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Discrete line integral on uniform grids: Probabilistic interpretation and applications

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Abstract. Following the methodology developed by (*Comput. Math. Appl.* **33** (1997) 81–104), we define a discrete version of gradient vector and associated line integral along arbitrary path connecting two nodes of uniform grid. An exponential representation of joint survival function of bivariate discrete non-negative integer-valued random variables in terms of discrete line integral is established. We apply it to generate a discrete analogue of the Sibuya-type aging property, incorporating many classical and new bivariate discrete models. Several characterizations and closure properties of this class of bivariate discrete distributions are presented.

1 Introduction

Let us recall the notion of line integral in the continuous case and show briefly its role when studying continuous distributions. Suppose $\mathbf{F}$ is a conservative continuous vector field on the plane R^2; that is, $\mathbf{F}$ is equal to the gradient of some scalar differentiable function f on a open connected region $D \subset R^2$. In other words,

$$\mathbf{F}(x, y) = \nabla f(x, y) = \left(\frac{\partial}{\partial x} f(x, y), \frac{\partial}{\partial y} f(x, y)\right) \quad \text{for all } (x, y) \in D \subset R^2.$$

A conservative vector field satisfies the fundamental theorem of calculus, which says that if two points are connected by a sufficiently smooth continuous path $\mathcal{C}$, lying entirely in D, parametrized by a differentiable function $\mathbf{r}(t)$ for $a \le t \le b$, then the line integral

$$\int_{\mathcal{C}} \mathbf{F}\, d\mathbf{r} = \int_{\mathcal{C}} \nabla f(\mathbf{r})\, d\mathbf{r} = f\big(\mathbf{r}(b)\big) - f\big(\mathbf{r}(a)\big), \tag{1.1}$$

consult Apostol (1969).

A consequence of the fundamental theorem of calculus for line integrals is that if $\mathbf{F}$ is a conservative vector field on D, then $\mathbf{F}$ is path independent on D, meaning that the value of line integral of $\mathbf{F}$ along the path $\mathcal{C}$ only depends on the start and end points of $\mathcal{C}$, and not on the path in between. Another conclusion from (1.1) is that the line integral of conservative vector field $\mathbf{F}$ along any closed path in D is always equal to 0. It turns out that the property of a vector field $\mathbf{F}$ being conservative on D is actually equivalent to path-independence on D. A probability interpretation of line integral relation (1.1) is provided by Marshall (1975), who established an exponential representation of the joint survival function of a non-negative continuous random vector. We will discuss briefly this characterization and its importance for survival analysis in Section 2.2.1. A similar tool in the discrete multivariate case is missing in literature and our goal is to develop it and to suggest applications.

When dealing with discrete data, the vector field $\mathbf{G}$, that is, the gradient ∇g of a scalar function g, is not available in analytical form. It is defined at discrete locations. For instance,

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if we consider the bivariate case and the set of discrete locations is composed by points (x, y) for $x = 1, 2, \ldots, M$ and $y = 1, 2, \ldots, N$, then the discrete gradient vector $\mathbf{G}$ is determined by MN pairs of values of associated scalar function g. For this reason, many authors define the domain of scalar function as the nodes of some grid. The space of discrete scalar functions consists of all discrete scalar functions with the same domain. On the other side, a discrete vector function is a discrete analog of continuous vector function. It has two components associated to each node, and can be viewed as discrete scalar function.

An important contribution to the theory of vector fields discretization and its applications is developed by Hyman and Shashkov (1997). The authors created discrete versions of vector fields for 2-D logically rectangular grids introducing both cell-valued and nodal discretization for scalar functions. In particular, the natural discrete gradient operator has been defined. Hyman and Shashkov (1997) introduced a discrete surface integrals of scalar functions and a discrete analogue of the line integral of gradient vector along an arbitrary path connecting two nodes and proved discrete versions of the main theorems relating these objects. For example, a discrete counterpart of the fundamental theorem of calculus for line integrals has been provided.

The theory developed by Hyman and Shashkov (1997) fits perfectly for our needs: to establish an exponential representation of the joint survival function of non-negative integer-valued random variables through the discrete line integral. We adapt the corresponding definitions and statements to the case of uniform grids. As a result, the important conclusion is that the components of discrete gradient vector $\mathbf{G}$ uniquely determine the joint distribution. We are convinced that the methodology proposed in this paper is a new and powerful tool to generate discrete probability models given the knowledge of the underlying vector field.

The article is organized as follows. To make the exposition self-containing, we outline in Section 2 basic reliability relations for univariate continuous and discrete distributions. After that we consider the bivariate case. We briefly discuss main properties of the line integral in continuous case and consider several typical examples. Following Hyman and Shashkov (1997), we introduce a discrete version of the line integral of gradient vector along a path connecting two nodes of uniform grid in Section 3 and we establish in Section 4 its exponential relation to the joint survival function of non-negative integer-valued random variables. In Section 5, we apply the discrete line integral to construct a discrete analogue of the Sibuya-type aging property introduced by Pinto and Kolev (2016). We present several characterizations of this class of bivariate discrete distributions which incorporates several classical models, see, for example, examples in Section 5.3. In Section 5.4, we present closure properties of the class and show how to use them as a tool to generate new bivariate discrete distributions. Finally, we conclude by suggesting a weaker version of the discrete Sibuya-type aging property.

2 Exponential representation of survival function

We begin with necessary notations and facts involving the failure rate and its connection with the corresponding survival functions for univariate continuous and discrete models. Then, we will consider the bivariate case.

2.1 Univariate case

To proceed, consider first a non-negative continuous random variable X defined by its survival function $S_X(x) = P(X > x)$, or equivalently, by its failure (hazard) rate

$$r_X(x) = \frac{d}{dx}[-\ln S_X(x)] = \frac{f_X(x)}{S_X(x)} > 0 \quad \text{for all } x \geq 0,$$

where $f_X(x)$ is the corresponding density. The survival function $S_X(x)$ and the cumulative hazard function $R_X(x) = \int_0^x r_X(u)\,du$ are connected by the exponential representation

$$S_X(x) = \exp\{-R_X(x)\} = \exp\left\{-\int_0^x r_X(u)\,du\right\} \quad \text{for all } x \geq 0, \tag{2.1}$$

see Barlow and Proschan (1965).

Now, let X be an integer-valued random variable taking values $0, 1, 2, \dots$ with survival function $S_X(x) = P(X \geq x)$. By analogy with the continuous case, define the discrete failure (hazard) rate by

$$h_X(x) = \frac{P(X = x)}{S_X(x)} = 1 - \frac{S_X(x+1)}{S_X(x)} \in [0, 1] \quad \text{for all } x = 0, 1, 2, \dots.$$

Iterating the equation $1 - h_X(x) = \frac{S_X(x+1)}{S_X(x)}$ for decreasing values of x yields

$$S_X(x) = \prod_{k=1}^{x} [1 - h_X(x-k)] \quad \text{for all } x = 0, 1, 2, \dots,$$

assuming that $\prod_{k=1}^{0}[\cdot] = 1$. To get an exponential representation of the survival function $S_X(x)$ (similar to the continuous case relation (2.1)), one can substitute $g_X(x) = -\ln[1 - h_X(x)]$ in the last equation, resulting in

$$S_X(x) = \exp\{-G_X(x)\} = \exp\left\{-\sum_{k=1}^{x} g_X(x-k)\right\} \quad \text{for all } x = 0, 1, 2, \dots, \tag{2.2}$$

where $G_X(x) = \sum_{k=1}^{x} g_X(x-k)$ is the discrete cumulative hazard rate. Roy and Gupta (1992) called $G_X(x)$ cumulative pseudo-hazard rate, consult Cox and Oakes (1984) for an alternative definition.

Thus, (2.2) is a discrete analogue of the continuous case exponential relation (2.1).

2.2 Bivariate case

Here we will review exponential representations of joint survival function for bivariate non-negative continuous and discrete distributions.

2.2.1 *Continuous models.* Let us consider a non-negative bivariate continuous random vector (X, Y) defined by its joint survival function $S(x, y) = P(X > x, Y > y)$ where $x, y \geq 0$. If the first partial derivatives of $S(x, y)$ exist, the quantities

$$r_1(x, y) = \frac{\partial}{\partial x}[-\ln S(x, y)] \quad \text{and} \quad r_2(x, y) = \frac{\partial}{\partial y}[-\ln S(x, y)],$$

can be interpreted as the univariate failure rates of conditional distributions of each variate, given certain inequality of the remainder, see Johnson and Kotz (1975). Observe that $r_1(x, 0) = r_X(x)$ and $r_2(0, y) = r_Y(y)$ where $r_X(x)$ and $r_Y(y)$ are the marginal failure rates.

When the joint survival function $S(x, y)$ has continuous second order partial derivatives at all points (x, y) in the first quadrant, the vector-valued function

$$\mathbf{R}(x, y) = \big(r_1(x, y), r_2(x, y)\big)$$

is called a *hazard gradient of the random vector* (X, Y). The components $r_1(x, y)$ and $r_2(x, y)$ can not be arbitrary, but must be related by equation

$$\frac{\partial}{\partial y} r_1(x, y) = \frac{\partial}{\partial x} r_2(x, y) \quad \text{for all } x, y \geq 0. \tag{2.3}$$

The hazard gradient vector $\mathbf{R}(x, y)$ uniquely determines the bivariate distribution by means of a line integral through exponential representation

$$S(x, y) = \exp\left\{-\int_{\mathcal{C}} \mathbf{R}(\mathbf{z})\, d\mathbf{z}\right\}, \tag{2.4}$$

where $\mathcal{C}$ is any sufficiently smooth continuous path beginning at $(0, 0)$ and terminating at (x, y). We use $d\mathbf{z}$ in (2.4) to acknowledge the fact that we are moving along the curve $\mathcal{C}$, instead of coordinate axes. The relation (2.4) holds provided that along the path of integration $S(x, y)$ is absolutely continuous, see Marshall (1975). Note that the equation (2.4) is a bivariate version of the exponential representation of univariate survival function in (2.1). As a consequence of the fundamental theorem of calculus for line integrals, it turns out that the line integral in the right-hand side of (2.4) is independent on the path $\mathcal{C}$ if the integrand is an exact differential of a function, a condition that always holds for the hazard gradient vector $\mathbf{R}(x, y)$.

