) families of probability distributions when it is known apriori that the corresponding location (scale) parameters $\theta_1, \theta_2, \dots, \theta_k$ satisfy the tree ordering ($\theta_1 \leq \theta_i$, $i = 2, 3, \dots, k$). Under the set-up considered by Kubokawa and Saleh (1994), the random variables X_1 and X_2 are independently distributed. In this paper, we extend the study of Kubokawa and Saleh (1994) to situations where X_1 and X_2 may be statistically dependent. As in Kubokawa and Saleh (1994), we will closely follow the IERD (Integral expression risk difference) approach of Kubokawa (1994), to obtain improvements over the BLEE/BSEE of θ_1 and θ_2. We also consider estimation of the larger location (scale) parameter θ_2 that has not been addressed by Kubokawa and Saleh (1994). To avoid some notational and presentation difficulties, throughout the paper, we extend the usual orders "$\leq$" and "$<$" in the real line to the extended real line $\Re \cup \{-\infty, \infty\}$ with the following convention. For any positive (negative) real number "b", we take $\frac{b}{0} = \infty$ ($-\infty$) and, for any real number "c", we take $-\infty < c < \infty$.

In Section 2 (3), we consider a general bivariate location (scale) family of distributions and deal with componentwise estimation of order restricted location (scale) parameters θ_1 and θ_2 under a quite general loss function. We derive sufficient conditions that guarantee improvements over the BLEE (BSEE). The explicit expressions of dominating estimators are obtained. In Section 2.3 (3.3), we provide applications of various results derived in the paper. In Section 2.4 (3.4), we consider a simulation study for comparing risk performances of various estimators of smaller location (scale) parameter under bivariate normal (Cheriyan and Ramabhadran's bivariate gamma) model.

2 Improving the Best Location Equivariant Estimators (BLEEs)

Firstly, we will introduce some notations in connection with the probability model (1.1). Let $Z_i = X_i - \theta_i$, $i = 1, 2$, and $\boldsymbol{Z} = (Z_1, Z_2)$, so that $\boldsymbol{Z}$ has the joint pdf $f(t_1, t_2)$, $(t_1, t_2) \in \Re^2$. Let S_i be the distributional support of $Z_i = X_i - \theta_i$, $i = 1, 2$. Let $Z = Z_2 - Z_1$ and f_i be the pdf of Z_i, $i = 1, 2$. Then

$$f_1(s) = \int_{-\infty}^{\infty} f(s,t)\,dt, \quad s \in \Re, \qquad f_2(s) = \int_{-\infty}^{\infty} f(t,s)\,dt, \quad s \in \Re.$$

For any $s \in S_i$, let $Z_s^{(i)}$ denote a random variable having the same distribution as conditional distribution of Z given $Z_i = s$, $i = 1, 2$. Then the pdf and the distribution function (df) of $Z_s^{(1)}$ $(s \in S_1)$ are given by

$$h_1(t|s) = \frac{f(s, t+s)}{f_1(s)}, \quad t \in \Re, \quad \text{and} \quad H_1(t|s) = \frac{\int_{-\infty}^{t} f(s, z+s)\,dz}{f_1(s)}, \quad t \in \Re,$$

respectively, and the pdf and the df of $Z_s^{(2)}$ $(s \in S_2)$ are given by

$$h_2(t|s) = \frac{f(s-t, s)}{f_2(s)}, \quad t \in \Re, \quad \text{and} \quad H_2(t|s) = \frac{\int_{-\infty}^{t} f(s-z, s)\,dz}{f_2(s)}, \quad t \in \Re,$$

respectively.

For the location model (1.1), consider estimation of θ_i under the loss function

$$L_i(\boldsymbol{\theta}, a) = W(a - \theta_i), \quad \boldsymbol{\theta} \in \Theta, a \in \mathcal{A} = \Re, i = 1, 2, \tag{2.1}$$

where $W : \Re \to [0, \infty)$ is a specified non-negative function. We make the following assumptions on the function $W(\cdot)$:

$\boldsymbol{A_1}$: $W : \Re \to [0, \infty)$ is absolutely continuous, $W(0) = 0$, $W(t)$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$. Further $W'(t)$ is non-decreasing on the set D_0 (the set of points at which $W(\cdot)$ is differentiable).

First consider estimation of θ_i, $i = 1, 2$, under the unrestricted parameter space $\Theta = \Re^2$ and the loss function (2.1). Under this set-up, the problem of estimating θ_i $(i = 1, 2)$ is invariant under the additive group of transformation $\mathcal{G} = \{g_{c_1,c_2} : (c_1, c_2) \in \Re^2\}$, where $g_{c_1,c_2}(x_1, x_2) = (x_1 + c_1, x_2 + c_2)$, $(x_1, x_2) \in \Re^2$, $(c_1, c_2) \in \Re^2$. Any (non-randomized) unrestricted location equivariant estimator δ_i of θ_i is of the form $\delta_{c,i}(X_1, X_2) = X_i - c$, $i = 1, 2$, for some constant $c \in \Re$. The risk function of $\delta_{c,i}$ is given by $R_i(\boldsymbol{\theta}, \delta_{c,i}) = E_{\boldsymbol{\theta}}[L_i(\boldsymbol{\theta}, \delta_{c,i}(\boldsymbol{X}))]$, $\boldsymbol{\theta} \in \Theta$, $i = 1, 2$.

The risk function of any unrestricted location invariant estimator $\delta_{c,i}$ of θ_i is constant (does not depend on $\boldsymbol{\theta} \in \Theta$). For the existence of unrestricted ($\boldsymbol{\theta} \in \Theta$) BLEE, we need the following assumption:

$\boldsymbol{A_2}$: The equation $E[W'(Z_i - c)] = 0$ has the unique solution, say $c = c_{0,i}$, $i = 1, 2$.

Since the risk function of any unrestricted equivariant estimator of θ_i is constant on Θ (see Theorem 1 on p. 245 of Berger (2013)), under assumptions A_1 and A_2, the unique BLEE of θ_i is

$$\delta_{c_{0,i},i}(\boldsymbol{X}) = X_i - c_{0,i}, \quad i = 1, 2, \tag{2.2}$$

where $c_{0,i}$ is the unique solution of the equation

$$\int_{-\infty}^{\infty} W'(z - c) f_i(z)\,dz = 0, \quad i = 1, 2. \tag{2.3}$$

Now consider estimation of location parameter θ_i, $i=1,2$, under the restricted parameter space $\Theta_0=\{(x_1,x_2)\in\Theta : x_1\le x_2\}$ and the loss function (2.1). Under the restricted parameter space Θ_0, the location family of distributions (1.1) is not invariant under the group of transformations $\mathcal{G}=\{g_{c_1,c_2} : (c_1,c_2)\in\Re^2\}$, considered above. An appropriate group of transformations ensuring invariance under restricted parameter space Θ_0 is $\mathcal{K}=\{k_c : c\in\Re\}$, where $k_c(x_1,x_2)=(x_1+c,x_2+c)$, $(x_1,x_2)\in\Re^2$, $c\in\Re$. Under the group of transformations $\mathcal{K}$, the problem of estimating θ_i, under $\boldsymbol{\theta}\in\Theta_0$ and the loss function (2.1), is invariant. Any location equivariant estimator of θ_i is of the form

$$\delta_{\psi_i}(\boldsymbol{X})=X_i-\psi_i(D), \tag{2.4}$$

for some function $\psi_i:\Re\to\Re$, $i=1,2$, where $D=X_2-X_1$. Here the risk function

$$R_i(\boldsymbol{\theta},\delta_{\psi_i})=E_{\boldsymbol{\theta}}\big[L_i\big(\boldsymbol{\theta},\delta_{\psi_i}(\boldsymbol{X})\big)\big],\quad \boldsymbol{\theta}\in\Theta_0, \tag{2.5}$$

of any location equivariant estimator δ_{ψ_i} of θ_i, $i=1,2$, may not be constant on Θ_0, and it depends on $\boldsymbol{\theta}\in\Theta_0$ only through $\lambda=\theta_2-\theta_1\in[0,\infty)$.

The following lemma will be useful in proving the main results of the paper. The proof of the lemma is straight forward and hence omitted.

Lemma 2.1. *Let $s_0\in\Re$ and let $M:\Re\to\Re$ be such that $M(s)\le 0$, $\forall\, s<s_0$, and $M(s)\ge 0$, $\forall\, s>s_0$. Let $M_i:\Re\to[0,\infty)$, $i=1,2$, be non-negative functions such that $M_1(s)M_2(s_0)\ge(\le)M_1(s_0)M_2(s)$, $\forall\, s<s_0$, and $M_1(s)M_2(s_0)\le(\ge)M_1(s_0)M_2(s)$, $\forall\, s>s_0$. Then,*

$$M_2(s_0)\int_{-\infty}^{\infty}M(s)M_1(s)\,ds\le(\ge)M_1(s_0)\int_{-\infty}^{\infty}M(s)M_2(s)\,ds.$$

The facts stated in the following lemma are well known in the theory of stochastic orders (see Shaked and Shanthikumar (2007)). The proof of the lemma is straight forward, hence skipped.

Lemma 2.2. *If, for any fixed $\Delta\ge 0$ and $t\in\Re$, $h_i(t-\Delta|s)/h_i(t|s)$ is non-decreasing (non-increasing) in $s\in S_i$, then $H_i(t-\Delta|s)/H_i(t|s)$ is non-decreasing (non-increasing) in $s\in S_i$ and $h_i(t|s)/H_i(t|s)$ is non-increasing (non-decreasing) in $s\in S_i$, $i=1,2$.*

In the next section, we consider equivariant estimation of location parameter θ_1 under the loss function L_1, defined by (2.1), when it is known apriori that $\boldsymbol{\theta}\in\Theta_0$. We aim to find estimators that dominate the BLEE $\delta_{c_{0,i},i}(\boldsymbol{X})$, $i=1,2$ (defined through (2.2) and (2.3)) by exploiting the prior information that $\boldsymbol{\theta}\in\Theta_0$.

2.1 Improvements over the BLEE of θ_1

Consider estimation of θ_1 under the loss function $L_1(\boldsymbol{\theta},a)=W(a-\theta_1)$, $\boldsymbol{\theta}\in\Theta_0$, $a\in\Re$, when it is known apriori that $\boldsymbol{\theta}\in\Theta_0$. Throughout this section, we will assume that the function $W(\cdot)$ satisfies assumptions A_1 and A_2.

In the following theorem, we provide a class of estimators that improve upon the BLEE $\delta_{c_{0,1},1}(\boldsymbol{X})=X_1-c_{0,1}$, defined by (2.2) and (2.3).

Theorem 2.1. *Consider a location equivariant estimator $\delta_{\psi_1}(\boldsymbol{X})=X_1-\psi_1(D)$ of θ_1, where $\psi_1(t)$ is non-increasing (non-decreasing) in t, $\lim_{t\to\infty}\psi_1(t)=c_{0,1}$ and $\int_{-\infty}^{\infty}W'(s-\psi_1(t))H_1(t|s)f_1(s)\,ds\ge(\le)\,0$, $\forall\, t$. Then*

$$R_1(\boldsymbol{\theta},\delta_{\psi_1})\le R_1(\boldsymbol{\theta},\delta_{c_{0,1},1}),\quad \forall\ \boldsymbol{\theta}\in\Theta_0.$$

Proof. Let us fix $\boldsymbol{\theta} \in \Theta_0$ and let $\lambda = \theta_2 - \theta_1$, so that $\lambda \geq 0$. Consider the risk difference

$$\begin{aligned}\Delta_1(\lambda) &= R_1(\boldsymbol{\theta}, \delta_{c_{0,1},1}) - R_1(\boldsymbol{\theta}, \delta_{\psi_1})\\ &= E_{\boldsymbol{\theta}}\big[W(Z_1 - c_{0,1}) - W\big(Z_1 - \psi_1(Z+\lambda)\big)\big]\\ &= E_{\boldsymbol{\theta}}\bigg[\int_{Z+\lambda}^{\infty}\bigg\{\frac{d}{dt}W\big(Z_1 - \psi_1(t)\big)\bigg\}\,dt\bigg]\\ &= E_{\boldsymbol{\theta}}\bigg[\int_{Z}^{\infty}\big(-\psi_1'(t+\lambda)\big)W\big(Z_1 - \psi_1(t+\lambda)\big)\,dt\bigg]\\ &= -\int_{-\infty}^{\infty}\psi_1'(t+\lambda)E_{\boldsymbol{\theta}}\big[W'\big(Z_1 - \psi_1(t+\lambda)\big)I_{(-\infty,t]}(Z)\big]\,dt,\end{aligned}$$

where, for any set A, $I_A(\cdot)$ denotes its indicator function. Since $\psi_1(t)$ is a non-increasing (non-decreasing) function of t, it suffices to show that, for every t and $\lambda \geq 0$,

$$E_{\boldsymbol{\theta}}\big[W'\big(Z_1 - \psi_1(t+\lambda)\big)I_{(-\infty,t]}(Z)\big] \geq (\leq)\, 0. \tag{2.6}$$

Since $W'(t)$ is non-decreasing function of t and $\psi_1(t)$ is a non-increasing (non-decreasing) function of t, for $\lambda \geq 0$, we have

$$\begin{aligned}E_{\boldsymbol{\theta}}\big[W'\big(Z_1 - \psi_1(t+\lambda)\big)I_{(-\infty,t]}(Z)\big] &\geq (\leq) E_{\boldsymbol{\theta}}\big[W'\big(Z_1 - \psi_1(t)\big)I_{(-\infty,t]}(Z)\big]\\ &= \int_{-\infty}^{\infty} W'\big(s - \psi_1(t)\big)H_1(t|s)f_1(s)\,ds,\end{aligned}$$

which, in turn, implies (2.6). □

Now we will prove two useful corollaries to the above theorem. The following corollary provides the Brewster–Zidek (1974) type (B–Z type) improvement over the BLEE $\delta_{c_{0,1},1}$.

