L-Logistic regression models: Prior sensitivity analysis, robustness to outliers and applications

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Abstract. Tadikamalla and Johnson [*Biometrika* **69** (1982) 461–465] developed the L_B distribution to variables with bounded support by considering a transformation of the standard Logistic distribution. In this manuscript, a convenient parametrization of this distribution is proposed in order to develop regression models. This distribution, referred to here as L-Logistic distribution, provides great flexibility and includes the uniform distribution as a particular case. Several properties of this distribution are studied, and a Bayesian approach is adopted for the parameter estimation. Simulation studies, considering prior sensitivity analysis, recovery of parameters and comparison of algorithms, and robustness to outliers are all discussed showing that the results are insensitive to the choice of priors, efficiency of the algorithm MCMC adopted, and robustness of the model when compared with the beta distribution. Applications to estimate the vulnerability to poverty and to explain the anxiety are performed. The results to applications show that the L-Logistic regression models provide a better fit than the corresponding beta regression models.

1 Introduction

Modeling data that are restricted to the interval (0, 1), as for example, the proportion of children vulnerable to poverty or anxiety as a function of the stress, is frequently encouraged by researchers. Many different regression models have been proposed in the past two decades for modeling this type of data. For example, Buckley (2003), Ferrari and Cribari-Neto (2004), Paz, Bazán and Milan (2015), Lemonte and Bazán (2016), Gómez-Déniz, Sordo and Calderín-Ojeda (2014) and Bayes, Bazán and Castro (2017) have all proposed regression models. Yet, there are still continuous distributions with bounded support that need further study. This is the case of the L-Logistic distribution, which was originally proposed by Tadikamalla and Johnson (1982) through a transformation of the standard Logistic distribution. This construction is similar to the S_B system proposed by Johnson (1949). This distribution was studied by, Tadikamalla and Johnson (1990) and

Johnson and Tadikamalla (1991), among others, who proposed the method of moments and the percentile point method to fit this distribution, and by Wang and Rennolls (2005), who considered the Maximum Likelihood estimation. However, regression models have not been studied based on this distribution.

In this work, we discuss the properties of this distribution by considering a new parametrization. Another parametrization is also preserved in the Online Suplementary Material. Here, we present the parametrization of the L-Logistic distribution that considers the median as one parameter and the dispersion as another parameter. Therefore, we propose a new regression model considering this distribution in the context of quantile regression (QR) models, which were introduced by Koenker and Bassett (1978). Specifically, we propose a median regression model, which may represent the relationship between the median (central location) of the response and a set of covariates as well as of the dispersion parameter and another or the same) set of covariates, by using a convenient link function. If the data are highly skewed, since the median is a natural robust measure of location, the conditional median modeling can be more useful than the usual conditional mean modeling adopted usually in beta regression models (Buckley, Ferrari and Cribari-Neto (2003, 2004)). Different from previous studies of L-Logistic distribution, here we propose a Bayesian approach employing a Markov chain Monte Carlo (MCMC) method for the modeling framework. The issues of model fitting are addressed by means of a hybrid algorithm that combines *Metropolis–Hasting* algorithm with *Gibbs sampling*. In the Online Supplementary Material (da Paz, Balakrishnan and Bazán (2018)), we report results from the first studies with simulated data sets to investigate prior sensitivity analysis of the dispersion parameter of the L-logistic distribution and the median L-logistic regression, parameter recovery and comparison of algorithms, and to evaluate the robustness to outliers of the L-logistic distribution in comparison with beta distribution. These results display that the proposed estimation method works well, and that the model proposed is more robust than the beta models in the presence of outliers. Also, real application of social and psychological data is considered to show the advantages of proposed approach. First, we show that the proportion of children vulnerable to poverty of the municipalities of the state of Alagoas in Brazil, for the 2010 season, is best fitted by the L-Logistic distribution as compared to the beta distribution. Additionally, we show that L-Logistic regression models provide a better fit than the corresponding beta regression models for analyzing the anxiety as a function of the stress using a sample of nonclinical women in Townsville, Queensland, Australia.

The rest of this manuscript is organized as follows. Section 2 is dedicated to L-Logistic distribution, and we study some characteristics of this distribution like alternative parametrizations, some related distributions, moments, the skewness and kurtosis measures. In Section 3, we propose the L-Logistic median regression model. Section 4 is dedicated to the Bayesian estimation of the distribution parameters, and to the parameter of the proposed median regression model. Section 5
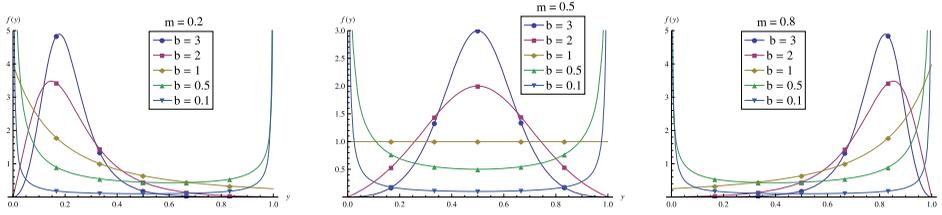
Figure 1 *L-Logistic probability density functions for m = 0.2, 0.5 and 0.8 and some choices of parameter b.*

presents the results of three simulation studies that examine a prior sensitivity analysis, parameter recovery, comparison of algorithms, and robustness to outliers of the L-Logistic distribution. Section 6 discusses applications of the proposed distribution, including the applicability of regression models to real data sets. Finally, some concluding remarks are made in Section 7.

2 The L-logistic distribution

We say that the random variable (r.v.) Y follows a L-Logistic distribution, denoted by $Y \sim LL(m, b)$, if its probability density function (pdf) is given by

$$f(y|m, b) = \frac{b(1-m)^b m^b y^{b-1}(1-y)^{b-1}}{[(1-m)^b y^b + m^b(1-y)^b]^2},$$

$$0 < y < 1, 0 < m < 1, b > 0.$$

(1)

Depending on the parameters m and b, the L-Logistic distribution takes on a variety of shapes (see, for example, Figures 1 and 2). Note that when we set $m = 0.5$ and $b = 1$ in (1), the pdf of the L-Logistic distribution simply becomes the pdf of the standard uniform distribution. Here, m is the median of the distribution, which scales the graph to the left or right on the horizontal axis, and consequently it is a location parameter. The L-Logistic density is uni-modal (or "uni-antimodal"), increasing, decreasing, or constant, depending on the values of its parameters. Additionally, we note that for a fixed value of the parameter m, the dispersion of the
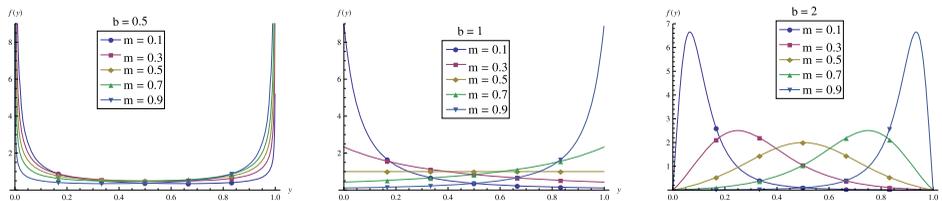

Figure 2 *L-Logistic probability density function for dispersion parameter b = 0.5, 1 and 2 and some choices of parameter m.*

distribution decreases as b increases, and so b is a parameter that governs the dispersion of the distribution. In the Online Supplementary Material, we show that the parameter b is in fact a dispersion parameter.

The cumulative distribution function (cdf) of the L-Logistic distribution is given by

$$F_Y(y|m,b) = \left(1 + \left(\frac{m(1-y)}{y(1-m)}\right)^b\right)^{-1}, \qquad 0 < y < 1, \tag{2}$$

which can be readily inverted to yield the quantile function

$$Q_Y(p) = F_Y^{-1}(p) = \frac{(\frac{m}{1-m})}{(\frac{m}{1-m}) + (\frac{1-p}{p})^{1/b}}, \qquad 0 \le p \le 1. \tag{3}$$

This would readily enable a quantile-based analysis of this model. Note that if $p = 1 - p = 0.5$, then $Q(p) = m$, which means that the parameter m is indeed the 50th percentile or the median of the L-Logistic distribution.