Since the line integral of gradient vector in (2.4) does not depend on the path, one can arbitrarily choose particular interesting smooth continuous connecting paths $\mathcal{C}_1$ from $(0, 0)$ to (x_0, y_0) and $\mathcal{C}_2$ from (x_0, y_0) to (x, y) such that

$$\int_{\mathcal{C}} \mathbf{R}(\mathbf{z})\, d\mathbf{z} = \int_{\mathcal{C}_1} \mathbf{R}(\mathbf{z})\, d\mathbf{z} + \int_{\mathcal{C}_2} \mathbf{R}(\mathbf{z})\, d\mathbf{z}. \tag{2.5}$$

The relation (2.5) may be generalized to any number $n \geq 2$ of sub-paths $\mathcal{C}_i$, such that the terminal point of $\mathcal{C}_i$ is the initial point of $\mathcal{C}_{i+1}$, $i = 1, 2, \ldots, n-1$.

The additive property (2.5) can be applied to get a useful computational form of the line integral in order to obtain a specific desired formula of the joint survival function $S(x, y)$ under the knowledge of analytical form of the components $r_1(x, y)$ and $r_2(x, y)$ of the gradient vector $\mathbf{R}(x, y)$.

Line integrals of the type appearing in (2.4) can be evaluated by expressing a path $\mathcal{C}$ in a parametric form. For example, one might parametrize the path (curve) $\mathcal{C}$ from $(0, 0)$ to (x, y) by $x = \psi_1(t)$ and $y = \psi_2(t)$ for $t \in [a, b]$ with $0 \leq a < b$, where $\psi_i(t)$ are differentiable functions such that $\psi_i(a) = 0$, $i = 1, 2$, $\psi_1(b) = x$ and $\psi_2(b) = y$. Hence,

$$\int_{\mathcal{C}} \mathbf{R}(\mathbf{z})\, d\mathbf{z} = \int_a^b \left[r_1\big(\psi_1(t), \psi_2(t)\big)\psi_1'(t) + r_2\big(\psi_1(t), \psi_2(t)\big)\psi_2'(t)\right] dt, \tag{2.6}$$

where $\psi_1'(t)$ and $\psi_2'(t)$ mean the corresponding first derivatives. Observe that the parametrization $x = \psi_1(t)$ and $y = \psi_2(t)$ determines an orientation for the curve $\mathcal{C}$ where the positive direction is the direction that is traced out as t increases.

Consider a path $\mathcal{C}$ from the point $(0, 0)$ to an arbitrary point (x, y) in the first quadrant. We give below three equivalent representations of $S(x, y)$ by using three different paths connecting these points.

- A way of choosing a path $\mathcal{C}$ is to move along the x-axis from $(0, 0)$ to the point $(x, 0)$ and let this path be $\mathcal{C}_1$. After that, move along the vertical line $l_1 = x$ from the point $(x, 0)$ to the point (x, y). Denote this second path by $\mathcal{C}_2$ and observe that $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2$. Applying (2.6), we obtain

$$S(x, y) = \exp\left\{-\int_0^x r_1(t, 0)\, dt - \int_0^y r_2(x, t)\, dt\right\} \quad \text{for all } x, y \geq 0.$$

  A multivariate version of this expression is given by Marshall (1975).
- If one selects an alternative path moving along the y-axis from $(0, 0)$ to $(0, y)$ and keeping the horizontal line $l_2 = y$ from $(0, y)$ to (x, y), then

$$S(x, y) = \exp\left\{-\int_0^y r_2(0, t)\, dt - \int_0^x r_1(t, y)\, dt\right\} \quad \text{for all } x, y \geq 0. \tag{2.7}$$

Note that the last relation is useful to obtain formulas for conditional survival functions. Indeed, since $r_2(0,t) = r_Y(t)$ and $S_Y(y) = \exp\{-\int_0^y r_Y(u)\,du\}$, we get

$$P(X > x \mid Y > y) = \exp\left\{-\int_0^x r_1(t,y)\,dt\right\}.$$

- Let $x \geq y$ and consider a path $\mathcal{C}$ as a union of two line segments $\mathcal{C}_1$ and $\mathcal{C}_2$ linking the points $(0,0)$ and $(x-y,0)$ and the point $(x-y,0)$ with (x,y), correspondingly. Using (2.6) leads to

$$S(x,y) = S_X(x-y)\exp\left\{-\int_0^y \left[r_1(x-y+t,t) + r_2(x-y+t,t)\right]dt\right\},$$

where S_X is the marginal survival function of the random variable X. By analogy, we can compute $S(x,y)$ when $x \leq y$. We link both expressions as follows

$$S(x,y) = \begin{cases} S_X(x-y)\exp\left\{-\int_0^y \left[r_1(x-y+t,t) + r_2(x-y+t,t)\right]dt\right\}, \\ \quad \text{if } x \geq y \geq 0, \\ S_Y(y-x)\exp\left\{-\int_0^x \left[r_1(t,y-x+t) + r_2(t,y-x+t)\right]dt\right\}, \\ \quad \text{if } y \geq x \geq 0. \end{cases} \tag{2.8}$$

Remark 2.1. There is another known exponential representation of the joint survival function $S(x,y)$ which involves the function $D(x,y)$ introduced by Sibuya (1960) as follows

$$D(x,y) = \ln\left[\frac{S(x,y)}{S_X(x)S_Y(y)}\right], \tag{2.9}$$

to be referred "Sibuya's dependence function", where $S_X(x)$ and $S_Y(y)$ are the marginal survival functions. The function $D(x,y)$ can be considered as a measure of local dependence and describes the amount of the association between variables X and Y for all $x, y \geq 0$, free of the marginal contributions, see Pinto and Kolev (2016) for more details and properties.

The importance of Sibuya's dependence function $D(x,y)$ defined by (2.9) is also justified by the exponential representation of the joint survival function written as

$$S(x,y) = \exp\{-R_X(x) - R_Y(y) + D(x,y)\}, \quad x, y \geq 0,$$

where $R_X(x)$ and $R_Y(y)$ are cumulative hazard rates of X and Y, correspondingly.

Remark 2.2. It can be seen that for all $u, v \geq 0$, the integrands in (2.8) are given by the sum $r(u,v) = r_1(u,v) + r_2(u,v)$ which has a natural interpretation as a directional derivative of the function $-\ln[S(u,v)]$ in direction the unit vectors $(0,1)$ and $(1,0)$. Depending on the real problem at hand, one might impose appropriate functional representations of the sum $r(u,v)$. For example, Pinto and Kolev (2016) postulated a linear form for the sum of elements of the hazard vector $\mathbf{R}(x,y)$, assuming that

$$r_1(x,y) + r_2(x,y) = a_0 + a_1 x + a_2 y \quad \text{for all } x, y \geq 0, \tag{2.10}$$

where a_0, a_1 and a_2 are non-negative parameters. As a result, (2.8) transforms into

$$S(x,y) = \begin{cases} S_X(x-y)\exp\{-a_0 y - a_1 xy - 0.5(a_2 - a_1)y^2\}, & \text{if } x \geq y \geq 0, \\ S_Y(y-x)\exp\{-a_0 x - a_2 xy - 0.5(a_1 - a_2)x^2\}, & \text{if } y \geq x \geq 0. \end{cases} \tag{2.11}$$

It turns out that relations (2.10) and (2.11) are equivalent. Moreover, the joint survival function specified by (2.11) characterizes the Sibuya-type bivariate aging property launched by Pinto and Kolev (2016) as follows.

Definition 2.1. Denote by $(X_t, Y_t) = [(X - t, Y - t) \mid X > t, Y > t]$ for $t \geq 0$ the residual lifetime vector corresponding to the non-negative continuous random vector (X, Y). We say that the vector (X, Y) possesses *Sibuya-type aging property*, if and only if

$$\frac{S_{X_t,Y_t}(x, y)}{S_{X_t}(x)S_{Y_t}(y)} = \frac{S(x, y)}{S_X(x)S_Y(y)} \quad \text{for all } x, y, t \geq 0,$$

where $S_{X_t,Y_t}(x, y)$ is the joint survival function of the residual lifetime vector (X_t, Y_t) with marginal survival functions fixed by the equations

$$S_{X_t}(x) = S_X(x)\exp\{-a_1 xt\} \quad \text{and} \quad S_{Y_t}(y) = S_Y(y)\exp\{-a_2 yt\},$$

where a_1 and a_2 are non-negative constants.

In fact, Definition 2.1 introduces a class of bivariate continuous distributions which are tail invariant with respect to Sibuya's dependence function of (X, Y) given by (2.9). In other words, the dependence between the variables X and Y may vary in time, but their Sibuya's dependence function remain the same. A discrete version of Sibuya-type aging property will be defined in Section 5.2.

The class of bivariate distributions introduced in Definition 2.1 is huge. We list below only three its important members with exponential marginals.

1. Set $a_0 = \lambda_1 + \lambda_2$ and $a_1 = a_2 = 0$ in (2.11) to get $S(x, y) = \exp\{-\lambda_1 x - \lambda_2 y\}$ for $\lambda_1, \lambda_2 > 0$, that is, X and Y are independent and exponentially distributed.
2. Put in (2.11) $a_1 = a_2 = 0$ and $a_0 = const \neq \lambda_1 + \lambda_2$ (depending on relation between x and y), to obtain the *Marshall–Olkin's (MO) bivariate exponential distribution* with survival function

$$S(x, y) = \exp\{-\lambda_1 x - \lambda_2 y - \lambda_3 \max(x, y)\}, \quad x, y \geq 0, \lambda_1, \lambda_2, \lambda_3 > 0, \tag{2.12}$$

see Marshall and Olkin (1967). Due to $P(X = Y) = \frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3} > 0$, the bivariate MO exponential distribution (2.12) exhibits singularity along the main diagonal in the first quadrant, that is, it is not absolutely continuous. Also, (2.12) is positive quadrant dependent since $S(x, y) \geq S_X(x)S_Y(y)$ for all $x, y \geq 0$.

The MO distribution (2.12) is the only bivariate distribution with exponential marginals solving the functional equation $S(x + t, y + t) = S(x, y)S(t, t)$ for all $x, y, t \geq 0$, characterizing the bivariate lack of memory property. It is well known that there exist many bivariate continuous distributions satisfying the last equation. The reader might consult Kulkarni (2006) for examples of distributions with non-exponential marginals.

A discrete version of MO distribution (2.12) will be discussed in Example 5.1.

3. If substitute $a_0 = \lambda_1 + \lambda_2$ and $a_1 = a_2 = \theta\lambda_1\lambda_2$ for $\theta \in [0, 1]$ and $\lambda_1, \lambda_2 > 0$ in (2.11), then we obtain the *local bivariate lack of memory property* introduced by Johnson and Kotz (1975) in continuous case. It preserves the univariate lack of memory property of the conditional distributions of $X \mid Y > y$ and $Y \mid X > x$, which should be exponential, therefore. The only absolutely continuous distribution with such a property is the *Gumbel's type I bivariate exponential distribution* given by

$$S(x, y) = \exp\{-\lambda_1 x - \lambda_2 y - \theta\lambda_1\lambda_2 xy\}, \quad x, y \geq 0, \tag{2.13}$$

see Gumbel (1960). Gumbel's distribution (2.13) is negative quadrant dependent since $S(x, y) \leq S_X(x)S_Y(y)$ for all $x, y \geq 0$.

