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Absolute continuity of the super-Brownian motion with infinite mean

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Abstract. In this work, we prove that for any dimension $d \geq 1$ and any $\gamma \in (0, 1)$ super-Brownian motion corresponding to the log-Laplace equation

$$v(t,x) = (S_t f)(x) - \int_0^t \big(S_{t-s} v^\gamma(s,\cdot)\big)(x)\,\mathrm{d}s, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d,$$

is absolutely continuous with respect to Lebesgue measure at any fixed time $t > 0$. $\{S_t\}_{t\geq 0}$ denotes a transition semigroup of a standard Brownian motion. Our proof is based on properties of solutions of the log-Laplace equation. We also prove that when initial datum $v(0,\cdot)$ is a finite, non-zero measure, then the log-Laplace equation has a unique, continuous solution. Moreover this solution continuously depends on initial data.

1 Introduction and main result

This paper is devoted to studying regularity properties of the super-Brownian motion with stable branching mechanism with infinite mean.

Let us start with some notation. For a measure μ on $\mathbb{R}^d$ and a function f on $\mathbb{R}^d$ let $\langle \mu, f\rangle$ or $\langle f, \mu\rangle$ denote the integral of a function f with respect to a measure μ (whenever it is well defined):

$$\langle f, \mu\rangle = \langle \mu, f\rangle \equiv \int_{\mathbb{R}^d} f(x)\mu(\mathrm{d}x).$$

Let $\gamma \in (0, 2] \setminus \{1\}$. The super-Brownian motion with γ-stable branching mechanism, $X = \{X_t, t \geq 0\}$, is a Markov measure-valued process on $\mathbb{R}^d$ which is characterized as follows: for any finite measure μ and a nonnegative not identically zero bounded continuous function f,

$$E_\mu\big(e^{-\langle X_t, f\rangle}\big) = E\big(e^{-\langle X_t, f\rangle}\,|X_0 = \mu\big) = e^{-\langle \mu, v(t,\cdot)\rangle}, \quad \forall t \geq 0. \tag{1.1}$$

Here v is a solution to the so-called log-Laplace equation:

$$v(t,x) = (S_t f)(x) - \int_0^t \big(S_{t-s} v^\gamma(s,\cdot)\big)(x)\,\mathrm{d}s, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \tag{1.2}$$

if $\gamma \in (1, 2]$, and for $\gamma \in (0, 1)$, the sign in front of the non-linear term is reversed:

$$v(t,x) = (S_t f)(x) + \int_0^t \big(S_{t-s} v^\gamma(s,\cdot)\big)(x)\,\mathrm{d}s, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d. \tag{1.3}$$

Here and for the rest of the paper $\{S_t\}_{t\geq 0}$ denotes the transition semigroup of the Brownian motion whose generator is Laplacian $\frac{1}{2}\Delta$ in $\mathbb{R}^d$. Clearly

$$S_t f(x) = \int_{\mathbb{R}^d} f(y) p_t(x-y)\,\mathrm{d}y, \quad t \geq 0,$$

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where $\{p_t(x), t \geq 0, x \in \mathbb{R}^d\}$ is the transition density of the Brownian motion. $\{S_t\}_{t\geq 0}$ describes the underlying Brownian motion of X, whereas its continuous-state branching mechanism is described by $v \mapsto \pm v^\gamma$, $v \geq 0$. The change of sign in front of v^γ between (1.2) and (1.3) corresponds to the change of sign in the Laplace transforms for spectrally positive stable random variables with stability indexes $\gamma \in (1, 2)$ and $\gamma \in (0, 1)$, respectively.

The above equations were considered in Watanabe (1968) for a more general "motion" operator and state space. Existence and uniqueness for such equations was derived in Watanabe (1968) for strictly positive sufficiently regular initial conditions. For the equations (1.2), (1.3) existence and uniqueness was established later for much more general class of initial conditions (see, e.g., Fleischmann (1988) for $\gamma \in (1, 2]$ and Aguirre and Escobedo (1986/87) for $\gamma \in (0, 1)$).

For the case of super-Brownian motion with γ-stable branching mechanism (in what follows we will call it γ-super-Brownian motion) it is well known that for $\gamma \in (1, 2]$ in dimensions $d < \frac{2}{\gamma-1}$ at any fixed time $T > 0$, the measure $X_t = X_t(\mathrm{d}x)$ is absolutely continuous with respect to Lebesgue measure (in what follows, we will often write just "absolutely continuous") with probability one (cf. Fleischmann (1988)). By an abuse of notation, we sometimes denote a version of the density function of the measure $X_t = X_t(\mathrm{d}x)$ by the same symbol, $X_t(\mathrm{d}x) = X_t(x)\,\mathrm{d}x$. It is even known that for $d = 1$, $\gamma \in (1, 2]$, at fixed times t, there is a continuous version of the density in x variable (see Mytnik and Perkins (2003)), and for $\gamma = 2$, and again $d = 1$, there even exists a jointly space-time continuous version of the density (see Konno and Shiga (1988), Reimers (1989)). More detailed regularity properties of the densities of superprocesses with stable branching mechanism with possibly more general motion have been studied in Fleischmann, Mytnik and Wachtel (2010, 2011), Mytnik and Wachtel (2015, 2016).

This paper is devoted to deriving absolute continuity of X for the case of $\gamma \in (0, 1)$. It is easy to check that in this case $E(\langle X_t, 1\rangle) = \infty$, for $t > 0$, which adds some technical difficulties for the proofs.

Before we state the main result of this paper we need to introduce some notation. Let E be any Polish space. Let $C(E)$ and $B(E)$ be respectively, the spaces of continuous and *Borel* measurable functions on space E. If $F(E)$ is a space of real-valued functions on E we define the following subspaces of $F(E)$. $F_b(E)$ (respectively, $F^+(E)$, $F_c(E)$, $F_{bc}(E)$) denotes the subspace of bounded (respectively positive, with compact support, bounded with compact support) functions. For example, $B^+_{bc}(\mathbb{R}^d)$ denotes a set of positive, bounded, Borel measurable functions with compact support on $\mathbb{R}^d$.

Now let us define the explosion time of the superprocess.

Definition 1.1 (Time of explosion). Let $d \geq 1$ and let $\{X_t\}_{t\geq 0}$ be a super-Brownian motion with non-random initial state X_0. Given nonnegative continuous function f on $\mathbb{R}^d$, we define the time of explosion $T(X_0, f)$ of X as follows

$$T(X_0, f) \equiv \inf\{t \geq 0 : \langle X_t, f\rangle = \infty\}.$$

Now we are able to state the main result of the paper.

Theorem 1.2 (Absolute continuity). *Let $d \geq 1$ and $0 < \gamma < 1$. Let $\{X_t\}_{t\geq 0}$ be a γ-super-Brownian motion with non-random initial state X_0 being a finite measure on $\mathbb{R}^d$. For each $t > 0$, $X_t(\mathrm{d}x)$ is $P - a.s.$ absolutely continuous on the event $\{t < T(X_0, 1)\}$.*

The proof of this theorem is based on the properties of solutions to the log-Laplace equation corresponding to the process $\{X_t\}_{t\geq 0}$. These properties are stated in Theorem 2.4. This theorem extends results of Aguirre and Escobedo (1986/87) for the case of non-zero measure-valued initial conditions. All of Section 4 is devoted to the proof of these properties. In Section 3.1, we will prove that for any nonnegative, non-zero continuous function f on $\mathbb{R}^d$,

$$T(X_0, f) = T(X_0, 1), \quad P\text{-a.s.}$$

This property allows us to define the density of the superprocess $\{X_t\}_{t\geq 0}$ for a fixed time $t > 0$. In Section 3.2, we conclude the proof of Theorem 1.2—the main result of the paper.

2 Semilinear heat equation

For the rest of the paper fix $\gamma \in (0, 1)$ and arbitrary dimension $d \geq 1$. One of the main tools for investigating the γ-super-Brownian motion is the log-Laplace equation

$$v(t, x) = (S_t f)(x) + \int_0^t \big(S_{t-s} v^{\gamma}(s, \cdot)\big)(x)\, \mathrm{d}s. \tag{2.1}$$

Usually in the literature (2.1) is studied for f being a non-negative function. In the sequel, we will consider (2.1) also with f being a measure.

Before we discuss properties of (2.1), we need to introduce some notation. For a topological space S, $\mathcal{B}(S)$ will denote the *Borel* σ-algebra on the space S.

We denote by $L^{p,w}(\mathbb{R}^d)$ (for $p = 1$ or $p = \infty$) a Banach space of (equivalence classes of) measurable functions on $\mathbb{R}^d$ with the norms:

$$\|f\|_{1,w} \equiv \int_{\mathbb{R}^d} \big|w(x) f(x)\big|\, \mathrm{d}x, \quad \text{for } p = 1$$

$$\|f\|_{\infty,\omega} \equiv \inf\big\{M : \mathrm{Leb}\big(x : \big|w(x) f(x)\big| > M\big) = 0\big\}, \quad \text{for } p = \infty,$$

where

$$w(x) \equiv C_w e^{-|x|}, \qquad \int_{\mathbb{R}^d} w(x)\, \mathrm{d}x = 1,$$

and Leb denotes Lebesgue measure on $\mathbb{R}^d$. $L_+^{p,w}(\mathbb{R}^d)$ (respectively, $L_+^p(\mathbb{R}^d)$) will denote the nonnegative elements of $L^{p,w}(\mathbb{R}^d)$ (respectively, $L^p(\mathbb{R}^d)$).