Equation (3) facilitates simple r.v. generation. Specifically, if $U \sim \text{uniform}(0, 1)$, then

$$X = Q(U) = \frac{(\frac{m}{1-m})}{(\frac{m}{1-m}) + (\frac{1-U}{U})^{1/b}} \sim LL(m, b). \tag{4}$$

Additionally, we can express the inter-quartile range (IQR) as

$$IQR = Q(0.75) - Q(0.25) = \frac{m3^{1/b}}{(1-m) + 3^{1/b}m} - \frac{m}{3^{1/b}(1-m) + m}. \tag{5}$$

The IQR has a breakdown point of 50%, and this measure is often preferred over range. When the distribution is symmetric, half IQR equals the median absolute deviation, and is often used in the detection of outliers in data.

2.1 Mode

Property 2.1. *For $b > 1$, the mode y_0 of the L-Logistic distribution is the solution of the equation*

$$\left(\frac{1-m}{m}\right)^b = \left(\frac{1-y_0}{y_0}\right)^b \frac{b + 2y_0 - 1}{b - 2y_0 + 1}. \tag{6}$$

Note that, upon taking $\delta = -b\log(\frac{m}{1-m})$, the mode y_0 can be obtained by solving the equation

$$\delta = \log\left(\left(\frac{1-y_0}{y_0}\right)^b \frac{b + 2y_0 - 1}{b - 2y_0 + 1}\right). \tag{7}$$

In addition, from (6) and (7), if $y_0 = m = 0.5$, then $\delta = 0$ for all values of b. Thus, we can study the behavior of the mode by studying the function in (7). For

this purpose, we take the derivative of the right-hand side of (7) with respect to y_0 to obtain the equation

$$\frac{\partial \delta}{\partial y_0} = \frac{b(b^2 - 1)}{(y_0 - 1)y_0\{(b^2 - 1) + 4y_0 - 4y_0^2\}}. \tag{8}$$

(6) is negative for $b > 1$, the situation where δ decreases as y_0 (mode) increases (first derivative test), then the mode lies in the interval $\langle 0, 0.5 \rangle$ if $\delta > 0$ (or $m < 0.5$) and for $\delta < 0$ (or $m > 0.5$) the mode is in the interval $\langle 0.5, 1 \rangle$. If $b < 1$, (8) is positive whenever $\{(b^2 - 1) + 4y_0 - 4y_0^2\} > 0$, that is, whenever $\frac{1-b}{2} < y < \frac{1+b}{2}$, the situation where δ increases as y increases. Thus, from (7) and (8), the minimum of the pdf lies in the interval $\langle \frac{1-b}{2}, 1/2 \rangle$ for $\delta < 0$ or $m > 0.5$, and in the interval $\langle 0.5, \frac{1+b}{2} \rangle$ for $\delta > 0$ or $m < 0.5$.

2.2 Moments

The following proposition gives an expression for the moments of the L-Logistic distribution.

Property 2.2. *If* $Y \sim LL(m, b)$, *then the moments of* Y *about zero are given by*

$$E[Y^t] = \int_0^1 \left[1 + \left(\frac{1-v}{v} \right)^{1/b} \left(\frac{1-m}{m} \right) \right]^{-t} dv. \tag{9}$$

The integral in (9) cannot be expressed in an analytical form. However, we can use numerical integration to evaluate some moments as $E_Y(Y)$, $E_Y(Y^2)$ and $\text{Var}_Y(Y) = E_Y(Y^2) - E_Y(Y)^2$. Table 1 shows some values of the first and second moments and the variance of the L-Logistic distribution. In addition, Figure 3 shows the graphs of the mean and variance as functions of the dispersion parameter b, for some choices of the parameter m. For this purpose, the integral in (9) was evaluated by the Gaussian quadrature.

2.3 Skewness and kurtosis measures

First, we have the following symmetry property.

Table 1 $E_Y[Y]$, $E_Y[Y^2]$, *and* $\text{Var}_Y(X)$ *of the L-Logistic distribution for some choices of b and m*

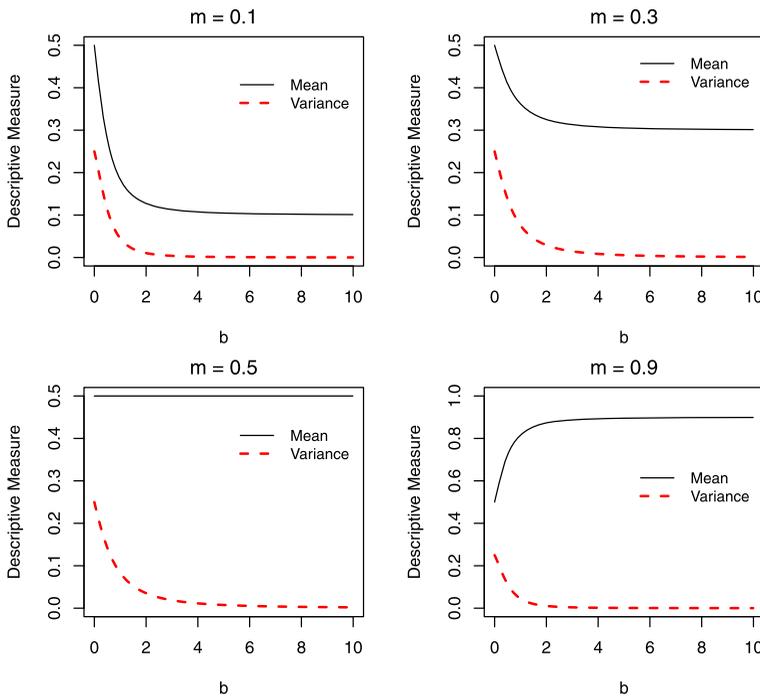| m | 0.2 | 0.5 | 0.8 | 0.2 | 0.5 | 0.8 |
b	1	1	1	3	3	3
$E_Y[Y]$	0.283	0.5	0.717	0.216	0.5	0.784
$E_Y[Y^2]$	0.145	0.333	0.579	0.056	0.269	0.625
$\text{Var}_Y[Y]$	0.065	0.083	0.065	0.010	0.019	0.01

Figure 3 *Descriptive measures of the L-Logistic distributions for some values of the parameters.*

Property 2.3. *The L-Logistic density is symmetric when m = 0.5 for all values of b.*

For the case when the L-Logistic density is asymmetric, the degree of skewness can be quantified by some measures of skewness. Since the L-Logistic distribution is related to the Logistic distribution, the skewness measure introduced by Arnold and Groeneveld (1995), denoted by γ_M, seems to be a suitable skewness measure. The measure γ_M is based on the mode of distribution and is given by

$$\gamma_M = 1 - 2F(M), \tag{10}$$

where M is the mode of the distribution and $F(\cdot)$ is the distribution function. The value of γ_M lies in $(-1, 1)$, and if γ_M is near 1, it indicates extreme right skewness. On the other hand, if γ_M is near -1, it indicates extreme left skewness.

We also consider another measure of skewness, called quantile skewness (denoted here by γ_p), first proposed by Hinkley (1975). This skewness measure is given by

$$\gamma_p = \frac{Q(1 - p) + Q(p) - 2m}{Q(1 - p) - Q(p)}, \tag{11}$$

which is a function of high and low percentiles defined by $p \in (0, 0.5)$ with $Q(\cdot)$ being as in (3). The maximum value of γ_p is 1, representing extreme right skewness, while the minimum is -1 representing extreme left skewness. This measure is also zero for any symmetric distribution. However, the function in (11) depends on the value of p. We can remove this dependence by integrating over p, or to decide which value of p is appropriate for use. In Brys, Hubert and Struyf (2003), there is a comparison between several robust skewness measures in which accuracy, robustness, and computational complexity are all considered. The most interesting skewness measure of all the measures investigated is octile skewness. Octile skewness takes $p = 0.125$ in (11), that is, it is given by

$$\gamma_O = \frac{O_7 - O_4 + O_1 - O_4}{O_7 - O_1} = \frac{Q(0.875) + Q(0.125) - 2m}{Q(0.875) - Q(0.125)}. \tag{12}$$

For the L-Logistic distribution, we made use of this particular skewness measure instead of removing the dependence over p through integration.