For a discrete analog of the Gumbel's distribution (2.13), see Example 5.2.

2.2.2 *Discrete models.* We will assume hereafter that X and Y are two non-negative integer valued random variables and the vector (X, Y) is defined by the set of masses $f(x, y)$ and/or the joint survival function $S(x, y)$ for all $x, y = 0, 1, 2, \dots$, where

$$f(x, y) = P(X = x, Y = y) \quad \text{and} \quad S(x, y) = P(X \geq x, Y \geq y).$$

Similarly to the definition in the continuous case, Nair and Nair (1990) define the bivariate hazard rate as a two component vector $\mathbf{H}(x, y) = (h_1(x, y), h_2(x, y))$ for all $x, y = 0, 1, 2, \dots$, where

$$h_1(x, y) = \frac{\sum_{v \geq y} f(x, v)}{S(x, y)} = \frac{P(X = x \mid Y \geq y)}{P(X \geq x \mid Y \geq y)}$$

and

$$h_2(x, y) = \frac{\sum_{u \geq x} f(u, y)}{S(x, y)} = \frac{P(Y = y \mid X \geq x)}{P(Y \geq y \mid X \geq x)}.$$

Considering a two-component system, one can interpret $h_1(x, y)$ as the conditional probability that component 1 fails at age x, given that components 1 and 2 survive age x and y, respectively. Similar interpretation can be given to $h_2(x, y)$.

The marginal failure rates $h_X(x)$ and $h_Y(y)$ can be obtained from the components of the hazard vector $\mathbf{H}(x, y)$ as $h_X(x) = h_1(x, 0)$ and $h_Y(y) = h_2(0, y)$.

Observe that

$$1 - h_1(x, y) = \frac{S(x+1, y)}{S(x, y)} \quad \text{and} \quad 1 - h_2(x, y) = \frac{S(x, y+1)}{S(x, y)}$$

for all $x, y = 0, 1, 2, \dots$. Iterating the relation

$$S(x+1, y) = [1 - h_1(x, y)]S(x, y)$$

for decreasing values of x yields

$$S(x, y) = S_Y(y) \prod_{k=1}^{x} [1 - h_1(x-k, y)].$$

Taking into account that $S_Y(y) = \prod_{k=1}^{y}[1 - h_2(0, y-k)]$, we conclude that the components of the discrete bivariate hazard rate vector $\mathbf{H}(x, y)$ determine uniquely the joint distribution of (X, Y) through the formula

$$S(x, y) = \prod_{k=1}^{y} [1 - h_2(0, y-k)] \prod_{k=1}^{x} [1 - h_1(x-k, y)] \quad \text{for all } x, y = 0, 1, 2, \dots. \tag{2.14}$$

A multivariate version of this relation can be found in Nair and Asha (1997). Consult Section 6.2.3 in the excellent recent monograph of Nair, Sankaran and Balakrishnan (2018) as well.

It is direct to check that

$$\begin{aligned}[1 - h_1(x, y+1)][1 - h_2(x, y)] &= [1 - h_1(x, y)][1 - h_2(x+1, y)] \\ &= \frac{S(x+1, y+1)}{S(x, y)}. \end{aligned} \tag{2.15}$$

Therefore, one can not choose the components $h_1(x, y)$ and $h_2(x, y)$ arbitrary as they should satisfy (2.15). The relation (2.15) is discussed in Remark 6.3 in Nair, Sankaran and Balakrishnan (2018) and it represents a discrete counterpart of (2.3).

Our aim is to get an exponential representation of $S(x, y)$. This can be achieved by substitution $g_i(x, y) = -\ln[1 - h_i(x, y)]$ in (2.14) for $i = 1, 2$. So, we obtain

$$S(x, y) = \exp\left\{-\sum_{k=1}^{y} g_2(0, y-k) - \sum_{k=1}^{x} g_1(x-k, y)\right\} \tag{2.16}$$

for all $x, y = 0, 1, 2, \ldots$. This expression for the joint survival function is a discrete analogue of the continuous case exponential representation (2.7).

Thus, instead of the gradient vector $\mathbf{H}(x, y)$, one can equivalently use the vector $\mathbf{G}(x, y) = (g_1(x, y), g_2(x, y))$ in order to identify the joint distribution of (X, Y).

In terms of the components $g_1(x, y)$ and $g_2(x, y)$, relation (2.15) can be rewritten as

$$g_1(x, y+1) + g_2(x, y) = g_1(x, y) + g_2(x+1, y) = \ln\left[\frac{S(x, y)}{S(x+1, y+1)}\right] \tag{2.17}$$

for all $x, y = 0, 1, 2, \ldots$. Finally, notice that the components of the gradient vector $\mathbf{G}(x, y)$ satisfy the relations

$$g_1(x, y) = \ln\left[\frac{S(x, y)}{S(x+1, y)}\right] \quad \text{and} \quad g_2(x, y) = \ln\left[\frac{S(x, y)}{S(x, y+1)}\right]. \tag{2.18}$$

3 Discrete line integral on uniform grids

Hyman and Shashkov (1997) introduced discrete versions of vector fields for 2-D logically rectangular grids considering both *cell-valued* and *nodal* discretizations for scalar functions. In particular, the authors defined a *discrete analog of the line integral of gradient vector* along arbitrary path connecting two nodes of the grid and proved a discrete counterpart of the fundamental theorem of calculus for line integrals.

First, we will adapt the corresponding definitions for uniform grids. By an uniform grid, we mean a collection of nodes and cells arranged on a regular quadratic lattice in the first quadrant. The edges of the lattice are parallel to the axes of the global coordinate system and the spacing between grid points in each direction is equal to 1. The nodes of uniform grid are indexed by (x, y), where the pairs (x, y) belong to the set $I_+^2 = \{(x, y) \mid x, y = 0, 1, 2, \ldots\}$. The quadrilateral (square) defined by the nodes (x, y), $(x+1, y)$, $(x, y+1)$ and $(x+1, y+1)$ is called the (x, y)-cell.

The two components of the vector $\mathbf{G}_{x,y} = (GX_{x,y}, GY_{x,y})$ associated to each node $(x, y) \in I_+^2$ are specified by

$$GX_{x,y} = g_{x+1,y} - g_{x,y} \quad \text{and} \quad GY_{x,y} = g_{x,y+1} - g_{x,y},$$

where the scalar function $g_{x,y}$ is given by its values at the nodes (x, y) of uniform grid. Finally, the discrete gradient vector $\mathbf{G}$ is defined by a collection of all pairs $(GX_{x,y}, GY_{x,y})$, that is,

$$\mathbf{G} = \{\mathbf{G}_{x,y}, (x, y) \in I_+^2\}.$$

Our main goal is to introduce a discrete analog of the line integral of gradient vector $\mathbf{G}$ along arbitrary path $\mathcal{L}$ connecting the nodes (x_1, y_1) and (x_2, y_2) of an uniform grid. First, we will define the discrete line integral of $\mathbf{G}$ over one edge as a difference of function values at the ends of this edge as follows. The one-step right-edge $(x, y) \to (x+1, y)$ line integral (to be denoted by $\mathbf{I}_{x+,y}$ for a fixed y), is determined by

$$\mathbf{I}_{x+,y} = g_{x+1,y} - g_{x,y} \quad \text{for all } (x, y) \in I_+^2,$$

being identical with equation (4.22) in Hyman and Shashkov (1997). By analogy, we define the one-step left-edge $(x+1, y) \to (x, y)$ line integral by $\mathbf{I}_{x-,y} = g_{x,y} - g_{x+1,y}$. Note that $\mathbf{I}_{x+,y} + \mathbf{I}_{x-,y} = 0$.

Similarly, the one-step up-edge $(x, y) \to (x, y+1)$ line integral $\mathbf{I}_{x,y+}$ and one-step down-edge $(x, y+1) \to (x, y)$ line integral $\mathbf{I}_{x,y-}$ are given, for fixed x, by

$$\mathbf{I}_{x,y+} = g_{x,y+1} - g_{x,y} \quad \text{and} \quad \mathbf{I}_{x,y-} = g_{x,y} - g_{x,y+1}.$$

The complete line integral of gradient vector $\mathbf{I}_{\mathcal{L}}(\mathbf{G})$ along the path $\mathcal{L} : (x_1, y_1) \to (x_2, y_2)$ can be expressed as the sum of the one-step integrals, that is,

$$\mathbf{I}_{\mathcal{L}}(\mathbf{G}) = \sum_{\text{right-edges}} \mathbf{I}_{x+,y} + \sum_{\text{left-edges}} \mathbf{I}_{x-,y} + \sum_{\text{up-edges}} \mathbf{I}_{x,y+} + \sum_{\text{down-edges}} \mathbf{I}_{x,y-},$$

where the summation is provided for the set of edges that determine the discrete path $\mathcal{L}$. Since the end of one edge is the beginning of next edge in the path $\mathcal{L}$, all the function values in the sum of these pieces cancel, except for the first and last node. Thus,

$$\mathbf{I}_{\mathcal{L}}(\mathbf{G}) = g_{x_2,y_2} - g_{x_1,y_1}. \tag{3.1}$$

In fact, we proved the following statement.

Lemma 3.1. *The discrete line integral* $\mathbf{I}_{\mathcal{L}}(\mathbf{G})$ *of gradient vector* $\mathbf{G}$ *over an arbitrary connected path* $\mathcal{L} : (x_1, y_1) \to (x_2, y_2)$ *on an uniform grid is given by* (3.1).

Therefore, the value of discrete line integral $\mathbf{I}_{\mathcal{L}}(\mathbf{G})$ specified by (3.1) does not depend on the path. In particular, $\mathbf{I}_{\mathcal{L}}(\mathbf{G})$ is zero over the closed paths on an uniform grid. Really, suppose that $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$, where $\mathcal{L}_1 : (x_1, y_1) \to (x_2, y_2)$ and $\mathcal{L}_2 : (x_2, y_2) \to (x_1, y_1)$, i.e., the path $\mathcal{L}_2$ has an inverse direction with respect to the path $\mathcal{L}_1$. Apply relation (3.1) to summands in $\mathbf{I}_{\mathcal{L}}(\mathbf{G}) = \mathbf{I}_{\mathcal{L}_1}(\mathbf{G}) + \mathbf{I}_{\mathcal{L}_2}(\mathbf{G})$ to conclude that $\mathbf{I}_{\mathcal{L}}(\mathbf{G}) = 0$. Hence, when we change the direction of the line integral, the sign changes. Thus, the discrete line integral preserves the properties of the line integral of gradient vector in continuous case.