Corollary 2.1. (i) *Suppose that, for any fixed $\Delta \geq 0$ and t, $H_1(t-\Delta|s)/H_1(t|s)$ is non-decreasing (non-increasing) in $s \in S_1$. Further suppose that, for every fixed t, the equation*

$$k_1(c|t) = \int_{-\infty}^{\infty} W'(s-c)H_1(t|s)f_1(s)\,ds = 0$$

has the unique solution $c \equiv \psi_{0,1}(t) \in S_1$. Then

$$R_1(\boldsymbol{\theta}, \delta_{\psi_{0,1}}) \leq R_1(\boldsymbol{\theta}, \delta_{c_{0,1},1}), \quad \forall\ \boldsymbol{\theta} \in \Theta_0,$$

where $\delta_{\psi_{0,1}}(\boldsymbol{X}) = X_1 - \psi_{0,1}(D)$.

(ii) In addition to assumptions of (i) above, suppose that $\psi_{1,1} : \Re \to \Re$ is such that $\psi_{1,1}(t) \leq (\geq) \psi_{0,1}(t)$, $\forall\ t$, $\psi_{1,1}(t)$ is non-increasing (non-decreasing) in t and $\lim_{t\to\infty}\psi_{1,1}(t) = c_{0,1}$. Then

$$R_1(\boldsymbol{\theta}, \delta_{\psi_{1,1}}) \leq R_1(\boldsymbol{\theta}, \delta_{c_{0,1},1}), \quad \forall\ \boldsymbol{\theta} \in \Theta_0,$$

where $\delta_{\psi_{1,1}}(\boldsymbol{X}) = X_1 - \psi_{1,1}(D)$.

Proof. It suffices to show that $\psi_{0,1}(t)$ satisfies conditions of Theorem 2.1. Note that the hypothesis of the corollary, along with the assumption A_2, ensure that $\lim_{t\to\infty}\psi_{0,1}(t) = c_{0,1}$. To show that $\psi_{0,1}(t)$ is a non-increasing (non-decreasing) function of t, suppose that, there exist numbers t_1 and t_2 such that $t_1 < t_2$ and $\psi_{0,1}(t_1) \neq \psi_{0,1}(t_2)$. We have $k_1(\psi_{0,1}(t_1)|t_1) = 0$. Also, using the hypotheses of the corollary and the assumption A_1, it follows that $\psi_{0,1}(t_2)$ is the unique solution of $k_1(c|t_2) = 0$ and $k_1(c|t_2)$ is a non-increasing function of c. Let

$s_0 = \psi_{0,1}(t_1)$, $M(s) = W'(s - s_0) f_1(s)$, $M_1(s) = H_1(t_2|s)$ and $M_2(s) = H_1(t_1|s)$, $s \in S_1$. Then, under assumption A_1, using Lemma 2.1, we get

$$H_1(t_1|\psi_{0,1}(t_1)) \int_{-\infty}^{\infty} W'\big(s - \psi_{0,1}(t_1)\big) H_1(t_2|s) f_1(s)\, ds$$

$$\leq (\geq) H_1\big(t_2|\psi_{0,1}(t_1)\big) \int_{-\infty}^{\infty} W'\big(s - \psi_{0,1}(t_1)\big) H_1(t_1|s) f_1(s)\, ds = 0.$$

$$\Longrightarrow \quad k_1\big(\psi_{0,1}(t_1)|t_2\big) = \int_{-\infty}^{\infty} W'\big(s - \psi_{0,1}(t_1)\big) H_1(t_2|s) f_1(s)\, ds \leq (\geq)\, 0.$$

This implies that $k_1(\psi_{0,1}(t_1)|t_2) < (>)\, 0$, as $k_1(c|t_2) = 0$ has the unique solution $c \equiv \psi_{0,1}(t_2)$ and $\psi_{0,1}(t_1) \neq \psi_{0,1}(t_2)$. Since $k_1(c|t_2)$ is a non-increasing function of c, $k_1(\psi_{0,1}(t_2)|t_2) = 0$ and $k_1(\psi_{0,1}(t_1)|t_2) < (>)\, 0$, it follows that $\psi_{0,1}(t_1) > (<)\, \psi_{0,1}(t_2)$.

The proof of part (ii) is an immediate by-product of Theorem 2.1 using the fact that, for any t, $k_1(c|t)$ is a non-increasing function of $c \in \Re$. □

In the following corollary, we provide the Stein (1964) type improvements over the BLEE $\delta_{c_{0,1},1}(\boldsymbol{X})$.

Corollary 2.2. (i) *Suppose that, for any fixed $\Delta \geq 0$ and t, $h_1(t - \Delta|s)/h_1(t|s)$ is non-decreasing (non-increasing) in $s \in S_1$. Let $\psi_{0,1}(t) \in S_1$ be as defined in Corollary 2.1. In addition suppose that, for any t, the equation*

$$k_2(c|t) = \int_{-\infty}^{\infty} W'(s - c) h_1(t|s) f_1(s)\, ds = 0$$

has the unique solution $c \equiv \psi_{2,1}(t) \in S_1$. Let $\psi_{2,1}^*(t) = \max\{c_{0,1}, \psi_{2,1}(t)\}$ ($\psi_{2,1}^*(t) = \min\{c_{0,1}, \psi_{2,1}(t)\}$) and $\delta_{\psi_{2,1}^*}(\boldsymbol{X}) = X_1 - \psi_{2,1}^*(D)$. Then

$$R_1(\boldsymbol{\theta}, \delta_{\psi_{2,1}^*}) \leq R_1(\boldsymbol{\theta}, \delta_{c_{0,1},1}), \quad \forall\ \boldsymbol{\theta} \in \Theta_0.$$

(ii) *In addition to assumptions of (i) above, suppose that $\psi_{3,1} : \Re \to \Re$ be such that $\psi_{3,1}(t) \leq (\geq) \psi_{2,1}(t)$, $\forall\, t$ and $\psi_{3,1}(t)$ is non-increasing (non-decreasing) in t. Define $\psi_{3,1}^*(t) = \max\{c_{0,1}, \psi_{3,1}(t)\}$ ($\psi_{3,1}^*(t) = \min\{c_{0,1}, \psi_{3,1}(t)\}$) and $\delta_{\psi_{3,1}^*}(\boldsymbol{X}) = X_1 - \psi_{3,1}^*(D)$. Then*

$$R_1(\boldsymbol{\theta}, \delta_{\psi_{3,1}^*}) \leq R_1(\boldsymbol{\theta}, \delta_{c_{0,1},1}), \quad \forall\ \boldsymbol{\theta} \in \Theta_0.$$

Proof. It suffices to show that $\psi_{2,1}^*(\cdot)$ satisfies conditions of Theorem 2.1. Under the assumption that, for any fixed $\Delta \geq 0$ and t, $h_1(t - \Delta|s)/h_1(t|s)$ is non-decreasing (non-increasing) in $s \in S_1$, on following the line of arguments used in proving Corollary 2.1, it can be concluded that $\psi_{2,1}(t)$ (and hence $\psi_{2,1}^*(t)$) is non-increasing (non-decreasing) in t. To show that $\lim_{t\to\infty} \psi_{2,1}^*(t) = c_{0,1}$, we will show that $\psi_{2,1}(t) \leq (\geq) \psi_{0,1}(t)$, $\forall\, t$. Let us fix t. Then $k_1(\psi_{0,1}(t)|t) = k_2(\psi_{2,1}(t)|t) = 0$.

The hypothesis of the corollary and Lemma 2.2, imply that, for every fixed t, $h_1(t|s)/H_1(t|s)$ is non-increasing (non-decreasing) in $s \in S_1$. Let $s_0 = \psi_{0,1}(t)$, $M(s) = W'(s - s_0) f_1(s)$, $M_1(s) = h_1(t|s)$ and $M_2(s) = H_1(t|s)$, $s \in S_1$. Using assumption A_1, the monotonicity of $h_1(t|s)/H_1(t|s)$, Lemma 2.1 and the fact that $k_1(\psi_{0,1}(t)|t) = 0$, we conclude that

$$H_1\big(t|\psi_{0,1}(t)\big) \int_{-\infty}^{\infty} W'\big(s - \psi_{0,1}(t)\big) h_1(t|s) f_1(s)\, ds$$

$$\leq (\geq) h_1\big(t|\psi_{0,1}(t)\big) \int_{-\infty}^{\infty} W'\big(s - \psi_{0,1}(t)\big) H_1(t|s) f_1(s)\, ds = 0$$

$$\Longrightarrow \quad k_2\big(\psi_{0,1}(t)|t\big) = \int_{-\infty}^{\infty} W'\big(s - \psi_{0,1}(t)\big) h_1(t|s) f_1(s)\, ds \leq (\geq)\, 0. \qquad (2.7)$$

Since $k_2(c|t)$ is a non-increasing function of c and $\psi_{2,1}(t)$ is the unique solution of $k_2(c|t) = 0$, using (2.7), we conclude that $\psi_{0,1}(t) \geq (\leq)\psi_{2,1}(t)$. Hence, $c_{0,1} = \lim_{t\to\infty} \psi_{0,1}(t) \geq (\leq) \lim_{t\to\infty} \psi_{2,1}(t)$ and $\lim_{t\to\infty} \psi^*_{2,1}(t) = \max\{c_{0,1}, \lim_{t\to\infty} \psi_{2,1}(t)\} = c_{0,1}$ ($\lim_{t\to\infty} \psi^*_{2,1}(t) = \min\{c_{0,1}, \lim_{t\to\infty} \psi_{2,1}(t)\} = c_{0,1}$). Note that $\psi^*_{2,1}(t) \leq (\geq)\psi_{0,1}(t)$, $\forall\, t$. Since $k_1(c|t)$ is a non-increasing function of c, we have

$$k_1\big(\psi^*_{2,1}(t)|t\big) \geq (\leq) k_1\big(\psi_{0,1}(t)|t\big) = 0, \quad \forall\, t.$$

Hence, the result follows. □

The proof of part (ii) of Corollary 2.2 is immediate from Theorem 2.1 on noting that $\psi^*_{3,1}(t) \leq (\geq)\psi^*_{2,1}(t)$, $\forall\, t$, and $k_1(c|t)$ is a non-increasing function of c, for every t.

Remark 2.1. It is straightforward to see that the Brewster–Zidek (1974) type estimator $\delta_{\psi_{0,1}}$, derived in Corollary 2.1 (i), is the generalized Bayes estimator with respect to the non-informative prior density $\pi(\theta_1, \theta_2) = 1$, $(\theta_1, \theta_2) \in \Theta_0$.

The results reported in Theorem 2.1, Corollary 2.1 (i) and Corollary 2.2 (i) are extensions of results proved by Kubokawa and Saleh (1994) for the special case when X_1 and X_2 are independently distributed. The results for estimating the larger location parameter θ_2 can be obtained along the same lines. For brevity, in the following section, we state these results without providing their proofs.

2.2 Improvements over the BLEE of θ_2

Under assumptions A_1 and A_2, consider estimation of θ_2 under the loss function $L_2(\boldsymbol{\theta}, a) = W(a - \theta_2)$, $\boldsymbol{\theta} \in \Theta_0$, $a \in \Re$, when it is known apriori that $\boldsymbol{\theta} \in \Theta_0$.

The following theorem provides a class of estimators that improve upon the BLEE, $\delta_{c_{0,2},2}(\boldsymbol{X}) = X_2 - c_{0,2}$, of θ_2, defined by (2.2) and (2.3).