Moreover, the kurtosis of the L-Logistic distribution can also be derived easily by using the quantiles. The kurtosis measure introduced by Moors (1988) is given by

$$\begin{aligned}
\kappa_O &= \frac{O_7 - O_6 + O_3 - O_1}{O_6 - O_2} \\
&= \frac{Q(0.875) - Q(0.625) + Q(0.375) - Q(0.125)}{Q(0.75) - Q(0.25)},
\end{aligned} \tag{13}$$

with $\kappa_O \in (0, \infty)$. Moors (1988) justified the use of the kurtosis measure in (13) by the interpretation that the two terms in the numerator of (13) are large (small) if relatively little (much) probability mass is concentrated in the neighborhood of $Q(0.75)$ and $Q(0.25)$. This corresponds to large (small) dispersion around (roughly) $E_Y[Y] \pm \text{Var}_Y[Y]$ where $E_Y[Y]$ and $\text{Var}_Y[Y]$ are the mean and variance of Y, respectively.

Figure 4 presents the results of the measures of skewness and kurtosis described here for some values of the parameter m as a function of the dispersion parameter b, for $b > 1$. From this figure, we can see that when $m < 0.5$, the two measures of skewness decrease as b increases. However, when $m > 0.5$, the two measures of skewness increase as b increases. For $m = 0.5$, we can see that these two measures are constant. In addition, the mode of the L-Logistic distribution increases as b increases when $m < 0.5$, is constant when $m = 0.5$, and decreases as b increases when $m > 0.5$. For the measure of kurtosis, we see no pattern in this figure.

3 L-Logistic median regression model

Regression analysis estimates the potential differential effect of a covariate on the mean or quantiles of the conditional distribution (Hao and Naiman (2007)). Here,
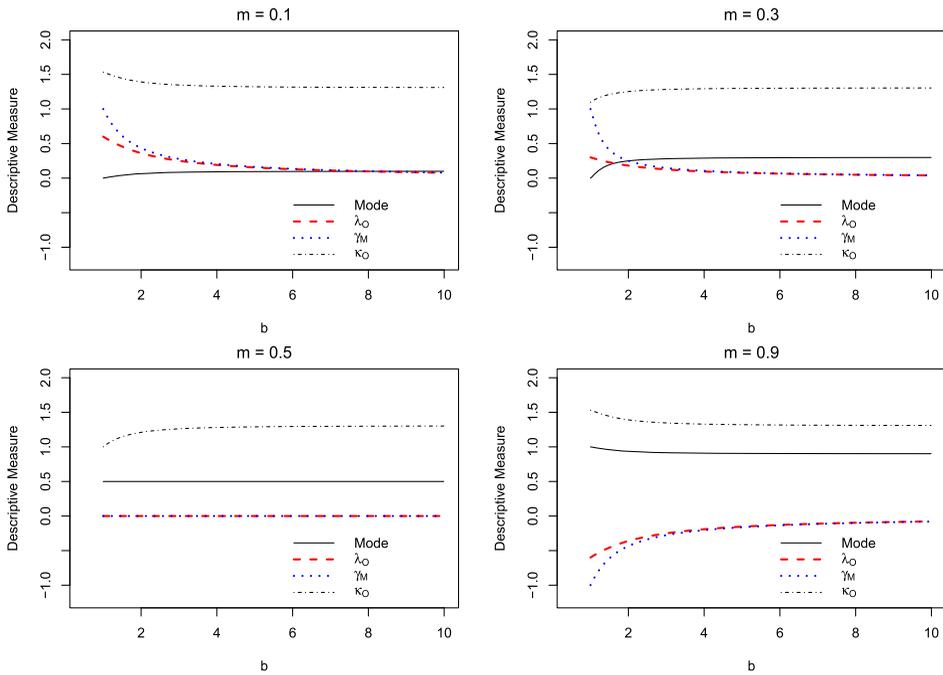
Figure 4 *The mode, two measure of skewness (γ_M, γ_O), and kurtosis (κ_O) of the L-Logistic distribution for some choice of the parameters.*

we are interested in studying the conditional (or regression) median as a function of the covariates when the response variable takes values in a bounded interval. Our goal is to define a median regression model for a r.v. that assumes values in the standard unit interval. Let $Y = (Y_1, \ldots, Y_n)$ be a vector of independent r.v.'s following the distribution in (1) with median m_i and dispersion parameter b_i, and consider that x_{1i}^T and x_{2i}^T are q- and d-dimensional vectors, respectively, containing the explanatory variables both with 1 as the first component. Thus, in the regression analysis with the L-Logistic distribution, we assume that conditional on the explanatory variables (covariates), the r.v.'s Y_i, $i = 1, \ldots, n$, are mutually independent with L-Logistic distribution, that is,

$$Y_i \sim LL(m_i, b_i), \tag{14}$$

with

$$h_1(m_i) = x_{1i}^T \boldsymbol{\beta} \quad \text{and} \quad h_2(b_i) = x_{2i}^T \boldsymbol{\delta}, \tag{15}$$

where $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{q-1})$ and $\boldsymbol{\delta} = (\delta_0, \ldots, \delta_{d-1})$ ($\beta_i, \delta_j \in \mathbb{R}$ for $i = 0, \ldots,$ $(q-1)$ and $j = 0, \ldots, (d-1)$) represent, respectively, q- and d-dimensional vectors of unknown regression parameters. In (15), h_1 and h_2 are strictly monotone and twice differentiable real link functions. This method allows to fit the model

adequately with a variety of link functions, which ensures parameter m is in the interval $(0, 1)$ and the dispersion parameter b is in the interval $(0, \infty)$. Some choices of link functions for a parameter are discussed in Ferrari and Cribari-Neto (2004). A common link function for the parameter m is the logit function,

$$\text{logit}(m_i) = \boldsymbol{x}_{1i}^T \boldsymbol{\beta} \quad \Rightarrow \quad m_i = \frac{\exp\{\boldsymbol{x}_{1i}^T \boldsymbol{\beta}\}}{1 + \exp\{\boldsymbol{x}_{1i}^T \boldsymbol{\beta}\}}. \tag{16}$$

For the dispersion parameter, a common link function is the log-linear link function. For easy interpretation, here we follow Smithson and Verkuilen (2006) and take $h_2 = -\log(b_i)$, that is,

$$\log(b_i) = -\boldsymbol{x}_{2i}^T \boldsymbol{\delta} \quad \text{or} \quad b_i = \exp\{-\boldsymbol{x}_{2i}^T \boldsymbol{\delta}\}. \tag{17}$$

4 Bayesian estimation

In this section, we describe the Bayesian approach for the estimation of parameters of the L-Logistic distribution, and also of the L-Logistic regression model.

4.1 Bayesian estimation of the L-logistic distribution

If we consider a random sample $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ from the distribution in (2), then the likelihood function is given by

$$L(m, b|\boldsymbol{y}) = \prod_{i=1}^n \frac{b(1-m)^b m^b y_i^{b-1}(1-y_i)^{b-1}}{[(1-m)^b y_i^b + m^b(1-y_i)^b]^2}, \tag{18}$$

where $0 < m < 1$ and $b > 0$. To complete the Bayesian specification of the model, since parameters m and b have different behavior, we assume independence between them, and the following structure is then considered:

$$\pi(m, b) = \pi(m)\pi(b), \tag{19}$$

where $\pi(m)$ and $\pi(b)$ are the prior densities for m and b, respectively.