4 Probability meaning of the discrete line integral

Since the value of the discrete line integral is zero over the closed paths on an uniform grid, without loss of generality, we will consider hereafter increasing paths along the edges of the grid (consisting of moving up or right but not down or left) that start at node $(0, 0)$ and end at node (x, y) for $(x, y) \in I_+^2$. In this case, we will denote the line integral $\mathbf{I}_{\mathcal{L}}(\mathbf{G})$ by $\mathbf{I}_{(0,0)\to(x,y)}(\mathbf{G})$.

In the following statement we give a probability interpretation of the discrete line integral formula (3.1) for uniform grids in the case of bivariate discrete distributions in the support of I_+^2 in terms of their joint survival function $S(x, y)$.

Theorem 4.1. *The joint survival function* $S(x, y)$ *of a bivariate discrete distribution defined in the set* I_+^2 *can be specified by*

$$S(x, y) = \exp\{-\mathbf{I}_{(0,0)\to(x,y)}(\mathbf{G})\}, \tag{4.1}$$

where $\mathbf{I}_{(0,0)\to(x,y)}(\mathbf{G})$ *is the line integral over an increasing connected path starting at* $(0, 0)$ *and terminating at* (x, y), *where the components of the hazard vector* $\mathbf{G}$ *are defined by* (2.18).

Proof. Really, set $(x_1, y_1) = (0, 0)$, $(x_2, y_2) = (x, y)$ and substitute $g_{x,y} = -\ln[S(x, y)]$ for $x, y = 0, 1, 2, \ldots$ in (3.1) to conclude that the discrete line integral along the increasing path $\mathcal{L} : (0, 0) \to (x, y)$ can be represented as

$$\mathbf{I}_{\mathcal{L}}(\mathbf{G}) = \mathbf{I}_{(0,0)\to(x,y)}(\mathbf{G}) = -\ln\big[S(x, y)\big]$$

since $S(0, 0) = 1$. □

Therefore, we got the exponential representation (4.1) of the joint survival function in discrete setting, similar to the continuous case relation (2.4). We will apply formula (4.1) in examples below to find counterparts of the corresponding expressions for the joint survival function in continuous case.

At first, we will obtain useful formulas for the line integral along the zigzag path $(x, y) \to (x + 1, y + 1)$ for $x, y \in I_+^2$.

Example 4.1. Consider a path $(x, y) \to (x + 1, y + 1)$ for arbitrary $x, y = 0, 1, 2, \ldots$. It can be realized in two alternative ways using the "down" path $\mathcal{L}_D$ or the "up" path $\mathcal{L}_U$ written in symbolic form as follows

$$\mathcal{L}_D = (x, y) \to (x + 1, y) \to (x + 1, y + 1)$$

and

$$\mathcal{L}_U = (x, y) \to (x, y + 1) \to (x + 1, y + 1).$$

The one-step right-edge $(x, y) \to (x + 1, y)$ line integral is given by

$$\mathbf{I}_{(x,y)\to(x+1,y)} = -\ln\big[S(x + 1, y)\big] + \ln\big[S(x, y)\big] = \ln\left[\frac{S(x, y)}{S(x + 1, y)}\right],$$

which is exactly the first component $g_1(x, y)$ of the gradient vector $\mathbf{G}$, see (2.18).

By analogy, the one-step up-edge $(x, y) \to (x, y + 1)$ line integral can be computed as

$$\mathbf{I}_{(x,y)\to(x,y+1)} = \ln\left[\frac{S(x, y)}{S(x, y + 1)}\right] = g_2(x, y).$$

Thus, the line integral along the path $\mathcal{L}_D$ is determined by

$$\mathbf{I}_{\mathcal{L}_D} = \mathbf{I}_{(x,y)\to(x+1,y)} + \mathbf{I}_{(x+1,y)\to(x+1,y+1)}.$$

Using the above relations, we get

$$\mathbf{I}_{\mathcal{L}_D} = g_1(x, y) + g_2(x + 1, y).$$

Similarly, the line integral along the path $\mathcal{L}_U$ is given by

$$\mathbf{I}_{\mathcal{L}_U} = g_1(x, y + 1) + g_2(x, y).$$

But $\mathbf{I}_{\mathcal{L}_U} = \mathbf{I}_{\mathcal{L}_D}$ because of the path independence of the line integral, hence we arrive at equation (2.17), which is equivalent to (2.15).

Therefore, the components of the hazard gradient vector $\mathbf{G}$ specified by (2.18) represent just the one-step right-edge and up-edge line integrals contributing to the computation of the complete integral in (4.1). Thus, the pair $(g_1(x, y), g_2(x, y))$, or equivalently, the pair $(-\ln[1 - h_1(x, y)], -\ln[1 - h_2(x, y)])$, uniquely determines the joint distribution of (X, Y) in the support of I_+^2 by formula (4.1).

The number of the shortest paths along the edges of uniform grid (consisting of moving up or right but not down or left) that start at $(0, 0)$ and end at (x, y) is equal to $\frac{(x+y)!}{x!y!}$ and each one of them generates equivalent expressions for the joint survival function $S(x, y)$ according to Theorem 4.1. In the next example we will consider four particularly simple paths.

Example 4.2. Consider an increasing path $\mathcal{L}$ from $(0,0)$ to (x,y) for $x, y \in I_+^2$.

A. A simple way to calculate the line integral $\mathbf{I}_{(0,0)\to(x,y)}(\mathbf{G})$ is to follow vertical and horizontal sub-paths:

$$\mathcal{V}_1 = (0,0) \to (0,1) \to \cdots \to (0, y-1) \to (0, y)$$

and

$$\mathcal{H}_1 = (0,y) \to (1,y) \to \cdots \to (x-1, y) \to (x, y).$$

Using iteratively the one-step up-edge and right-edge line integral representations via the components $g_1(x,y)$ and $g_2(x,y)$ from Example 4.1, we obtain

$$\mathbf{I}_{\mathcal{V}_1} = \sum_{r=1}^{y} g_2(0, y-r) \quad \text{and} \quad \mathbf{I}_{\mathcal{H}_1} = \sum_{r=1}^{x} g_1(x-r, y).$$

Inserting $\mathbf{I}_{(0,0)\to(x,y)}(\mathbf{G}) = \mathbf{I}_{\mathcal{V}_1} + \mathbf{I}_{\mathcal{H}_1}$ in (4.1) yields the exponential representation (2.16) for the joint survival function $S(x,y)$.

B. If one decides to use an alternative path from $(0,0)$ to (x,y), first moving horizontally

$$\mathcal{H}_2 = (0,0) \to (1,0) \to \cdots \to (x-1,0) \to (x,0)$$

and then following the vertical sub-path

$$\mathcal{V}_2 = (x,0) \to (x,1) \to \cdots \to (x, y-1) \to (x,y),$$

yields

$$\mathbf{I}_{\mathcal{H}_2} = \sum_{r=1}^{x} g_1(x-r, 0) \quad \text{and} \quad \mathbf{I}_{\mathcal{V}_2} = \sum_{k=1}^{y} g_2(x, y-r).$$

Thus, the exponential representation (4.1) generates the following joint survival function

$$\begin{aligned} S(x,y) &= \exp\left\{-\sum_{r=1}^{x} g_1(x-r,0) - \sum_{k=1}^{y} g_2(x, y-k)\right\} \\ &= S_X(x)\exp\left\{-\sum_{k=1}^{y} g_2(x,y-k)\right\}. \end{aligned}$$

C. Assume that $x \geq y$. Let us examine a particular path $\mathcal{L}$ being a union of a horizontal part

$$\mathcal{H} = (0,0) \to (1,0) \to \cdots \to (x-y-1, 0) \to (x-y, 0)$$

and a zigzag section

$$\mathcal{Z} = (x-y, 0) \to (x-y+1, 1) \to \cdots \to (x-y-1, y-1) \to (x,y).$$

Applying the formula for the line integral along the horizontal path $\mathcal{H}_2$ obtained in *B*, we have

$$\mathbf{I}_{(0,0)\to(x-y,0)}(\mathbf{G}) = \mathbf{I}_{\mathcal{H}} = \sum_{r=1}^{y} g_1(x-r, 0) = S_X(x-y).$$

Observe that the path $\mathcal{Z}$ is composed by y consecutive "down" sub-paths of the form

$$(u,v) \to (u+1, v) \to (u+1, v+1) \quad \text{starting with } u = x-y \text{ and } v = 0.$$

Iterating the expression for $\mathbf{I}_{\mathcal{L}_D}$ from Example 4.1, we obtain

$$\mathbf{I}_{\mathcal{Z}} = \sum_{r=1}^{y} [g_1(x-r, y-r) + g_2(x-r+1, y-r)].$$

Since $\mathbf{I}_{\mathcal{L}} = \mathbf{I}_{\mathcal{H}} + \mathbf{I}_{\mathcal{Z}}$, from (4.1), we arrive at the expression

$$S(x, y) = S_X(x-y) \exp\left\{-\sum_{r=1}^{y} [g_1(x-r, y-r) + g_2(x-r+1, y-r)]\right\}.$$

By analogy, for $x \leq y$ we get

$$S(x, y) = S_Y(y-x) \exp\left\{-\sum_{r=1}^{x} [g_1(y-r, x-r) + g_2(y-r, x-r+1)]\right\}.$$

Joining the last two relations yields

$$S(x, y) = \begin{cases} S_X(x-y) \exp\left\{-\sum_{r=1}^{y} [g_1(x-r, y-r) + g_2(x-r+1, y-r)]\right\}, \\ \quad \text{if } x \geq y \geq 0, \\ S_Y(y-x) \exp\left\{-\sum_{r=1}^{x} [g_1(y-r, x-r) + g_2(y-r, x-r+1)]\right\}, \\ \quad \text{if } y \geq x \geq 0. \end{cases} \tag{4.2}$$

Formula (4.2) is a discrete counterpart of the joint survival function given by (2.8). It will serve as a base of our investigations in Section 5.