Given $L^{p,w}(\mathbb{R}^d)$ (for $p = 1$ or $p = \infty$) we define the Banach space $L^{\infty}_{\mathrm{loc}}((0, \infty), L^{p,w}(\mathbb{R}^d))$ of (equivalent classes of) measurable functions on $(0, \infty) \times \mathbb{R}^d$ as follows: $f \in L^{\infty}_{\mathrm{loc}}((0, \infty), L^{p,w}(\mathbb{R}^d))$ if and only if $f(t, \cdot) \in L^{p,w}(\mathbb{R}^d)$ for any fixed t and

$$t \to \big\|f(t)\big\|_{p,w} \in L^{\infty}\big([a, b]\big)$$

for any compact interval $[a, b] \in (0, \infty)$. Similarly, $L^{\infty}_{\mathrm{loc}}((0, \infty), L_+^{p,w}(\mathbb{R}^d))$ is defined.

By $M_F(E)$ (respectively, $M_{F,S}(E)$) we denote the space of finite (respectively, finite signed) measures on a Polish space E equipped with the topology of the weak convergence. We write

$$\mu_n \stackrel{w}{\Longrightarrow} \mu, \quad \text{as } n \to \infty,$$

if the sequence $\{\mu\}_{n=1}^{\infty}$ of finite measures or finite signed measures weakly converges to a finite measure μ.

If F is a set of functions or measures then $\overset{\circ}{F}$ denotes this set without zero element, that is $\overset{\circ}{F} = F \setminus \{0\}$. If F is a topological space then, topology of $\overset{\circ}{F}$ is inherited from F. For example, $\overset{\circ}{M}_F(E)$ is a space of finite non-zero measures on Polish space E with the topology inherited from $M_F(E)$.

With all this notation at hand we can get back to (2.1). Equation (2.1) was studied in Aguirre and Escobedo (1986/87)). We state some of their results in the following theorem.

Theorem 2.1 (Aguirre and Escobedo (1986/87)). *For any not identically zero* $f \in L_+^{\infty,w}(\mathbb{R}^d)$ *there exists the unique solution* $v(t,x,f)$ *of equation* (2.1) *such that*

(1) $v(\cdot,\cdot,f) \in C^+((0,\infty)\times\mathbb{R}^d) \cap L^\infty_{\mathrm{loc}}((0,\infty), L_+^{\infty,w}(\mathbb{R}^d))$;
(2) $v(t,x,f) > ((1-\gamma)t)^{1/(1-\gamma)}$, $(t,x)\in(0,\infty)\times\mathbb{R}^d$,
(3) *for* $i=1,2$, *let* $v(t,x,f_i)$ *be the solution to* (2.1) *with initial condition* $v(0,\cdot,f_i)=f_i$. *If* $f_1(x)\le f_2(x)$, *a.e.x, then*

$$v(t,x,f_1)\le v(t,x,f_2), \quad \forall (t,x)\in(0,\infty)\times\mathbb{R}^d.$$

(4) $\lim_{t\to 0} v(t,\cdot,f)=f$, *for a.e.* $x\in\mathbb{R}^d$;
(5) *for any fixed* $(t,x)\in(0,\infty)\times\mathbb{R}^d$ *the mapping*

$$v(t,x,\cdot): L_+^{\infty,w}(\mathbb{R}^d)\mapsto \mathbb{R}_{++}$$

is continuous. Here $\mathbb{R}_{++}\equiv(0,\infty)$.

Remark 2.2. In fact, Aguirre and Escobedo prove the above theorem for a more general class of initial data.

Remark 2.3. Note that $(t,x)\mapsto((1-\gamma)t)^{1/(1-\gamma)}$ is a non-trivial solution to the equation (2.1) with the initial condition $f\equiv 0$. This together with the comparison result (see Theorem 2.8 in Aguirre and Escobedo (1986/87)) explains the conclusion (2) of the theorem: any solution starting from non-null nonnegative initial condition should be bounded from below by $((1-\gamma)t)^{1/(1-\gamma)}$. The strict inequality follows since for non-null $f\ge 0$, we have $(S_t f)(x)>0$, for all $x\in\mathbb{R}^d$.

We extend the results in Theorem 2.4 for the case of not identically zero measure-valued initial conditions. Consider the following equation:

$$v(t,x)=(S_t\mu)(x)+\int_0^t \big(S_{t-s}v^\gamma(\cdot,s)\big)\,\mathrm{d}s, \quad x\in\mathbb{R}^d, t>0, \tag{2.2}$$

where $\mu\in \mathring{M}_F(\mathbb{R}^d)$, and again $d\ge 1$ is an arbitrary dimension. We set $S_t\mu(x)=\int_{\mathbb{R}}^d p_t(x-y)\mu(\mathrm{d}y)$, $x\in\mathbb{R}^d$. In order to stress dependence of the solutions of this equation on initial data, we will sometimes write $v(t,x,\mu)$. In what follows, we will also use the following notation for solutions of (2.2):

$$V_t(\mu)(x)\equiv v(t,x,\mu), \quad t>0, x\in\mathbb{R}^d, \tag{2.3}$$

for $\mu\in\mathring{M}_F(\mathbb{R}^d)$ or being a non-negative, not identically zero function.

Before we state the main result of this section, let us define the constant γ' in terms of γ as follows:

$$\gamma'=\frac{1}{1-\gamma}.$$

Theorem 2.4 (Existence, uniqueness and dependence on initial data). *For any* $\mu\in\mathring{M}_F(\mathbb{R}^d)$, *equation* (2.2) *has the unique solution* $v(t,x)$ *such that*

$$v(\cdot,\cdot,\mu)\in L^\infty_{\mathrm{loc}}\big((0,\infty), L_+^{1,w}(\mathbb{R}^d)\big)\cap C^+\big((0,\infty)\times\mathbb{R}^d\big)$$

and

$$\big((1-\gamma)t\big)^{\gamma'}<v(t,x,\mu)\le e^t(S_t\mu)(x)+e^t, \quad 0<t<\infty. \tag{2.4}$$

Moreover, this solution continuously depends on initial data: if a sequence $\{\mu_n\}_{n=1}^{\infty}$ *from* $\overset{\circ}{M}_F(\mathbb{R}^d)$ *converges weakly to* $\mu \in \overset{\circ}{M}_F(\mathbb{R}^d)$ *then*

$$\lim_{n\to\infty} v(t,x,\mu_n) \to v(t,x,\mu),$$

for any $(t,x) \in (0,\infty) \times \mathbb{R}^d$.

Remark 2.5. Since for any $t \in (0,\infty)$,

$$e^t(S_t\mu)(x) + e^t \in L^{1,w}(\mathbb{R}^d),$$

it easily follows from inequality (2.4) that the sequence of solutions $\{v(\cdot,\cdot,\mu_n)\}_{n=1}^{\infty}$ which also converges to $v(\cdot,\cdot,\mu)$ in $L^{\infty}_{\mathrm{loc}}((0,\infty), L^{1,w}_{+}(\mathbb{R}^d)) \cap C^+((0,\infty)\times\mathbb{R}^d)$.

The proof of the next lemma is trivial and hence it is omitted.

Lemma 2.6. *Let* $\mu \in M_F(\mathbb{R}^d)$. *Then, for any* $t \in (0,\infty)$,

$$(S_t\mu)(x) \le \frac{\mu(\mathbb{R}^d)}{(2\pi t)^{d/2}}, \quad \forall x \in \mathbb{R}^d.$$

Now we are ready to state the corollary to Theorem 2.4.

Corollary 2.7. *Let* $\{\mu_n\}_{n=1}^{\infty} \subset \overset{\circ}{M}_F(\mathbb{R}^d)$ *be a sequence of measures that converges weakly to* $\mu \in \overset{\circ}{M}_F(\mathbb{R}^d)$ *and let* $v(\cdot,\cdot,\mu_n)$ *be the corresponding solutions of* (2.2). *Then, for any* $\chi \in M_F(\mathbb{R}^d)$ *and* $t \in (0,\infty)$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} v(t,x,\mu_n)\chi(\mathrm{d}x) = \int_{\mathbb{R}^d} v(t,x,\mu)\chi(\mathrm{d}x). \tag{2.5}$$

Proof. By Theorem 2.4, we have

$$v(t,x,\mu) \le e^t(S_t\mu)(x) + e^t, \quad n = 1,2,\ldots, 0 < t < \infty.$$

Since the sequence $\{\mu_n\}_{n=1}^{\infty}$ converges weakly to μ, then by Lemma 2.6

$$\sup_{x\in\mathbb{R}^d} \left(e^t(S_t\mu)(x) + e^t\right) \le e^t\left(\frac{1}{(2\pi t)^{d/2}} \sup_{n\ge 1} \mu_n(\mathbb{R}^d) + 1\right) < \infty.$$

Thus, we conclude that the sequence $\{v(t,\cdot,\mu_n)\}_{n=1}^{\infty}$ is bounded. Also by Theorem 2.4 $\{v(t,\cdot,\mu_n)\}_{n=1}^{\infty}$ converges pointwise to $v(t,\cdot,\mu)$. Hence, by the bounded convergence theorem, (2.5) follows. □

Theorem 2.4 will be proved in Section 4.

3 Proof of Theorem 1.2

In this section, we prove the main result of this paper—absolute continuity of the super-Brownian motion X with the branching mechanism $v \mapsto v^{\gamma}$, for $\gamma \in (0,1)$.

In Section 3.1, we investigate the explosion time for the γ-super-Brownian motion: this is necessary for the proof of Theorem 1.2 that will be concluded in Section 3.2.

