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Some unified results on stochastic properties of residual lifetimes at random times

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Abstract. The residual life of a random variable X at random time Θ is defined to be a random variable X_Θ having the same distribution as the conditional distribution of $X - \Theta$ given $X > \Theta$ (denoted by $X_\Theta = (X - \Theta | X > \Theta)$). Let (X, Θ_1) and (Y, Θ_2) be two pairs of jointly distributed random variables, where X and Θ_1 (and, Y and Θ_2) are not necessarily independent. In this paper, we compare random variables X_{Θ_1} and Y_{Θ_2} by providing sufficient conditions under which X_{Θ_1} and Y_{Θ_2} are stochastically ordered with respect to various stochastic orderings. These comparisons have been made with respect to hazard rate, likelihood ratio and mean residual life orders. We also study various ageing properties of random variable X_{Θ_1}. By considering this generalized model, we generalize and unify several results in the literature on stochastic properties of residual lifetimes at random times. Some examples to illustrate the application of the results derived in the paper are also presented.

1 Introduction

Let the lifetime of a component be represented by a non-negative random variable (r.v.) X having absolutely continuous distribution function (d.f.) $F(\cdot)$, probability density function (p.d.f.) $f(\cdot)$ and survival function (s.f.) $\bar{F}(\cdot) = 1 - F(\cdot)$. Then, the residual life of the component which has survived up to time t, $t > 0$, is given by r.v. X_t having the same distribution as conditional distribution of $X - t$ given $X > t$ (denoted by $X_t = (X - t | X > t)$). If t is replaced by a r.v. Θ, then $X_\Theta = (X - \Theta | X > \Theta)$ represents the residual lifetime of r.v. X at random time Θ. The following situations illustrate the interpretation of r.v. X_Θ:

- In clinical trials, it often happens that the time at which a person goes to clinic for examination of a disease is actually different from the time he got infected. In this scenario, the latent period of the disease can be estimated by $X_\Theta = (X - \Theta | X > \Theta)$ (Cha and Finkelstein (2014), Finkelstein and Vaupel (2015)).
- Consider a series system with two components C_1 and C_2, having lifetimes Θ and X, respectively. If C_1 fails before C_2, then the system will fail to work but C_2 may still be in working condition. In this situation, X_Θ can be used to determine the residual life of C_2 after the failure of C_1.

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The concept of residual life at a random time (RLRT), or at fixed time, has been studied and discussed extensively in the literature (Stoyan (1983), Guess and Proschan (1988), Shaked and Shanthikumar (2007), Cai and Zheng (2012)). Various researchers have presented results on stochastic comparisons of RLRT and have discussed its ageing properties (Yue and Cao (2000, 2001), Li and Zuo (2004), Misra, Gupta and Dhariyal (2008), Eryilmaz (2013), Dewan and Khaledi (2014)). These results have been derived either by assuming that $X \stackrel{d}{=} Y$ or $\Theta_1 \stackrel{d}{=} \Theta_2$; here, $\stackrel{d}{=}$ means equality in distribution. Moreover, all the studies carried out so far have assumed that X and Θ_1 (and, Y and Θ_2) are independently distributed. Under the assumptions that $X \stackrel{d}{=} Y$, X and Θ_1, and Y and Θ_2 are independently distributed, the following results are available in the literature:

(i) Yue and Cao (2000) established that if $\Theta_1 \leq_{\mathrm{rh}} \Theta_2$ and X has decreasing (increasing) failure rate, then $X_{\Theta_1} \leq_{\mathrm{st}} (\geq_{\mathrm{st}})\ X_{\Theta_2}$. This result was further strengthened by Misra, Gupta and Dhariyal (2008) where, under the same assumptions as in Yue and Cao (2000), it is proved that $X_{\Theta_1} \leq_{\mathrm{hr}} (\geq_{\mathrm{hr}})\ X_{\Theta_2}$. Recently, Dewan and Khaledi (2014) gave a different proof of this result of Misra, Gupta and Dhariyal (2008).

(ii) Under the assumptions that $\Theta_1 \leq_{\mathrm{rh}} \Theta_2$ and X has decreasing (increasing) mean residual life, Yue and Cao (2000) proved that $E(X_{\Theta_1}) \leq (\geq)\ E(X_{\Theta_2})$. This result was generalized by Li and Zuo (2004) who, under the same assumptions as in Yue and Cao (2000), established increasing convex order between X_{Θ_1} and X_{Θ_2}. Later on, Misra, Gupta and Dhariyal (2008), in their Theorem 3.2, further strengthened the result of Li and Zuo (2004) by establishing the mean residual life order between X_{Θ_1} and X_{Θ_2}. Dewan and Khaledi (2014) in their Theorem 2.8 (d) provided an alternate proof of the above result proved by Misra, Gupta and Dhariyal (2008).

Dewan and Khaledi (2014) assumed that $\Theta_1 \stackrel{d}{=} \Theta_2$, X and Θ_1, and Y and Θ_2 are independently distributed, and proved the following results:

(i) If $X \leq_{\mathrm{rh}} Y$ and either X or Y has increasing reversed failure rate, then $X_\Theta \leq_{\mathrm{rh}} Y_\Theta$.

(ii) If $X \leq_{\mathrm{hr}} Y$ and either X or Y has decreasing failure rate, then $X_\Theta \leq_{\mathrm{hr}} Y_\Theta$.

(iii) If $X \leq_{\mathrm{mrl}} Y$ and either X or Y has increasing mean residual life, then $X_\Theta \leq_{\mathrm{mrl}} Y_\Theta$.