Theorem 2.2. *Let $\delta_{\psi_2}(\boldsymbol{X}) = X_2 - \psi_2(D)$ be a location equivariant estimator of θ_2 such that $\psi_2(t)$ is non-decreasing (non-increasing) in t, $\lim_{t\to\infty} \psi_2(t) = c_{0,2}$ and $\int_{-\infty}^{\infty} W'(s - \psi_2(t))H_2(t|s) f_2(s)\, ds \leq (\geq)\, 0$, $\forall\, t$. Then*

$$R_2(\boldsymbol{\theta}, \delta_{\psi_2}) \leq R_2(\boldsymbol{\theta}, \delta_{c_{0,2},2}), \quad \forall\ \boldsymbol{\theta} \in \Theta_0.$$

The following corollary provides the B–Z type improvements over the BLEE $\delta_{c_{0,2},2}(\cdot)$.

Corollary 2.3. *Suppose that, for any fixed $\Delta \geq 0$ and t, $H_2(t - \Delta|s)/H_2(t|s)$ is non-increasing (non-decreasing) in $s \in S_2$. Further suppose that, for every fixed t, the equation*

$$k_3(c|t) = \int_{-\infty}^{\infty} W'(s - c)H_2(t|s) f_2(s)\, ds = 0$$

has the unique solution $c \equiv \psi_{0,2}(t)$.

(i) *Let $\delta_{\psi_{0,2}}(\boldsymbol{X}) = X_2 - \psi_{0,2}(D)$. Then*

$$R_2(\boldsymbol{\theta}, \delta_{\psi_{0,2}}) \leq R_2(\boldsymbol{\theta}, \delta_{c_{0,2},2}), \quad \forall\ \boldsymbol{\theta} \in \Theta_0.$$

(ii) *Let $\psi_{1,2} : \Re \to \Re$ be such that $\psi_{1,2}(t) \geq (\leq)\psi_{0,2}(t)$, $\forall\, t$, $\psi_{1,2}(t)$ is non-decreasing (non-increasing) in t and $\lim_{t\to\infty} \psi_{1,2}(t) = c_{0,2}$. Then*

$$R_2(\boldsymbol{\theta}, \delta_{\psi_{1,2}}) \leq R_2(\boldsymbol{\theta}, \delta_{c_{0,2},2}), \quad \forall\ \boldsymbol{\theta} \in \Theta_0,$$

where $\delta_{\psi_{1,2}}(\boldsymbol{X}) = X_2 - \psi_{1,2}(D)$.

In the following corollary we provide the Stein type improvement over the BLEE $\delta_{c_{0,2},2}(\boldsymbol{X})$.

Corollary 2.4. *Suppose that, for any fixed* $\Delta \geq 0$ *and* t, $h_2(t-\Delta|s)/h_2(t|s)$ *is non-increasing (non-decreasing) in* $s \in S_2$ *and let* $\psi_{0,2}(t)$ *be as defined in Corollary 2.3. Further suppose that, for every* t, *the equation*

$$k_4(c|t) = \int_{-\infty}^{\infty} W'(s-c)h_2(t|s)f_2(s)\,ds = 0$$

has the unique solution $c \equiv \psi_{2,2}(t)$.

(i) *Let* $\psi^*_{2,2}(t) = \min\{c_{0,2}, \psi_{2,2}(t)\}$ $(\psi^*_{2,2}(t) = \max\{c_{0,2}, \psi_{2,2}(t)\})$ *and* $\delta_{\psi^*_{2,2}}(\boldsymbol{X}) = X_2 - \psi^*_{2,2}(D)$. *Then*

$$R_2(\boldsymbol{\theta}, \delta_{\psi^*_{2,2}}) \leq R_2(\boldsymbol{\theta}, \delta_{c_{0,2},2}), \quad \forall\ \boldsymbol{\theta} \in \Theta_0.$$

(ii) *Suppose that* $\psi_{3,2} : \Re \to \Re$ *is such that* $\psi_{3,2}(t) \geq (\leq)\psi_{2,2}(t)$, $\forall\, t$ *and* $\psi_{3,2}(t)$ *is non-decreasing (non-increasing) in* t. *Define* $\psi^*_{3,2}(t) = \min\{c_{0,2}, \psi_{3,2}(t)\}$ $(\psi^*_{3,2}(t) = \max\{c_{0,2}, \psi_{3,2}(t)\})$ *and* $\delta_{\psi^*_{3,2}}(\boldsymbol{X}) = X_2 - \psi^*_{3,2}(D)$. *Then*

$$R_2(\boldsymbol{\theta}, \delta_{\psi^*_{3,2}}) \leq R_2(\boldsymbol{\theta}, \delta_{c_{0,2},2}), \quad \forall\ \boldsymbol{\theta} \in \Theta_0.$$

Proofs of Theorem 2.2 and Corollaries 2.3–2.4 are on the same lines as proofs of Theorem 2.1 and Corollaries 2.1–2.2, respectively. For brevity, these proofs have been provided in Section 2 of the supplementary material (Garg and Misra (2022)).

It is straightforward to see that the B–Z type estimator $\delta_{\psi_{0,2}}(\cdot)$, derived in Corollary 2.3 (i), is the generalised Bayes estimator with respect to the non-informative prior density $\pi(\theta_1, \theta_2) = 1$, $(\theta_1, \theta_2) \in \Theta_0$. Theorems 2.1–2.2 (or Corollaries 2.1–2.2 and Corollaries 2.3–2.4) are applicable to a variety of situations studied in the literature for specific probability models, having independent marginals, and specific loss functions (e.g., Kushary and Cohen (1989), Misra and Singh (1994), Vijayasree, Misra and Singh (1995), Misra, Iyer and Singh (2004), etc.). Theorems 2.1–2.2 (or Corollaries 2.1–2.2 and Corollaries 2.3–2.4) also extend the results of Kubokawa and Saleh (1994) to general bivariate location models.

2.3 Applications

In the sequel we demonstrate an application of Theorems 2.1–2.2 (or Corollaries 2.1–2.2 and Corollaries 2.3–2.4) to a situation where results of Kubokawa and Saleh (1994) are not applicable.

Example 2.1. Let $\boldsymbol{X} = (X_1, X_2)$ have the bivariate normal distribution with joint pdf given by (1.1), where, for known $\sigma_i > 0$, $i = 1, 2$, and $\rho \in (-1, 1)$,

$$f(z_1, z_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{z_1^2}{\sigma_1^2} - 2\rho\frac{z_1 z_2}{\sigma_1\sigma_2} + \frac{z_2^2}{\sigma_2^2}\right]}, \quad \boldsymbol{z} = (z_1, z_2) \in \Re^2.$$

Consider estimation of location parameter θ_i, $i = 1, 2$, under the squared error loss function (i.e., $W(t) = t^2$, $t \in \Re$). Here the BLEE of θ_i is $\delta_{0,i}(\boldsymbol{X}) = X_i$ (i.e., $c_{0,i} = 0$), $i = 1, 2$. Also, for any $s \in \Re$, and $t \in \Re$, $h_i(t|s) = \frac{1}{\xi_i}\phi\left(\frac{t - \frac{s\mu_i}{\sigma_i}}{\xi_i}\right)$ and $H_i(t|s) = \Phi\left(\frac{t - \frac{s\mu_i}{\sigma_i}}{\xi_i}\right)$, $i = 1, 2$, where $\mu_1 = \rho\sigma_2 - \sigma_1$, $\mu_2 = \sigma_2 - \rho\sigma_1$, $\xi_1^2 = (1-\rho^2)\sigma_2^2$ and $\xi_2^2 = (1-\rho^2)\sigma_1^2$. For $\mu_i < (>)0$, it is easy to verify that, for any fixed $\Delta \geq 0$ and $t \in \Re$, $h_i(t-\Delta|s)/h_i(t|s)$ is non-decreasing (non-increasing) in $s \in \Re$, $i = 1, 2$. Using Lemma 2.2, this ensures that, for any fixed $\Delta \geq 0$

and $t \in \Re$, $H_i(t-\Delta|s)/H_i(t|s)$ is non-decreasing (non-increasing) in $s \in \Re$, $i = 1, 2$. For any $t \in \Re$,

$$\psi_{0,i}(t) = \frac{\int_{-\infty}^{\infty} s H_i(t|s) f_i(s)\, ds}{\int_{-\infty}^{\infty} H_i(t|s) f_i(s)\, ds} = \frac{\int_{-\infty}^{\infty} s \Phi(\frac{t-\frac{s\mu_i}{\sigma_i}}{\xi_i}) \frac{1}{\sigma_i}\phi(\frac{s}{\sigma_i})\, ds}{\int_{-\infty}^{\infty} \Phi(\frac{t-\frac{s\mu_i}{\sigma_i}}{\xi_i}) \frac{1}{\sigma_i}\phi(\frac{s}{\sigma_i})\, ds}, \quad i = 1, 2,$$

$$\text{and} \quad \psi_{2,i}(t) = \frac{\int_{-\infty}^{\infty} s h_i(t|s) f_i(s)\, ds}{\int_{-\infty}^{\infty} h_i(t|s) f_i(s)\, ds} = \frac{\int_{-\infty}^{\infty} \frac{s}{\xi_i} \phi(\frac{t-\frac{s\mu_i}{\sigma_i}}{\xi_i}) \frac{1}{\sigma_i}\phi(\frac{s}{\sigma_i})\, ds}{\int_{-\infty}^{\infty} \frac{1}{\xi_i} \phi(\frac{t-\frac{s\mu_i}{\sigma_i}}{\xi_i}) \frac{1}{\sigma_i}\phi(\frac{s}{\sigma_i})\, ds}, \quad i = 1, 2.$$

It is easy to verify that $\psi_{0,1}(t) = -(\beta_0 - 1)\tau \frac{\phi(\frac{t}{\tau})}{\Phi(\frac{t}{\tau})}$, $t \in \Re$, $\psi_{0,2}(t) = -\beta_0 \tau \frac{\phi(\frac{t}{\tau})}{\Phi(\frac{t}{\tau})}$, $t \in \Re$, $\psi_{2,1}(t) = (\beta_0 - 1)t$, $t \in \Re$, and $\psi_{2,2}(t) = \beta_0 t$, $t \in \Re$, where $\tau^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$ and $\beta_0 = 1 + \frac{\sigma_1\mu_1}{\tau^2} = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2} = \frac{\sigma_2\mu_2}{\tau^2}$.

Estimation of θ_1:

For $\mu_1 < 0$, i.e., $\rho < \frac{\sigma_1}{\sigma_2}$ ($\mu_1 > 0$, i.e., $\rho > \frac{\sigma_1}{\sigma_2}$), we have $\beta_0 < (>)\, 1$, $\lim_{t\to\infty} \psi_{0,1}(t) = 0 = c_{0,1}$ and, $\psi_{0,1}(t)$ and $\psi_{2,1}(t)$ are non-increasing (non-decreasing) functions of $t \in \Re$. Thus, functions $\psi_{0,1}(t)$ and $\psi_{2,1}(t)$ satisfy hypotheses of Theorem 2.1 and Corollaries 2.1 and 2.2. For $\mu_1 < 0$, i.e., $\rho < \frac{\sigma_1}{\sigma_2}$ ($\mu_1 > 0$, i.e., $\rho > \frac{\sigma_1}{\sigma_2}$), we have

$$\psi_{2,1}^*(t) = \max\{0, \psi_{2,1}(t)\} = \begin{cases} (\beta_0 - 1)t, & \text{if } t < 0 \\ 0, & \text{if } t \geq 0 \end{cases}$$

$$\left(\psi_{2,1}^*(t) = \min\{0, \psi_{2,1}(t)\} = \begin{cases} (\beta_0 - 1)t, & \text{if } t \leq 0 \\ 0, & \text{if } t > 0 \end{cases} \right).$$

Using Corollaries 2.1 (i) and 2.2 (i), we obtain the B–Z type and the Stein type improvements over the BLEE $\delta_{\psi_{0,1}}(\boldsymbol{X}) = X_1$ as

$$\delta_{\psi_{0,1}}(\boldsymbol{X}) = X_1 - \psi_{0,1}(D) = X_1 + \frac{\sigma_1\mu_1}{\tau} \frac{\phi(\frac{D}{\tau})}{\Phi(\frac{D}{\tau})} = X_1 + (\beta_0 - 1)\tau \frac{\phi(\frac{D}{\tau})}{\Phi(\frac{D}{\tau})} \tag{2.8}$$

$$\text{and} \quad \delta_{\psi_{2,1}^*}(\boldsymbol{X}) = X_1 - \psi_{2,1}^*(D) = \begin{cases} X_1, & \text{if } X_1 \leq X_2 \\ \beta_0 X_1 + (1-\beta_0)X_2, & \text{if } X_1 > X_2 \end{cases}, \tag{2.9}$$

respectively, where $\beta_0 = 1 + \frac{\sigma_1\mu_1}{\tau^2} = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}$.

It is worth mentioning here that $\delta_{\psi_{0,1}}(\cdot)$ is the generalized Bayes estimator θ_1 with respect to non-informative prior on Θ_0 and $\delta_{\psi_{2,1}^*}(\cdot)$ is the restricted maximum likelihood estimator of θ_1 (see Patra and Kumar (2017)).