3.1 Explosion times

As we will see for any $t > 0$, the γ-super-Brownian motion $X = \{X_t\}_{t\geq 0}$ explodes by time t with non-zero probability. In this section, we investigate the distribution of the explosion times. We assume that X is defined on probability space $(\Omega, P, \mathcal{F}, \mathcal{F}_t)$ and adapted to filtration $\{\mathcal{F}_t\}_{t\geq 0}$. We also assume that the initial state X_0 of X is a non-random finite measure.

Remark 3.1. By Corollary 4.3.2, in Dawson (1992), it is easy to show that $\{X_t\}_{t\geq 0}$ is a Feller process and therefore it has a strong Markov property.

The next lemma states the elementary properties of the explosion times. The proofs are simple and easily follow from the definition, so they are omitted.

Lemma 3.2.

(1) *For any function* $f \in \overset{\circ}{L}{}^{\infty}_{+}(\mathbb{R}^d)$, *and any* $a \in (0, \infty)$,

$$T(X_0, f) = T(X_0, af), \quad P\text{-a.s.}$$

(2) *For any* $f \in \overset{\circ}{L}{}^{\infty}_{+}(\mathbb{R}^d)$,

$$T(X_0, 1) \leq T(X_0, f), \quad P\text{-a.s.}$$

In the next lemma, we will show that for any $t \geq T(X_0, f)$, one has $X_t(f) = \infty$. Before we proceed, let us recall from (1.1) that the Laplace transform of $\{X_t\}_{t\geq 0}$ is given by

$$E\big(e^{-\langle X_t, f\rangle}\big) = e^{-\langle X_0, V_t(f)\rangle}, \quad f \in \overset{\circ}{L}{}^{\infty}_{+}(\mathbb{R}^d), \tag{3.1}$$

where $\{V_t(f)\}_{t\geq 0}$ solves log-Laplace equation (2.1).

Lemma 3.3. *For any* $t > 0$, $f \in \overset{\circ}{L}{}^{\infty}_{+}(\mathbb{R}^d)$,

$$\big\{T(X_0, f) \leq t\big\} = \big\{X_t(f) = \infty\big\}, \quad P\text{-a.s.}$$

Proof. The P-a.s. inclusion $\{X_t(f) = \infty\} \subset \{T(X_0, f) \leq t\}$ is trivial. Now let us show $\{T(X_0, f) \leq t\} \subset \{X_t(f) = \infty\}$, P-a.s. We define $e^{-\infty}$ to be 0. Thus, it is enough to verify that

$$E\big(e^{-\langle X_t, f\rangle} 1_{\{T(X_0,f)\leq t\}}\big) = 0. \tag{3.2}$$

Define the stopping time

$$T_n(X_0, f) \equiv \inf\big\{t \geq 0, X_t(f) = n\big\}.$$

Clearly $T_n(X_0, f) \to T(X_0, f)$, P-a.s., as $n \to \infty$. Then, for any $\delta \in (0, t)$ arbitrarily small,

$$\begin{aligned}
&E\big(e^{-\langle X_t, f\rangle} 1_{\{T(X_0,f)\leq t-\delta\}}\big)\\
&\quad = E\big(e^{-\langle X_t, f\rangle} 1_{\{T(X_0,f)\leq t-\delta\}} 1_{\{T_n(X_0,f)\leq t-\delta\}}\big)\\
&\quad \leq E\big(e^{-\langle X_t, f\rangle} 1_{\{T_n(X_0,f)\leq t-\delta\}}\big)\\
&\quad = E\big(E\big(e^{-\langle X_t, f\rangle} | \mathcal{F}_{T_n(X_0,f)}\big) 1_{\{T_n(X_0,f)\leq t-\delta\}}\big) &(3.3)\\
&\quad = E\big(e^{-\langle X_{T_n(X_0,f)}, V_{t-T_n(X_0,f)}(f)\rangle} 1_{\{T_n(X_0,f)\leq t-\delta\}}\big). &(3.4)
\end{aligned}$$

Here, in (3.3), we used the strong Markov Property (see Remark 3.1). Fix $c(\delta) > 0$ sufficiently small such that $c(\delta) f(x) \le ((1-\gamma)t)^{\gamma'}$ for all $t \ge \delta$, $x \in \mathbb{R}^d$. Then by Theorem 2.1 (see also Lemma 2.2 in Aguirre and Escobedo (1986/87)) we have

$$c(\delta) f(x) \le \big((1-\gamma)t\big)^{\gamma'} \le V_t(f)(x), \quad \forall t \ge \delta, x \in \mathbb{R}^d.$$

Therefore the expression (3.4) can be bounded from the above by

$$E\big(e^{-c(\delta)n} 1_{\{T_n(X_0,f) \le t-\delta\}}\big).$$

By the dominated convergence theorem this expression tends to zero as $n \to \infty$ and we get

$$E\big(e^{-\langle X_t, f\rangle} 1_{\{T(X_0,f) \le t-\delta\}}\big) = 0, \quad \forall t > 0.$$

Now take $\delta \searrow 0$ and by the monotone convergence theorem we get (3.2) and this completes the proof. □

The following corollary is immediate.

Corollary 3.4. *Let* $f \in \overset{\circ}{L}{}^{\infty}_{+}(\mathbb{R}^d)$. *Then*

$$E\big(e^{-\langle X_t, f\rangle}\big) = E\big(e^{-\langle X_t, f\rangle} 1_{\{T(X_0,f) > t\}}\big), \quad \forall t > 0. \tag{3.5}$$

Now, let us calculate the distribution of $T(X_0, 1)$—the distribution of the explosion time of the total mass of the super-Brownian motion X. By Corollary 3.4 and (3.1), we get

$$\begin{aligned} P\big(t < T(1, X_0)\big) &= \lim_{a \searrow 0} E\big(1_{\{t < T(1,X_0)\}} e^{-a\langle X_t, 1\rangle}\big) \\ &= \lim_{a \searrow 0} e^{-\langle V_t(a), X_0\rangle} \\ &= \lim_{a \searrow 0} \exp\big(-\langle X_0, 1\rangle \big(a^{1-\gamma} + t(1-\gamma)\big)^{\gamma'}\big) \\ &= \exp\big(-\langle X_0, 1\rangle \big(t(1-\gamma)\big)^{\gamma'}\big), \end{aligned} \tag{3.6}$$

where the third equality follows from the fact that $V_t(a)$ is a solution of the ordinary differential equation

$$\begin{cases} \dfrac{\mathrm{d}v(t)}{\mathrm{d}t} = v^{\gamma}(t), & t \ge 0, \\ v(0) = a, \end{cases}$$

and hence

$$V_t(a)(x) = \big(a^{1-\gamma} + t(1-\gamma)\big)^{\gamma'}, \quad \forall x \in \mathbb{R}^d, t \ge 0. \tag{3.7}$$

Then

$$F_{T(1,X_0)}(t) = P(t \ge T) = 1 - \exp\big(-\langle X_0, 1\rangle \big(t(1-\gamma)\big)^{\gamma'}\big). \tag{3.8}$$

But what about other test functions f? What is the law of $\langle X_t, f\rangle$ for a general $f \in \overset{\circ}{L}{}^{\infty}_{+}(\mathbb{R}^d)$? The answer is given in the following lemma. In what follows, in order to simplify notation, we often write $T(f)$ instead of $T(f, X_0)$.

Lemma 3.5. *For any* $f \in \overset{\circ}{L}{}^{\infty}_{+}(\mathbb{R}^d)$ *the random variable* $T(f)$ *has the same distribution as* $T(1)$:

$$F_{T(f)}(t) = P\big(t \ge T(f)\big) = 1 - \exp\big(-\langle X_0, 1\rangle \big(t(1-\gamma)\big)^{\gamma'}\big), \quad t \ge 0.$$

Proof. Fix an arbitrary $f \in \overset{\circ}{L}{}^{\infty}_{+}(\mathbb{R}^d)$, and $t > 0$. Then we have

$$\begin{aligned} P\big(t < T(f)\big) &= \lim_{a \searrow 0} E\big(1_{\{t<T(f)\}} e^{-\langle X_t, af\rangle}\big) \\ &= \lim_{a \searrow 0} E\big(e^{-\langle X_t, af\rangle}\big) \\ &= \lim_{a \searrow 0} e^{-\langle V_t(af), X_0\rangle}, \end{aligned} \tag{3.9}$$

where the second equality follows by Corollary 3.4. By Theorem 2.1 (see also Lemma 2.2 in Aguirre and Escobedo (1986/87)), we get

$$\big((1-\gamma)t\big)^{\gamma'} < V_t(af)(x), \quad \forall a > 0, t > 0, x \in \mathbb{R}^d. \tag{3.10}$$

Using (3.10), we get

$$\exp\big(-\langle X_0, V_t(af)\rangle\big) \le \exp\big(-\langle X_0, 1\rangle\big(t(1-\gamma)\big)^{\gamma'}\big). \tag{3.11}$$

By Lemma 3.2(2) we have $T(1) \le T(f)$, P-a.s. By this, (3.8), (3.9), and (3.11) we obtain

$$\begin{aligned} P\big(t < T(1)\big) &\le P\big(t < T(f)\big) \\ &\le \exp\big(-\langle X_0, 1\rangle\big(t(1-\gamma)\big)^{\gamma'}\big) \\ &= P\big(t < T(1)\big). \end{aligned}$$

Thus, we get $P(t < T(f)) = P(t < T(1))$. Since $t > 0$ was arbitrary, we are done. □

The next lemma is a consequence of the first two lemmas in this section.