The purpose of this study is to unify and generalize aforementioned results by considering stochastic comparison of X_{Θ_1} and Y_{Θ_2}, without assuming that $X \stackrel{d}{=} Y$ or $\Theta_1 \stackrel{d}{=} \Theta_2$, and also without assuming that X and Θ_1 (and, Y and Θ_2) are independently distributed. The general layout of the model considered in this paper is as follows: Let X, Y, Θ_1, and Θ_2 be non negative r.v.s with Θ_i, $i = 1, 2$, having p.d.f. h_i, d.f. H_i and s.f. $\bar{H}_i$. Let (X, Θ_1) and (Y, Θ_2) be two pairs of jointly distributed r.v.s with the common support $[0, \infty) \times [0, \infty)$. For any fixed $\theta > 0$, let X_θ (Y_θ) denote the r.v. having the same distribution as the conditional distribution of X (Y) given that $\Theta_1 = \theta$ ($\Theta_2 = \theta$). Let $f_\theta(\cdot)$, $F_\theta(\cdot)$, $\bar{F}_\theta(\cdot)$ ($g_\theta(\cdot)$, $G_\theta(\cdot)$, $\bar{G}_\theta(\cdot)$), respectively be the p.d.f., d.f. and s.f. of X_θ (Y_θ), $\theta > 0$. The residual life of r.v. X (Y) at random time Θ_1 (Θ_2) is given by $X_{\Theta_1} = (X - \Theta_1 | X > \Theta_1)$ ($Y_{\Theta_2} = (Y - \Theta_2 | Y > \Theta_2)$). The s.f.s of X_{Θ_1} and Y_{Θ_2} are given by

$$\begin{aligned} \bar{M}_1(x) &= \frac{\int_0^\infty \bar{F}_\theta(x+\theta) h_1(\theta)\, d\theta}{\int_0^\infty \bar{F}_\theta(\theta) h_1(\theta)\, d\theta}, \qquad \text{if } x \geq 0, \quad \text{and} \\ \bar{M}_2(x) &= \frac{\int_0^\infty \bar{G}_\theta(x+\theta) h_2(\theta)\, d\theta}{\int_0^\infty \bar{G}_\theta(\theta) h_2(\theta)\, d\theta}, \qquad \text{if } x \geq 0, \end{aligned} \tag{1.1}$$

respectively. The density functions of X_{Θ_1} and Y_{Θ_2} are given by

$$\begin{aligned} m_1(x) &= \frac{\int_0^\infty f_\theta(x+\theta) h_1(\theta)\, d\theta}{\int_0^\infty \bar{F}_\theta(\theta) h_1(\theta)\, d\theta}, \qquad \text{if } x > 0, \quad \text{and} \\ m_2(x) &= \frac{\int_0^\infty g_\theta(x+\theta) h_2(\theta)\, d\theta}{\int_0^\infty \bar{G}_\theta(\theta) h_2(\theta)\, d\theta}, \qquad \text{if } x > 0, \end{aligned} \tag{1.2}$$

respectively.

Note that models (1.1) and (1.2) are conditionally dependent mixture models. Recently, various authors have studied stochastic properties of conditionally independent mixture models. Some of the contributions in this direction are due to Gupta, Dhariyal and Misra (2011), Gupta and Kirmani A (2006) and Misra, Gupta and Gupta (2009).

This paper is organized as follows: In Section 2, we mention some auxiliary results which will be used in proving the main results of the paper. Section 3 presents results on stochastic comparison of X_{Θ_1} and Y_{Θ_2} with respect to various stochastic orders. Specifically, we focus on stochastic comparisons with respect to hazard rate, likelihood ratio and mean residual life orders. In Section 4, we discuss ageing properties of random variable X_{Θ_1}, and finally, in Section 5, we present some examples to illustrate applications of the results derived in the paper.

2 Preliminaries

Throughout the paper, the terms *increasing* and *decreasing* will imply *non-decreasing* and *non-increasing*, respectively. Consider a r.v. X_i, $i = 1, 2$, having absolutely continuous d.f. F_i, s.f. $\bar{F}_i$ and the Lebesgue p.d.f. f_i, $i = 1, 2$. Further, for the sake of simplicity, assume that distributions of X_1 and X_2 have the common support $[0, \infty) = \{t \in \mathbb{R} : f_i(t) > 0\}$, $i = 1, 2$. For ease of reference, we first review some standard notations and definitions before stating our main results. We begin with definitions of some standard stochastic orders and ageing notions (see Shaked and Shanthikumar (2007), Lai and Xie (2006), Li and Li (2013)).

Definition 2.1. The r.v. X_1 is said to be smaller than the r.v. X_2 in

(i) hazard rate order (denoted by $X_1 \leq_{\mathrm{hr}} X_2$) if $\bar{F}_2(x)/\bar{F}_1(x)$ increases in $x > 0$;
(ii) reversed hazard rate order (denoted by $X_1 \leq_{\mathrm{rh}} X_2$) if $F_2(x)/F_1(x)$ increases in $x > 0$;
(iii) likelihood ratio order (denoted by $X_1 \leq_{\mathrm{lr}} X_2$) if $f_2(x)/f_1(x)$ increases in $x > 0$;
(iv) mean residual life order (denoted by $X_1 \leq_{\mathrm{mrl}} X_2$) if $\int_x^\infty \bar{F}_2(u)\,du / \int_x^\infty \bar{F}_1(u)\,du$ increases in $x > 0$.

Definition 2.2. The r.v. X is said to have the

(i) increasing (decreasing) likelihood ratio (ILR (DLR)) if $f(x)$ is log-concave (log-convex) on $(0, \infty)$;
(ii) increasing (decreasing) failure rate (IFR (DFR)) if $\bar{F}(x)$ is log-concave (log-convex) on $(0, \infty)$;
(iii) decreasing reversed failure rate (DRFR) if $F(x)$ is log-concave on $(0, \infty)$;
(iv) increasing (decreasing) mean residual life (IMRL (DMRL)) if $\int_x^\infty \bar{F}(t)\,dt$ is log-convex (log-concave) on $(0, \infty)$.

The next definition on totally positive and reverse regular functions may be found in Karlin (1968).

Definition 2.3. Let $S_1, S_2 \subseteq \mathbb{R}$ and let $k : S_1 \times S_2 \to [0, \infty)$ be a non-negative function, where $\mathbb{R}$ denotes the real line. The function $k(x, y)$ is said to be totally positive (reverse regular) of order 2, denoted by $\mathrm{TP}_2(\mathrm{RR}_2)$, if $k(x_1, y_1)k(x_2, y_2) \geq (\leq)\ k(x_1, y_2)k(x_2, y_1)$, whenever $x_1 \leq x_2$, $y_1 \leq y_2$, $x_1, x_2 \in S_1$ and $y_1, y_2 \in S_2$.