Using relations $g_i(x, y) = -\ln[1 - h_i(x, y)]$ in (4.2) for $i = 1, 2$, one will get the following equivalent representation:

$$S(x, y) = \begin{cases} S_X(x-y) \prod_{r=1}^{y} [1 - h_1(x-r, y-r)][1 - h_2(x-r+1, y-r)], \\ \quad \text{if } x \geq y \geq 0, \\ S_Y(y-x) \prod_{r=1}^{x} [1 - h_1(y-r, x-r)][1 - h_2(y-r, x-r+1)], \\ \quad \text{if } y \geq x \geq 0. \end{cases}$$

D. Similarly, if one prefers to follow the vertical path

$$\mathcal{V} = (0, 0) \to (0, 1) \to \cdots \to (0, x-y-1) \to (0, x-y)$$

and the zigzag path containing y consecutive "up" sub-paths $(u, v) \to (u, v+1) \to (u+1, v+1)$, an application of the exponential representation (4.1) would imply alternative expression

$$S(x, y) = \begin{cases} S_X(x-y) \exp\left\{-\sum_{r=1}^{y} [g_1(x-r, y-r+1) + g_2(x-r, y-r)]\right\}, \\ \quad \text{if } x \geq y \geq 0, \\ S_Y(y-x) \exp\left\{-\sum_{r=1}^{x} [g_1(y-r+1, x-r) + g_2(y-r, x-r)]\right\}, \\ \quad \text{if } y \geq x \geq 0 \end{cases}$$

for all $(x, y) \in I_+^2$. Another way to obtain the last relation is to substitute (2.17) in (4.2).

To mention only, that Konstantopoulos and Yuan (2017) presented a probability interpretation of the area of the region under the path from (0, 0) and (x, y) in I_+^2.

5 Discrete Sibuya-type bivariate aging property

Let us consider the joint survival function specified by (4.2). One can observe that the sum of the components of the gradient vector **G** have similar arguments. This fact motivates us to introduce a class of discrete bivariate distributions such that

$$g_1(x, y) + g_2(x+1, y) = a_0 + a_1 x + a_2 y \quad \text{for all } x, y = 0, 1, 2, \ldots, \tag{5.1}$$

where a_0, a_1 and a_2 are non-negative parameters. Let us denote by $\mathcal{DS}(x, y; \mathbf{a})$ the class specified by (5.1), where $\mathbf{a} = (a_0, a_1, a_2)$ is a parameter vector. Note that (5.1) can be considered as a discrete version of relation (2.10).

First, we will characterize the class $\mathcal{DS}(x, y; \mathbf{a})$ and obtain its parameter space. In Section 5.2, we will determine and investigate a discrete version of bivariate distributions possessing Sibuya-type bivariate aging property. The reader will recognize that it is an equivalent representation of the class $\mathcal{DS}(x, y; \mathbf{a})$ specified by (5.1). We will exhibit some typical members of $\mathcal{DS}(x, y; \mathbf{a})$ and its closure properties, being a powerful tool to generate new bivariate models belonging to the same class

5.1 Distributions with linear sum of gradient components

We begin with the following theorem characterizing the class $\mathcal{DS}(x, y; \mathbf{a})$.

Theorem 5.1. *The relation (5.1) is satisfied if and only if the joint survival function of non-negative discrete random vector (X, Y) can be represented by*

$$S(x, y) = \begin{cases} S_X(x-y)\exp\{-0.5(2a_0 - a_1 - a_2)y - a_1 xy - 0.5(a_2 - a_1)y^2\}, \\ \quad \text{if } x > y \geq 0, \\ \exp\{-0.5(2a_0 - a_1 - a_2)x - 0.5(a_1 + a_2)x^2\}, \\ \quad \text{if } x = y \geq 0, \\ S_Y(y-x)\exp\{-0.5(2a_0 - a_1 - a_2)x - a_2 xy - 0.5(a_1 - a_2)x^2\}, \\ \quad \text{if } y > x \geq 0 \end{cases} \tag{5.2}$$

for all $x, y = 0, 1, 2, \ldots$, where $2a_0 - a_1 - a_2 \geq 0$.

Proof. Apply iteratively relation (5.1) in (4.2) and perform the corresponding summation to get formula (5.2).

Conversely, let relation (5.2) be true. After some algebra we obtain

$$\frac{S(x+1, y+1)}{S(x, y)} = \exp\{-a_0 - a_1 x - a_2 y\} \quad \text{for all } x, y = 0, 1, 2, \ldots. \tag{5.3}$$

Taking logarithm of both sides of (5.2) and using the equality

$$\ln\left[\frac{S(x, y)}{S(x+1, y+1)}\right] = g_1(x, y) + g_2(x+1, y)$$

(see relation (2.17)), we confirm the linear representation (5.1). □

Note that the joint survival function determined by (5.2) is a discrete analogue of relation (2.11).

The non-negative parameters a_0, a_1 and a_2 in (5.1) can be calculated using the first several values of $S(x, y)$ specified by (5.2), as the following statement shows.

Corollary 5.1. *The parameters* a_0, a_1 *and* a_2 *in* (5.2) *can be computed as*

$$a_0 = -\ln[S(1,1)], \qquad a_1 = -\ln\left[\frac{S(2,1)}{S(1,0)S(1,1)}\right] \quad \text{and}$$
$$a_2 = -\ln\left[\frac{S(1,2)}{S(0,1)S(1,1)}\right]. \tag{5.4}$$

Proof. Exercise relation (5.3) for the pairs $(0,0)$, $(1,0)$ and $(0,1)$ to get the expressions

$$S(1,1) = \exp(-a_0), \qquad S(2,1) = S(1,0)\exp(-a_0 - a_1)$$

and

$$S(1,2) = S(0,1)\exp(-a_0 - a_2),$$

correspondingly, which lead to (5.4). □

Remark 5.1. In fact, we suggest the following simple practical rule to check if some bivariate discrete distribution defined on I_+^2 belongs to the class $\mathcal{DS}(x, y; \mathbf{a})$: if the analytical expression of the joint survival function $S(x, y)$ is known, one can verify the validity of the linear representation (5.1) by applying relation (5.3). The parameters a_0, a_1 and a_2 can be calculated alternatively via formulas (5.4).

In order $S(x, y)$ given by (5.2) to be a proper joint survival function, it must satisfy the following conditions:

- $S(0,0) = 1$;
- $S(x, y)$ is monotone non-increasing in each argument;
- $\lim_{x\to\infty} S(x, y) = 0$, $\lim_{y\to\infty} S(x, y) = 0$ and

$$S(x+1, y+1) - S(x+1, y) - S(x, y+1) + S(x, y) \geq 0, \quad (x, y) \in I_+^2. \tag{5.5}$$

Note that inequality (5.5) is a discrete analog of the restriction $\frac{\partial^2}{\partial x\,\partial y} S(x, y) \geq 0$, being a necessary condition for $S(x, y)$ to be a valid joint survival function in the continuous case.

The parameter space of the class $\mathcal{DS}(x, y; \mathbf{a})$ is given in the next corollary.

Corollary 5.2. *The parameter space* $\boldsymbol{a} = (a_0, a_1, a_2)$ *of the joint survival function specified by* (5.2) *satisfies the inequalities*

$$\max\left\{\ln\left[\frac{P(X=0)}{P(X=1)}\right], \ln\left[\frac{P(Y=0)}{P(Y=1)}\right]\right\} \leq a_0 \leq B,$$

where $B = \min\{-\ln[S_X(1) + S_Y(1) - 1], -\ln[P(X=1)], -\ln[P(Y=1)]\}$ *and*

$$a_1 \leq -\ln\left[\frac{1 - P(X=1)\exp(a_0)}{S_X(1)}\right] \quad \text{and} \quad a_2 \leq -\ln\left[\frac{1 - P(Y=1)\exp(a_0)}{S_Y(1)}\right],$$

such that $2a_0 \geq a_1 + a_2$.

Proof. Use inequality (5.5) with $x = y = 0$ to get

$$S(1,1) - S(1,0) - S(0,1) + S(0,0) \geq 0, \quad \text{that is, } S(1,1) \geq S_X(1) + S_Y(1) - 1.$$

Recall that from (5.4), $a_0 = -\ln[S(1,1)]$, hence we obtain

$$a_0 \leq -\ln[S_X(1) + S_Y(1) - 1].$$

Since $a_0 \geq 0$, the marginal survival functions of X and Y should satisfy inequalities $1 \geq S_X(1) + S_Y(1) - 1 \geq 0$. But always $2 \geq S_X(1) + S_Y(1)$, so the marginal distributions corresponding to the survival function $S(x, y)$ specified by (5.2) are such that

$$P(X \geq 1) \geq P(Y = 0) \quad \text{and} \quad P(Y \geq 1) \geq P(X = 0).$$

Apply (5.5) with $x = 1$ and $y = 0$ to obtain

$$S(2, 1) - S(2, 0) - S(1, 1) + S(1, 0) \geq 0.$$

Since $S(2, 1) = S(1, 0)\exp(-a_0 - a_1)$, $S(1, 1) = \exp(-a_0)$ and $S(1, 0) - S(2, 0) = P(X = 1)$, the last inequality can be written as

$$S_X(1)\exp(-a_0 - a_1) \geq \exp(-a_0) - P(X = 1),$$

i.e.,

$$a_1 \leq -\ln\left[\frac{1 - P(X = 1)\exp(a_0)}{S_X(1)}\right],$$

which implies that $0 \leq 1 - P(X = 1)\exp(a_0) \leq S_X(1)$ and therefore,

$$\ln\left[\frac{P(X = 0)}{P(X = 1)}\right] \leq a_0 \leq -\ln\left[P(X = 1)\right].$$

Now, exercise (5.5) with $x = 0$ and $y = 1$ yielding

$$S(1, 2) - S(0, 2) - S(1, 1) + S(0, 1) \geq 0.$$

By analogy with the previous case we arrive at the restrictions

$$a_2 \leq -\ln\left[\frac{1 - P(Y = 1)\exp(a_0)}{S_Y(1)}\right]$$

and

$$\ln\left[\frac{P(Y = 0)}{P(Y = 1)}\right] \leq a_0 \leq -\ln\left[P(Y = 1)\right].$$

Finally, linking all above inequalities we obtain the parameter space announced. □

5.2 Discrete Sibuya-type aging property

First, we will adapt Definition 2.1 to the discrete case as follows.

Definition 5.1. Let $(X_t, Y_t) = [(X - t, Y - t) \mid X > t, Y > t]$ for $t = 0, 1, 2, \ldots$ be the residual lifetime vector corresponding to the non-negative integer-valued random vector (X, Y). The vector (X, Y) possesses *discrete Sibuya-type bivariate aging property* (to be denoted DS-BAP), if and only if

$$\frac{S_{X_t,Y_t}(x, y)}{S_{X_t}(x)S_{Y_t}(y)} = \frac{S(x, y)}{S_X(x)S_Y(y)} \tag{5.6}$$

and the marginal survival functions of (X_t, Y_t) are specified by

$$S_{X_t}(x) = S_X(x)\exp\{-a_1xt\} \quad \text{and} \quad S_{Y_t}(y) = S_Y(y)\exp\{-a_2yt\}, \tag{5.7}$$

for all $x, y, t = 0, 1, 2 \ldots$ and $a_1, a_2 \geq 0$.