Lemma 3.6. *For any* $f \in \overset{\circ}{L}{}^{\infty}_{+}(\mathbb{R}^d)$,

$$P\big(T(1)\big) \ne T(f)\big) = 0.$$

Proof. By Lemma 3.2(2), $T(1) \le T(f)$, P-a.s. and, by Lemma 3.5, $T(1)$ and $T(f)$ have the same distribution, hence the result follows. □

Corollary 3.7. *For any* $f \in \overset{\circ}{L}{}^{\infty}_{+}(\mathbb{R}^d)$,

$$E\big(e^{-\langle X_t, f\rangle}\big) = E\big(e^{-\langle X_t, f\rangle} 1_{\{t<T(1)\}}\big) = e^{-\langle X_0, V_t(f)\rangle}, \quad t > 0.$$

3.2 Proof of Theorem 1.2

We begin this subsection with the following remark.

Remark 3.8. By Lemma 3.4.2.1 in Dawson (1993), any random measure $Y \in M_F(\mathbb{R}^d)$ can be decomposed into its absolutely continuous Y^{ac} and singular Y^s parts with respect to Lebesgue measure: $Y(\omega, \mathrm{d}x) = Y^{ac}(\omega, \mathrm{d}x) + Y^s(\omega, \mathrm{d}x)$. By the definition of $T(1)$, X_t is a finite measure on $\{t < T(1)\}$. Hence on the event $\{t < T(1)\}$, X_t can be decomposed into absolutely continuous and singular parts

$$X_t(\omega, \mathrm{d}x) = X_t^{ac}(\omega, \mathrm{d}x) + X_t^s(\omega, \mathrm{d}x).$$

Define the σ-algebra on $\{t < T(1)\}$:

$$\mathcal{F}^{\{t<T(1)\}} = \big\{A \cap \{t < T(1)\} : A \in \mathcal{F}\big\}.$$

The next lemma is used in the proof of measurability of density. Its proof is is standard (see Chapter 1 of Li (2011)) and therefore it is omitted.

Lemma 3.9. *For any $f \in B_{bc}^{+}(\mathbb{R}^d)$ and any fixed $t \in (0, \infty)$, the map $(\omega, z) \mapsto \langle X_t(\omega), f(z - \cdot)\rangle$ is a measurable map from $(\{t < T(1)\}, \mathcal{F}^{\{t<T(1)\}}) \times (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ to $\mathbb{R}_+$.*

For studying differentiability properties of X_t, let us introduce a sequence of functions $\{\delta^n(\cdot)\}_{n=1}^{\infty}$ defined as

$$\delta^n(x) = \begin{cases} 1/\operatorname{Leb}(B_{1/n}(0)), & \text{if } |x| \le \dfrac{1}{n}, \\ 0, & \text{otherwise.} \end{cases}$$

Here $B_{1/n}(0)$ is a closed ball of radius $1/n$, centered at the origin. Notice that the sequence $\{\delta^n(z - \cdot)\}_{n=1}^{\infty}$ converges to Dirac δ-function with support at point z.

Lemma 3.10. *On $\{t < T(1)\} \times \mathbb{R}^d$, $P(\mathrm{d}\omega)\,\mathrm{d}z$-a.e. there exists a limit*

$$\widetilde{\eta}_t^{ac}(\omega, z) = \lim_{n\to\infty} \langle X_t(\omega), \delta^n(z - \cdot)\rangle.$$

The random function $\widetilde{\eta}_t^{ac}$ is a version of the Radon–Nikodym derivative of X_t on $\{t < T(1)\}$. Moreover $\widetilde{\eta}_t^{ac}$ is a measurable map from $(\{t < T(1)\}, \mathcal{F}^{\{t<T(1)\}}) \times (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ to $\mathbb{R}_+$.

Proof. By the Lebesgue density theorem (see Rudin ((1987), Theorem 7.14)), for P-a.s. $\omega \in \{t < T(1)\}$, there exists a limit

$$\widetilde{\eta}_t^{ac}(\omega, z) = \lim_{n\to\infty} \langle X_t(\omega), \delta^n(z - \cdot)\rangle \tag{3.12}$$

for all $z \in R^d \setminus N(\omega)$ where $N(\omega)$ is a Borel subset of Lebesgue measure zero and $\widetilde{\eta}_t^{ac}$ is a Radon–Nikodym derivative with respect to Lebesgue measure. It is easy to see that convergence in (3.12) takes place $P(\mathrm{d}\omega)\,\mathrm{d}z$-a.e. We set $\widetilde{\eta}_t^{ac}(\omega, z)$ to be zero at points (ω, z) where the limit does not exist.

By Lemma 3.9, for each $n = 1, 2, \dots$, $\langle X_t(\omega), \delta^n(z - \cdot)\rangle$ is measurable and the measurability of $\widetilde{\eta}_t^{ac}(\omega, z)$ follows from $P(\mathrm{d}\omega)\,\mathrm{d}z$-a.e. convergence. □

The function $\widetilde{\eta}_t^{ac}(\omega, z)$ is defined on $\{t < T(1)\}$. The function $\eta_t^{ac}(\omega, z)$ is an extension of the function $\widetilde{\eta}_t^{ac}(\omega, z)$ to entire Ω:

$$\eta_t^{ac}(\omega, z) = \begin{cases} \widetilde{\eta}_t^{ac}(\omega, z) & \text{if } \omega \in \{t < T(1)\}, \\ \infty & \text{otherwise.} \end{cases} \tag{3.13}$$

Recall that for any $\mu \in \overset{\circ}{M}_F(\mathbb{R}^d)$, $\{V_t(\mu)\}_{t>0}$ denotes the solution to (2.2).

Lemma 3.11. *For every $t \in (0, \infty)$ the equality*

$$E\left(1_{\{t<T(1)\}} \exp\left(-\sum_{i=1}^{N} a_i \eta^{ac}(z_i)\right)\right) = \exp(-\left(\left\langle X_0, V_t\left(\sum_{i=0}^{N} a_i \delta(z_i - \cdot)\right)\right\rangle\right)$$

holds for almost every $\{z_i\}_{i=1}^{N} \subset \mathbb{R}^d$ and any $\{a_i\}_{i=1}^{N} \subset \mathbb{R}_{++}$.

Proof. Let $\phi(z_1, z_2, \dots, z_N)$ be any function in $C_b^+(\mathbb{R}^{d\times N}) \cap L^1(\mathbb{R}^{d\times N})$. By Corollary 3.7, we have

$$E\left(1_{\{t<T(1)\}} e^{-\langle X_t, \sum_{i=1}^{N} a_i \delta^n(z_i - \cdot)\rangle}\right) = e^{-\langle X_0, V_t(\sum_{i=1}^{N} a_i \delta^n(z_i - \cdot))\rangle}.$$

Let us multiply both parts of this equation by the function $\phi(z_1, z_2, \dots, z_N)$, integrate over $\mathbb{R}^{d\times N}$ and take the limit

$$\lim_{n\to\infty}\int_{\mathbb{R}^{d\times N}} E\big(1_{\{t<T(1)\}}e^{-X_t(\sum_{i=1}^N a_i\delta^n(z_i-\cdot))}\big)\phi(z_1, z_2, \dots, z_N)\,\mathrm{d}z_1\,\mathrm{d}z_2\cdots\mathrm{d}z_N$$
$$= \lim_{n\to\infty}\int_{\mathbb{R}^{d\times N}} e^{-\langle V_t(\sum_{i=1}^N a_i\delta^n(z_i-\cdot)), X_0\rangle}\phi(z_1, z_2, \dots, z_N)\,\mathrm{d}z_1\,\mathrm{d}z_2\cdots\mathrm{d}z_N. \tag{3.14}$$

By Lemma 3.10, the limit

$$\lim_{n\to\infty}\left\langle X_t(\omega), \sum_{i=1}^N a_i\delta^n(z_i-\cdot)\right\rangle = \sum_{i=1}^N a_i\eta_t^{ac}(\omega, z_i)$$

exists almost everywhere on $\{t < T(1)\}\times\mathbb{R}^{N\times d}$ with respect to the measure $P(\mathrm{d}\omega)\phi(z_1, z_2, \dots, z_N)\,\mathrm{d}z_1\,\mathrm{d}z_2\cdots\mathrm{d}z_N$. Therefore, by the bounded convergence theorem, we get following limit on the left-hand side of (3.14):

$$\lim_{n\to\infty}\int_{\mathbb{R}^{d\times N}} E\big(1_{\{t<T(1)\}}e^{-\langle X_t, \sum_{i=1}^N a_i\delta^n(z_i-\cdot)\rangle}\big)\phi(z_1, z_2, \dots, z_N)\,\mathrm{d}z_1\,\mathrm{d}z_2\cdots\mathrm{d}z_N$$
$$= \int_{\mathbb{R}^{d\times N}} E\big(1_{\{t<T(1)\}}e^{-\sum_{i=1}^N a_i\eta_t^{ac}(z_i)}\big)\phi(z_1, z_2, \dots, z_N)\,\mathrm{d}z_1\,\mathrm{d}z_2\cdots\mathrm{d}z_N. \tag{3.15}$$