The prior $\pi(b)$ can be, for example, the pdf of the Gamma distribution with parameter vector (ϵ, ϵ), ϵ being a small value. The choice of this prior seems to be reasonable as a Gamma prior has large variance. The prior sensitivity analysis considering other priors for the parameter b presented in Section 7 shows the robustness of results in the selection of prior distributions. For the parameter m, we can take $m \sim \text{unifom}(0, 1)$ or $m \sim \text{beta}(1, 1)$, where $\text{beta}(a, b)$ represents the beta distribution with parameters a and b. This is a "flat" prior where all values in the range are equally likely. This choice can be considered weakly informative because in other cases (i.e., when a or $b \neq 1$) we will expect a value of the median m to be greater than 0.5 or otherwise. Then, a more informative, subjective or expert prior by interviewing experts can be considered in order to elicit parameters in a

parametric family of priors following methods discussed, for example, by Albert et al. (2012). Another choice for m can be through specifying other values of the hyper parameters a and b in the beta distribution. From our results in simulation studies, we did not find any problem with this choice for the model without co-variate. By considering the specifications above, the joint posterior distribution for (m, b) is given by

$$\pi(m, b|\mathbf{y}) \propto \prod_{i=1}^{n} \frac{b(1-m)^{b} m^{b} y_i^{b-1} (1-y_i)^{b-1}}{[(1-m)^{b} y_i^{b} + m^{b} (1-y_i)^{b}]^2} \pi(b), \tag{20}$$

where $0 < m < 1$ and $b > 0$.

Since the posterior distribution is not available in a closed-form, the Markov Chain Monte Carlo (MCMC) approach (Gelman et al. (2013, pp. 259–349)) is used to estimate the model parameters. Initially, we consider the full conditional posterior distributions for the parameters (m, b) given by

$$\pi(m|b, \mathbf{y}) = K_1^{-1} \frac{(1-m)^{nb} m^{nb}}{\prod_{i=1}^{n} [(1-m)^{b} y_i^{b} + m^{b} (1-y_i)^{b}]^2}, \tag{21}$$

$$\pi(b|m, \mathbf{y}) = K_2^{-1} \prod_{i=1}^{n} \left(\frac{b(1-m)^{b} m^{b} y_i^{b-1} (1-y_i)^{b-1}}{[(1-m)^{b} y_i^{b} + m^{b} (1-y_i)^{b}]^2} \right) \pi(b), \tag{22}$$

where $0 < m < 1$ and $b > 0$ with K_1 and K_2 being normalizing constants.

Thus, a hybrid algorithm that combines Metropolis–Hastings algorithm and *Gibbs sampling* was implemented in R language (R Development Core Team (2015)) to obtain a sample from the posterior distribution of model parameters (m, b). These codes are available upon request from the authors.

As suggested by a referee, we also implemented a MH algorithm. Based on different scenarios, we did not find difference in the recovery of parameters based on this two algorithms.

4.2 Bayesian estimation of the L-logistic regression model

Now, let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be a vector of independent r.v.'s following the L-Logistic distribution with median m_i and dispersion parameter b_i given by (16) and (17), respectively. Then, the likelihood of the observed sample $\mathbf{y} = (y_1, \ldots, y_n)$ of $\mathbf{Y}$ can be written as

$$L(\boldsymbol{\beta}, \boldsymbol{\delta}|\mathbf{y}, X) = \prod_{i=1}^{n} \frac{b_i(1-m_i)^{b_i} m_i^{b_i} y_i^{b_i-1} (1-y_i)^{b_i-1}}{[(1-m_i)^{b_i} y_i^{b} + m^{b_i} (1-y_i)^{b_i}]^2}, \tag{23}$$

where X is the matrix containing all the explanatory variables, and $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$ are the regression parameters such that

$$\text{logit}(m_i) = \mathbf{x}_{1i}^{T} \boldsymbol{\beta} \quad \text{and} \quad \log(b_i) = -\mathbf{x}_{2i}^{T} \boldsymbol{\delta}. \tag{24}$$

Suppose we have no prior information from historical data for the regression parameters. In many applications, a normal prior distribution centered at zero with a standard error equal to 100 for the regression coefficient will be sufficiently noninformative. So, we assign these weakly informative prior distributions to the parameters, that is, we adopt prior normal distributions with large variance such that

$$\beta_j \sim \text{normal}(0, 100), \qquad \text{for } j = 0, \ldots, q-1,$$
$$\delta_l \sim \text{normal}(0, 100), \qquad \text{for } l = 0, \ldots, d-1. \tag{25}$$

The prior distributions of parameters are chosen here under the assumption that they are independent of each other.

Assuming the prior distributions in (25) for the parameters, the posterior density takes on the form

$$\pi(\boldsymbol{b}, \boldsymbol{\beta}, \boldsymbol{\delta}|\boldsymbol{y}) \propto L(\boldsymbol{\beta}, \varphi|\boldsymbol{y}, \boldsymbol{X})\pi(\boldsymbol{\beta})\pi(\delta) = \prod_{i=1}^{n} f(y_i|\boldsymbol{x}_i, \boldsymbol{\beta}, \delta)\pi(\boldsymbol{\beta})\pi(\boldsymbol{\delta}). \tag{26}$$

Therefore, the full conditional posterior distributions for $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$ can be written as

$$\pi(\boldsymbol{\beta}|\boldsymbol{x}_i, \boldsymbol{\delta}) \propto \prod_{i=1}^{n} f(y_i|\boldsymbol{x}_i, \boldsymbol{\beta}, \boldsymbol{\delta})\pi(\boldsymbol{\beta}),$$
$$\pi(\boldsymbol{\delta}|\boldsymbol{x}_{ij}, \boldsymbol{\beta}) \propto \prod_{i=1}^{n} f(y_i|\boldsymbol{x}_i, \boldsymbol{\beta}, \boldsymbol{\delta})\pi(\boldsymbol{\delta}). \tag{27}$$

Using the posterior distribution in (26), we can use several Bayesian mechanisms for estimating the parameters. Here, a *Gibbs sampling* algorithm with *Metropolis–Hasting* step inside becomes useful since the full conditional posterior distribution for $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$ in (27) does not have a closed-form. The algorithm described above was implemented in R language (R Development Core Team (2015)), and these codes are available upon request from the authors.

4.3 Model comparison criteria and Posterior predictive checking

In order to compare alternative models, we made use of some model comparison criteria. Specifically, we considered the Expected Akaike information criterion (EAIC), the expected Bayesian information criterion (EBIC), the deviance information criterion (DIC), and the Watanabe–Akaike information criterion, which are all described in Section 2 of Suplementary Material. Additionally, in this section, using MCMC techniques, we show how to simulate values of the posterior predictive distribution for checking the model.

5 Simulation studies

This section presents three simulation studies, one that examines a prior sensitivity analysis, another investigates the recovery of the parameters of the model by the proposed estimation method and, finally, another that compares the proposed approach and existing approaches to model data with outliers. For this purpose, the Bayesian method is applied on simulated data sets from the L-Logistic distribution, considering different scenarios. For the estimation of parameters, we generated 20,000 values from the posterior distribution in (20), then the first 10,000 values were discarded and sequences of 10 observations were eliminated, and finally the resulting sample of size 1000 were used for inference.

5.1 Prior sensitivity analysis

Prior sensitivity analysis plays an important role in applied Bayesian analysis. This is especially true for Bayesian models used for new distribution, wherein the interpretability of the corresponding parameters becomes important. In this section, we consider a simulation study to evaluate the sensitivity of different choices of prior distributions for parameter b since this is different from parameter m, which is clearer in its interpretation. Specifically, we assume prior independence between parameters b and m, considering a unit uniform distribution for parameter m.

We considered five different prior distributions for b, considering simulated data sets from the L-Logistic distribution for some pairs of parameters m and b. The values of m and b used are as follows: $b \in \{0.5, 1, 5\}$ and $m \in \{0.2, 0.5, 0.9\}$, leading to nine scenarios or pairs of parameters, corresponding to nine models simulated. We simulated samples of size $n = 100$, $y_1, \ldots y_n$, from the L-Logistic distribution based on these pairs of parameters, then nine distinct simulated datasets were considered in the analysis.