Note that, when $\mu_1 = 0$ (i.e., $\rho = \frac{\sigma_1}{\sigma_2}$ and $\beta_0 = 1$), we have $\psi_{0,1}(t) = \psi_{2,1}(t) = \psi_{2,1}^*(t) = c_{0,1} = 0$, $\forall\, t \in \Re$. Thus, for $\rho = \frac{\sigma_1}{\sigma_2}$, we are not able to get improvements over the BLEE using our results. Interestingly, in this case, the BLEE is also the restricted maximum likelihood estimator and the generalized Bayes estimator with respect to non-informative prior on Θ_0.

From the above discussion we conclude that, for $\rho \neq \frac{\sigma_1}{\sigma_2}$, the generalized Bayes estimator $\delta_{\psi_{0,1}}(\cdot)$ and the restricted MLE $\delta_{\psi_{2,1}^*}(\cdot)$ dominate the BLEE $\delta_{0,1}(\boldsymbol{X}) = X_1$.

Now, we will illustrate an application of Corollary 2.1 (ii). Define

$$\psi_{1,1,\alpha}(t) = \tau(1-\alpha)\frac{\phi(\frac{t}{\tau})}{\Phi(\frac{t}{\tau})}, \quad t \in \Re, \alpha \in \Re.$$

For $\mu_1 < 0$ and $\beta_0 \le \alpha < 1$ ($\mu_1 > 0$ and $1 < \alpha \le \beta_0$), note that $\beta_0 < 1$ ($\beta_0 > 1$), $\psi_{1,1,\alpha}(t)$ is a non-increasing (non-decreasing) functions of $t \in \Re$, $\lim_{t\to\infty} \psi_{1,1,\alpha}(t) = 0 = c_{0,1}$ and $\psi_{1,1,\alpha}(t) \le (\ge) \psi_{0,1}(t)$, $\forall\, t \in \Re$. Let

$$\delta_{\psi_{1,1,\alpha}}(\boldsymbol{X}) = X_1 - \psi_{1,1,\alpha}(D) = X_1 - \tau(1-\alpha)\frac{\phi(\frac{D}{\tau})}{\Phi(\frac{D}{\tau})}, \quad \alpha \in \Re.$$

Using Corollary 2.1 (ii) it follows that, for $\mu_1 < (>) 0$ (i.e., $\rho < (>) \frac{\sigma_1}{\sigma_2}$), the estimators $\{\delta_{\psi_{1,1,\alpha}} : \beta_0 \le \alpha < 1\}$ ($\{\delta_{\psi_{1,1,\alpha}} : 1 < \alpha \le \beta_0\}$) dominate the BLEE $\delta_{0,1}(\boldsymbol{X}) = X_1$.

To see an application of Corollary 2.2 (ii), let

$$\psi_{3,1,\alpha}(t) = \begin{cases} (\alpha-1)t, & \text{if } t < 0 \\ (\beta_0 - 1)t, & \text{if } t \ge 0 \end{cases}, \quad \alpha \in \Re.$$

For $\mu_1 < 0$ and $\beta_0 \le \alpha < 1$ ($\mu_1 > 0$ and $1 < \alpha \le \beta_0$), note that $\beta_0 < 1$ ($\beta_0 > 1$), $\psi_{3,1,\alpha}(t) \le (\ge) \psi_{2,1}(t) = (\beta_0 - 1)t$, $\forall\, t \in \Re$, and $\psi_{3,1,\alpha}(t)$ is a non-increasing (non-decreasing) function of $t \in \Re$. Let

$$\psi^*_{3,1,\alpha}(t) = \max\{0, \psi_{3,1,\alpha}(t)\}(\min\{0, \psi_{3,1,\alpha}(t)\}) = \begin{cases} (\alpha-1)t, & \text{if } t < 0 \\ 0, & \text{if } t \ge 0 \end{cases}$$

$$\text{and} \quad \delta_{\psi^*_{3,1,\alpha}}(\boldsymbol{X}) = X_1 - \psi^*_{3,1,\alpha}(D) = \begin{cases} \alpha X_1 + (\alpha-1)X_2, & \text{if } X_2 < X_1 \\ X_1, & \text{if } X_2 \ge X_1 \end{cases}, \quad \alpha \in \Re.$$

Using Corollary 2.2 (ii), it follows that, for $\rho < (>) \frac{\sigma_1}{\sigma_2}$, the estimators $\{\delta_{\psi^*_{3,1,\alpha}} : \beta_0 \le \alpha < 1\}$ ($\{\delta_{\psi^*_{3,1,\alpha}} : 1 < \alpha \le \beta_0\}$) dominate the BLEE $\delta_{0,1}(\boldsymbol{X}) = X_1$.

Estimation of θ_2:

For $\mu_2 < 0$, that is, $\rho > \frac{\sigma_2}{\sigma_1}$ ($\mu_2 > 0$, that is, $\rho < \frac{\sigma_2}{\sigma_1}$), we have $\beta_0 < (>) 0$, $\lim_{t\to\infty} \psi_{0,2}(t) = 0 = c_{0,2}$ and, $\psi_{0,2}(t)$ and $\psi_{2,2}(t)$ are non-increasing (non-decreasing) functions of $t \in \Re$. Let

$$\psi^*_{2,2}(t) = \max\{0, \psi_{2,2}(t)\} = \begin{cases} \beta_0 t, & \text{if } t \le 0 \\ 0, & \text{if } t > 0 \end{cases}$$

$$\left(\psi^*_{2,2}(t) = \min\{0, \psi_{2,2}(t)\} = \begin{cases} \beta_0 t, & \text{if } t \le 0 \\ 0, & \text{if } t > 0 \end{cases}\right).$$

Applications of Corollaries 2.3 (i) and 2.4 (i), yield the B–Z type and the Stein type improvements over the BLEE $\delta_{0,2}(\boldsymbol{X}) = X_2$ as

$$\delta_{\psi_{0,2}}(\boldsymbol{X}) = X_2 - \psi_{0,2}(D) = X_2 + \frac{\sigma_2 \mu_2}{\tau}\frac{\phi(\frac{D}{\tau})}{\Phi(\frac{D}{\tau})} = X_2 + \beta_0 \tau \frac{\phi(\frac{D}{\tau})}{\Phi(\frac{D}{\tau})} \tag{2.10}$$

$$\text{and} \quad \delta_{\psi^*_{2,2}}(\boldsymbol{X}) = X_2 - \psi^*_{2,2}(D) = \begin{cases} \beta_0 X_1 + (1-\beta_0)X_2, & \text{if } X_2 \le X_1 \\ X_2, & \text{if } X_2 > X_1 \end{cases}, \tag{2.11}$$

respectively. Note that $\delta_{\psi_{0,2}}(\cdot)$ is the generalized Bayes estimators of θ_2 under the non-informative prior on Θ_0 and $\delta_{\psi^*_{2,2}}(\cdot)$ is the restricted MLE of θ_2.

For $\mu_2 = 0$ (that is, $\rho = \frac{\sigma_2}{\sigma_1}$), we have $\beta_0 = 0$ and $\psi_{0,2}(t) = \psi_{2,2}(t) = \psi^*_{2,2}(t) = 0$, $\forall\, t \in \Re$. Thus, for $\mu_2 = 0$, our results do not provide improvements over the BLEE $\delta_{\psi_{0,2}}(\boldsymbol{X}) = X_2$. In this case, the BLEE is also the restricted maximum likelihood estimator and the generalized Bayes estimator with respect to non-informative prior on Θ_0.

From the above discussion we conclude that, for $\rho \ne \frac{\sigma_2}{\sigma_1}$, the generalized Bayes estimator $\delta_{\psi_{0,2}}(\cdot)$ and the restricted MLE $\delta_{\psi^*_{2,2}}(\cdot)$ dominate the BLEE $\delta_{0,2}(\boldsymbol{X}) = X_2$.

To see an application of Corollary 2.3 (ii), define

$$\psi_{1,2,\alpha}(t) = -\alpha\tau \frac{\phi(\frac{t}{\tau})}{\Phi(\frac{t}{\tau})}, \quad t \in \Re, \alpha \in \Re.$$

For $\mu_2 < 0$ and $\beta_0 \leq \alpha < 0$ ($\mu_2 > 0$ and $0 < \alpha \leq \beta_0$), note that $\beta_0 < (>)\, 0$, $\psi_{1,2,\alpha}(t)$ is a non-increasing (non-decreasing) function of $t \in \Re$, $\lim_{t\to\infty} \psi_{1,2,\alpha}(t) = 0 = c_{0,2}$ and $\psi_{1,2,\alpha}(t) \leq (\geq) \psi_{0,2}(t) = \beta_0 \tau \frac{\phi(\frac{t}{\tau})}{\Phi(\frac{t}{\tau})}$, $\forall\, t \in \Re$. Define

$$\delta_{\psi_{1,2,\alpha}}(\boldsymbol{X}) = X_2 - \psi_{2,2,\alpha}(D) = X_2 + \alpha\tau \frac{\phi(\frac{D}{\tau})}{\Phi(\frac{D}{\tau})}, \quad \alpha \in \Re.$$

Using Corollary 2.3 (ii), for $\rho > (<) \frac{\sigma_2}{\sigma_1}$, it follows that the estimators $\{\delta_{\psi_{1,2,\alpha}} : \beta_0 \leq \alpha < 0\}$ ($\{\delta_{\psi_{1,2,\alpha}} : 0 < \alpha \leq \beta_0\}$) dominate the BLEE $\delta_{0,2}(\boldsymbol{X}) = X_2$.

Now consider an application of Corollary 3.4 (ii). Define

$$\psi_{3,2,\alpha}(t) = \begin{cases} \alpha t, & \text{if } t < 0 \\ \beta_0 t, & \text{if } t \geq 0 \end{cases}, \quad \alpha \in \Re.$$

For $\mu_2 < 0$ and $\beta_0 \leq \alpha < 0$ ($\mu_2 > 0$ and $0 < \alpha \leq \beta_0$) note that, $\beta_0 < (>)\, 0$, $\psi_{3,2,\alpha}(t)$ is non-increasing (non-decreasing) in $t \in \Re$ and $\psi_{3,2,\alpha}(t) \leq (\geq) \psi_{2,2}(t) = \beta_0 t$, $\forall\, t \in \Re$. Let

$$\psi^*_{3,2,\alpha}(t) = \max\{0, \psi_{3,2,\alpha}(t)\}\left(\min\{0, \psi_{3,2,\alpha}(t)\}\right) = \begin{cases} \alpha t, & \text{if } t < 0 \\ 0, & \text{if } t \geq 0 \end{cases}$$

$$\text{and} \quad \delta_{\psi^*_{3,2,\alpha}}(\boldsymbol{X}) = X_2 - \psi^*_{3,2,\alpha}(D) = \begin{cases} \alpha X_1 + (\alpha-1)X_2, & \text{if } X_2 < X_1 \\ X_2, & \text{if } X_2 \geq X_1. \end{cases}$$