Now let us take care of the right-hand side of (3.14). Since X_0 is a finite, non-random measure, then by Corollary 2.7, the right-hand side of equation (3.15) also converges:

$$\lim_{n\to\infty}\int_{\mathbb{R}^{d\times N}} e^{-\langle V_t(\sum_{i=1}^N a_i\delta^n(z_i-\cdot)), X_0\rangle}\phi(z_1, z_2, \dots, z_N)\,\mathrm{d}z_1\,\mathrm{d}z_2\cdots\mathrm{d}z_N$$
$$= \int_{\mathbb{R}^{d\times N}} e^{-\langle V_t(\sum_{i=1}^N a_i\eta^{ac}(z_i-\cdot)), X_0\rangle}\phi(z_1, z_2, \dots, z_N)\,\mathrm{d}z_1\,\mathrm{d}z_2\cdots\mathrm{d}z_N. \tag{3.16}$$

Now, since ϕ was chosen arbitrarily, we can combine (3.14), (3.15) and (3.16) and get

$$E\big(1_{\{t<T(1)\}}e^{-\sum_{i=1}^N a_i\eta_t^{ac}(z_i)}\big) = e^{-\langle V_t(\sum_{i=1}^N a_i\delta(z_i-\cdot)), X_0\rangle}$$

for Lebesgue almost every $\{z_i\}_{i=1}^N$ in $\mathbb{R}^d$. □

Lemma 3.12. *Let $\phi \in C_b^+(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$ and let $\{\xi_n\}_{n=1}^\infty$ be a sequence of i.i.d. random variables defined on some probability space $(\Omega', \mathcal{F}', P')$ with the probability density function*

$$g_\xi^r(x) = \begin{cases} 1/\operatorname{Leb}(B_r(0)), & \textit{if } |x|\le r,\\ 0, & \textit{elsewhere}.\end{cases}$$

Then, for any $f \in L^1(\mathbb{R}^d)$,

$$\lim_{N\to\infty}\frac{\operatorname{Leb}(B_r(0))}{N}\sum_{i=1}^N\phi(\xi_i)f(\xi_i) = \operatorname{Leb}\big(B_r(0)\big)\int_{\Omega'}\phi(\omega')f\big(\xi_1(\omega)\big)P'\big(\mathrm{d}\omega'\big)$$
$$= \int_{\mathbb{R}^d}\phi(x)f(x)1_{B_r(0)}(x)\,\mathrm{d}x, \quad P'\text{-a.s.}$$

This also implies that

$$\lim_{N\to\infty}\frac{\operatorname{Leb}(B_r(0))}{N}\sum_{i=1}^N\phi(\xi_i)\delta(\xi_i-\cdot)\overset{w}{\Longrightarrow}\phi(x)1_{B_r(0)}(x)\,\mathrm{d}x, \quad P'\text{-a.s.}$$

Proof. It is obvious that $\phi f \in L^1(\mathbb{R}^d)$ and the rest follows from the law of large numbers. □

Lemma 3.13. *For any* $f \in \overset{\circ}{C}{}_b^+(\mathbb{R}^d)$, $t > 0$,

$$E\Big(1_{\{t<T(1)\}} \exp\Big(-\int_{\mathbb{R}^d} \eta_t^{ac}(z) f(x)\,\mathrm{d}x\Big)\Big) = E\Big(1_{\{t<T(1)\}} \exp\Big(-\int_{\mathbb{R}^d} X_t(\mathrm{d}x) f(x)\Big)\Big).$$

Proof. We augment our probability space $(\Omega, \mathcal{F}, P(\mathrm{d}\omega))$ by taking the Cartesian product with another probability space $(\Omega', \mathcal{F}', P'(\mathrm{d}\omega'))$:

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}) \equiv \big(\Omega \times \Omega', \mathcal{F} \times \mathcal{F}', P(\mathrm{d}\omega)P'(\omega)\big).$$

We also denote expectations on these spaces by E, E' and $\tilde{E}$ respectively. Let $C_b^{++}(\mathbb{R}^d)$ denote the space of bounded continuous functions on $\mathbb{R}^d$ such that for any $f \in C_b^{++}(\mathbb{R}^d)$, we have $\inf_{x\in\mathbb{R}^d} f(x) > 0$. Let us fix an arbitrary $f \in C_b^{++}(\mathbb{R}^d)$ and a positive integer n.

By the Borel theorem (see Kallenberg (2002), Thm 3.19, p. 55), for each $n \geq 1$ we can build on the probability space $(\Omega', \mathcal{F}', P'(\mathrm{d}\omega'))$ a sequence $\{\xi_i^n(\omega)\}_{i=1}^\infty$ of i.i.d. random variables with the density function

$$g_\xi^n(x) = \begin{cases} 1/\mathrm{Leb}\big(B_n(0)\big) & \text{if } |x| \leq n, \\ 0 & \text{elsewhere.} \end{cases}$$

By Lemma 3.11 we get, that the equality

$$E\Big(1_{\{t<T(1)\}} \exp\Big(-\frac{\mathrm{Leb}(B_n(0))}{N} \sum_{i=1}^N f(z_i)\eta^{ac}(z_i)\Big)\Big)$$
$$= \exp\Big(-\Big\langle X_0, V_t\Big(\frac{\mathrm{Leb}(B_n(0))}{N} \sum_{i=1}^N f(z_i)\delta(z_i - \cdot)\Big)\Big\rangle\Big)$$

holds for Lebesgue almost every $\{z_i\}_{i=1}^N$ in $\mathbb{R}^d$. By changing $\{z_i\}_{i=1}^N$ to $\{\xi_i^n\}_{i=1}^N$, we obtain

$$E\Big(1_{\{t<T(1)\}} \exp\Big(-\frac{\mathrm{Leb}(B_n(0))}{N} \sum_{i=1}^N f(\xi_i) 1_{B_n(0)}(\xi_i^n)\eta^{ac}(\xi_i^n)\Big)\Big)$$
$$= \exp\Big(-\Big\langle X_0, V_t\Big(\frac{\mathrm{Leb}(B_n(0))}{N} \sum_{i=1}^N f(\xi_i) 1_{B_n(0)}(\xi_i^n)\delta(\xi_i^n - \cdot)\Big)\Big\rangle\Big), \quad P'\text{-a.s.} \tag{3.17}$$

By taking limits $N \to \infty$ on both sides of (3.17), as well as using Corollary 2.7 and Lemma 3.12 we get the equality

$$E\Big(1_{\{t<T(1)\}} \exp\Big(-\int_{\mathbb{R}^d} \eta_t^{ac}(x) f(x) 1_{B_n(0)}(x)\,\mathrm{d}x\Big)\Big)$$
$$= \exp\big(-\langle X_0, V_t(f 1_{B_n(0)})\rangle\big), \quad P'\text{-a.s.}$$

Since both sides of the above equation are constants, we can drop P'-a.s., and get

$$E\Big(1_{\{t<T(1)\}} \exp\Big(-\int_{\mathbb{R}^d} \eta^{ac}(x) f(x) 1_{B_n(0)}(x)\,\mathrm{d}x\Big)\Big) = \exp\big(-\langle X_0, V_t(f 1_{B_n(0)})\rangle\big). \tag{3.18}$$

By Theorem 2.8 in Aguirre and Escobedo (1986/87),

$$V_t(f 1_{B_n(0)}) \leq V_t(f 1_{B_{n+1}(0)})$$

and

$$\lim_{n\to\infty} V_t(f 1_{B_n(0)}) = V_t(f). \tag{3.19}$$

Now we take limits, as $n \to \infty$ on both sides of (3.18), use the monotone convergence theorem and (3.19) to get

$$E\biggl(1_{\{t<T(1)\}} \exp\biggl(-\int_{\mathbb{R}^d} \eta_t^{ac}(z) f(x)\,\mathrm{d}x\biggr)\biggr) = \exp\bigl(-\langle X_0, V_t(f)\rangle\bigr). \tag{3.20}$$

Since any function in $\overset{\circ}{C}{}_b^+(\mathbb{R}^d)$ can be approximated boundedly pointwise by functions from $C_b^{++}(\mathbb{R}^d)$, we can again apply the dominated convergence theorem and obtain that the equality (3.20) holds for any $f \in \overset{\circ}{C}{}_b^+(\mathbb{R}^d)$. Recall, that

$$E\bigl(1_{\{t<T(1)\}} \exp(-\langle X_t, f\rangle)\bigr) = \exp\bigl(-\langle V_t(f), X_0\rangle\bigr),$$

and we are done. □

Now we are ready to conclude the proof of the main result.

Proof of Theorem 1.2. Fix an arbitrary real number $t > 0$. By Corollary 3.7, Lemma 3.13, for every $f \in \overset{\circ}{C}{}_b^+(\mathbb{R}^d)$,

$$\begin{aligned} &E\bigl(1_{\{t<T(1)\}} \exp(-\langle X_t, f\rangle)\bigr) \\ &\quad = E\biggl(1_{\{t<T(1)\}} \exp\biggl(-\int_{\mathbb{R}^d} \eta_t^{ac}(x) f(x)\,\mathrm{d}x\biggr)\biggr). \end{aligned} \tag{3.21}$$

This equation implies, that, on the event $\{t < T(1)\}$,

$$\int_{\mathbb{R}^d} X_t(\mathrm{d}x) f(x) \overset{d}{=} \int_{\mathbb{R}^d} \eta_t^{ac}(x) f(x)\,\mathrm{d}x,$$

where $\overset{d}{=}$ means equality in distribution. By Lemma 3.10 and the definition of η_t^{ac}, η_t^{ac} is a version of the Radon–Nikodym derivative of $X_t(\mathrm{d}x)$ on $\{t < T(1)\}$. Therefore, on $\{t < T(1)\}$,

$$\int_{\mathbb{R}^d} X_t(\mathrm{d}x) f(x) \ge \int_{\mathbb{R}^d} \eta_t^{ac}(x) f(x)\,\mathrm{d}x, \quad P\text{-a.s.} \tag{3.22}$$

Equations (3.22) and (3.2) imply that

$$\int_{\mathbb{R}^d} X_t(\mathrm{d}x) f(x) = \int_{\mathbb{R}^d} \eta_t^{ac}(x) f(x)\,\mathrm{d}x, \quad P\text{-a.s. on } \bigl\{t < T(1)\bigr\}.$$

Since $f \in \overset{\circ}{C}{}_b^+(\mathbb{R}^d)$ was arbitrary, this completes the proof of the theorem. □

4 Proof of Theorem 2.4

Many steps in the proof follow the lines from Aguirre and Escobedo (1986/87). However, since the initial conditions are measures, modifications are required.