Based on the works of Figueroa-Zúñiga, Arellano-Valle and Ferrari (2013), we consider for the parameter b three relatively non-informative and two informative prior distributions. The non-informative prior distributions are the gamma distribution with parameter vector $(0.001, 0.001)$ ($b \sim \text{Gamma}(0.001, 0.001)$), denoted by prior A, the uniform distribution with parameter vector $(0, 100)$ for U ($U \sim \text{uniform}(0, 100)$) with $b = U^2$, denoted by prior B, and the central Student t distribution with parameter vector $(10, 0, 2)$ ($L \sim St(10, 0, 2)$) for L with $\log(b) = L$, denoted by prior C. The prior B is chosen because it is less informative than the usual gamma with parameter vector (ϵ, ϵ). For the informative prior distributions, we consider $b \sim \text{Gamma}(2.5, 1)$, denoted by prior D, and $b \sim \text{Gamma}(50, 1)$, denoted by prior E. Note that prior E provides incorrect information about parameter b, while prior D provides almost correct information. In all the cases, the prior distribution for parameter m is taken as the uniform distribution with parameters 0 and 1, that is, $m \sim \text{uniform}(0, 1)$.

In order to compare the models with different prior distributions, we made use of some model comparison criteria. Specifically, we considered the Expected Akaike

information criterion (EAIC), the expected Bayesian information criterion (EBIC), the deviance information criterion (DIC), and the Watanabe–Akaike information criterion. For a review of these criteria, one may refer to Gelman et al. (2013) (a brief description of these criteria is also given in the Online Supplementary Material 2). Based on these criteria, for all the simulated datasets in the nine considered scenarios, we found that prior E provided the model with the worst fit among all the fitted models. However, for the models using all other prior distributions, the values of WAIC, EAIC, EBIC, and DIC are all quite close, showing no significant difference, giving evidence that the estimated models provide almost the same quality of fit for the analyzed samples. Thus, for these cases, the posterior distribution does not seem to be sensitive with respect to the specification of these prior distributions. The values of WAIC, EAIC, EBIC and DIC for the fitted models, considering these different prior distributions, are presented in Tables A and B of the Supplementary Material 3.1.

For a more detailed analysis, additionally, we choose the non-informative priors A and C, and the worst informative prior E to present HPD intervals and point estimates. Prior A was chosen for this second analysis because it is simplest among the non-informative priors considered before, while prior C presents lower values for EAIC, EBIC and DIC than prior B in most of the studied cases. In this analysis, we observe that when prior E is used, the HPD interval does not contain the true value of b. On the other hand, the non-informative A and C priors provide intervals containing the true value of the parameters for all the cases analyzed. However, prior A provides the estimated values for the parameter b (posterior mean) closer to the true value than prior C in most cases. The posterior mean and the 95% HPD interval (obtained from the package of Martin, Quinn and Park (2011)) can be seen in Table C of the Supplementary Material 3.1.

Considering the results discussed before, we choose priors A and C for developing a sensitivity analysis in the context of the median regression model. Here, we also consider prior independence between the parameters in which $\beta_j \sim N(0, 100)$, for $j = 0, 1, \ldots, q - 1$. The simulated data sets for this analysis were generated from L-Logistic distribution such that

$$Y_i \sim LL(m_i, b),$$
$$\text{logit}(m_i) = \boldsymbol{x}_i^T \boldsymbol{\beta}, \tag{28}$$

for $i = 1, \ldots, n$, where $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{q-1})$. The $x_i's$ were generated independently from beta distributions (beta(2, 5), beta(5, 1) and beta(1, 3), respectively) and their values were centered at their respective sample averages, in order to improve the convergence of the MCMC algorithm. We considered two different median regression models, the first with just one covariate ($q = 2$) and the second with three covariates ($q = 4$). For each median regression model, we considered three true values for the dispersion parameter b leading to six scenarios which are presented in Table 2. For each scenario, we simulated samples of size

Table 2 *Statistics for model comparison with different prior distributions for parameter b, using sample size of* 100, 50 *and* 20 *observations simulated from L-Logistic distribution for different values of b and the coefficient of the median regression model*

Coefficient $\beta = (\beta_0, \ldots, \beta_p)$	Dispersion b	Prior A				Prior C			
		WAIC	EAIC	EBIC	DIC	WAIC	EAIC	EBIC	DIC
		Sample size $n = 100$							
$\beta = (-1.5, 1.5)$	0.5	99.60	−192.25	−182.44	−193.20	99.64	−192.37	−182.56	−193.44
	1	69.77	−132.59	−122.77	−133.52	69.81	−132.62	−122.81	−133.58
	5	187.88	−368.84	−359.03	−369.82	187.89	−368.73	−358.92	−369.60
$\beta = (-3, -1.5, 1.5, 3)$	0.5	206.55	−400.50	−385.48	−403.45	206.65	−400.41	−385.38	−403.19
	1	220.62	−428.65	−413.62	−431.61	220.72	−428.73	−413.70	−431.70
	5	366.60	−720.56	−705.53	−723.46	366.60	−720.72	−705.69	−723.79
		Sample size $n = 50$							
$\beta = (-1.5, 1.5)$	0.5	42.56	−78.25	−70.52	−79.21	42.63	−78.36	−70.62	−79.40
	1	35.25	−63.59	−55.86	−64.52	35.30	−63.75	−56.01	−64.80
	5	98.36	−189.83	−182.10	−190.78	98.35	−189.92	−182.18	−190.96
$\beta = (-3, -1.5, 1.5, 3)$	0.5	112.62	−212.86	−201.30	−215.78	112.81	−213.01	−201.45	−215.94
	1	124.40	−236.41	−224.85	−239.28	124.56	−236.63	−225.07	−239.61
	5	198.21	−384.03	−372.47	−386.89	198.22	−383.97	−372.41	−386.79
		Sample size $n = 20$							
$\beta = (-1.5, 1.5)$	0.5	30.10	−52.91	−47.92	−53.80	30.38	−53.04	−48.05	−54.00
	1	19.22	−31.18	−26.20	−32.10	19.46	−31.40	−26.42	−32.43
	5	39.81	−72.35	−67.36	−73.21	39.90	−72.47	−67.48	−73.43
$\beta = (-3, -1.5, 1.5, 3)$	0.5	39.34	−66.33	−59.35	−69.17	39.85	−66.64	−59.66	−69.51
	1	33.03	−53.67	−46.69	−56.48	33.48	−54.10	−47.12	−57.06
	5	56.97	−101.49	−94.51	−104.19	57.10	−101.66	−94.68	−104.48

$n \in \{20, 50, 100\}$ from these models. The true values of the coefficients and the dispersion parameter for each considered model are shown in Table 2. As in the previous analysis, considering the results presented in Table 2, we can see no difference between the models fitted with priors A and C for the dispersion parameter b.

5.2 Parameter recovery and comparison of algorithms

A study of parameter recovery for the parameters of the L-Logistic distribution using prior A for the dispersion parameter b and the unit uniform prior for the parameter m was conducted and can be seen in Supplementary Material 3.2. This study showed that the proposed estimation method for the parameters of the L-Logistic distribution works quite well. Additionally, following a recommendation of the Associate Editor, we compared the proposed hybrid algorithm (Metropolis–Hastings algorithm within the Gibbs sampler) with an adaptive Metropolis–Hastings algorithm in order to estimate the median L-logistic regression. The results presented in Supplementary Material reveal that there is no difference in the recovery of the parameters of the model by both these methods.

5.3 Robustness to outliers of L-Logistic distribution

Now, we discuss a simulation study carried out to examine the robustness to outliers of the L-Logistic distribution, that is, we discuss a study of the relative performance of the procedure for estimating the beta and L-Logistic models, with data coming from a beta distribution with outliers.