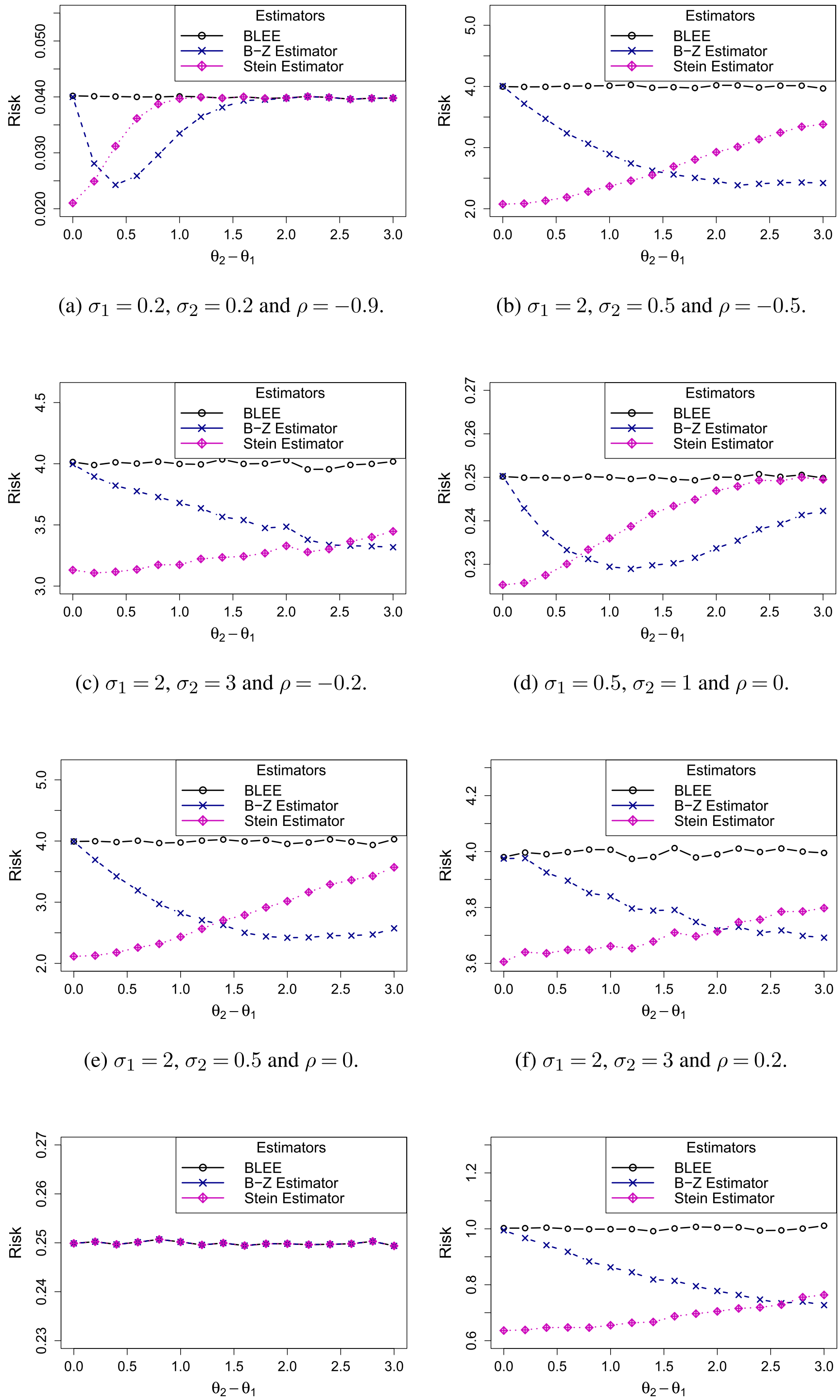
Using Corollary 2.4 (ii), it follows that, for $\rho > (<) \frac{\sigma_2}{\sigma_1}$, the class of the estimators $\{\delta_{\psi^*_{3,2,\alpha}} : \beta_0 \leq \alpha < 0\}$ ($\{\delta_{\psi^*_{3,2,\alpha}} : 0 < \alpha \leq \beta_0\}$) dominate the BLEE $\delta_{0,2}(\boldsymbol{X}) = X_2$.

2.4 Simulation study for estimation of location parameters $\boldsymbol{\theta_1}$ and $\boldsymbol{\theta_2}$

In Example 2.1, under the squared error loss function, we have considered component-wise estimation of the smaller mean θ_1 and the larger mean θ_2 of a bivariate normal distribution with unknown order restricted means (that is, $\theta_1 \leq \theta_2$), known variances (σ_1^2 and σ_2^2) and known correlation coefficient (ρ), and obtained improvement over the BLEEs $\delta_{0,1}(\boldsymbol{X}) = X_1$ and $\delta_{0,2}(\boldsymbol{X}) = X_2$ of θ_1 and θ_2, receptively. To further evaluate the performances of various estimators of θ_1 under the squared error loss function, in this section, we compare the risk performances of estimators BLEE $\delta_{0,1}(\boldsymbol{X}) = X_1$, the B–Z estimator $\delta_{\psi_{0,1}}$ and the Stein (1964) type estimator $\delta_{\psi^*_{2,1}}$ (as defined in (2.8) and (2.9)), numerically, through Monte Carlo simulations. Similarly, to evaluate the performances of various estimators of θ_2 under the squared error loss function, we compare the risk performances of estimators BLEE $\delta_{0,2}(\boldsymbol{X}) = X_2$, the B–Z estimator $\delta_{\psi_{0,2}}$ and the Stein (1964) type estimator $\delta_{\psi^*_{2,2}}$ (as defined in (2.10) and (2.11)), numerically, through Monte Carlo simulations. The simulated risks of the BLEE, the B–Z estimator and the Stein estimator (restricted MLE) have been computed based on 50,000 simulations from relevant distributions. Note that the B–Z estimator is the generalized Bayes estimator and the Stein estimator is the restricted MLE under $\boldsymbol{\theta} \in \Theta_0$.

The simulated values of risks of various estimators of θ_1 are plotted in Figure 1. The following observations are evident from Figure 1:

(i) The risk function values of the B–Z type and the Stein type estimators are nowhere larger than the risk function values of the BLEE, which is in conformity with theoretical findings of Example 2.1.



(a) $\sigma_1 = 0.2$, $\sigma_2 = 0.2$ and $\rho = -0.9$.

(b) $\sigma_1 = 2$, $\sigma_2 = 0.5$ and $\rho = -0.5$.

(c) $\sigma_1 = 2$, $\sigma_2 = 3$ and $\rho = -0.2$.

(d) $\sigma_1 = 0.5$, $\sigma_2 = 1$ and $\rho = 0$.

(e) $\sigma_1 = 2$, $\sigma_2 = 0.5$ and $\rho = 0$.

(f) $\sigma_1 = 2$, $\sigma_2 = 3$ and $\rho = 0.2$.

(g) $\sigma_1 = 0.5$, $\sigma_2 = 1$ and $\rho = 0.5$.

(h) $\sigma_1 = 1$, $\sigma_2 = 5$ and $\rho = 0.9$.

Figure 1 *Estimators of θ_1: Risk plots of $\delta_{0,1}$ (BLEE), $\delta_{\psi_{0,1}}$ (B–Z type estimator) and $\delta_{\psi^*_{2,1}}$ (Stein type estimator) estimators against the values of $\theta_2 - \theta_1$.*

(ii) There is no clear cut winner between the B–Z type estimator $\delta_{\psi_{0,1}}$ and the Stein type estimator $\delta_{\psi_{2,1}^*}$. The Stein type estimator performs better than the B–Z type estimator, for small values of $\theta_2 - \theta_1$, and the B–Z type estimator dominates the Stein type estimator for the large values of $\theta_2 - \theta_1$.

The simulated values of risks of various estimators of θ_2 are plotted in Figure 2. Similar observations, as mentioned above for three estimators of θ_1, are evident from Figure 2.

3 Improving the Best Scale Equivariant Estimators (BSEEs)

In this section, we consider the bivariate scale model (1.2) and deal with the problem of estimating scale parameters θ_i, $i = 1, 2$, when it is known apriori that $\boldsymbol{\theta} \in \Theta_0 = \{(x, y) \in \Re^2_{++} : x \leq y\}$. The following notations will be used throughout this section. Let $Z_i = \frac{X_i}{\theta_i}$, $i = 1, 2$, $\boldsymbol{Z} = (Z_1, Z_2)$ and $Z = \frac{Z_2}{Z_1}$. The pdf of $\boldsymbol{Z} = (Z_1, Z_2)$ is $f(z_1, z_2)$, $(z_1, z_2) \in \Re^2$. Let S_i denote the support of random variable Z_i, $i = 1, 2$. Under the above notations, assume that $\{(z_1, z_2) \in \Re^2 : f(z_1, z_2) > 0\} \subseteq \Re^2_{++}$, so that $S_1 \subseteq \Re_{++}$ and $S_2 \subseteq \Re_{++}$. Let f_i denote the pdf of Z_i, $i = 1, 2$, so that $f_1(s) = \int_0^\infty f(s, t)\,dt$, $s \in \Re_{++}$ and $f_2(s) = \int_0^\infty f(t, s)\,dt$, $s \in \Re_{++}$.

For any $s \in S_i$, let $Z_s^{(i)}$ denote a random variable having the same distribution as the conditional distribution of Z given $Z_i = s$, $i = 1, 2$. Then, the pdf and the df of $Z_s^{(1)}$ $(s \in S_1)$ are given by

$$h_1(t|s) = s\frac{f(s, st)}{f_1(s)}, \quad t \in \Re_{++}, \quad \text{and} \quad H_1(t|s) = \int_0^t h_1(z|s)\,dz, \quad t \in \Re_{++},$$

respectively, and the pdf and the df of $Z_s^{(2)}$ $(s \in S_2)$ are given by

$$h_2(t|s) = \frac{s}{t^2}\frac{f(\frac{s}{t}, s)}{f_2(s)}, \quad t \in \Re_{++}, \quad \text{and} \quad H_2(t|s) = \int_0^t h_2(z|s)\,dz, \quad t \in \Re_{++},$$

respectively.

For the scale model (1.2), consider estimation of the scale parameter θ_i under the loss function

$$L_i(\boldsymbol{\theta}, a) = W\left(\frac{a}{\theta_i}\right), \quad \boldsymbol{\theta} \in \Theta, a \in \mathcal{A} = \Re_{++}, i = 1, 2, \tag{3.1}$$

where $W : \Re \to [0, \infty)$ is a specified non-negative function. Throughout, we make the following assumptions on the function $W(\cdot)$:

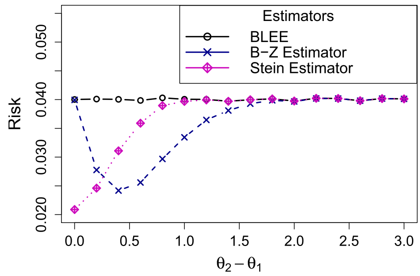
$\boldsymbol{A_3}$: $W : \Re \to [0, \infty)$ is absolute continuous, $W(1) = 0$, $W(t)$ is decreasing on $(-\infty, 1)$ and increasing on $(1, \infty)$. Further $W'(t)$ is non-decreasing on the set D_0 (the set of points at which $W(\cdot)$ is differentiable).

$\boldsymbol{A_4}$: The equation $E[Z_i W'(cZ_i)] = 0$ has the unique solution, say $c = c_{0,i}$, $i = 1, 2$.

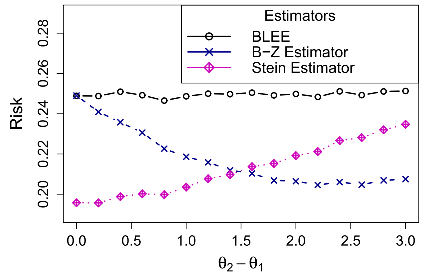
Under the unrestricted case $(\Theta = \Re_{++})$, the problem of estimating θ_i, under the loss function (3.1), is invariant under the multiplicative group of transformations $\mathcal{G}_0 = \{g_{b_1,b_2} : (b_1, b_2) \in \Re^2_{++}\}$, where $g_{b_1,b_2}(x_1, x_2) = (b_1x_1, b_2x_2)$, $(x_1, x_2) \in \Re^2$, $(b_1, b_2) \in \Re^2_{++}$, and the best scale equivariant estimator of θ_i is $\delta_{c_{0,i},i}(\boldsymbol{X}) = c_{0,i}X_i$, $i = 1, 2$, where $c_{0,i}$ is the unique solution of the equation $\int_0^\infty sW'(cs)f_i(s)\,ds = 0$, $i = 1, 2$.

Under the restricted parameter space Θ_0, the problem of estimating θ_i, under the loss function (3.1), is invariant under the group of transformations $\mathcal{G} = \{g_b : b \in (0, \infty)\}$, where $g_b(x_1, x_2) = (bx_1, bx_2)$, $(x_1, x_2) \in \Re^2$, $b \in (0, \infty)$. Any scale equivariant estimator of θ_i has the form

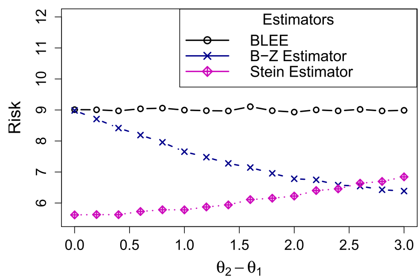
$$\delta_{\psi_i}(\boldsymbol{X}) = \psi_i(D)X_i, \tag{3.2}$$



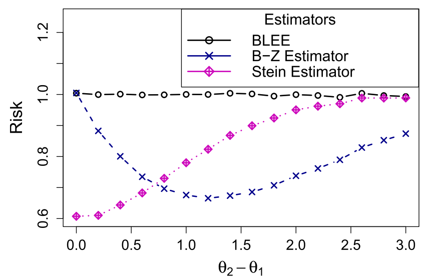
(a) $\sigma_1 = 0.2$, $\sigma_2 = 0.2$ and $\rho = -0.9$.



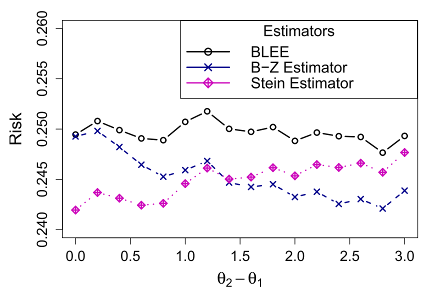
(b) $\sigma_1 = 2$, $\sigma_2 = 0.5$ and $\rho = -0.5$.