4.1 Existence of solutions

We now prove the existence of a solution to equation (2.2) by the Picard iterations. Let $\mu \in \overset{\circ}{M}_F(\mathbb{R}^d)$, and

$$w(x,t,\mu) = S_t\mu + \int_0^t \bigl(S_{t-s}\Psi\bigl(w(s,\cdot,\mu)\bigr)\bigr)(x)\,\mathrm{d}s, \quad x \in \mathbb{R}^d, t > 0, \tag{4.1}$$

be an integral evolution equation such that Ψ is some non-negative function defined on $\mathbb{R}_+$. Recall that the Picard iterations for this equation are defined by induction as follows:

$$\begin{aligned} w_1(x,t,\mu) &= (S_t\mu)(x), \\ w_{n+1}(x,t,\mu) &= (S_t\mu)(x) + \int_0^t \bigl(S_{t-s}\Psi\bigl(w_n(s,\cdot,\mu)\bigr)\bigr)(x)\,\mathrm{d}s, \\ & x\in\mathbb{R}^d, t>0, n=1,2,\ldots. \end{aligned} \tag{4.2}$$

Notice that (2.2) is a particular case of (4.1) with $\Psi(\lambda)=\lambda^\gamma$. It is obvious that for $0<t<\infty$, the Picard iterations (4.2) form the non-decreasing sequence: $w_n(x,t,\mu)\le w_{n+1}(x,t,\mu)$. In the next lemma we derive some properties of the Picard iterations.

Lemma 4.1. *Let $\{v_n(x,t,\mu)\}_{n=1}^\infty$ be a sequence of Picard iterations corresponding to* (2.2). *Then for every $n=1,2,\ldots$ and any $0<t<\infty$, $x\in\mathbb{R}^d$, the following inequalities hold*:

(1) $0\le v_n(t,x,\mu)\le e^t(S_t\mu)(x)+e^t$,
(2) $v_n^\gamma(t,x,\mu)\le e^t(S_t\mu)(x)+e^t$.

Proof. First note that $v_n, n\ge 1$, are non-negative by construction. Let $\mu\in M_F(\mathbb{R}^d)$ and let us consider a linear integral equation

$$u(t,x,\mu) = (S_t\mu)(x) + \int_0^t \bigl(S_{t-s}\bigl(u(s,\cdot,\mu)+1\bigr)\bigr)(x)\,\mathrm{d}s, \quad x\in\mathbb{R}^d, t>0. \tag{4.3}$$

Let $\{u_n(x,t,\mu)\}_{n=1}^\infty$ be corresponding Picard iterations. Note that (4.3) is a particular case of (4.1) with $\Psi(\lambda)=\lambda+1$. Since $\lambda^\gamma\le\lambda+1$ for $\gamma\in(0,1)$, one can easily see that $v_n(t,x,\mu)\le u_n(t,x,\mu)$, for all $n\ge 1$.

On the other hand by direct calculations, one gets that

$$\lim_{n\to\infty} u_n(x,t,\mu) \nearrow e^t(S_t\mu)(x)+e^t-1, \quad \text{as } n\to\infty, \forall(t,x)\in(0,\infty)\times\mathbb{R}^d, \tag{4.4}$$

and the first inequality of the lemma follows. The second inequality is a consequence of $\lambda^\gamma\le\lambda+1$, $v_n\le u_n$ and (4.4). □

Proposition 4.2 (Existence). *Let $\mu\in \mathring{M}_F(\mathbb{R}^d)$. Then the integral equation* (2.2) *has a solution $v(\cdot,\cdot)$ which is a limit of Picard iterations and, for any $\phi\in L^1(\mathbb{R}^d)\cap C_b(\mathbb{R}^d)$*

$$\lim_{t\to 0}\int_{\mathbb{R}^d} v(t,x)\phi(x)\,\mathrm{d}x = \int_{\mathbb{R}^d}\phi(x)\mu(\mathrm{d}x). \tag{4.5}$$

Moreover $v(\cdot,\cdot)$ satisfies the following inequalities for any $(t,x)\in(0,\infty)\times\mathbb{R}^d$:

$$v(t,x)\le e^t(S_t\mu)(x)+e^t, \tag{4.6}$$

$$v^\gamma(t,x)\le e^t(S_t\mu)(x)+e^t. \tag{4.7}$$

Proof. Let $\{v_n(t,x)\}_{n=1}^\infty$ be a sequence of Picard iterations corresponding to equation (2.2). By the previous discussion for any $t\in(0,\infty)$, $\{v_n(t,\cdot)\}_{n=1}^\infty$ form a non-decreasing sequence and by Lemma 4.1 we have

$$v_n(t,x), v_n^\gamma(t,x)\le e^t(S_t\mu)(x)+e^t, \quad \forall n\ge 1, \forall(t,x)\in(0,\infty)\times\mathbb{R}^d. \tag{4.8}$$

Lemma 2.6 tells us that for every $t>0$, $(S_t\mu)(\cdot)$ is bounded. Thus, for any (t,x) in $(0,\infty)\times\mathbb{R}^d$, the sequence $\{v_n(t,x)\}_{n=1}^\infty$ is non-decreasing and bounded. Consequently, there exists a bounded limit $v(t,x)=\lim_{n\to\infty}v_n(t,x)$. Inequalities (4.6) and (4.7) follow from existence of the limit and (4.8).

Now consider the sequence of equations which defines the Picard iterations:

$$v_{l+1}(x,t) = \int_{\mathbb{R}^d} p_{t-s}(x-y)\mu(\mathrm{d}y) + \int_0^t \int_{\mathbb{R}^d} p_t(x-y) v_l^\gamma(s,y)\,\mathrm{d}y\,\mathrm{d}s,$$
$$l = 1, 2, \ldots. \tag{4.9}$$

We have already proved that the left-hand side of (4.9) converges boundedly pointwise to $v(t,x)$. From the monotone convergence theorem, it follows that the right-hand side converges to

$$\int_{\mathbb{R}^d} p_{t-s}(x-y)\mu(\mathrm{d}y) + \int_0^t \int_{\mathbb{R}^d} p_t(x-y) v^\gamma(s,y))\,\mathrm{d}y\,\mathrm{d}s.$$

Thus $v(t,x)$ satisfies equation (2.2) for all $x \in \mathbb{R}^d$, $t > 0$.

Now let us verify (4.5). In the following discussion, we can assume without loss of generality that $\phi \in C_b^+(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Since the family of functions $\{p_t(\cdot)\}_{t>0}$ builds up the *Dirac family* (see Lang ((1997), pages 284–287, 348)), we can easily conclude that

$$\lim_{t \searrow 0} \langle S_t\mu, \phi\rangle = \langle \mu, \phi\rangle. \tag{4.10}$$

Now let us prove

$$\lim_{t\to 0} \int_{\mathbb{R}^d} \biggl(\int_0^t (S_{t-s} v^\gamma(s,\cdot))(x) \biggr)\,\mathrm{d}s) \phi(x)\,\mathrm{d}x = 0. \tag{4.11}$$

First, using the inequality (4.7) we obtain

$$\begin{aligned} \int_0^t (S_{t-s} v^\gamma(s))(x)\,\mathrm{d}s &\le \int_0^t (S_{t-s}(e^s S_s\mu + e^s)\,\mathrm{d}s \\ &= (e^t - 1)(S_t\mu)(x) + (e^t - 1), \end{aligned}$$

and verifying (4.11) from this is, and easy exercise. Now (4.11) and (4.10) imply (4.5) and this completes the proof of the proposition. □

Corollary 4.3. *Let $\mu \in \overset{\circ}{M}_F(\mathbb{R}^d)$ and let $v(\cdot,\cdot,\mu)$ be a solution of (2.2) obtained as a limit of the Picard iterations in Proposition 4.2. Then $v(\cdot,\cdot,\mu) \in L^\infty_{\mathrm{loc}}(((0,\infty), L_+^{1,w}(\mathbb{R}^d))$.*

Proof. Let $v(\cdot,\cdot,\mu)$ be a solution constructed in Proposition 4.2. Using the bound (4.6), it is easy to derive the result by standard Gaussian bounds. □

4.2 Continuity of solutions

In this section, we will prove the continuity of the solution obtained in Proposition 4.2.

We start with the technical lemma, whose proof is pretty standard, and therefore it is omitted.

Lemma 4.4. *Fix $0 < T_1 < T_2$ and $r > 0$. Let $\{p_s(\cdot + z), s \in [T_1, T_2], |z| \le r\}$ be a family of functions, where $p_s(\cdot)$ is a standard Gaussian kernel on $\mathbb{R}^d$. Then, there exists a constant K, such that*

$$p_s(x+z) \le K p_{2T_2}(x), \quad \forall s \in [T_1, T_2], |z| \le r, x \in \mathbb{R}^d.$$

Now we are ready to state and prove the main proposition of Section 4.2.