Definition 5.1 indicates that the random vector (X, Y) and its residual lifetime vector (X_t, Y_t) should share, for all $t = 0, 1, 2, \ldots$, the same Sibuya's dependence function, that is, it has to be tail invariant under marginal restrictions (5.7).

The marginal survival functions of the residual lifetime vector (X_t, Y_t) are given by

$$S_{X_t}(x) = \frac{S(x+t,t)}{S(t,t)} \quad \text{and} \quad S_{Y_t}(y) = \frac{S(t,y+t)}{S(t,t)}.$$

Hence, the class of bivariate discrete distributions specified by (5.6) can be equivalently defined by the functional equation

$$S(x+t, y+t) = S(x,y)S(t,t)G(x,y;t) \tag{5.8}$$

for all $x, y, t = 0, 1, 2, \ldots$, where the function $G(x, y; t) = \frac{S_{X_t}(x)S_{Y_t}(y)}{S_X(x)S_Y(y)}$ might be interpreted as an "aging" factor. Therefore, if only condition (5.6) in Definition 5.1 is fulfilled, we obtain a general function $G(x, y; t)$ depending of x, y and t in (5.8), that should satisfy the boundary conditions $G(x, y; 0) = G(0, 0; t) = 1$.

For practical needs one is urged to consider appropriate and simple expression for the function $G(x, y; t)$, with a reasonable reliability interpretation. For this reason, the additional marginal conditions (5.7) determine an aging function in (5.8) given by $G(x, y; t) = \exp\{-(a_1x + a_2y)t\}$. So, our basic equation hereafter will be

$$S(x+t, y+t) = S(x,y)S(t,t)\exp\{-(a_1x+a_2y)t\}, \quad x, y, t = 0, 1, 2, \ldots. \tag{5.9}$$

In the next characterization statements, we establish the equivalence between Definition 5.1 introducing the DS-BAP, joint survival function $S(x, y)$ given by (5.2) representing the class $\mathcal{DS}(x, y; \mathbf{a})$, and the functional equation (5.9).

Theorem 5.2. *The discrete bivariate distribution of the random vector (X, Y) possesses DS-BAP determined by (5.6) and (5.7) if and only if its joint survival function solves the functional equation (5.9) for all $x, y, t = 0, 1, 2, \ldots$.*

Proof. Let (5.9) be true. Put $x = 0$ in (5.9) to get

$$\frac{S(t,y+t)}{S(t,t)} = S(0,y)\exp\{-a_2yt\}, \quad \text{that is,} \quad \frac{S_{Y_t}(y)}{S_Y(y)} = \exp\{-a_2yt\}.$$

By analogy, set $y = 0$ in (5.9) to conclude that $\frac{S_{X_t}(x)}{S_X(x)} = \exp\{-a_1xt\}$. The last two expressions are given by (5.7). Substitute both exponents in (5.9) to restore (5.6).

Conversely, let (5.6) and (5.7) be fulfilled. Substitute both relations (5.7) in the left side of (5.6) to obtain (5.9). □

In Theorem 5.1, we established that relations (5.1) and (5.2) are equivalent and in Theorem 5.2 we proved that DS-BAP is characterized by the functional equation (5.9). To close the circle, the next statement shows that equations (5.9) and (5.2) are equivalent as well.

Theorem 5.3. *The joint survival function $S(x, y)$ is given by (5.2) if and only if the functional equation (5.9) is fulfilled for all $x, y, t = 0, 1, 2, \ldots$.*

Proof. It follows step by step the proof of Theorem 4 in Pinto and Kolev (2015), taking into account that the possible values of x, y and t belong to the set of non-negative integers. □

Thus, the class $\mathcal{DS}(x, y; \mathbf{a})$ specified by (5.1) coincides with DS-BAP notion introduced by Definition 5.1.

5.3 Examples

Here we will present several members of the class $\mathcal{DS}(x, y; \mathbf{a})$, that is, bivariate discrete distributions possessing DS-BAP in a sense of Definition 5.1.

First, observe that under substitutions

$$p_i = \exp(-a_i) \in (0, 1] \quad \text{for } i = 0, 1, 2, \tag{5.10}$$

where the parameter vector $\mathbf{a} = (a_0, a_1, a_2)$ satisfies conditions listed in Corollary 5.2, the joint survival function given by (5.2) can be alternatively represented as

$$S(x, y) = \begin{cases} S_X(x-y) p_0^y p_1^{y(2x-y-1)/2} p_2^{y(y-1)/2}, & \text{if } x > y \geq 0, \\ p_0^x (p_1 p_2)^{x(x-1)/2}, & \text{if } x = y \geq 0, \\ S_Y(y-x) p_0^x p_1^{x(x-1)/2} p_2^{x(2y-x-1)/2}, & \text{if } y > x \geq 0 \end{cases}$$

for all $x, y = 0, 1, 2, \ldots$. Perhaps, this expression for $S(x, y)$ characterizing DS-BAP and the class $\mathcal{DS}(x, y; \mathbf{a})$ is more convenient for the reader. Note that using (5.10), relation (5.3) transforms into $\frac{S(x+1,y+1)}{S(x,y)} = p_0 p_1^x p_2^y$.

In the examples below, the joint survival function $S(x, y)$ is given explicitly. Following Remark 5.1, we will check if the linear representation (5.1) is fulfilled, by applying relation (5.3). In such a case, the parameters a_0, a_1 and a_2 can be computed using equations (5.4) as well.

Example 5.1 (MO-type bivariate geometric distribution). Let the joint survival function of non-negative integer valued random variables X and Y be given by

$$S(x, y) = \begin{cases} p_0^y p_1^{x-y}, & \text{if } x \geq y, \\ p_0^x p_2^{y-x}, & \text{if } y \geq x \end{cases} \tag{5.11}$$

for $x, y = 0, 1, 2, \ldots$, where $p_1 + p_2 \leq 1 + p_0$ and $0 < p_0 \leq p_i < 1, i = 1, 2$.

The distribution related to (5.11) is known as *MO-type bivariate geometric distribution*, being a discrete analog of the MO bivariate exponential distribution (2.12). The marginals X and Y are geometric random variables with survival functions $S_X(x) = p_1^x$ and $S_Y(y) = p_2^y$.

Applying (5.3) we obtain that $\frac{S(x+1,y+1)}{S(x,y)} = \exp\{-a_0\}$, that is, the parameters in (5.1) satisfy $a_1 = a_2 = 0$ and $a_0 = -\ln p_0$. Hence, the distribution specified by (5.11) belongs to the class $\mathcal{DS}(x, y; \mathbf{a})$, that is, it possesses DS-BAP according to Definition 5.1.

The joint survival function $S(x, y)$ given by (5.11) solves the functional equation

$$S(x+t, y+t) = S(x, y) S(t, t) \quad \text{for all } x, y, t = 0, 1, 2, \ldots, \tag{5.12}$$

being a discrete analogue of bivariate lack of memory property (DBLMP). Moreover, if the marginals X and Y are geometric random variables, the discrete vector (X, Y) possesses DBLMP if and only if its joint survival function is specified by (5.11).

For history, properties and multivariate version of (5.11) consult Section 8.2.2 in Nair, Sankaran and Balakrishnan (2018) and references therein. Let us note that (depending on the sign of $p_0 - p_1 p_2$), the distribution (5.11) is neither positive quadrant dependent nor negative quadrant dependent. Observe the difference with continuous case.

Now, we will generate the joint survival function (5.11) using an alternative approach. Let us consider the stochastic representation

$$(X, Y) = \left[\min(T_0, T_1), \min(T_0, T_2)\right], \tag{5.13}$$

where T_0, T_1 and T_2 are three independent geometric variates with parameters α_0, α_1 and α_2, respectively. Then $S(x, y) = \alpha_0^{\max(x,y)} \alpha_1^x \alpha_2^y$ for all $x, y = 0, 1, 2, \ldots$ and the last expression transforms into (5.11) by parametrization $p_0 = \alpha_0 \alpha_1 \alpha_2$, $p_1 = \alpha_0 \alpha_1$ and $p_2 = \alpha_0 \alpha_2$.

Example 5.2 (Gumbel's bivariate geometric distribution). A discrete version of Gumbel's distribution (2.13) has been introduced by Mi (1993) through

$$S(x, y) = p_1^x p_2^y \theta^{xy},$$
$$p_1, p_2 \in (0, 1), \theta \in (0, 1], 1 + p_1 p_2 \theta \geq p_1 + p_2, x, y = 0, 1, 2, \ldots \quad (5.14)$$

and it also has geometric marginals given by $S_X(x) = p_1^x$ and $S_Y(y) = p_2^y$. See Section 8.2.1 in Nair, Sankaran and Balakrishnan (2018) for more information.

The Gumbel's discrete distribution (5.14) satisfies relation (5.1) with $a_0 = -\ln(p_1 p_2 \theta)$ and $a_1 = a_2 = -\ln\theta$, i.e., it is a member of the class $\mathcal{DS}(x, y; \mathbf{a})$ as well.

Let us mention, that by analogy to the continuous case, the Gumbel's-type bivariate geometric distribution (5.14) is negative quadrant dependent and satisfies local lack of memory property (i.e., conditional distributions $X \mid Y \geq y$ and $Y \mid X \geq x$ are geometrically distributed).

Example 5.3 (Bivariate geometric distribution of Nair and Asha (1997)). The authors define the joint survival function by

$$S(x, y) = \begin{cases} p_1^x, & \text{if } p_1^x \leq p_2^y; \\ p_2^y, & \text{if } p_1^x \geq p_2^y, \end{cases} \quad (5.15)$$

where $p_1, p_2 \in (0, 1)$ and $x, y = 0, 1, 2, \ldots$ Its marginals are geometric also.