Dewan and Khaledi (2014) and Khaledi (2014) have listed a few results due to Karlin (1968) and Joag-Dev, Kochar and Proschan (1995) on TP_2 (RR_2) functions. We now state the following lemma, proved in Naqvi (2017), which will be helpful in deriving the main results of this paper and may also be of independent interest to researchers. This lemma extends the results mentioned in Dewan and Khaledi (2014) (also see Khaledi (2014) and Misra and van der Meulen (2003)).

Let $\psi_i : [0, \infty) \times [0, \infty) \to \mathbb{R}$, $i = 1, 2$, be a function and let $g_i(\theta)$ be the Lebesgue p.d.f. of a r.v. T_i, $i = 1, 2$. In many branches of statistics, one often encounters the problem of verifying the monotonicity of function of the type

$$\psi(x) = \frac{\int_0^\infty \psi_2(x, \theta) g_2(\theta)\,d\theta}{\int_0^\infty \psi_1(x, \theta) g_1(\theta)\,d\theta}, \qquad x > 0. \tag{2.1}$$

In the following lemma, we provide sufficient conditions on $\psi_i(\cdot, \cdot)$, $i = 1, 2$, for the function in (2.1) to be monotone.

Lemma 2.4. *Suppose that* $\frac{\psi_2(x,\theta)}{\psi_1(x,\theta)}$ *increases* (*decreases*) *in* $x \in (0, \infty)$ *and increases in* $\theta \in (0, \infty)$. *Further suppose that any of the following three conditions hold*:

(i) $T_1 \leq_{\rm lr} T_2$ *and* $\psi_1(x, \theta)$ *or* $\psi_2(x, \theta)$ *is* TP_2 (RR_2) *in* $(x, \theta) \in (0, \infty) \times (0, \infty)$;

(ii) $T_1 \leq_{\rm hr} T_2$ *and*
$\psi_1(x, \theta)$ *is* TP_2 (RR_2) *in* $(x, \theta) \in (0, \infty) \times (0, \infty)$ *and is increasing in* $\theta \in (0, \infty)$ *or,*
$\psi_2(x, \theta)$ *is* TP_2 (RR_2) *in* $(x, \theta) \in (0, \infty) \times (0, \infty)$ *and is increasing in* $\theta \in (0, \infty)$;

(iii) $T_1 \leq_{\rm rh} T_2$ *and*
$\psi_1(x, \theta)$ *is* TP_2 (RR_2) *in* $(x, \theta) \in (0, \infty) \times (0, \infty)$ *and is decreasing in* $\theta \in (0, \infty)$ *or,*
$\psi_2(x, \theta)$ *is* TP_2 (RR_2) *in* $(x, \theta) \in (0, \infty) \times (0, \infty)$ *and is decreasing in* $\theta \in (0, \infty)$.

Then, *the function* $\psi(x)$, *as defined in* (2.1), *increases* (*decreases*) *in* $x \in (0, \infty)$.

3 Main results

In this section, we carry out stochastic comparisons of residual lifetimes at random times. The first result presents sufficient conditions for stochastic monotonicity in terms of the likelihood ratio order.

Theorem 3.1. *Suppose that* $\Theta_1 \leq_{\rm lr} \Theta_2$ *and that the following assumptions are fulfilled*:

(i) $f_\theta(x + \theta)$ *or* $g_\theta(x + \theta)$ *is* TP_2 (RR_2) *in* $(x, \theta) \in (0, \infty) \times (0, \infty)$;

(ii) *For every fixed* $\theta > 0$, $g_\theta(x + \theta)/f_\theta(x + \theta)$ *is increasing* (*decreasing*) *in* $x \in (0, \infty)$;

(iii) *For every fixed* $x > 0$, $g_\theta(x + \theta)/f_\theta(x + \theta)$ *is increasing in* $\theta \in (0, \infty)$.

Then, $X_{\Theta_1} \leq_{\rm lr} (\geq_{\rm lr})\ Y_{\Theta_2}$.

Proof. From (1.2), it suffices to prove that

$$\psi_1^*(x) = \frac{m_2(x)}{m_1(x)} = \frac{\int_0^\infty g_\theta(x+\theta)h_2(\theta)\,d\theta}{\int_0^\infty f_\theta(x+\theta)h_1(\theta)\,d\theta} \tag{3.1}$$

is increasing (decreasing) in $x \in (0, \infty)$. Define $\psi_1(x, \theta) = f_\theta(x + \theta)$ and $\psi_2(x, \theta) = g_\theta(x+\theta)$, $(x, \theta) \in (0, \infty) \times (0, \infty)$. Now, upon applying Lemma 2.4(i) with $T_1 \stackrel{d}{=} \Theta_1$ and $T_2 \stackrel{d}{=} \Theta_2$, it can be proved that the function $\psi_1^*(x)$ increases (decreases) in $x \in (0, \infty)$. Hence the theorem follows. □

Remark 3.2. It is useful to observe that condition (i) of Theorem 3.1, which is equivalent to saying that $f_{\theta_2}(x_2+\theta_2)f_{\theta_1}(x_1+\theta_1) \geq (\leq)\ f_{\theta_2}(x_1+\theta_2)f_{\theta_1}(x_2+\theta_1)$, for $0 < x_1 < x_2 < \infty$ and $0 < \theta_1 < \theta_2 < \infty$ is satisfied if X_θ has *DLR* (*ILR*) and $X_{\theta_1} \leq_{\mathrm{lr}} (\geq_{\mathrm{lr}})\ X_{\theta_2}, \forall\ 0 < \theta_1 \leq \theta_2$. Also, it can be easily seen that condition (ii) of Theorem 3.1 holds if $X_\theta \leq_{\mathrm{lr}} (\geq_{\mathrm{lr}})\ Y_\theta, \forall\ \theta > 0$.

The above remark leads to the following corollary.