Proposition 4.5 (Continuity). *Let $\mu \in \overset{\circ}{M}_F(\mathbb{R}^d)$ and let $v(\cdot,\cdot)$ be a solution of (2.2) obtained as a limit of Picard iterations in Proposition 4.2. Then $v(\cdot,\cdot) \in C^+((0,\infty) \times \mathbb{R}^d)$.*

Proof. By construction the solution is clearly non-negative. Now, let us fix a point $(t,x)\in(0,\infty)\times\mathbb{R}^d$ and an arbitrary $\epsilon>0$. Let $\delta_i>0$, $i=1,2,3$, $\delta_1<\delta_3<t/10$. In what follows we will show that δ_1, δ_2 and δ_3 can be chosen sufficiently small so that if $|\Delta t|<\delta_1$ and $|\Delta x|<\delta_2$ then

$$\left|v(t+\Delta t,x+\Delta x)-v(t,x)\right|\le\epsilon. \tag{4.12}$$

We will bound absolute value of difference $v(t+\Delta t,x+\Delta x)-v(t,x)$ only for the case of $\Delta t\ge 0$, since the case of $\Delta t<0$ can be treated similarly. We split the difference $v(t+\Delta t,x+\Delta x)-v(t,x)$ as follows:

$$\begin{aligned}v(t+\Delta t,x+\Delta x)-v(x,t)&=I(\Delta t,\Delta x)+J_1(\Delta t,\Delta x)-J_2(\Delta t,\Delta x)+J_3(\Delta t,\Delta x)\\&\quad+J_4(\Delta t,\Delta x)-J_5(\Delta t,\Delta x).\end{aligned}$$

Here

$$\begin{aligned}I(\Delta t,\Delta x)&=(S_{t+\Delta t}\mu)(x+\Delta x)-(S_t\mu)(x),\\J_1(\Delta t,\Delta x)&=\int_{t-\delta_3}^{t+\Delta t}\big(S_{t+\Delta t-s}v^\gamma(s)\big)(x+\Delta x)\,\mathrm{d}s,\\J_2(\Delta t,\Delta x)&=\int_{t-\delta_3}^{t}\big(S_{t-s}v^\gamma(s)\big)(x)\,\mathrm{d}s,\\J_3(\Delta t,\Delta x)&=\int_{\delta_3}^{t-\delta_3}\big(S_{t+\Delta t-s}v^\gamma(s)\big)(x+\Delta x)\,\mathrm{d}s-\int_{\delta_3}^{t-\delta_3}\big(S_{t-s}v^\gamma(s)\big)(x)\,\mathrm{d}s,\\J_4(\Delta t,\Delta x)&=\int_0^{\delta_3}\big(S_{t+\Delta t-s}v^\gamma(s)\big)(x+\Delta x)\,\mathrm{d}s,\\J_5(\Delta t,\Delta x)&=\int_0^{\delta_3}\big(S_{t-s}v^\gamma(s)\big)(x)\,\mathrm{d}s.\end{aligned}$$

Note that the integrals J_1, J_2, J_4 and J_5 have the same form:

$$J_*=\int_{t_1}^{t_2}\big(S_{t_3-s}v^\gamma(s)\big)(z)\,\mathrm{d}s, \tag{4.13}$$

for appropriate $t_1,t_2,t_3\ge 0$ and $z\in\mathbb{R}^d$. From the definitions of δ_1, δ_3 and Δt, it follows that t_1, t_2 and t_3 in (4.13) can vary but satisfy the inequalities $t_1<t_2$, $t\le t_3$ and $t_2-t_1\le 2\delta_3$ hold. Let us bound J_* from above. By Lemma 2.6 and Proposition 4.2, we easily get

$$\begin{aligned}J_*&=\int_{t_1}^{t_2}\big(S_{t_3-s}v^\gamma(s)\big)(z)\,\mathrm{d}s\\&\le\int_{t_1}^{t_2}\big(S_{t_3-s}\big(e^s(S_s\mu+1)\big)\big)(z)\,\mathrm{d}s\\&=\int_{t_1}^{t_2}e^s\big((S_{t_3}\mu)(z)+1\big)\,\mathrm{d}s\\&=\big(e^{t_2}-e^{t_1}\big)\big((S_{t_3}\mu)(z)+1\big)\\&\le\frac{(e^{t_2}-e^{t_1})(\mu(\mathbb{R}^d)+1)}{(2\pi t_3)^{d/2}}\\&\le\frac{(e^{t_2}-e^{t_1})(\mu(\mathbb{R}^d)+1)}{(2\pi t)^{d/2}},\end{aligned} \tag{4.14}$$

where the last inequality follows from $t \le t_3$. Recall that $t_2 - t_1 \le 2\delta_3$, and so by (4.14) we can choose δ_3 sufficiently small so that, for $i = 1, 2, 4, 5$

$$J_i(\Delta t, \Delta x) \le \epsilon/10, \quad \Delta t < \delta_1 < \delta_3. \tag{4.15}$$

Let us fix such δ_3. Let us recall that $\Delta t < \delta_1 < \delta_3$. Now we will handle $J_3(\Delta t, \Delta x)$. Write J_3 as $J_3(\Delta t, \Delta x) = J_{31}(\Delta t, \Delta x) - J_{32}$, where

$$J_{31}(\Delta t, \Delta x) = \int_{\delta_3}^{t-\delta_3} \int_{\mathbb{R}^d} p_{t+\Delta t-s}(x + \Delta x - y) v^\gamma(s, y)\, \mathrm{d}y\, \mathrm{d}s, \tag{4.16}$$

$$J_{32} = \int_{\delta_3}^{t-\delta_3} \int_{\mathbb{R}^d} p_{t-s}(x - y) v^\gamma(s, y)\, \mathrm{d}y\, \mathrm{d}s. \tag{4.17}$$

By Lemma 4.4 and Proposition 4.2, we immediately get that there exists $K = K(\delta_1, \delta_1, \delta_3, t)$ such that

$$\begin{aligned} &p_{t+\Delta t-s}(x + \Delta x - y) v^\gamma(s, y) \le K p_{2t}(x - y) e^s \big((S_s\mu)(y) + 1\big) \\ &\quad \forall \Delta t \in (0, \delta_1), s \in (\delta_3, t - \delta_3), |\Delta x| < \delta_2. \end{aligned}$$

It is easy to verify that

$$\int_{\delta_3}^{t-\delta_3} \int_{\mathbb{R}^d} K p_{2t}(x - y) e^s \big((S_s\mu)(y) + 1\big)\, \mathrm{d}y\, \mathrm{d}s < \infty.$$

Therefore we can use the dominated convergence theorem and obtain:

$$\begin{aligned} \lim_{\substack{\Delta t \to 0 \\ \Delta x \to 0}} J_{31}(\Delta t, \Delta x) &= \lim_{\substack{\Delta t \longrightarrow 0 \\ \Delta x \longrightarrow 0}} \int_{\delta_3}^{t-\delta_3} \int_{\mathbb{R}^d} p_{t+\Delta t-s}(x + \Delta x - y) v^\gamma(s, y)\, \mathrm{d}y\, \mathrm{d}s \\ &= \int_{\delta_3}^{t-\delta_3} \int_{\mathbb{R}^d} p_{t-s}(x - y) v^\gamma(s, y)\, \mathrm{d}y\, \mathrm{d}s \\ &= J_{32}. \end{aligned} \tag{4.18}$$

Similarly we show

$$\lim_{\substack{\Delta t \longrightarrow 0 \\ \Delta x \longrightarrow 0}} (S_{t+\Delta t}\mu)(x + \Delta x) = (S_t\mu)(x), \tag{4.19}$$

and thus from (4.18), (4.19) and the definition of $I(\Delta t, \Delta x)$, $J_3(\Delta t, \Delta x)$ we get

$$\lim_{\substack{\Delta t \longrightarrow 0 \\ \Delta x \longrightarrow 0}} I(\Delta t, \Delta x) + J_3(\Delta t, \Delta x) = 0. \tag{4.20}$$

This implies that there exist $\delta_1, \delta_2 \in (0, \delta_3)$ sufficiently small such that for $|\Delta t| < \delta_1$ and $|\Delta x| < \delta_2$

$$\big|J_3(\Delta t, \Delta x)\big| + \big|J_3(\Delta t, \Delta x)\big| \le \epsilon/2. \tag{4.21}$$

Thus we get from (4.15) and (4.21) that

$$\begin{aligned} &\big|v(t + \Delta t, x + \Delta x) - v(t, x)\big| \\ &\quad = \big|I(t + \Delta t, x + \Delta x) + J_1(t + \Delta t, x + \Delta x) - J_2(t + \Delta t, x + \Delta x) \\ &\qquad + J_3(t + \Delta t, x + \Delta x) + J_4(t + \Delta t, x + \Delta x) - J_5(t + \Delta t, x + \Delta x)\big| \\ &\quad \le \big|I(t + \Delta t, x + \Delta x)\big| + \big|J_1(t + \Delta t, x + \Delta x)\big| + \big|J_2(t + \Delta t, x + \Delta x)\big| \\ &\qquad + \big|J_3(t + \Delta t, x + \Delta x)\big| + \big|J_4(t + \Delta t, x + \Delta x)\big| + \big|J_5(t + \Delta t, x + \Delta x)\big| \\ &\quad \le \epsilon, \quad \forall \Delta t, \Delta x : \Delta t \in (0, \delta_1), |\Delta x| < \delta_2. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary we are done. $\square$

4.3 Uniqueness of solutions

The proof of uniqueness is again based on proofs in Aguirre and Escobedo (1986/87) which are adjusted to our case. Let us recall that $\gamma' = 1/(1-\gamma)$.