Calculating coefficients a_0, a_1 and a_2 via (5.3) and using the results in Nair and Asha (1997) one can write

$$g_1(x, y) + g_2(x+1, y) = \begin{cases} -\ln p_1, & \text{if } p_1^x \leq p_2^y, p_1^x \leq p_2^{y+1}; \\ -\ln p_2 + (\ln p_1)x - (\ln p_2)y, & \text{if } p_1^x \leq p_2^y, p_1^x \geq p_2^{y+1}; \\ -\ln p_1 - (\ln p_1)x + (\ln p_2)y, & \text{if } p_1^x \geq p_2^y, p_1^{x+1} \leq p_2^y; \\ -\ln p_2, & \text{if } p_1^x \geq p_2^y, p_1^{x+1} \geq p_2^y. \end{cases} \quad (5.16)$$

As we see, depending on values p_1, p_2 and realizations $x, y = 0, 1, 2, \ldots$, the sum $g_1(x, y) + g_2(x + 1, y)$ might be a constant $a_0 = -\ln p_1$ or $a_0 = -\ln p_2$ (with $a_1 = a_2 = 0$), or with $a_1 \neq a_2$ in relation (5.1) governing the class $\mathcal{DS}(x, y; \mathbf{a})$.

Of course, there are many bivariate discrete distributions with geometric marginals that do need to satisfy Definition 5.1 or equation (5.1), that is, do not belong to the class $\mathcal{DS}(x, y; \mathbf{a})$. An example is the distribution given by

$$S(x, y) = \begin{cases} \alpha p_0^{x+y} + (1-\alpha) p_0^x, & \text{if } x \geq y; \\ \alpha p_0^{x+y} + (1-\alpha) p_0^y, & \text{if } x \leq y, \end{cases}$$

where $p_0 \in (0, 1), \alpha \in (0, 1]$ and $x, y = 0, 1, 2, \ldots$ If $\alpha = 1$, we obtain that X and Y are independent and identically distributed geometric random variables.

On the other side, there are many bivariate discrete distributions which follow Definition 5.1 or relation (5.1), but their marginals are not geometrically distributed. The next two examples are confirmation.

Example 5.4 (Mixture of MO geometric distributions of Asha, Sankaran and Nair (2003)). The joint survival function for $x, y = 0, 1, 2, \ldots$ is given by

$$S(x, y) = \begin{cases} p_0^y [\alpha p_0^{x-y} + (1-\alpha) p_1^{x-y}], & \text{if } x \geq y; \\ p_0^x [\alpha p_0^{y-x} + (1-\alpha) p_2^{y-x}], & \text{if } x \leq y, \end{cases} \quad (5.17)$$

where $0 < p_0 < p_1, p_2 < 1$, $p_0 \neq p_1, p_2$, $\alpha \in [0, 1]$ and $1 - p_0 + \alpha(2p_0 - p_1 - p_2) > 0$. The marginal distributions of (5.17) are mixtures of two geometric distributions

$$S_X(x) = \alpha p_0^x + (1-\alpha)p_1^x \quad \text{and} \quad S_Y(y) = \alpha p_0^y + (1-\alpha)p_2^y.$$

Moreover, $S(x, y)$ satisfies Definition 5.1 and (5.1) with $a_0 = -\ln p_0$ and $a_1 = a_2 = 0$. If substitute $\alpha = 1$ in (5.17) one will get the MO geometric distribution distribution (5.11) considered in Example 5.1 and the independence model can be obtained when $\alpha = 0$. Two more bivariate discrete distributions, being mixtures of MO-type distributions given via (5.11) are listed in Asha, Sankaran and Nair (2003).

Example 5.5 (Generalized MO distributions). Let us consider the stochastic representation (5.13) keeping the independence between discrete variables T_0, T_1 and T_2 defined on $0, 1, 2, \dots$, but allowing that their distribution is arbitrary, not necessarily geometric. Such a construction has been introduced by Li and Pellerey (2011) in continuous case (where the T_i's are independent non-negative random variables).

For instance, let us assume that variables T_1 and T_2 in stochastic relation (5.13) are independent and discrete Rayleigh distributed, that is, $P(T_i \geq x) = p_i^{x^2}$ for $x = 0, 1, 2, \dots$ and $p_i \in (0, 1)$, $i = 1, 2$. If T_0 is independent of T_1 and T_2 and geometrically distributed with parameter $p_0 \in (0, 1)$, the joint survival function of X and Y can be written as

$$S(x, y) = p_0^{\max(x,y)} p_1^{x^2} p_2^{y^2}, \quad x, y = 0, 1, 2, \dots \tag{5.18}$$

The marginal survival functions are $S_X(x) = p_0^x p_1^{x^2}$ and $S_Y(y) = p_0^y p_2^{y^2}$, and from (2.18) we obtain the corresponding hazard rates

$$g_X(x) = -\ln\left[\frac{S_X(x+1)}{S_X(x)}\right] = \alpha_1 + a_1 x$$

and

$$g_Y(y) = -\ln\left[\frac{S_Y(y+1)}{S_Y(y)}\right] = \alpha_2 + a_2 y,$$

where $\alpha_i = -\ln(p_0 p_i)$ and $a_i = -2\ln p_i$, $i = 1, 2$. Thus, the bivariate distribution (5.18) has linear marginal hazard/failure rates, corresponding to continuous distributions of the same type, see Kodlin (1967).

It is direct to verify that the joint distribution (5.18) satisfies Definition 5.1 and relation (5.1) with $a_0 = -\ln(p_0 p_1 p_2)$, $a_1 = -2\ln p_1$ and $a_2 = -2\ln p_2$, but it does not exhibit *DBLMP* advocated by functional equation (5.12).

Example 5.6 (Bivariate discrete distributions possessing DBLMP). If substitute $a_1 = a_2 = 0$ in (5.1), one will get a subclass of DS-BAP. Taking into account Definition 5.1, from relations (5.7) we have $S_{X_t}(x) = S_X(x)$ and $S_{Y_t}(y) = S_Y(y)$ for all $x, y, t = 0, 1, 2, \dots$, that is, the joint distribution of the vector (X, Y) and its residual lifetime vector (X_t, Y_t) coincide. But this is the familiar feature of the *DBLMP* specified by functional equation (5.12), which preserves the joint distributions of the vectors (X_t, Y_t) and (X, Y) and their marginal distributions as well.

It is well known that many bivariate discrete distributions follow *DBLMP* even when their marginals are not geometrically distributed. For instance, consult Example 5.4 and the model considered by Dhar (1998), who derived a bivariate geometric distribution which is a discrete analog of the continuous model introduced by Freund (1961). We will mention two more geometric type bivariate models introduced by Omey and Minkova (2014) and Lee, Cha and Pulcini (2017). Other examples can be found in Nair, Sankaran and Balakrishnan (2018).

Example 5.7 (Independent members of the class). $\mathcal{DS}(x, y; \mathbf{a})$. Here we will identify members of the class $\mathcal{DS}(x, y; \mathbf{a})$ with independent marginals. Hence, when $(X, Y) \in \mathcal{DS}(x, y; \mathbf{a})$, we are urged to find solutions of functional equation

$$S(x, y) = S_X(x)S_Y(y) \quad \text{for all } x, y = 0, 1, 2, \ldots,$$

where $S(x, y)$ is given by (5.2).

Let $g_X(x)$ and $g_Y(y)$ be hazard rates of random variables X and Y, correspondingly, that is,

$$g_X(x) = -\ln\left[\frac{S(x+1, 0)}{S(x, 0)}\right] \quad \text{and} \quad g_Y(y) = -\ln\left[\frac{S(0, y+1)}{S(0, y)}\right].$$

The independence between X and Y implies that $g_1(x, y) = g_X(x)$ and $g_2(x, y) = g_Y(y)$ for all $x, y = 0, 1, 2, \ldots$. Therefore, relation (5.1) transforms into

$$g_1(x, y) + g_2(x+1, y) = g_X(x) + g_Y(y) = a_0 + a_1x + a_2y.$$

The last functional equation is equivalent to both

$$g_X(x) = \beta_1 + a_1x \quad \text{and} \quad g_Y(y) = \beta_2 + a_2y,$$

where $\beta_1 \in [0, a_0]$ and $\beta_2 = a_0 - \beta_1$, that is, the marginal hazard rates of random variables X and Y are linear functions corresponding to the joint survival function considered in Example 5.5. If $\beta_1 = \beta_2 = 0$, we obtain independent geometric marginals. All possible combinations indicate nine members of the class $\mathcal{DS}(x, y; \mathbf{a})$ with independent marginals.

Therefore, the class of bivariate discrete distributions possessing DS-BAP is huge: linking *DBLMP* and local lack of memory property. Its members can exhibit positive quadrant dependence or/nor negative quadrant dependence. The distributions in above examples belong to different bivariate aging notions as well, see a detailed analysis in Nair, Sankaran and Balakrishnan (2018), Chapter 7.

5.4 Closure properties

The solutions $S(x, y)$ of functional equation (5.6) which satisfy (5.7) (i.e., distributions possessing DS-BAP being members of the class $\mathcal{DS}(x, y; \mathbf{a})$ as well), are given by (5.2) and the parameter vector $\mathbf{a}$ have to satisfy the inequalities listed in Corollary 5.2. These bivariate discrete distributions can be used as building blocks to construct other distributions that satisfy relation (5.1).

We will present in the next theorem several *closure properties* of the class $\mathcal{DS}(x, y; \mathbf{a})$.

Theorem 5.4. *Denote by $\mathcal{S}_1$ and $\mathcal{S}_2$ joint survival functions belonging to the class $\mathcal{DS}(x, y; \mathbf{a})$. The following closure properties are fulfilled*:

(CP1) *If $\mathcal{S}_1, \mathcal{S}_2 \in \mathcal{DS}(x, y; \mathbf{a})$, then their product $\mathcal{S}_1\mathcal{S}_2 \in \mathcal{DS}(x, y; \mathbf{a})$.*
(CP2) *If $\mathcal{S}_1 \in \mathcal{DS}(x, y; \mathbf{a})$, then $[\mathcal{S}_1]^q \in \mathcal{DS}(x, y; \mathbf{a})$ for some $q \geq 1$.*
(CP3) *If $\mathcal{S}_1, \mathcal{S}_2 \in \mathcal{DS}(x, y; \mathbf{a})$, then $[\mathcal{S}_1]^{q_1}[\mathcal{S}_2]^{q_2} \in \mathcal{DS}(x, y; \mathbf{a})$ for some $q_1, q_2 \geq 1$.*
(CP4) *If $\mathcal{S}_1(x, y) \in \mathcal{DS}(x, y; \mathbf{a})$ and $q > 0$, then $\mathcal{S}_1(qx, qy) \in \mathcal{DS}(x, y; \mathbf{a})$.*

Proof. To check (CP1), let us suppose that the independent vectors (X_i, Y_i), with joint survival function $\mathcal{S}_i$, are members of the class $\mathcal{DS}(x, y; \mathbf{a})$, $i = 1, 2$. Then, the joint distribution of vector $[(X_1, Y_1), (X_2, Y_2)]$ has a survival function $\mathcal{S}_1\mathcal{S}_2$ which is proper and also belongs to the class $\mathcal{DS}(x, y; \mathbf{a})$.