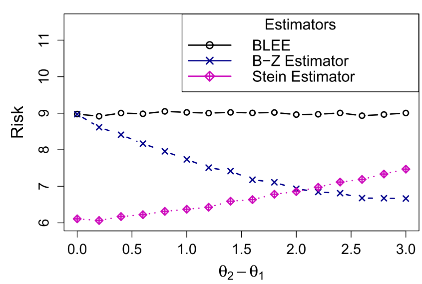
(c) $\sigma_1 = 2$, $\sigma_2 = 3$ and $\rho = -0.2$.



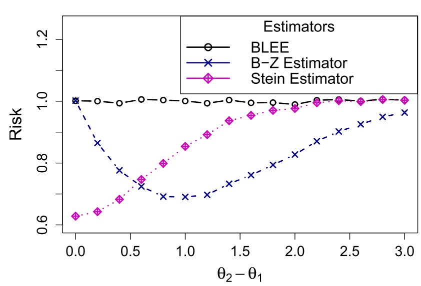
(d) $\sigma_1 = 0.5$, $\sigma_2 = 1$ and $\rho = 0$.



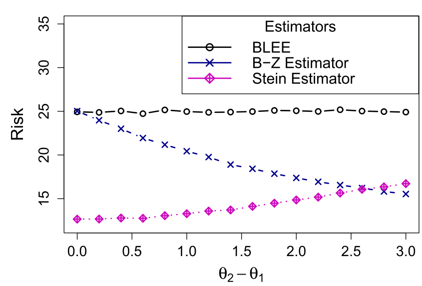
(e) $\sigma_1 = 2$, $\sigma_2 = 0.5$ and $\rho = 0$.



(f) $\sigma_1 = 2$, $\sigma_2 = 3$ and $\rho = 0.2$.



(g) $\sigma_1 = 0.5$, $\sigma_2 = 1$ and $\rho = 0.5$.



(h) $\sigma_1 = 1$, $\sigma_2 = 5$ and $\rho = 0.9$.

Figure 2 *Estimators of θ_2: Risk plots of $\delta_{0,2}$ (BLEE), $\delta_{\psi_{0,2}}$ (B–Z type estimator) and $\delta_{\psi^*_{2,2}}$ (Stein type estimator) estimators against the values of $\theta_2 - \theta_1$.*

for some function $\psi_i : \Re_{++} \to \Re$, $i = 1, 2$, where $D = \frac{X_2}{X_1}$. The risk function

$$R_i(\boldsymbol{\theta}, \delta_{\psi_i}) = E_{\boldsymbol{\theta}}\big[L_i\big(\boldsymbol{\theta}, \delta_{\psi_i}(\boldsymbol{X})\big)\big], \quad \boldsymbol{\theta} \in \Theta_0, \tag{3.3}$$

of any scale equivariant estimator δ_{ψ_i} of θ_i, $i = 1, 2$, depends on $\boldsymbol{\theta} \in \Theta_0$ only through $\lambda = \frac{\theta_2}{\theta_1} \in [1, \infty)$.

The following dual of Lemma 2.2 will be useful in proving the results of this section.

Lemma 3.1. *If, for any fixed $\Delta \geq 1$ and t, $h_i(\frac{t}{\Delta}|s)/h_i(t|s)$ is non-decreasing (non-increasing) in $s \in S_i$, then $H_i(\frac{t}{\Delta}|s)/H_i(t|s)$ is non-decreasing (non-increasing) in $s \in S_i$ and $h_i(t|s)/H_i(t|s)$ is also non-increasing (non-decreasing) in $s \in S_i$, $i = 1, 2$.*

In Section 3.1 (3.2), we consider the equivariant estimation of scale parameter θ_1 (θ_2) under the loss function L_1 (L_2), defined by (3.1), when it is known apriori that $\boldsymbol{\theta} \in \Theta_0$. In Section 3.3, we provide an application of our results to a bivariate gamma distribution, not studied before in the literature. In Section 3.4, we report a simulation study on comparison of various competing estimators for smaller scale parameter in the Cheriyan and Ramabhadran's bivariate gamma distribution.

3.1 Improvements over the BSEE of θ_1

The following theorem provides a class of estimators that improve upon the BSEE $\delta_{c_{0,1},1}(\boldsymbol{X}) = c_{0,1}X_1$, where $c_{0,1}$ is the unique solution of the equation $\int_{-\infty}^{\infty} zW'(cz) \times f_1(z)\,dz = 0$.

Theorem 3.1. *Let $\delta_{\psi_1}(\boldsymbol{X}) = \psi_1(D)X_1$ be a scale equivariant estimator for estimating θ_1 such that $\lim_{t\to\infty}\psi_1(t) = c_{0,1}$, $\psi_1(t)$ is a non-decreasing (non-increasing) function of t and $\int_0^{\infty} sW'(\psi_1(t)s)H_1(t|s)f_1(s)\,ds \geq (\leq) 0$, $\forall\, t$. Then, $\forall\, \boldsymbol{\theta} \in \Theta_0$, the estimator $\delta_{\psi_1}(\boldsymbol{X})$ dominates the BSEE $\delta_{c_{0,1},1}(\boldsymbol{X}) = c_{0,1}X_1$.*

Proof. For $\boldsymbol{\theta} \in \Theta_0$ and $\lambda = \frac{\theta_2}{\theta_1}$ (so that $\lambda \geq 1$), the risk difference can be written as

$$\begin{aligned}
\Delta_1(\lambda) &= E_{\boldsymbol{\theta}}\left[W\left(\frac{c_{0,1}X_1}{\theta_1}\right)\right] - E_{\boldsymbol{\theta}}\left[W\left(\frac{\psi_1(D)X_1}{\theta_1}\right)\right] \\
&= E_{\boldsymbol{\theta}}\left[\int_{\lambda Z}^{\infty} \psi_1'(t)Z_1W'\big(\psi_1(t)Z_1\big)\,dt\right] \\
&= \lambda E_{\boldsymbol{\theta}}\left[\int_{Z}^{\infty} \psi_1'(\lambda t)Z_1W'\big(\psi_1(\lambda t)Z_1\big)\,dt\right] \\
&= \lambda \int_0^{\infty} \psi_1'(\lambda t)E_{\boldsymbol{\theta}}\big[Z_1W'\big(\psi_1(\lambda t)Z_1\big)I_{(0,t]}(Z)\big]\,dt.
\end{aligned}$$

In light of the hypotheses of the theorem, it is enough to prove that, for every fixed t and $\lambda \geq 1$,

$$E_{\boldsymbol{\theta}}\big[Z_1W'\big(\psi_1(\lambda t)Z_1\big)I_{(0,t]}(Z)\big] \geq (\leq)\, 0. \tag{3.4}$$

Since $W'(t)$ is non-decreasing function of t and $\psi_1(t)$ is non-decreasing (non-increasing) function of t, for $\lambda \geq 1$, we have

$$\begin{aligned}
E_{\boldsymbol{\theta}}\big[Z_1W'\big(\psi_1(\lambda t)Z_1\big)I_{(0,t]}(Z)\big] &\geq (\leq) E_{\boldsymbol{\theta}}\big[Z_1W'\big(\psi_1(t)Z_1\big)I_{(0,t]}(Z)\big] \\
&= \int_0^{\infty} sW'\big(\psi_1(t)s\big)H_1(t|s)f_1(s)\,ds
\end{aligned}$$

which, in turn, implies (3.4). □

The following corollary gives us the B–Z type improvements over the BSEE $\delta_{c_{0,1},1}(\boldsymbol{X})$.

Corollary 3.1. *Suppose that, for any fixed* $\Delta \geq 1$ *and* t, $H_1(\frac{t}{\Delta}|s)/H_1(t|s)$ *is non-decreasing (non-increasing) in* $s \in S_1$. *Further suppose that, for every fixed* t, *the equation*

$$l_1(c|t) = \int_0^\infty sW'(cs)H_1(t|s)f_1(s)\,ds = 0$$

has the unique solution $c \equiv \psi_{0,1}(t)$.

(i) *Then the estimator* $\delta_{\psi_{0,1}}(\boldsymbol{X}) = \psi_{0,1}(D)X_1$ *dominates the BSEE* $\delta_{c_{0,1},1}(\boldsymbol{X})$, $\forall\, \boldsymbol{\theta} \in \Theta_0$.

(ii) *Suppose that* $\psi_{1,1} : \Re_{++} \to \Re$ *is such that* $\psi_{1,1}(t) \geq (\leq) \psi_{0,1}(t)$, $\forall\, t$, $\psi_{1,1}(t)$ *is non-decreasing (non-increasing) in* t *and* $\lim_{t\to\infty} \psi_{1,1}(t) = c_{0,1}$. *Then the estimator* $\delta_{\psi_{1,1}}(\boldsymbol{X}) = \psi_{1,1}(D)X_1$ *dominates the BSEE* $\delta_{c_{0,1},1}(\boldsymbol{X})$, $\forall\, \boldsymbol{\theta} \in \Theta_0$.

Proof. It is sufficient to prove that $\psi_{0,1}(t)$ satisfies conditions of Theorem 3.1. To prove that $\psi_{0,1}(t)$ is an non-decreasing (non-increasing) function of t, suppose that, there exist positive numbers t_1 and t_2 such that $t_1 < t_2$ and $\psi_{0,1}(t_1) \neq \psi_{0,1}(t_2)$. Then $l_1(\psi_{0,1}(t_1)|t_1) = 0$. Since $W'(t)$ is an non-decreasing function of $t \in \Re$, it follows that $l_1(c|t_2)$ is a non-decreasing function of c and $\psi_{0,1}(t_2)$ is the unique solution of $l_1(c|t_2) = 0$. Let $s_0 = \frac{1}{\psi_{0,1}(t_1)}$, $M(s) = sW'(\frac{s}{s_0})f_1(s)$, $M_1(s) = H_1(t_2|s)$ and $M_2(s) = H_1(t_1|s)$, $s \in S_1$. Then, using hypotheses of the corollary and the Lemma 2.1, we get

$$l_1\big(\psi_{0,1}(t_1)|t_2\big) = \int_0^\infty sW'\big(\psi_{0,1}(t_1)s\big)H_1(t_2|s)f_1(s)\,ds < (>)\, 0,$$

as $l_1(c|t_2) = 0$ has the unique solution $c = \psi_{0,1}(t_2)$ and $\psi_{0,1}(t_1) \neq \psi_{0,1}(t_2)$. Since $l_1(c|t_2)$ is a non-decreasing function of c and $l_1(\psi_{0,1}(t_2)|t_2) = 0$, this implies that $\psi_{0,1}(t_1) < (>) \psi_{0,1}(t_2)$.

Also, $l_1(\psi_{0,1}(t)|t) = 0$ and the assumption A_2 ensures that $\lim_{t\to\infty} \psi_{0,1}(t) = c_{0,1}$. Hence, the assertion follows.

The proof of part (ii) is immediate from Theorem 3.1, since $l_1(c|t)$ is a non-decreasing function of $c \in \Re$, $\forall\, t$. □

The following corollary gives us the Stein type improvements over the BSEE $\delta_{c_{0,1},1}(\boldsymbol{X})$.

Corollary 3.2. *Suppose that, for any fixed* $\Delta \geq 1$ *and* t, $h_1(\frac{t}{\Delta}|s)/h_1(t|s)$ *is non-decreasing (non-increasing) in* $s \in S_1$. *In addition suppose that, for any* t, *the equation*

$$l_2(c|t) = \int_0^\infty sW'(cs)h_1(t|s)f_1(s)\,ds = 0$$

has the unique solution $c \equiv \psi_{2,1}(t)$.

(i) *Let* $\psi^*_{2,1}(t) = \min\{c_{0,1}, \psi_{2,1}(t)\}$ $(\psi^*_{2,1}(t) = \max\{c_{0,1}, \psi_{2,1}(t)\})$ *and* $\delta_{\psi^*_{2,1}}(\boldsymbol{X}) = \psi^*_{2,1}(D)X_1$. *Then,* $\forall\, \boldsymbol{\theta} \in \Theta_0$, *the estimator* $\delta_{\psi^*_{2,1}}(\boldsymbol{X})$ *dominates the BSEE* $\delta_{c_{0,1},1}(\boldsymbol{X}) = c_{0,1}X_1$.

(ii) *Let* $\psi_{3,1} : \Re_{++} \to \Re$ *be such that* $\psi_{3,1}(t) \geq (\leq) \psi_{2,1}(t)$, $\forall\, t$ *and* $\psi_{3,1}(t)$ *is non-decreasing (non-increasing) in* t. *Define* $\psi^*_{3,1}(t) = \min\{c_{0,1}, \psi_{3,1}(t)\}(\psi^*_{3,1}(t) = \max\{c_{0,1}, \psi_{3,1}(t)\})$. *Then,* $\forall\, \boldsymbol{\theta} \in \Theta_0$, *the estimator* $\delta_{\psi^*_{3,1}}(\boldsymbol{X}) = \psi^*_{3,1}(D)X_1$ *dominates the BSEE* $\delta_{c_{0,1},1}(\boldsymbol{X})$.