Corollary 3.3. *Suppose that* $\Theta_1 \leq_{\mathrm{lr}} \Theta_2$ *and that the following assumptions are fulfilled*:

(i) *For every* $\theta > 0$, X_θ *has DLR* (*ILR*) *and* $X_{\theta_1} \leq_{\mathrm{lr}} (\geq_{\mathrm{lr}})\ X_{\theta_2}, \forall\ 0 < \theta_1 \leq \theta_2$; *or*,

For every $\theta > 0$, Y_θ *has DLR* (*ILR*) *and* $Y_{\theta_1} \leq_{\mathrm{lr}} (\geq_{\mathrm{lr}})\ Y_{\theta_2}, \forall\ 0 < \theta_1 \leq \theta_2$;

(ii) $X_\theta \leq_{\mathrm{lr}} (\geq_{\mathrm{lr}})\ Y_\theta, \forall\ \theta > 0$;

(iii) *Condition* (iii) *of Theorem* 3.1 *holds*.

Then, $X_{\Theta_1} \leq_{\mathrm{lr}} (\geq_{\mathrm{lr}})\ Y_{\Theta_2}$.

As a consequence of Theorem 3.1, we immediately obtain the following corollary, which compares the residual lifetimes at random times in terms of the likelihood ratio order for the case when X_θ and Y_θ are identically distributed.

Corollary 3.4. *Assume that* $X_\theta \stackrel{d}{=} Y_\theta, \forall\ \theta > 0$, $f_\theta(x+\theta)$ *is* TP_2 (RR_2) *in* $(x,\theta) \in (0,\infty)\times(0,\infty)$, *and* $\Theta_1 \leq_{\mathrm{lr}} \Theta_2$. *Then*, $X_{\Theta_1} \leq_{\mathrm{lr}} (\geq_{\mathrm{lr}})\ X_{\Theta_2}$.

It is to be noted here that when X and Θ_1 are independently distributed, and Y and Θ_2 are independently distributed, we have $X_\theta \stackrel{d}{=} X$ and $Y_\theta \stackrel{d}{=} Y$, i.e., $f_\theta \equiv f$ and $g_\theta \equiv g$, $\theta > 0$. In addition, if $X \stackrel{d}{=} Y$, then $f \equiv g$. Thus, we have the following corollary to Theorem 3.1.

Corollary 3.5. *Let* X *and* Θ_1 *be independently distributed*, *and let* Y *and* Θ_2 *be also independently distributed*.

(i) *If* X *or* Y *has DLR*, $X \leq_{lr} Y$, *and* $\Theta_1 \leq_{lr} \Theta_2$, *then* $X_{\Theta_1} \leq_{lr} Y_{\Theta_2}$.

(ii) *Let* $X \stackrel{d}{=} Y$. *If* X *has DLR* (*ILR*), *and* $\Theta_1 \leq_{lr} \Theta_2$, *then* $X_{\Theta_1} \leq_{lr} (\geq_{lr})\ X_{\Theta_2}$.

(iii) *Let* $\Theta_1 \stackrel{d}{=} \Theta_2$. *If* X *or* Y *has DLR*, *and* $X \leq_{lr} Y$, *then* $X_\Theta \leq_{lr} Y_\Theta$.

Remark 3.6. It is worth mentioning here that there are typos in Theorem 2.2(a) and Theorem 2.8(a) of Dewan and Khaledi (2014). In these theorems, ILR should be replaced by DLR, and vice-versa. Under this correction, Corollary 3.5(ii) and (iii) above are the corresponding versions of Theorem 2.2(a) and Theorem 2.8(a) of Dewan and Khaledi (2014).

In what follows, we would like to compare two residual lifetimes with respect to hazard rate ordering. In the following theorem, likelihood ratio order is assumed between Θ_1 and Θ_2 and sufficient conditions are obtained to establish hazard rate ordering between X_{Θ_1} and Y_{Θ_2}. The proof follows on using Lemma 2.4(i) with $T_1 \stackrel{d}{=} \Theta_1$, $T_2 \stackrel{d}{=} \Theta_2$, $\psi_1(x,\theta) = \bar{F}_\theta(x+\theta)$ and $\psi_2(x,\theta) = \bar{G}_\theta(x+\theta)$, $(x,\theta) \in (0,\infty) \times (0,\infty)$.

Theorem 3.7. *Suppose that* $\Theta_1 \leq_{lr} \Theta_2$ *and that the following assumptions are fulfilled*:

(i) $\bar{F}_\theta(x+\theta)$ *or* $\bar{G}_\theta(x+\theta)$ *is* TP_2 (RR_2) *in* $(x,\theta) \in (0,\infty) \times (0,\infty)$;
(ii) *For every fixed* $\theta > 0$, $\bar{G}_\theta(x+\theta)/\bar{F}_\theta(x+\theta)$ *is increasing* (*decreasing*) *in* $x \in (0,\infty)$;
(iii) *For every fixed* $x > 0$, $\bar{G}_\theta(x+\theta)/\bar{F}_\theta(x+\theta)$ *is increasing in* $\theta \in (0,\infty)$.

Then, $X_{\Theta_1} \leq_{\mathrm{hr}} (\geq_{\mathrm{hr}})\ Y_{\Theta_2}$.

Remark 3.8. It needs to be mentioned that condition (i) of Theorem 3.7 is satisfied if, $\forall\ \theta > 0$, X_θ has *DFR* (*IFR*) and $X_{\theta_1} \leq_{\mathrm{hr}} (\geq_{\mathrm{hr}})\ X_{\theta_2}$, $\forall\ 0 < \theta_1 \leq \theta_2$. Also, it can be easily seen that condition (ii) of Theorem 3.7 holds if $X_\theta \leq_{\mathrm{hr}} (\geq_{\mathrm{hr}})\ Y_\theta$, $\forall\ \theta > 0$. As a consequence, we obtain the following result for hazard rate ordering between X_{Θ_1} and Y_{Θ_2}.