The contaminated beta data were generated following Bayes, Bazán and García (2012) in two steps. First, the datasets were generated from a beta distribution with location parameter $\mu = 0.2$, considering two values of the dispersion parameter, $\phi = 10, 30$, and three sample sizes, $n = 50, 100, 200$. Second, these data were contaminated with outliers generated from a uniform distribution with parameters 0.999 and 1. The proportions of outliers considered were 0.02, 0.05 and 0.08 for each dataset, that is, $r = 2\%, 5\%, 8\%$ of the data in each dataset were randomly replaced by outliers. This gave $r \times n/100$ total outliers in each dataset containing n values. The combination of values of ϕ, n and r provides $2 \times 3 \times 3 = 18$ scenarios to be analyzed.

In order to compare the fit of beta and L-Logistic models to each of the contaminated datasets, WAIC, EAIC, EBIC and DIC were obtained for beta and L-Logistic models for 100 replications in each scenario. Thus, the percentage of cases in which the L-Logistic model achieved a lower value for WAIC, EAIC, EBIC and DIC than the beta model was determined. We found no significant difference between the two analyzed models when the DIC is used to select the model. However, the L-Logistic model performed better than beta models in all analyzed cases based on WAIC, EAIC and EBIC (see Table F of Suplementary Material 3.3).
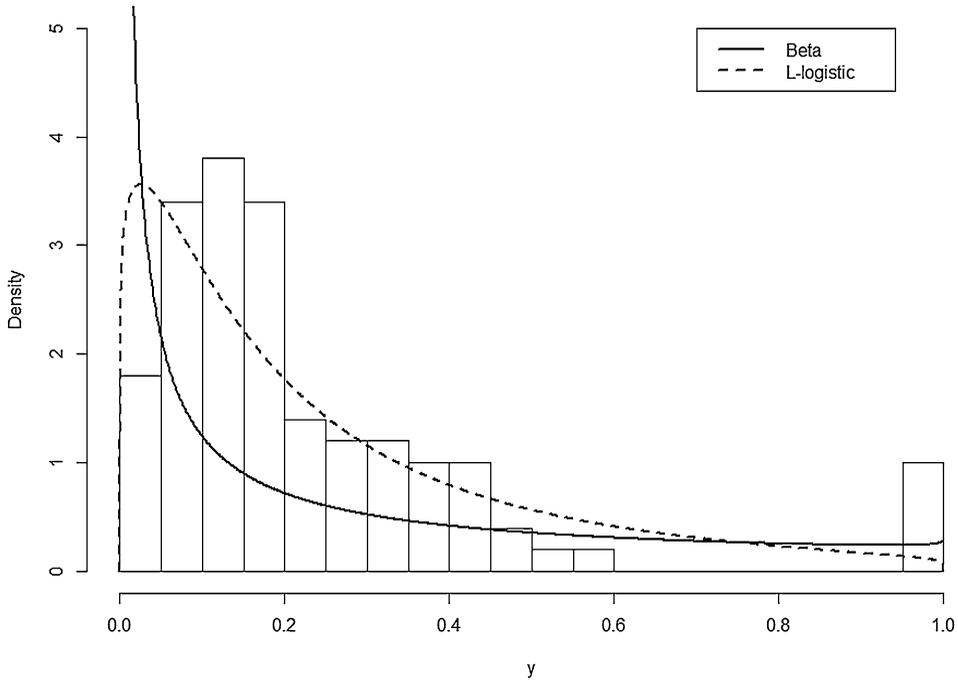
Figure 5 *Estimated densities for beta and L-Logistic models for the scenario in which $n = 100$, $\phi = 10$ and $r = 5\%$.*

The bias and MSE of the estimators of m and μ obtained by replicating in each scenario, considering 0.2 as the true value for the parameters m and μ, were obtained (see Table F of Supplementary Material 3.3). The bias and MSE were always smaller for the m estimator than the μ estimator showing that for any scenario with outliers, there was an improvement in the accuracy (bias and MSE decrease) for the estimation of the model parameters when using an L-Logistic model rather than the beta model for a contaminated dataset. In order to illustrate the results, the estimated densities for the scenario in which $n = 100$, $r = 5\%$ and $\phi = 10$ is shown in Figure 5, where the L-Logistic model is seem to fit the data better than the beta model.

6 Applications of L-Logistic distribution to real data

In order to illustrate the advantages of the use of the L-Logistic distribution in comparison to beta distribution, in Section 6.1 we estimate the distribution of the vulnerability to poverty in Alagoas state, Brazil. Later in Section 6.2, we propose different regression models to explain the anxiety as a function of stress, considering a know data set in the literature.

6.1 Estimating the distribution of the vulnerability to poverty in Alagoas state

In this subsection, we consider a real dataset, which contains the proportion of children (0–14 year olds) vulnerable to poverty. The data came from the municipalities of the state of Alagoas in Brazil, and was collected in 2010. The state of Alagoas is located in the eastern part of the Northeastern Region of Brazil and is made up of 102 municipalities. This state is one of the poorest states of Brazil and its HDI (Human Development Index) is the country's worst, based on information available in PNUD, IPEA and FJP (2013). Thus, we are interested in modeling the proportion of children vulnerable to poverty (PCVP). Here, a child is considered vulnerable to poverty if the per capita household income is at most BRL 255, in 2010. The PCVP data set comprises 102 observations and is modeled here using the L-Logistic distribution and the beta distribution that is often used to model data when a distribution over some finite interval is needed; see Gupta and Nadarajah (2004). Here, we use the re-parametrized beta distribution discussed by Ferrari and Cribari-Neto (2004) in the context of regression analysis.

The Bayesian methodology was used to estimate the parameters of both models. For the L-Logistic distribution with parameters m and b, we considered prior A discussed earlier in Section 5. Since the beta distribution has parameters $0 < \mu < 1$ and $\phi > 0$, we considered the same prior A in this model as well.

The final result on the estimation is presented in Table 3. This table also shows the values of statistics for model comparison in order to evaluate the ability of L-Logistic and beta models to fit the data. According to this table, it is clear that the L-Logistic model is better for modeling the PCVP data than the beta model. In addition, Figure 6 shows two graphs with the mean values and error bars with 95% credible intervals plotted against the corresponding observed value of the data. The error bars were constructed from 1000 values (ordered, and of size 102) generated from the L-Logistic and beta distributions, respectively, for each graph, with the estimated parameters. In the case of the L-Logistic model, the bars crossed by the diagonal line $y = x$ indicate that the model is quite suitable for the data. On the other hand, in the case of the beta model, we observe high deviations between

Table 3 *Estimates and* 95% *HPD intervals for the parameters of the L-Logistic and beta models, and model comparison criteria*

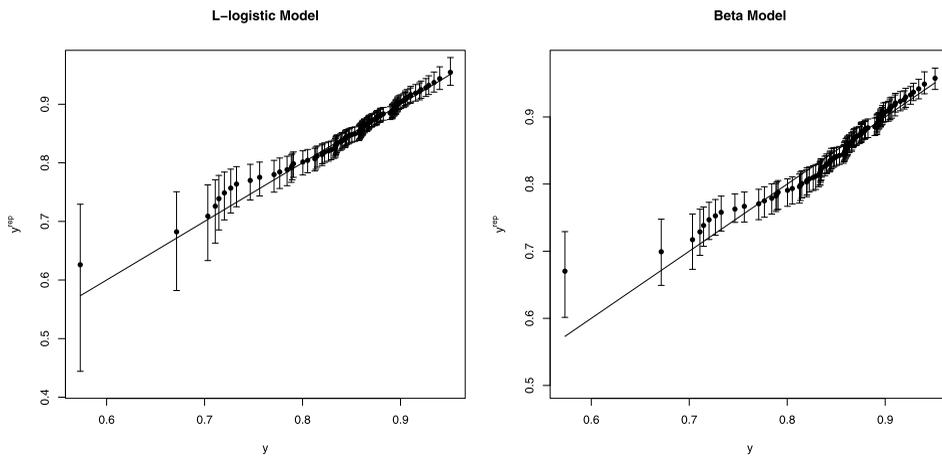| Model		Parameter	Criteria			
			WAIC	EAIC	EBIC	DIC
L-Logistic	m	0.86(0.85, 0.87)	155.1322	−304.2996	−299.0496	−306.3422
	b	4.04(3.42, 4.72)				
beta	μ	0.85(0.84, 0.86)	150.8993	−295.3312	−290.0813	−297.3437
	ϕ	37.81(27.55, 47.83)				