One can verify that the distributions obtained via operations (CP2) to (CP4) have valid joint survival functions which satisfy equation (5.6).

Note that the parameter vector $\mathbf{a}$ will be updated with additional parameters due to closure properties listed in Theorem 5.4. □

Example 5.8 (Application of closure property (CP1)). Let the distribution of independent vectors (X_i, Y_i) be given by Example 5.1, $i = 1, 2$. Closure property (CP1) results in a new member exhibiting DA-BAP with joint survival function

$$S(x, y) = \begin{cases} p_0^{2y} p_1^{2(x-y)}, & \text{if } x \geq y, \\ p_0^{2x} p_2^{2(y-x)}, & \text{if } y \geq x \end{cases} \tag{5.19}$$

for $x, y = 0, 1, 2, \ldots$, where $p_1 + p_2 \leq 1 + p_0$ and $0 < p_0 \leq p_i < 1, i = 1, 2$.

The closure properties listed in Theorem 5.4 are efficient tools to generate new members of the class $\mathcal{DS}(x, y; \mathbf{a})$. We will offer an alternative constructive method in the last example.

Example 5.9 (Discrete version of the Extended MO model). Let us examine the stochastic representation (5.13), where T_1 and T_2 are dependent random variables, defined by the joint survival function $S_{T_1,T_2}(x, y)$, and both T_1 and T_2 are independent of random variable T_0 having a survival function S_{T_0}. This construction extends the classical fatal shock model introduced in Marshall and Olkin (1967) in continuous case, where T_i's are assumed to be independent and exponentially distributed. Therefore, the joint survival function of the vector (X, Y) is given by

$$S(x, y) = S_{T_1,T_2}(x, y) \min\{S_{T_0}(x), S_{T_0}(y)\}, \quad x, y \geq 0. \tag{5.20}$$

The right-hand side in (5.20) is a product of two bivariate distributions. The first one is defined by $S_{T_1,T_2}(x, y)$, and the second one refers to a bivariate random vector with comonotonic components sharing the same marginal distribution as T_0. The class of bivariate continuous models specified by (5.20) has been introduced by Kolev and Pinto (2015b) under the name *Extended Marshall–Olkin model*.

Let us consider a discrete version of (5.20), assuming that (T_1, T_2) follows the Gumbel's bivariate geomeric distribution (5.14) and T_0 is geometrically distributed with parameter $p_0 \geq 0$. Hence, the resulting joint survival function reads

$$S(x, y) = p_0^{\max(x,y)} p_1^x p_2^y \theta^{xy}, \quad p_0, p_1, p_2 \in (0, 1), \theta \in (0, 1], 1 + p_1 p_2 \theta \geq p_1 + p_2 \tag{5.21}$$

for all $x, y = 0, 1, 2, \ldots$. Therefore, a new bivariate discrete distribution has been constructed: the joint survival function specified by (5.21) might be named *extended bivariate Gumbel-type geometric model*. Observe, that in this case the coefficients in (5.1) are given by $a_0 = -\ln(p_0 p_1 p_2 \theta)$ and $a_1 = a_2 = -\ln \theta$.

6 Conclusions

In this article, we introduced a probabilistic interpretation of discrete line integral of gradient vector on the uniform grid in the support of $I_+^2 = \{(x, y)|x, y = 0, 1, 2, \ldots\}$. In Lemma 3.1, we present a discrete analog of the fundamental theorem of calculus for the line integrals on uniform grids. In Theorem 4.1, we established an exponential representation of the joint survival function $S(x, y)$ through the line integral along the increasing path connecting the nodes $(0, 0)$ and (x, y). It seems that such a statement and related probability interpretation appears in literature for a first time. Since the components of the discrete gradient vector uniquely determine the joint distribution of the vector (X, Y) via exponential representation (4.1), all bivariate discrete distributions defined on I_+^2 can be obtained for specific (pre-specified) expressions of the gradient vector elements, taking into account the physical nature of the process to be analyzed.

Using a particular increasing path between $(0,0)$ and (x,y) we got a general expression (4.2) for $S(x,y)$. In Definition 5.1, we introduced a new aging notion: the discrete Sibuya-type bivariate aging property (DS-BAP), specified by equations (5.6) and (5.7). We established several characterizations connecting the class $\mathcal{DS}(x,y;\mathbf{a})$ and DS-BAP by the following equivalent relations

$$(5.1) \Leftrightarrow (5.2) \Leftrightarrow (5.6)+(5.7) \Leftrightarrow (5.9),$$

see Theorems 5.1, 5.2 and 5.3. Moreover, in Theorem 5.4 we obtained closure properties of bivariate discrete distributions belonging to the class $\mathcal{DS}(x,y;\mathbf{a})$ which help to generate plenty of new distributions from the same class. Many known bivariate discrete models possess DS-BAP (see Examples 5.1 to 5.8), but also new ones can be obtained using the extended MO construction used in Example 5.9.

Several authors define the joint survival function by $S(x,y)=P(X>x,Y>y)$ for $x,y=0,1,2,\dots$, see Shaked, Shanthikumar and Torres (1995). In such a case, all above results are valid with a minor adjustment of corresponding relations.

Of course, other analytical forms (even "non-aging") of the function $G(x,y;t)$ in (5.8) are possible and desirable, depending on the real problem at hand. For instance, a weak version of the DS-BAP can be investigated if we relax the marginal equations (5.7), that is, considering only the functional equation (5.6) in Definition 5.1. In this case, one would obtain new bivariate discrete models which includes DS-BAP as a particular case. For example, consider the joint survival function of X and Y given by

$$S(x,y)=p_0^{\max(x,y)}p_1^{x^3}p_2^{y^3}, \quad x,y=0,1,2,\dots$$

for $p_0,p_1,p_2\in(0,1)$. One can verify that the distribution satisfies partially Definition 5.1: relation (5.6) is fulfilled but (5.7) fails. Really, in this case we have

$$g_1(x,y)+g_2(x+1,y)=a_0+3a_1x(x+1)+3a_2y(y+1),$$

that is, the sum does not have a linear form advocated by (5.1), being a particular case of the relation

$$g_1(x,y)+g_2(x+1,y)=A_1(x)+A_2(y), \quad x,y=0,1,2,\dots,$$

where $A_1(x)$ and $A_2(y)$ are pre-specified non-negative increasing functions with properties reflecting the evolution of the gradient vector corresponding to problem at hand to be modeled. An investigation of the bivariate discrete model based on last equation would be a valuable future research.

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References

Apostol, T. M. (1969). *Calculus: Multi-Variable Calculus and Linear Algebra, with Applications to Differential Equations*, 2nd ed. New York: Wiley. MR0248290

Asha, G., Sankaran, P. G. and Nair, N. U. (2003). Probability models with constant total failure rate. *Communications in Statistics Theory and Methods* **32**, 1089–1099. MR1983233

Barlow, R. and Proschan, F. (1965). *Mathematical Theory of Reliability*. New York: Wiley. MR0195566
Cox, D. and Oakes, D. (1984). *Analysis of Survival Data*. London: Chapman & Hall. MR0751780
Dhar, S. K. (1998). Data analysis with discrete analog of Freund's model. *Journal of Applied Statistical Science* **7**, 169–183. MR1650507
Freund, J. E. (1961). A bivariate extension of the exponential distribution. *Journal of the American Statistical Association* **56**, 971–977. MR0132625
Gumbel, E. J. (1960). Bivariate exponential distributions. *Journal of the American Statistical Association* **55**, 698–707. MR0116403
Hyman, J. M. and Shashkov, M. (1997). Natural discretizations for the divergence, gradient, and curl on logically rectangular grids. *Computers & Mathematics with Applications* **33**, 81–104. MR1433461
Johnson, N. L. and Kotz, S. (1975). A vector multivariate hazard rate. *Journal of Multivariate Analysis* **5**, 53–66. MR0365901
Kodlin, D. (1967). A new response time distribution. *Biometrics* **23**, 227–239. MR0215398
Konstantopoulos, T. and Yuan, L. (2017). A probabilistic interpretation of the Gaussian binomial coefficients. *Journal of Applied Probability* **54**, 1295–1298. MR3731296
Kulkarni, H. V. (2006). Characterizations and modelling of multivariate lack of memory property. *Metrika* **64**, 167–180. MR2259220
Lee, H., Cha, J.-H. and Pulcini, G. (2017). Modeling discrete bivariate data with applications to failure and count data. *Quality and Reliability Engineering International* **33**, 1455–1473.
Li, X. and Pellerey, F. (2011). Generalized Marshall–Olkin distributions and related bivariate aging properties. *Journal of Multivariate Analysis* **102**, 1399–1409. MR2819957
Marshall, A. W. (1975). Some comments on the hazard gradient. *Stochastic Processes and Their Applications* **3**, 293–300. MR0400600
Marshall, A. W. and Olkin, I. (1967). A multivariate exponential distribution. *Journal of the American Statistical Association* **62**, 30–44. MR0215400
Mi, J. (1993). Discrete bathtub failure rate and upside-down bathtub mean residual life. *Naval Research Logistics* **40**, 361–371. MR1206196
Nair, N. U. and Asha, G. (1997). Some classes of multivariate life distributions in discrete time. *Journal of Multivariate Analysis* **62**, 181–189. MR1473872
Nair, N. U. and Nair, K. (1990). Characterizations of a bivariate geometric distribution. *Statistica* **50**, 247–253. MR1143754
Nair, N. U., Sankaran, P. G. and Balakrishnan, N. (2018). *Reliability Modelling and Analysis in Discrete Time*. London: Academic Press. MR3793634
Omey, E. and Minkova, L. (2014). Bivariate geometric distributions. *Comptes rendus de l'Academie bulgare des Sciences* **67**, 1201–1210. MR3308476
Pinto, J. and Kolev, N. (2015). Sibuya-type bivariate lack of memory property. *Journal of Multivariate Analysis* **134**, 119–128. MR3296038
Pinto, J. and Kolev, N. (2016). A class of continuous bivariate distributions with linear sum of hazard gradient components. *Journal of Statistical Distributions and Applications* **3**, Article ID 10.
Roy, D. and Gupta, R. (1992). Classifications of discrete lives. *Microelectronics and Reliability* **32**, 1459–1473.
Shaked, M., Shanthikumar, J. and Torres, J. (1995). Discrete hazard rate functions. *Computers & Operations Research* **22**, 391–402.
Sibuya, M. (1960). Bivariate extreme statistics I. *Annals of the Institute of Statistical Mathematics* **11**, 195–210. MR0115241

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