Corollary 3.9. *Suppose that* $\Theta_1 \leq_{lr} \Theta_2$ *and that the following assumptions are fulfilled*:

(i) *For every* $\theta > 0$, X_θ *has DFR* (*IFR*) *and* $X_{\theta_1} \leq_{\mathrm{hr}} (\geq_{\mathrm{hr}})\ X_{\theta_2}$, $\forall\ 0 < \theta_1 \leq \theta_2$ *or*,
For every $\theta > 0$, Y_θ *has DFR* (*IFR*) *and* $Y_{\theta_1} \leq_{\mathrm{hr}} (\geq_{\mathrm{hr}})\ Y_{\theta_2}$, $\forall\ 0 < \theta_1 \leq \theta_2$;
(ii) $X_\theta \leq_{\mathrm{hr}} (\geq_{\mathrm{hr}})\ Y_\theta$, $\forall\ \theta > 0$;
(iii) *Condition* (iii) *of Theorem* 3.7 *holds*.

Then, $X_{\Theta_1} \leq_{\mathrm{hr}} (\geq_{\mathrm{hr}})\ Y_{\Theta_2}$.

It is of interest to know whether conclusions of Theorem 3.7 may still hold if Θ_1 and Θ_2 are ordered with respect to hazard rate or reversed hazard rate order. The following theorem gives an answer.

Theorem 3.10. *Suppose that conditions* (ii) *and* (iii) *of Theorem* 3.7 *hold true*. *In addition*, *suppose that either of the following two assumptions are fulfilled*:

(i) $\Theta_1 \leq_{\mathrm{hr}} \Theta_2$ *and*
$\bar{F}_\theta(x+\theta)$ *is* TP_2 (RR_2) *in* $(x,\theta) \in (0,\infty) \times (0,\infty)$ *and is increasing in* $\theta \in (0,\infty)$ *or*,
$\bar{G}_\theta(x+\theta)$ *is* TP_2 (RR_2) *in* $(x,\theta) \in (0,\infty) \times (0,\infty)$ *and is increasing in* $\theta \in (0,\infty)$;

(ii) $\Theta_1 \leq_{rh} \Theta_2$ *and*

$\bar{F}_\theta(x+\theta)$ *is* TP_2 (RR_2) *in* $(x,\theta) \in (0,\infty) \times (0,\infty)$ *and is decreasing in* $\theta \in (0,\infty)$ *or*,

$\bar{G}_\theta(x+\theta)$ *is* TP_2 (RR_2) *in* $(x,\theta) \in (0,\infty) \times (0,\infty)$ *and is decreasing in* $\theta \in (0,\infty)$.

Then, $X_{\Theta_1} \leq_{\mathrm{hr}} (\geq_{\mathrm{hr}})\ Y_{\Theta_2}$.

Remark 3.11. It is useful to observe that when the assumption of stronger stochastic order between Θ_1 and Θ_2 (i.e., $\Theta_1 \leq_{lr} \Theta_2$) in Theorem 3.7 is replaced by a weaker stochastic ordering $(\Theta_1 \leq_{\mathrm{hr}} (\leq_{rh})\ \Theta_2)$ in Theorem 3.10, then an extra condition on $\bar{F}_\theta(x+\theta)$ or $\bar{G}_\theta(x+\theta)$ is required for the same result to hold. Thus, there exists a trade-off between these two set of conditions.

The following corollary, which readily follows from Theorem 3.7 and Theorem 3.10, explains how one can compare X_{Θ_1} and X_{Θ_2} in terms of hazard rate ordering based on the likelihood ratio, hazard rate and reversed hazard order between Θ_1 and Θ_2.

Corollary 3.12. *Assume that* $X_\theta \stackrel{d}{=} Y_\theta, \forall\, \theta > 0$. *Further suppose that either of the following three assumptions are fulfilled*:

(i) $\Theta_1 \leq_{lr} \Theta_2$ *and* $\bar{F}_\theta(x+\theta)$ *is* TP_2 (RR_2) *in* $(x,\theta) \in (0,\infty) \times (0,\infty)$;

(ii) $\Theta_1 \leq_{\mathrm{hr}} \Theta_2$ *and* $\bar{F}_\theta(x+\theta)$ *is* TP_2 (RR_2) *in* $(x,\theta) \in (0,\infty) \times (0,\infty)$ *and is increasing in* $\theta \in (0,\infty)$;

(iii) $\Theta_1 \leq_{rh} \Theta_2$ *and* $\bar{F}_\theta(x+\theta)$ *is* TP_2 (RR_2) *in* $(x,\theta) \in (0,\infty) \times (0,\infty)$ *and is decreasing in* $\theta \in (0,\infty)$.

Then, $X_{\Theta_1} \leq_{\mathrm{hr}} (\geq_{\mathrm{hr}})\ X_{\Theta_2}$.

As an immediate consequence of Theorem 3.10(ii), we have the following corollary.

Corollary 3.13. *Assume that* X *and* Θ_1 *are independently distributed*, *and* Y *and* Θ_2 *are also independently distributed.*

(i) *If* X *or* Y *has DFR*, $X \leq_{\mathrm{hr}} Y$, *and* $\Theta_1 \leq_{rh} \Theta_2$, *then* $X_{\Theta_1} \leq_{\mathrm{hr}} X_{\Theta_2}$.

(ii) *Let* $X \stackrel{d}{=} Y$. *If* X *has DFR* (*IFR*), *and* $\Theta_1 \leq_{rh} \Theta_2$, *then* $X_{\Theta_1} \leq_{\mathrm{hr}} (\geq_{\mathrm{hr}})\ X_{\Theta_2}$.

(iii) *Let* $\Theta_1 \stackrel{d}{=} \Theta_2$. *If* X *or* Y *has DFR*, *and* $X \leq_{\mathrm{hr}} Y$, *then* $X_\Theta \leq_{\mathrm{hr}} Y_\Theta$.

Remark 3.14. Misra, Gupta and Dhariyal (2008) and Dewan and Khaledi (2014) presented the result of Corollary 3.13(ii) as Theorem 3.1 and Theorem 2.8(c), respectively. Dewan and Khaledi (2014) also reported the result of Corollary 3.13(iii) as Theorem 2.2 (c).