Proof. It suffices to show that $\psi^*_{2,1}(\cdot)$ satisfies conditions of Theorem 3.1. On using arguments similar to the ones used in the proof of Corollary 3.1, it can be shown that $\psi_{2,1}(t)$ (and hence $\psi^*_{2,1}(t)$) is non-decreasing (non-increasing) in t. Now to show that $\lim_{t\to\infty} \psi^*_{2,1}(t) = c_{0,1}$, we will show that $\psi_{2,1}(t) \geq (\leq) \psi_{0,1}(t)$, $\forall\, t$. Let us fix t, then $l_1(\psi_{0,1}(t)|t) = 0$ and $l_2(\psi_{2,1}(t)|t) = 0$.

Let $s_0 = \frac{1}{\psi_{0,1}(t)}$, $M(s) = sW'(\frac{s}{s_0}) f_1(s)$, $M_1(s) = h_1(t|s)$ and $M_2(s) = H_1(t|s)$, $s \in \Re_{++}$. Using hypotheses of the corollary, Lemma 3.1 and Lemma 2.1, we conclude that

$$l_2\big(\psi_{0,1}(t)|t\big) \leq (\geq) \frac{h_1(t|s_0)}{H_1(t|s_0)} l_1\big(\psi_{0,1}(t)|t\big) = 0.$$

Since $l_2(c|t)$ is a non-decreasing function of c (using A_3) and $\psi_{2,1}(t)$ is the unique solution of $l_2(c|t) = 0$, we conclude that $\psi_{0,1}(t) \leq (\geq) \psi_{2,1}(t)$. Hence, $c_{0,1} = \lim_{t\to\infty} \psi_{0,1}(t) \leq (\geq) \lim_{t\to\infty} \psi_{2,1}(t)$ and $\lim_{t\to\infty} \psi^*_{2,1}(t) = \min\{c_{0,1}, \lim_{t\to\infty} \psi_{2,1}(t)\} = c_{0,1}$ ($\lim_{t\to\infty} \psi^*_{2,1}(t) = \max\{c_{0,1}, \lim_{t\to\infty} \psi_{2,1}(t)\} = c_{0,1}$). Note that $\psi^*_{2,1}(t) \geq (\leq) \psi_{0,1}(t)$, $\forall\, t$. Since $l_1(c|t)$ is a non-decreasing function of c, we have

$$l_1\big(\psi^*_{2,1}(t)|t\big) \geq (\leq) l_1\big(\psi_{0,1}(t)|t\big) = 0, \quad \forall\, t.$$

Hence, the result follows.

The proof of part (ii) is immediate using Theorem 3.1 and the fact that $l_1(c|t)$ is a non-decreasing function of c. □

It is straightforward to see that the estimator $\delta_{\psi_{0,1}}$, defined in Corollary 3.1 (i), is the generalized Bayes estimator with respect to the non-informative density $\pi(\theta_1, \theta_2) = \frac{1}{\theta_1\theta_2}$, $(\theta_1, \theta_2) \in \Theta_0$.

3.2 Improvements over the BSEE of θ_2

As proofs of various results stated in this section are similar to the proofs of similar results of the last section, they are being provided in supplementary material. The following theorem provides a class of estimators that improve upon the BSEE $\delta_{c_{0,2},2}(\boldsymbol{X}) = c_{0,2}X_2$, where $c_{0,2}$ is the unique solution of the equation $\int_{-\infty}^{\infty} zW'(cz) f_2(z)\, dz = 0$.

Theorem 3.2. *Let $\delta_{\psi_2}(\boldsymbol{X}) = \psi_2(D)X_2$ be a scale equivariant estimator for estimating θ_2 such that* $\lim_{t\to\infty} \psi_2(t) = c_{0,2}$, *$\psi_2(t)$ is an non-increasing (non-decreasing) function of t and $\int_0^{\infty} sW'(\psi_2(t)s) H_2(t|s) f_2(s)\, ds \leq (\geq) 0$, $\forall\, t$. Then, $\forall\, \boldsymbol{\theta} \in \Theta_0$, the estimator $\delta_{\psi_2}(\boldsymbol{X})$ dominates the BSEE $\delta_{c_{0,2},2}(\boldsymbol{X}) = c_{0,2}X_2$.*

The following corollary provides the B–Z type improvements over the BSEE $\delta_{c_{0,2},2}(\boldsymbol{X}) = c_{0,2}X_2$.

Corollary 3.3. (i) *Suppose that, for any fixed $\Delta \geq 1$ and t, $H_2(\frac{t}{\Delta}|s)/H_2(t|s)$ is non-increasing (non-decreasing) in $s \in S_2$. Further suppose that, for every fixed t, the equation*

$$l_3(c|t) = \int_0^{\infty} sW'(cs) H_2(t|s) f_2(s)\, ds = 0$$

has the unique solution $c \equiv \psi_{0,2}(t)$. Then, the estimator $\delta_{\psi_{0,2}}(\boldsymbol{X}) = \psi_{0,2}(D)X_2$ improves upon the BSEE $\delta_{c_{0,2},2}(\boldsymbol{X}) = c_{0,2}X_2$, $\forall\, \boldsymbol{\theta} \in \Theta_0$.

(ii) *In addition to assumptions of (i) above, suppose that $\psi_{1,2} : \Re_{++} \to \Re$ is such that $\psi_{1,2}(t) \leq (\geq) \psi_{0,2}(t)$, $\forall\, t$, $\psi_{1,2}(t)$ is non-increasing (non-decreasing) in t and* $\lim_{t\to\infty} \psi_{1,2}(t) = c_{0,2}$. *Then, $\forall\, \boldsymbol{\theta} \in \Theta_0$, the estimator $\delta_{\psi_{1,2}}(\boldsymbol{X}) = \psi_{1,2}(D)X_2$ dominates the BSEE $\delta_{c_{0,2},2}(\boldsymbol{X}) = c_{0,2}X_2$.*

In the following corollary we provide the Stein type improvements over the BSEE $\delta_{c_{0,2},2}(\boldsymbol{X})$.

Corollary 3.4. (i) *Suppose that for any fixed* $\Delta \geq 1$ *and* t, $h_2(\frac{t}{\Delta}|s)/h_2(t|s)$ *is non-increasing* (*non-decreasing*) *in* $s \in S_2$ *and let* $\psi_{0,2}(t)$ *be as defined in Corollary* 3.3. *In addition suppose that, for any* t, *the equation*

$$l_4(c|t) = \int_0^\infty sW'(cs)h_2(t|s)f_2(s)\,ds = 0$$

has the unique solution $c \equiv \psi_{2,2}(t)$. *Let* $\psi^*_{2,2}(t) = \max\{c_{0,2}, \psi_{2,2}(t)\}$ ($\psi^*_{2,2}(t) = \min\{c_{0,2}, \psi_{2,2}(t)\}$) *and* $\delta_{\psi^*_{2,2}}(\boldsymbol{X}) = \psi^*_{2,2}(D)X_2$. *Then*

$$R_2(\boldsymbol{\theta}, \delta_{\psi^*_{2,2}}) \leq R_2(\boldsymbol{\theta}, \delta_{c_{0,2},2}), \quad \forall\, \boldsymbol{\theta} \in \Theta_0.$$

(ii) *In addition to assumptions of* (*i*) *above, suppose that* $\psi_{3,2} : \Re_{++} \to \Re$ *is such that* $\psi_{3,2}(t) \leq (\geq) \psi_{2,2}(t), \forall\, t$ *and* $\psi_{3,2}(t)$ *is non-increasing* (*non-decreasing*) *in* t. *For fixed* t, *define* $\psi^*_{3,2}(t) = \max\{c_{0,2}, \psi_{3,2}(t)\}(\psi^*_{3,2}(t) = \min\{c_{0,2}, \psi_{3,2}(t)\})$ *and* $\delta_{\psi^*_{3,2}}(\boldsymbol{X}) = \psi^*_{3,2}(D)X_2$. *Then*

$$R_2(\boldsymbol{\theta}, \delta_{\psi^*_{3,2}}) \leq R_2(\boldsymbol{\theta}, \delta_{c_{0,2},2}), \quad \forall\, \boldsymbol{\theta} \in \Theta_0.$$

It is easy to verify that the B–Z type estimator $\delta_{\psi_{0,2}}(\cdot)$, derived in Corollary 3.3 (i), is the generalized Bayes estimator with respect to the non-informative prior density $\pi(\theta_1, \theta_2) = \frac{1}{\theta_1\theta_2}$, $(\theta_1, \theta_2) \in \Theta_0$.

The results of Theorems 3.1–3.2 (or Corollaries 3.1–3.2 and Corollaries 3.3–3.4) are applicable to various studies carried out in the literature for specific bivariate probability models, having independent marginals, and specific loss function (e.g., Misra and Dhariyal (1995), Vijayasree, Misra and Singh (1995), etc.). These results also extend the study of Kubokawa and Saleh (1994) to general bivariate scale models.

Now we provide an application of the results derived in Sections 3.1–3.2 to a situation where results of Kubokawa and Saleh (1994) are not applicable.

3.3 Applications

In the following example, we consider a bivariate model due to Cheriyan and Ramabhadran's (see Kotz, Balakrishnan and Johnson (2000)) and study estimation of order restricted scale parameters.

Example 3.1. Let X_1 and X_2 be two dependent random variables with joint pdf (1.2), where $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \Theta_0$ and

$$f(z_1, z_2) = \begin{cases} e^{-z_1}(1 - e^{-z_2}), & \text{if } 0 < z_2 < z_1 \\ e^{-z_2}(1 - e^{-z_1}), & \text{if } 0 < z_1 < z_2 \\ 0, & \text{otherwise.} \end{cases}$$

The above bivariate distribution is a special case of Cheriyan and Ramabhadran's bivariate gamma distribution (see Kotz, Balakrishnan and Johnson (2000)). Here random variable X_i follows Gamma distribution with pdf $f_i(\frac{x}{\theta_i}) = \frac{x}{\theta_i^2} e^{-\frac{x}{\theta_i}}$, $x > 0$, $i = 1, 2$.

For estimation of θ_i, $i = 1, 2$, consider the squared error loss function $L_i(\boldsymbol{\theta}, a) = (\frac{a}{\theta_i} - 1)^2$, $\boldsymbol{\theta} \in \Theta_0$, $a \in \Re_{++}$, $i = 1, 2$. The BSEE of θ_i is $\delta_{c_{0,i},i}(\boldsymbol{X}) = \frac{1}{3}X_i$, $i = 1, 2$ ($c_{0,i} = \frac{1}{3}$, $i = 1, 2$). We have $S_1 = S_2 = [0, \infty)$.

<u>*Estimation of θ_1*</u>:

For any $s \in S_1$, the pdf and df of $Z_s^{(1)}$, respectively, are

$$h_1(t|s) = \begin{cases} 1 - e^{-st}, & \text{if } 0 < t < 1 \\ e^{-s(t-1)}(1 - e^{-s}), & \text{if } 1 \le t < \infty \\ 0, & \text{otherwise} \end{cases}$$

$$\text{and } H_1(t|s) = \begin{cases} 0, & \text{if } t < 0 \\ t - \dfrac{1}{s} + \dfrac{e^{-st}}{s}, & \text{if } 0 \le t < 1 \\ 1 - \dfrac{1}{s} + \dfrac{e^{-s}}{s} + \dfrac{(1-e^{-s})(1-e^{-s(t-1)})}{s}, & \text{if } t \ge 1 \end{cases}.$$

It is easy to see that, for any fixed $\Delta \ge 1$ and t, $h_1(\frac{t}{\Delta}|s)/h_1(t|s)$ (and hence, $H_1(\frac{t}{\Delta}|s)/H_1(t|s)$) is non-decreasing in $s \in S_1 = \Re_{++}$. We have

$$\psi_{0,1}(t) = \frac{\int_{-\infty}^{\infty} s H_1(t|s) f_1(s)\,ds}{\int_{-\infty}^{\infty} s^2 H_1(t|s) f_1(s)\,ds} = \begin{cases} \dfrac{2t - 1 + \frac{1}{(t+1)^2}}{6t - 2 + \frac{2}{(t+1)^3}}, & \text{if } 0 < t < 1 \\ \dfrac{2 - \frac{1}{t^2} + \frac{1}{(t+1)^2}}{6 - \frac{2}{t^3} + \frac{2}{(t+1)^3}}, & \text{if } t \ge 1 \end{cases},$$

$$\psi_{2,1}(t) = \frac{\int_{-\infty}^{\infty} s h_1(t|s) f_1(s)\,ds}{\int_{-\infty}^{\infty} s^2 h_1(t|s) f_1(s)\,ds} = \begin{cases} \dfrac{1}{3}\dfrac{1 - \frac{1}{(t+1)^3}}{1 - \frac{1}{(t+1)^4}}, & \text{if } 0 < t < 1 \\ \dfrac{1}{3}\dfrac{\frac{1}{t^3} - \frac{1}{(t+1)^3}}{\frac{1}{t^4} - \frac{1}{(t+1)^4}}, & \text{if } t \ge 1 \end{cases}$$

$$\text{and} \quad \psi_{2,1}^*(t) = \min\{c_{0,1}, \psi_{2,1}(t)\} = \begin{cases} \dfrac{1}{3}\dfrac{1 - \frac{1}{(t+1)^3}}{1 - \frac{1}{(t+1)^4}}, & \text{if } 0 < t < 1 \\ \dfrac{1}{3}\min\left\{1, \dfrac{\frac{1}{t^3} - \frac{1}{(t+1)^3}}{\frac{1}{t^4} - \frac{1}{(t+1)^4}}\right\}, & \text{if } t \ge 1 \end{cases}.$$

Here $\psi_{0,1}(t)$ and $\psi_{2,1}(t)$ are non-decreasing in $t \in \Re_{++}$ and $\lim_{t\to\infty} \psi_{0,1}(t) = \frac{1}{3} = c_{0,1}$.