Figure 6 *Posterior predictive error bars with 95% confidence intervals of the generated values $y_{(i)}^{\text{rep}}$ versus ordered observed data $y_{(i)}$ for the PCVP data, using L-Logistic and beta models.*

the predicted and observed data, mainly in the tail of the distribution. In this case, an observation is flagged as an outlier, since the corresponding posterior interval does not contain the observed value. Thus, Figure 6 provides evidence that the beta model is not suitable for these data. Finally, the estimated and the observed histograms of the PCVP data are presented in Figure 7, which confirms that the L-Logistic model provides a better fit for these data than the beta model.

Assuming the L-Logistic model as a final model, we can see in Table 3 that the median of proportion of children vulnerable to poverty is close to 1 with low dispersion. Therefore, based on these results we can conclude that the children vulnerable to poverty in the Alagoas state is higher and systematically present lower dispersion.

6.2 Application of the L-Logistic regression model

In order to illustrate the regression analysis considering the L-Logistic distribution, we analyzed a known data set in the literature that come from a sample of nonclinical women in Townsville, Queensland, Australia. The data set contains 166 observations on two variables, namely, the stress score and the anxiety score. Both variables were assessed on the Depression Anxiety Stress Scales, ranging from 0 to 42, but linearly transformed to the open unit interval by Smithson and Verkuilen (2006). The scatter plot of the anxiety versus stress variable, and the histograms of the data, are presented in Figure 8. The histogram given in this figure suggest that the anxiety is strongly skewed.

For this data, we propose four possible regression sub-models using L-Logistic distribution. We consider a null regression model without any covariate, a regression model considering only covariate effects in parameter m, a dispersion regres-
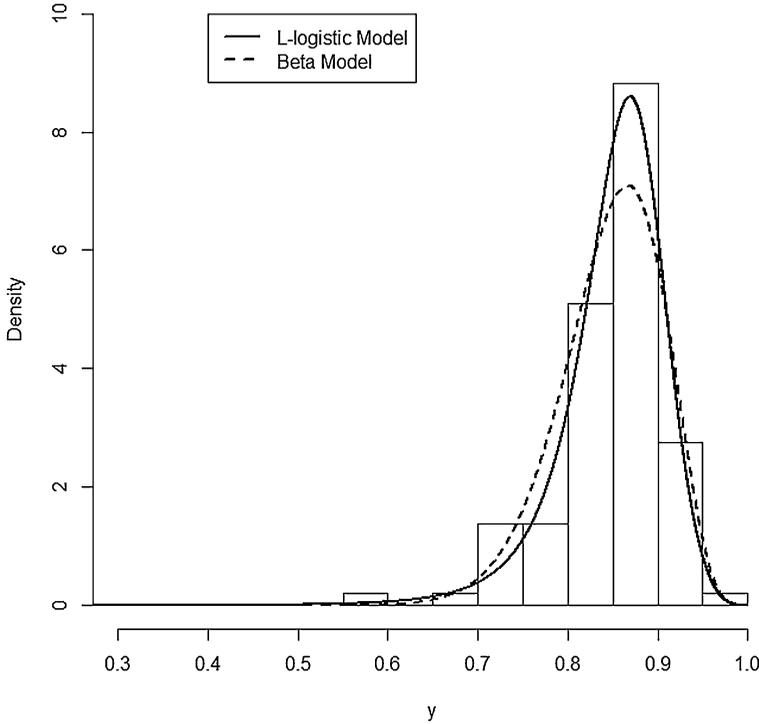
Figure 7 *Observed and estimated density of PCVP data.*

sion model considering only covariate effects in the dispersion parameter b, and full regression model considering both effects, as follows:

$$Y_i \sim LL(m_i, b_i) \quad \text{and}$$

$$\begin{cases} \text{null model } (L_0): & \text{logit}(m_i) = \beta_0 \text{ and } \log(b_i) = -\boldsymbol{\delta}, \\ \text{median-model } (L_1): & \text{logit}(m_i) = \boldsymbol{x}_{1i}^T \boldsymbol{\beta} \text{ and } \log(b_i) = -\delta_0, \\ \text{dispersion model } (L_2): & \text{logit}(m_i) = \beta_0 \text{ and } \log(b_i) = -\boldsymbol{x}_{1i}^T \boldsymbol{\delta}, \\ \text{full model } (L_3): & \text{logit}(m_i) = \boldsymbol{x}_{1i}^T \boldsymbol{\beta} \text{ and } \log(b_i) = -\boldsymbol{x}_{2i}^T \boldsymbol{\delta}, \end{cases}$$

for $i = 1, \ldots, 166$. In addition, we also consider equivalent regression models using the beta distribution as follows:

$$Y_i \sim \text{beta}(\mu_i, \phi_i) \quad \text{and}$$

$$\begin{cases} \text{null model } (B_0): & \text{logit}(\mu_i) = \beta_0 \text{ and } \log(\phi_i) = -\boldsymbol{\delta}, \\ \text{mean-model } (B_1): & \text{logit}(\mu_i) = \boldsymbol{x}_{1i}^T \boldsymbol{\beta} \text{ and } \log(b_i) = -\delta_0, \\ \text{dispersion model } (B_2): & \text{logit}(\mu_i) = \beta_0 \text{ and } \log(\phi_i) = -\boldsymbol{x}_{1i}^T \boldsymbol{\delta}, \\ \text{full model } (B_3): & \text{logit}(\mu_i) = \boldsymbol{x}_{1i}^T \boldsymbol{\beta} \text{ and } \log(\phi_i) = -\boldsymbol{x}_{2i}^T \boldsymbol{\delta}, \end{cases}$$
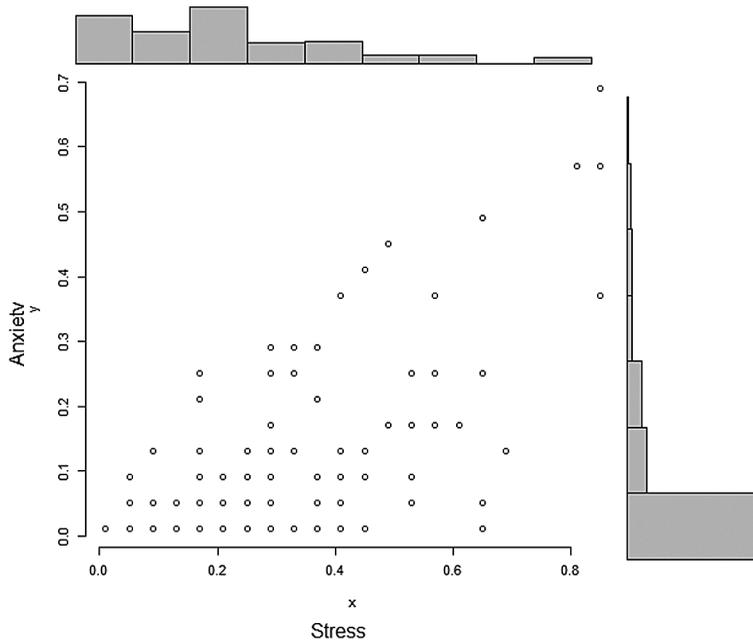
for $i = 1, \ldots, 166$.

Figure 8　*Scatterplot and histograms of the data in Application* 6.2.