In the next lemma, we prove an important lower bound.

Lemma 4.6. *Let $\mu \in \overset{\circ}{M}_F(\mathbb{R}^d)$ and $v(t,x)$ be a non-negative function on $(0,\infty)\times\mathbb{R}^d$ such that, for any $t \in (0,\infty)$ and any $x \in \mathbb{R}^d$:*

$$v(t,x) \geq S_t\mu + \int_0^t \big(S_{t-s}v^\gamma(s)\big)(x)\,\mathrm{d}s.$$

Then

$$v(t,x) > \big((1-\gamma)t\big)^{\gamma'}, \quad \forall (t,x) \in (0,\infty)\times\mathbb{R}^d. \tag{4.22}$$

Proof. Let us fix an arbitrary $t_0 > 0$ and define

$$\tilde{v}(t) \equiv v(t+t_0), \quad \forall t \geq 0.$$

Using this definition one can easily check that

$$\tilde{v}(t) \geq S_t\tilde{v}_0 + \int_0^t S_{t-s}\tilde{v}^\gamma(s)\,\mathrm{d}s, \quad t \geq 0,$$

where $\tilde{v}_0 = \tilde{v}(0) \geq S_{t_0}\mu$, and the last inequality follows by definition of $\tilde{v}$ and assumptions on v. Since $S_{t_0}\mu \in \overset{\circ}{C}{}_b^+(\mathbb{R}^d)$, we can apply Lemma 2.2 from Aguirre and Escobedo (1986/87) to get

$$\tilde{v}(t,x) \geq \big((1-\gamma)t\big)^{\gamma'}, \quad \forall (t,x) \in (0,\infty)\times\mathbb{R}^d.$$

Since $t_0 > 0$ was arbitrary, we have

$$v(t,x) \geq \big((1-\gamma)t\big)^{\gamma'}, \quad \forall (t,x) \in (0,\infty)\times\mathbb{R}^d,$$

and we are done. □

Lemma 4.7 (Comparison lemma). *Let*

$$v, u \in L^\infty_{\mathrm{loc}}\big((0,\infty), L^{1,w}(\mathbb{R}^d)\big) \cap C\big((0,\infty)\times\mathbb{R}^d\big)$$

be non-negative functions such that, for all $t > 0$,

$$u(t) \geq S_t\nu + \int_0^t S_{t-s}u^\gamma(s)\,\mathrm{d}s,$$

$$v(t) \leq S_t\mu + \int_0^t S_{t-s}v^\gamma(s)\,\mathrm{d}s.$$

Here $\mu, \nu \in \overset{\circ}{M}_F(\mathbb{R}^d)$ are such that

$$\nu(f) \geq \mu(f), \quad \forall f \in C_b^+(\mathbb{R}^d).$$

Then

$$u(t,x) \geq v(t,x), \quad \text{for all } (t,x) \in (0,\infty)\times\mathbb{R}^d.$$

Proof. Define

$$g(t) \equiv v(t) - u(t).$$

We will now prove that $g_+(t) \equiv \max(g(t), 0) = 0$.

Fix arbitrary $T > 0$. We use the condition $\nu \geq \mu$ and an elementary inequality $(a^\gamma - b^\gamma) \leq ((a-b)_+)^\gamma$ to get

$$\begin{aligned} g(t) &\leq S_t(\mu - \nu) + \int_0^t S_{t-s}\big(v^\gamma(s) - u^\gamma(s)\big)\,\mathrm{d}s \\ &\leq \int_0^t S_{t-s}\big(v^\gamma(s) - u^\gamma(s)\big)_+\,\mathrm{d}s \\ &\leq \int_0^t S_{t-s}\big((g_+(s))^\gamma\big)\,\mathrm{d}s. \end{aligned} \tag{4.23}$$

From this point, the proof follows the proof of Theorem 2.8 in Aguirre and Escobedo (1986/87) while using Lemma 4.6 whenever necessary. We left the details to the reader. □

The uniqueness for (2.2) follows easily from the above comparison Lemma 4.7.

Proposition 4.8 (Uniqueness). *Let* $\mu \in \overset{\circ}{M}_F(\mathbb{R}^d)$. *There is at most one solution to* (2.2) *which belongs to* $L^\infty_{\mathrm{loc}}([0,\infty), L^{1,w}_+(\mathbb{R}^d)) \cap C^+((0,\infty)\times\mathbb{R}^d)$.

Proof. Suppose there exist two functions $v, u \in L^\infty_{\mathrm{loc}}([0,\infty), L^{1,w}_+(\mathbb{R}^d)) \cap C^+((0,\infty)\times\mathbb{R}^d)$ that solve equation (2.2) for the same initial measure μ. Then by Lemma 4.7, $v(t,x) \geq u(t,x)$ and $u(t,x) \geq v(t,x)$ for any $(t,x) \in (0,\infty)\times\mathbb{R}^d$, and thus $u = v$ and we are done. □

4.4 Continuous dependence of solutions on initial data

In the previous sections, we proved the existence and uniqueness of solutions to equation (2.2), or looking from different perspective we proved for every $(t,x) \in (0,\infty)\times\mathbb{R}^d$ the existence of the mapping

$$v(t,x,\cdot) : \overset{\circ}{M}_F(\mathbb{R}^d) \to \mathbb{R}_{++}.$$

Here $v(t,x,\mu)$ is a solution to equation (2.2) with initial datum μ.

In this section, we will prove the continuity of this mapping.

Lemma 4.9. *For any* $(t,x) \in (0,\infty)\times\mathbb{R}^d$, *the mapping*

$$v(t,x,\cdot) : \overset{\circ}{M}_F(\mathbb{R}^d) \mapsto \mathbb{R}_{++}$$

is concave, that is,

$$\begin{gathered} v\big(t,x,\lambda\mu + (1-\lambda)\nu\big) \geq \lambda v(t,x,\mu) + (1-\lambda)v(t,x,\nu), \\ \forall\lambda \in (0,1), \forall(t,x) \in (0,\infty)\times\mathbb{R}^d. \end{gathered}$$

Proof. Since the function $x \to x^\gamma$ is concave, for any positive a, b and $\lambda \in (0,1)$, we have

$$\lambda a^\gamma + (1-\lambda)b^\gamma \leq \big(\lambda a + (1-\lambda)b\big)^\gamma. \tag{4.24}$$

Let us fix an arbitrary $\lambda \in (0,1)$ and define $u(t,z,\mu,\nu,\lambda)$ as follows

$$u(t,x,\mu,\nu,\lambda) \triangleq \lambda v(t,\mu) + (1-\lambda)v(t,\nu).$$

Then we have

$$\begin{aligned} u(t,x,\mu,\nu,\lambda) &= \lambda v(t,x,\mu) + (1-\lambda)v(t,x,\nu) \\ &= (S_t\sigma)(x) + \int_0^t \big(S_{t-s}\big(\lambda v^\gamma(s,\mu) + (1-\lambda)v^\gamma(s,\nu)\big)\big)(x)\,\mathrm{d}s \\ &\le (S_t\sigma)(x) + \int_0^t \big(S_{t-s}\big(\lambda v(s,\mu) + (1-\lambda)v(s,\nu)\big)^\gamma\big)(x)\,\mathrm{d}s \\ &= (S_t\sigma)(x) + \int_0^t \big(S_{t-s}\big(u^\gamma(s,\mu,\nu,\lambda)\big)(x)\,\mathrm{d}s, \end{aligned}$$

where the above inequality follows from (4.24), and we set $\sigma = \lambda\mu + (1-\lambda)\nu$. Hence, we obtained

$$u(t,x) \le (S_t\sigma)(x) + \int_0^t \big(S_{t-s}u^\gamma(s)\big)(x)\,\mathrm{d}s. \tag{4.25}$$

We now recall that by definition

$$v(t,x,\sigma) = (S_t\sigma)(x) + \int_0^t \big(S_{t-s}v(s,\sigma)^\gamma\big)(x)\,\mathrm{d}s. \tag{4.26}$$

and it is left to use comparison Lemma 4.7. □

Proposition 4.10. *For any fixed* $(t,x) \in (0,\infty)\times\mathbb{R}^d$, *the mapping*

$$v(t,x,\cdot) : \overset{\circ}{M}_F(\mathbb{R}^d) \mapsto \mathbb{R}_{++}$$

is continuous.

Remark 4.11. It follows from the above proposition that the weak convergence of initial measures implies pointwise convergence of solutions to equation (2.2).

Proof. By Lemma 4.9 for any $(t,x)\in(0,\infty)\times\mathbb{R}^d$, the mapping $\mu\mapsto v(t,x,\cdot)$ is concave, and $v(t,x,\mu)\ge 0$ for any $\mu\in\overset{\circ}{M}_F(\mathbb{R}^d)$. Hence, by Lemma 2.1 in Ekeland and Témam (1999), mapping $\mu\mapsto v(t,x,\mu)$ is continuous. □

Now we are ready to finish the proof of Theorem 2.4.

Proof of Theorem 2.4. The statement of the theorem follows from Propositions 4.2, 4.5, 4.8, Corollary 4.3 and Proposition 4.10. □

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