Now, we provide results concerning comparison of residual lifetimes with respect to mean residual life order. The following result follows on using Lemma 2.4(i) with $T_1 \stackrel{d}{=} \Theta_1$, $T_2 \stackrel{d}{=} \Theta_2$, $\psi_1(x,\theta) = \int_x^\infty \bar{F}_\theta(u+\theta)\,du$ and $\psi_2(x,\theta) = \int_x^\infty \bar{G}_\theta(u+\theta)\,du$, $(x,\theta) \in (0,\infty) \times (0,\infty)$.

Theorem 3.15. *Suppose that* $\Theta_1 \leq_{lr} \Theta_2$ *and that the following assumptions hold*:

(i) $\int_x^\infty \bar{F}_\theta(u+\theta)\,du$ *or* $\int_x^\infty \bar{G}_\theta(u+\theta)\,du$ *is* TP_2 (RR_2) *in* $(x,\theta) \in (0,\infty) \times (0,\infty)$;

(ii) *For every fixed* $\theta > 0$, $\int_x^\infty \bar{G}_\theta(u+\theta)\,du / \int_x^\infty \bar{F}_\theta(u+\theta)\,du$ *is increasing* (*decreasing*) *in* $x \in (0,\infty)$;

(iii) *For every fixed* $x > 0$, $\int_x^\infty \bar{G}_\theta(u+\theta)\,du / \int_x^\infty \bar{F}_\theta(u+\theta)\,du$ *is increasing in* $\theta \in (0,\infty)$.

Then, $X_{\Theta_1} \leq_{mrl} (\geq_{mrl})\ Y_{\Theta_2}$.

Remark 3.16. It should be noted here that condition (i) holds true if X_θ has *IMRL* (*DMRL*), $\forall\ \theta > 0$ and $X_{\theta_1} \leq_{mrl} (\geq_{mrl})\ X_{\theta_2}$, $\forall\ 0 < \theta_1 \leq \theta_2$. In addition, it can be readily seen that condition (ii) is satisfied if $X_\theta \leq_{mrl} (\geq_{mrl})\ Y_\theta$, $\forall\ \theta > 0$. As a consequence, we obtain the following corollary to Theorem 3.15.

Corollary 3.17. *Suppose that* $\Theta_1 \leq_{lr} \Theta_2$ *and that the following assumptions hold*:

(i) *For every* $\theta > 0$, X_θ *has IMRL* (*DMRL*) *and* $X_{\theta_1} \leq_{mrl} (\geq_{mrl})\ X_{\theta_2}$, $\forall\ 0 < \theta_1 \leq \theta_2$ *or*,
For every $\theta > 0$, Y_θ *has IMRL* (*DMRL*) *and* $Y_{\theta_1} \leq_{mrl} (\geq_{mrl})\ Y_{\theta_2}$, $\forall\ 0 < \theta_1 \leq \theta_2$;

(ii) $X_\theta \leq_{mrl} (\geq_{mrl})\ Y_\theta$, $\forall\ \theta > 0$;

(iii) *condition* (iii) *of Theorem* 3.15 *holds*.

Then, $X_{\Theta_1} \leq_{mrl} (\geq_{mrl})\ Y_{\Theta_2}$.

The following theorem, provides some sufficient conditions for mean residual life order between X_{Θ_1} and Y_{Θ_2} under the assumption of hazard rate (reversed hazard rate) order between Θ_1 and Θ_2

Theorem 3.18. *Suppose that conditions* (ii) *and* (iii) *of Theorem* 3.15 *hold true*. *In addition*, *suppose that either of the following two assumptions are fulfilled*:

(i) $\Theta_1 \leq_{\mathrm{hr}} \Theta_2$ *and*
$\int_x^\infty \bar{F}_\theta(u+\theta)\,du$ *is* TP_2 (RR_2) *in* $(x,\theta) \in (0,\infty) \times (0,\infty)$ *and is increasing in* $\theta \in (0,\infty)$ *or*,
$\int_x^\infty \bar{G}_\theta(u+\theta)\,du$ *is* TP_2 (RR_2) *in* $(x,\theta) \in (0,\infty) \times (0,\infty)$ *and is increasing in* $\theta \in (0,\infty)$;

(ii) $\Theta_1 \leq_{rh} \Theta_2$ *and*
$\int_x^\infty \bar{F}_\theta(u+\theta)\,du$ *is* TP_2 (RR_2) *in* $(x,\theta) \in (0,\infty) \times (0,\infty)$ *and is decreasing in* $\theta \in (0,\infty)$ *or,*
$\int_x^\infty \bar{G}_\theta(u+\theta)\,du$ *is* TP_2 (RR_2) *in* $(x,\theta) \in (0,\infty) \times (0,\infty)$ *and is decreasing in* $\theta \in (0,\infty)$.

Then, $X_{\Theta_1} \leq_{mrl} (\geq_{mrl})\ Y_{\Theta_2}$.

Proof. The proof of the theorem follows from Lemma 2.4(ii) and (iii) with $T_1 \stackrel{d}{=} \Theta_1$, $T_2 \stackrel{d}{=} \Theta_2$, $\psi_1(x,\theta) = \int_x^\infty \bar{F}_\theta(u+\theta)\,du$ and $\psi_2(x,\theta) = \int_x^\infty \bar{G}_\theta(u+\theta)\,du$, $(x,\theta) \in (0,\infty) \times (0,\infty)$. □

Remark 3.19. As discussed in Remark 3.11, there exists a trade-off between conditions of Theorem 3.15 and Theorem 3.18.

Upon applying an argument similar to that used in Theorem 3.15 and Theorem 3.18, an analogue of Corollary 3.12 on the mean residual life order can also be established.