Here, the Bayesian approach is considered for the inference process with a prior distribution for the unknown regression parameters, as given in (25). All the algorithms were prepared in the R language and we report the results corresponding to 10,000 iterations following a burn-in period also of 10,000 iterations. In order to eliminate dependence, we eliminated a sequence of 10 observations every 11 simulations in the sample of size 10,000, resulting in a final sample of 1000 elements. Finally, the convergence of MCMC chain was assessed by using the separated partial means test of Geweke (1992), which provided evidence for the chains to have converged.

The regression models investigated were compared by the use of EAIC, EBIC, DIC and WAIC criteria, and the obtained results are shown in Table 4. The param-

Table 4　*Model comparison criteria for model comparison*

Sub model	L-Logistic model				beta model			
	WAIC	EAIC	EBIC	DIC	WAIC	EAIC	EBIC	DIC
0	259.34	−512.94	−506.72	−514.92	239.45	−472.90	−466.67	−474.90
1	277.67	−545.51	−536.17	−548.56	243.28	−478.06	−468.72	−481.07
2	316.57	−624.78	−615.44	−627.83	283.41	−556.95	−547.62	−559.92
3	319.65	−627.47	−615.02	−631.48	301.91	−591.85	−579.41	−595.82

eter estimates for these models are shown in Table 5. Considering Table 4, we can observe that the regression models considering L-Logistic distribution provides a better fit than the corresponding beta regression models, for all the criteria considered. These results also give evidence that L_3 and L_2 are the best models among the ones based on the L-Logistic distribution. Though there is no significant difference between the L_2 and L_3 regressions, we consider the L_3 regression model to be a reasonable choice for this data set, due to the expected influence on covariates in the dispersion parameter (Smithson and Verkuilen (2006)).

Moreover, a posterior distribution of residuals was obtained and a posterior mean of this distribution was computed (Gelman et al. (2014)). That is, for $i = 1, \ldots, 166$, we have $\hat{r}_i = G^{-1} \sum_{g=1}^{G} \frac{y_i - \hat{y}^g}{SD(Y_i|\boldsymbol{\beta}^g)}$, where $\boldsymbol{\beta}^1, \ldots, \boldsymbol{\beta}^G$ are obtained from the posterior distribution, $\hat{y}$ is the estimated value of a data point y_i, and $SD(Y|\boldsymbol{\beta}^g)$ is the standard deviation of posterior values of Y, both obtained given a single random draw $\boldsymbol{\beta}^g$ of the posterior distribution. Figure 9 shows the standard residual versus the estimated values in which we can see that the L_3 regression model provides a better fit than the B_3 model, which confirms that L_3 model are better than the corresponding B_3 model.

Finally, from Table 5 giving the 95% HPD intervals for all the coefficients of the models under analysis, we can see that the estimates are quite precise. For the model chosen, that is, L_3 model, we observe that the HPD intervals for the estimates of the parameters β_1 and δ_1 do not contain zero giving evidence that the parameters in the model are significant in the model. In other words, stress is important in both parameters of the distribution of anxiety.

7 Final remarks

The L-Logistic distribution, introduced by Tadikamalla and Johnson (1982), is a bounded continuous distribution that possesses some nice properties, as discussed in Section 2. Considering the parametrization introduced in this manuscript, we propose a Bayesian estimator by considering an MCMC method as an alternative to the moment and maximum likelihood methods developed previously in the literature. In the Bayesian context, a non-informative prior distribution can be adopted for the parameter m since it lies in the unit interval, enabling the use of unit uniform distribution as a non-informative prior distribution.

The main motivation of the parametrization introduced here is the development of regression models based on the L-Logistic distribution. We also introduce conditional median regression models, which is a special case of quantile regression wherein a conditional quantile is modeled as a function of covariates.

Two applications have been considered in this work. First, we consider an application to social data, wherein the proportion of children vulnerable to poverty of the municipalities of the state of Alagoas in Brazil, for the 2010 season, is modeled. Second, we analyze a known data set, previously analyzed using beta distribution

Table 5 *Parameter estimates and 95% HPD intervals for the L-Logistic and beta models*

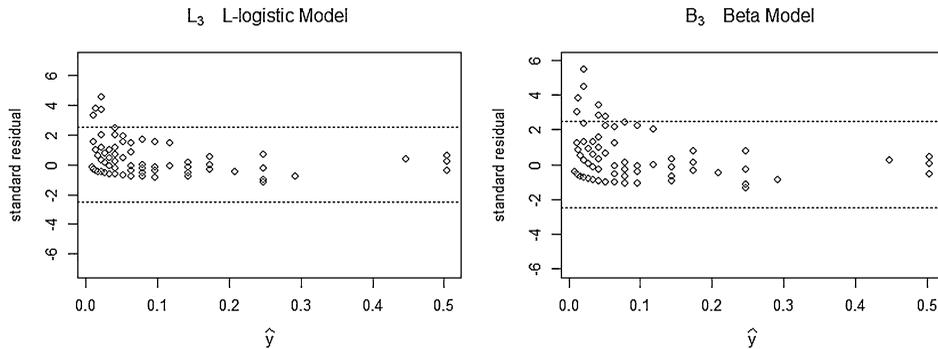| Model		$\hat{\beta_0}$(HPD)	$\hat{\beta_1}$(HPD)	$\hat{\delta_0}$(HPD)	$\hat{\delta_1}$(HPD)
				Coefficient	
L-Logistic	L_0	−3.354 (−3.615, −3.106)	–	−0.08 (−0.20, 0.04)	–
	L_1	−4.78 (−5.04, −4.52)	5.78 (4.99, 6.58)	−0.44 (−0.57, −0.31)	–
	L_2	−4.03 (−4.27, −3.78)	–	−0.87 (−1.14, −0.58)	2.56 (1.62, 3.42)
	L_3	−4.77 (−5.00, −4.53)	5.64 (4.68, 6.61)	−0.76 (−1.03, −0.48)	1.14 (0.19, 2.05)
beta	B_0	−2.239 (−2.430, −2.04)	–	−1.78 (−2.02, −1.54)	–
	B_1	−3.47 (−3.75, −3.18)	3.74 (3.11, 4.37)	−2.44 (−2.7, −2.20)	–
	B_2	−2.54 (−2.80, −2.27)	–	−2.49 (−2.98, −1.95)	1.53 (0.42, 2.55)
	B_3	−4.02 (−4.30, −3.72)	4.95 (4.09, 5.83)	−3.94 (−4.45, −3.44)	4.28 (2.78, 5.79)

Figure 9 *Standard residual versus adjusted values for the L-Logistic and beta models.*

by Smithson and Verkuilen (2006), which contains the stress score and the anxiety score. Here, the anxiety variable is modeled as a function of the stress. In the case of the L-Logistic distribution, we use a regression model proposed in this work. In these applications, we observe that the L-Logistic distribution seems to fit better than the beta model for both this analyzed cases. Considering the application to the anxiety data set (Anxiety explained by stress), we show that the L-Logistic regression models can be a good alternative to the beta model. An advantage of this approach is the possibility of modeling other quantiles in order to describe a non-central position of a distribution. So, one may choose a position specifically for his/her needs. For example, it is possible to consider a regression model to explain other quantiles to the Anxiety considering the influence of the stress in our application. Thus, conditional quantile models offer the flexibility to focus on these population segments, whereas conditional mean models do not. However, since quantile regression curves are estimated individually, the quantile curves can cross, leading to an invalid distribution for the response. Thus, this problem, referred to as crossing in the literature, needs to be studied carefully. Some authors have proposed methods to deal with this problem; see, for example, Cai and Jiang (2015).

In the future, we aim to develop techniques for mixed quantile regression for the L-Logistic distribution. Moreover, we intend to explore mixtures of L-Logistic distributions in a Bayesian framework as well as a multivariate version of this distribution.

Acknowledgments

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Supplementary Material

Supplement to "L-Logistic regression models: Prior sensitivity analysis, robustness to outliers and applications" (DOI: 10.1214/18-BJPS397SUPP; .pdf). Supplementary information.

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