Using Corollary 3.1 (i), the B–Z type estimator dominating the BSEE $\delta_{c_{0,1},1}(\boldsymbol{X}) = \frac{1}{3}X_1$ is

$$\delta_{\psi_{0,1}}(\boldsymbol{X}) = \psi_{0,1}(D)X_1 = \begin{cases} \dfrac{2D - 1 + \frac{1}{(D+1)^2}}{6D - 2 + \frac{2}{(D+1)^3}} X_1, & \text{if } X_1 > X_2 \\ \dfrac{2 - \frac{1}{D^2} + \frac{1}{(D+1)^2}}{6 - \frac{2}{D^3} + \frac{2}{(D+1)^3}} X_1, & \text{if } X_1 \le X_2 \end{cases}. \tag{3.5}$$

Here $\delta_{\psi_{0,1}}(\cdot)$ is also the generalized Bayes estimator with respect to the non-informative prior density on Θ_0.

Using Corollary 3.2 (i), the Stein type estimator dominating the BSEE $\delta_{c_{0,1},1}(\boldsymbol{X}) = \frac{1}{3}X_1$ is

$$\delta_{\psi_{2,1}^*}(\boldsymbol{X}) = \psi_{2,1}^*(D)X_1 = \begin{cases} \dfrac{1}{3}\dfrac{1 - \frac{1}{(D+1)^3}}{1 - \frac{1}{(D+1)^4}} X_1, & \text{if } X_1 > X_2 \\ \dfrac{1}{3}\min\left\{1, \dfrac{\frac{1}{D^3} - \frac{1}{(D+1)^3}}{\frac{1}{D^4} - \frac{1}{(D+1)^4}}\right\} X_1, & \text{if } X_1 \le X_2 \end{cases}. \tag{3.6}$$

<u>*Estimation of θ_2*</u>:

For any $s \in S_2 = [0, \infty)$, the pdf and df of $Z_s^{(2)}$, respectively, are

$$h_2(t|s) = \begin{cases} \dfrac{e^{-\frac{s}{t}} e^{s}(1-e^{-s})}{t^2}, & \text{if } 0 < t < 1 \\ \dfrac{(1-e^{-\frac{s}{t}})}{t^2}, & \text{if } 1 \le t < \infty \\ 0, & \text{elsewhere} \end{cases}$$

$$\text{and} \quad H_2(t|s) = \begin{cases} 0, & \text{if } t \le 0 \\ \dfrac{e^{-s(\frac{1}{t}-1)} - e^{-\frac{s}{t}}}{s}, & \text{if } 0 < t < 1 \\ 1 - \dfrac{1}{t} + \dfrac{1}{s} - \dfrac{e^{-\frac{s}{t}}}{s}, & \text{if } t \ge 1 \end{cases}.$$

One can easily see that, for any fixed $\Delta \ge 1$ and t, $h_2(\frac{t}{\Delta}|s)/h_2(t|s)$ (and hence $H_2(\frac{t}{\Delta}|s)/H_2(t|s)$) is non-increasing in $s \in (0, \infty)$. Let $\psi_{0,2}(\cdot)$ and $\psi_{2,2}(\cdot)$ be as defined in Corollaries 3.3 (i) and 3.4 (i), respectively, so that for fixed t, we have

$$\psi_{0,2}(t) = \frac{\int_{-\infty}^{\infty} s H_2(t|s) f_2(s)\, ds}{\int_{-\infty}^{\infty} s^2 H_2(t|s) f_2(s)\, ds} = \begin{cases} \dfrac{1}{2t} \dfrac{1 - \frac{1}{(1+t)^2}}{1 - \frac{1}{(1+t)^3}}, & \text{if } 0 < t < 1 \\ \dfrac{3 - \frac{2}{t} - \frac{t^2}{(1+t)^2}}{8 - \frac{6}{t} - \frac{2t^3}{(1+t)^3}}, & \text{if } t \ge 1 \end{cases},$$

$$\psi_{2,2}(t) = \frac{\int_{-\infty}^{\infty} s h_2(t|s) f_2(s)\, ds}{\int_{-\infty}^{\infty} s^2 h_2(t|s) f_2(s)\, ds} = \begin{cases} \dfrac{1}{3t} \dfrac{1 - \frac{1}{(1+t)^3}}{1 - \frac{1}{(1+t)^4}}, & \text{if } 0 < t < 1 \\ \dfrac{1}{3} \dfrac{1 - \frac{t^3}{(1+t)^3}}{1 - \frac{t^4}{(1+t)^4}}, & \text{if } t \ge 1 \end{cases}$$

$$\text{and} \quad \psi_{2,2}^*(t) = \max\{c_{0,2}, \psi_{2,2}(t)\} = \begin{cases} \dfrac{1}{3} \max\left\{1, \dfrac{1}{t} \dfrac{1 - \frac{1}{(1+t)^3}}{1 - \frac{1}{(1+t)^4}}\right\}, & \text{if } 0 < t < 1 \\ \dfrac{1}{3}, & \text{if } t \ge 1 \end{cases}.$$

Here $\psi_{0,2}(t)$ and $\psi_{2,2}(t)$ are non-increasing in $t \in \Re_{++}$ and $\lim_{t\to\infty} \psi_{0,2}(t) = \frac{1}{3}$.

Using Corollary 3.3 (i), the B–Z type estimator dominating the BSEE $\delta_{c_{0,2},2}(\boldsymbol{X}) = \frac{1}{3} X_2$ is

$$\delta_{\psi_{0,2}}(\boldsymbol{X}) = \psi_{0,2}(D) X_2 = \begin{cases} \dfrac{1}{2D} \dfrac{1 - \frac{1}{(1+D)^2}}{1 - \frac{1}{(1+D)^3}} X_2, & \text{if } X_1 > X_2 \\ \dfrac{3 - \frac{2}{D} - \frac{D^2}{(1+D)^2}}{8 - \frac{6}{D} - \frac{2D^3}{(1+D)^3}} X_2, & \text{if } X_1 \le X_2 \end{cases}. \tag{3.7}$$

Using Corollary 3.4 (i), the Stein type estimator dominating the BSEE $\delta_{c_{0,2},2}(\boldsymbol{X}) = \frac{1}{3} X_2$ is

$$\delta_{\psi_{2,2}^*}(\boldsymbol{X}) = \psi_{2,2}^*(D) X_2 = \begin{cases} \dfrac{1}{3} \max\left\{1, \dfrac{1}{D} \dfrac{1 - \frac{1}{(1+D)^3}}{1 - \frac{1}{(1+D)^4}}\right\} X_2, & \text{if } X_1 > X_2 \\ \dfrac{1}{3} X_2, & \text{if } X_1 \le X_2 \end{cases}. \tag{3.8}$$

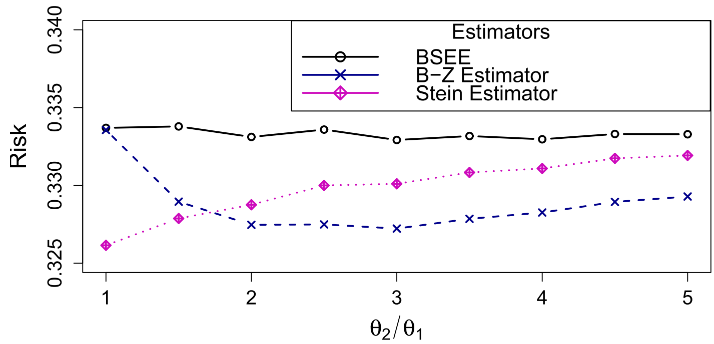


Figure 3 *Estimators of θ_1: Risk plot of $\delta_{c_{0,1},1}$ (BSEE), $\delta_{\psi_{0,1}}$ (B–Z type estimator) and $\delta_{\psi^*_{2,1}}$ (Stein type estimator) estimators against the values of $\frac{\theta_2}{\theta_1}$.*

3.4 Simulation study for estimation of scale parameters θ_1 and θ_2

In Example 3.1, we have considered a Cheriyan and Ramabhadran's bivariate gamma distribution with unknown order restricted scale parameters (that is, $\theta_1 \leq \theta_2$). To further evaluate the performances of various estimators of θ_1 under the scaled squared error loss function, in this section, we compare the risk performances of the BSEE $\delta_{c_{0,1},1}(\boldsymbol{X}) = \frac{X_1}{3}$, the B–Z estimator $\delta_{\psi_{0,1}}$ and the Stein (1964) type estimator $\delta_{\psi^*_{2,1}}$ (as defined in (3.5) and (3.6)), numerically, through Monte Carlo simulations. Also, to evaluate the performances of various estimators of θ_2 under the scaled squared error loss function, we compare the risk performances of the BSEE $\delta_{c_{0,2},2}(\boldsymbol{X}) = \frac{X_2}{3}$, the B–Z estimator $\delta_{\psi_{0,2}}$ and the Stein (1964) type estimator $\delta_{\psi^*_{2,2}}$ (as defined in (3.7) and (3.8)), numerically, through Monte Carlo simulations. The simulated risks of the BSEEs, the B–Z estimators and the Stein estimators have been computed based on 50,000 simulations from relevant distributions.

The simulated values of risks of various estimators of scale parameter θ_1 are plotted in Figure 3. The following observations are evident from Figure 3:

(i) The B–Z type and the Stein type estimators always perform better than the BSEE, which is in conformity with theoretical findings of Example 3.1.

(ii) There is no clear cut winner between the B–Z type estimator $\delta_{\psi_{0,1}}$ and the Stein type estimator $\delta_{\psi^*_{2,1}}$. The Stein type estimator performs better than the B–Z type estimator, for small values of $\frac{\theta_2}{\theta_1}$, and the B–Z type estimator dominates the Stein type estimator for the large values of $\frac{\theta_2}{\theta_1}$.

Figure 4 shows the simulated risks for different estimators of the scale parameter θ_2. Similar observations, as mentioned above for three estimators of θ_1, are evident from Figure 4.

4 Concluding remarks

The problem of estimation of order restricted location/scale parameters is widely studied for specific probability models, having independent marginals, and specific loss functions. In this paper, we unify these studies by considering a general bivariate location/scale model and a general loss function. We drive a class of estimators dominating over BLEE/BSEE using the IERD approach of Kubokawa (1994). We also obtain the Brewster and Zidek (1974) type and the Stein (1964) type estimators that dominate the BLEE/BSEE under the general loss function. We also demonstrate applications of our results to two bivariate probability models which have not been studied in the literature.

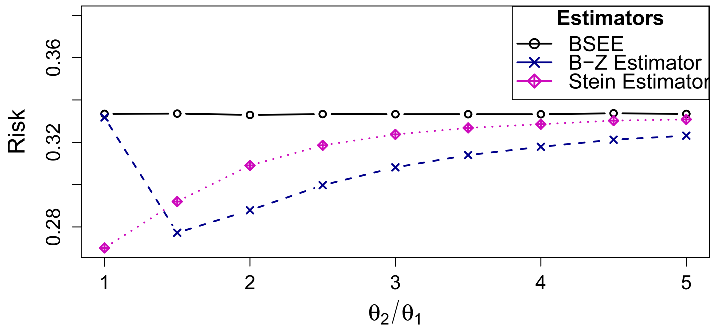


Figure 4 *Estimators of θ_2: Risk plot of $\delta_{c_{0,2},2}$ (BSEE), $\delta_{\psi_{0,2}}$ (B–Z type estimator) and $\delta_{\psi^*_{2,2}}$ (Stein type estimator) estimators against the values of $\frac{\theta_2}{\theta_1}$.*

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Supplementary Material

Supplement to "Componentwise equivariant estimation of order restricted location and scale parameters in bivariate models: A unified study" (DOI: 10.1214/23-BJPS562SUPP; .pdf). Supplementary information.

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