Corollary 3.20. *Assume that* $X_\theta \stackrel{d}{=} Y_\theta$, $\forall\ \theta > 0$. *Further, suppose that either of the following three assumptions are fulfilled*:

(i) $\Theta_1 \leq_{lr} \Theta_2$ *and* $\int_x^\infty \bar{F}_\theta(u+\theta)\,du$ *is* TP_2 (RR_2) *in* $(x,\theta) \in (0,\infty) \times (0,\infty)$;

(ii) $\Theta_1 \leq_{\mathrm{hr}} \Theta_2$ *and* $\int_x^\infty \bar{F}_\theta(u+\theta)\,du$ *is* TP_2 (RR_2) *in* $(x,\theta) \in (0,\infty) \times (0,\infty)$ *and is increasing in* $\theta \in (0,\infty)$;

(iii) $\Theta_1 \leq_{rh} \Theta_2$ *and* $\int_x^\infty \bar{F}_\theta(u+\theta)\,du$ *is* TP_2 (RR_2) *in* $(x,\theta) \in (0,\infty) \times (0,\infty)$ *and is decreasing in* $\theta \in (0,\infty)$.

Then, $X_{\Theta_1} \leq_{mrl} (\geq_{mrl})\ X_{\Theta_2}$.

As an immediate consequence of Theorem 3.18(ii), we have the following corollary.

Corollary 3.21. *Assume that* X *and* Θ_1 *are independently distributed, and that* Y *and* Θ_2 *are also independently distributed.*

(i) *If* X *or* Y *has IMRL,* $X \leq_{mrl} Y$, *and* $\Theta_1 \leq_{rh} \Theta_2$, *then* $X_{\Theta_1} \leq_{mrl} Y_{\Theta_2}$.

(ii) *Let* $X \stackrel{d}{=} Y$. *If* X *has IMRL (DMRL), and* $\Theta_1 \leq_{rh} \Theta_2$, *then* $X_{\Theta_1} \leq_{mrl} (\geq_{mrl}) X_{\Theta_2}$.

(iii) *Let* $\Theta_1 \stackrel{d}{=} \Theta_2$. *If* X *or* Y *has IMRL, and* $X \leq_{mrl} Y$, *then* $X_\Theta \leq_{mrl} Y_\Theta$.

Remark 3.22. Misra, Gupta and Dhariyal (2008) and Dewan and Khaledi (2014) obtained the result of Corollary 3.21(ii) as Theorem 3.2 and Theorem 2.8(d), respectively. Dewan and Khaledi (2014) also presented the result of Corollary 3.21(iii) as Theorem 2.2(d).

4 Ageing properties

In this section, we investigate under what conditions the ageing property of X_θ is preserved for X_{Θ_1}. In the first place, we discuss a result for *DFR* property of X_{Θ_1}.

Theorem 4.1. *Suppose that, for every fixed $x > 0$ and $t > 0$, $\frac{\bar{F}_\theta(\theta+x+t)}{\bar{F}_\theta(\theta+x)}$ increases in $\theta \in (0,\infty)$, X_θ has DFR (IFR), $\forall\ \theta > 0$, and $X_{\theta_1} \leq_{\mathrm{hr}} (\geq_{\mathrm{hr}})\ X_{\theta_2}$, $\forall\ 0 < \theta_1 \leq \theta_2$. Then X_{Θ_1} has DFR (IFR).*

Proof. To prove X_{Θ_1} has *DFR* (*IFR*), we need to show that $\bar{M}_1(x)$ is log-convex (log-concave) on $(0,\infty)$, i.e., for every fixed $t > 0$, $\frac{\bar{M}_1(x+t)}{\bar{M}_1(x)}$ increases (decreases) in $x \in (0,\infty)$. Consider

$$\psi_2(x) = \frac{\bar{M}_1(x+t)}{\bar{M}_1(x)} = \frac{\int_0^\infty \bar{F}_\theta(\theta+x+t)h_1(\theta)\,d\theta}{\int_0^\infty \bar{F}_\theta(\theta+x)h_1(\theta)\,d\theta}, \qquad x \in (0,\infty).$$

Let $\psi_1(x,\theta) = \bar{F}_\theta(\theta+x)$ and $\psi_2(x,\theta) = \bar{F}_\theta(\theta+x+t)$, $(x,\theta) \in (0,\infty)\times(0,\infty)$, it can be easily observed that, for every $\theta > 0$, $\frac{\psi_2(x,\theta)}{\psi_1(x,\theta)} = \frac{\bar{F}_\theta(\theta+x+t)}{\bar{F}_\theta(\theta+x)}$ increases (decreases) in $x \in (0,\infty)$, if X_θ has *DFR* (*IFR*). Also, it can be easily shown that $\bar{F}_\theta(\theta+x)$ is TP_2 (RR_2) in $(x,\theta) \in (0,\infty)\times(0,\infty)$ if X_θ has *DFR* (*IFR*), and $X_{\theta_1} \leq_{\mathrm{hr}} (\geq_{\mathrm{hr}})\ X_{\theta_2}$, $\forall\ 0 < \theta_1 \leq \theta_2$. From these observations, we can conclude that the assumptions of Lemma 2.4(i) are satisfied under the assumptions of Theorem 4.1. Hence, $\psi_2(x)$ is increasing (decreasing) in $x \in (0,\infty)$. □

As a consequence, we can obtain the following result immediately on assuming independence between X and Θ_1 in Theorem 4.1.

Corollary 4.2. *Assume that X and Θ_1 are independently distributed and X has DFR. Then X_{Θ_1} has DFR.*

Remark 4.3. Under the assumption of independence between X and Θ_1, Yue and Cao (2000) in their Theorem 4.1 showed that, if Θ_1 has *DRFR* and X has *DFR* (*IFR*), then X_{Θ_1} also has *DFR* (*IFR*). It is apparent that the result of Corollary 4.2 requires no condition on Θ_1.

In what follows, we will present the result on ageing notion of X_{Θ_1} in terms of mean residual life function. Applying Lemma 2.4(i) with $\psi_1(x,\theta) = \int_x^\infty \bar{F}_\theta(u+\theta)\,du$ and $\psi_2(x,\theta) = \int_{x+t}^\infty \bar{F}_\theta(u+\theta)\,du$, $(x,\theta) \in (0,\infty)\times(0,\infty)$, and employing arguments similar to that of Theorem 4.1, we have the following result.

Theorem 4.4. *Suppose that, for every fixed $x > 0$ and $t > 0$, $\frac{\int_{x+t}^\infty \bar{F}_\theta(u+\theta)\,du}{\int_x^\infty \bar{F}_\theta(u+\theta)\,du}$ increases in $\theta \in (0,\infty)$, X_θ has IMRL (DMRL), $\forall\ \theta > 0$ and $X_{\theta_1} \leq_{mrl} (\geq_{mrl})\ X_{\theta_2}$, $\forall\ 0 < \theta_1 \leq \theta_2$. Then X_{Θ_1} has IMRL (DMRL).*

Yue and Cao (2000) assumed that X and Θ_1 are independently distributed, and showed that if Θ_1 has *DRFR* and X has *DMRL* (*IMRL*), then X_{Θ_1} also has *DMRL* ($IMRL$). In the following corollary, which follows from Theorems 4.4, we obtain the same result without imposing any condition on Θ_1.

Corollary 4.5. *Assume that X and Θ_1 are independently distributed and X has IMRL. Then, X_{Θ_1} has IMRL.*

5 Examples

To illustrate the usefulness of results derived in this paper, we present below examples that cannot be dealt with existing results in the literature, but where our results can be applied.

Example 5.1. For $\theta > 0$, let the r.v.s X_θ and Y_θ follow Gamma distributions with p.d.f.s,

$$f_\theta(x) = \begin{cases} \dfrac{(1+\theta^2)^\alpha}{\theta^\alpha \Gamma\alpha}\, \mathrm{e}^{-(\frac{1+\theta^2}{\theta})x} x^{\alpha-1}, & \text{if } x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$g_\theta(x) = \begin{cases} \dfrac{1}{\theta^\alpha \Gamma\alpha} \mathrm{e}^{-\frac{x}{\theta}} x^{\alpha-1}, & \text{if } x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

respectively, where $\alpha \in (0, 1)$ is a fixed shape parameter. For $0 < x_1 \le x_2 < \infty$,

$$\frac{g_\theta(x_2+\theta)}{g_\theta(x_1+\theta)} = \mathrm{e}^{-(x_2-x_1)\frac{c}{\theta}} \left(\frac{x_2+\theta}{x_1+\theta}\right)^{\alpha-1}$$

increases in $\theta > 0$. Thus, $g_\theta(x+\theta)$ is TP_2 in $(x, \theta) \in (0, \infty) \times (0, \infty)$. Also, note that for $\theta > 0$,

$$\frac{g_\theta(x+\theta)}{f_\theta(x+\theta)} = \frac{1}{(1+\theta^2)^\alpha}\, \mathrm{e}^{\theta(x+\theta)}$$

increases in $x > 0$ and $\theta > 0$. Hence, the conditions of Theorem 3.1 are satisfied for any pair of r.v.s Θ_1 and Θ_2 such that $\Theta_1 \le_{lr} \Theta_2$.

Example 5.2. Suppose that the joint density of (X, Θ_1) is

$$g_1(x, \theta) = \begin{cases} \dfrac{1}{\theta^\alpha} \dfrac{1}{\Gamma\alpha\Gamma\beta_1} \mathrm{e}^{-(\frac{x}{\theta}+\theta)} x^{\alpha-1} \theta^{\beta_1-1}, & \text{if } x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

and the joint density of (Y, Θ_2) is

$$g_2(x,\theta) = \begin{cases} \dfrac{1}{\theta^\alpha}\dfrac{1}{\Gamma\alpha\Gamma\beta_2}\mathrm{e}^{-(\frac{x}{\theta}+\theta)}x^{\alpha-1}\theta^{\beta_2-1}, & \text{if } x>0,\\ 0, & \text{otherwise,}\end{cases}$$

where $\alpha \in (0,1)$ and $0 < \beta_1 < \beta_2 < \infty$ are fixed shape parameters. Clearly, for every $\theta > 0$, $X_\theta \overset{d}{=} Y_\theta$, X_θ has p.d.f.

$$f_\theta(x) = \frac{1}{\theta^\alpha \Gamma\alpha}\mathrm{e}^{-\frac{x}{\theta}}x^{\alpha-1},\ x > 0$$

and p.d.f. of r.v. Θ_i, $i = 1, 2$ is

$$g_i(\theta) = \frac{1}{\Gamma\beta_i}\mathrm{e}^{-\theta}\theta^{\beta_i-1},\ \theta > 0.$$

It can be seen that for $0 < x_1 \le x_2 < \infty$,

$$\frac{f_\theta(x_2+\theta)}{f_\theta(x_1+\theta)} = \mathrm{e}^{(\frac{x_1-x_2}{\theta})}\left(\frac{x_2+\theta}{x_1+\theta}\right)^{\alpha-1}$$

increases in $\theta \in (0,\infty)$, $\forall\ \alpha \in (0,1)$. Also,

$$\frac{h_2(\theta)}{h_1(\theta)} = \frac{\Gamma\beta_1}{\Gamma\beta_2}\theta^{(\beta_2-\beta_1)}$$

increases in $\theta > 0$, $\forall\ \beta_2 > \beta_1$. Thus, $\Theta_1 \le_{lr} \Theta_2$ and conditions of Corollary 3.4 are satisfied.

Now, we state the following example to illustrate applications of Corollary 3.12.

Example 5.3.

(i) For $\theta > 0$, let the random variable X_θ follow Exponential Distribution with survival function

$$\bar{F}_\theta(x) = e^{-\frac{x}{\theta}},\ x > 0$$

It can be easily seen that $\bar{F}_\theta(x+\theta)$ is TP_2 in $(x,\theta) \in (0,\infty)\times(0,\infty)$ and is also increasing in $\theta > 0$. Thus, the conditions of Corollary 3.12(ii) are satisfied for any pair if r.v.s Θ_1 and Θ_2 such that $\Theta_1 \le_{\mathrm{hr}} \Theta_2$.

(ii) For $\theta > 0$, let the random variable X_θ follow Exponential Distribution with survival function

$$\bar{F}_\theta(x) = e^{-\theta x},\ x > 0$$

It can be easily seen that $\bar{F}_\theta(x+\theta)$ is RR_2 in $(x,\theta) \in (0,\infty)\times(0,\infty)$ and is also decreasing in $\theta > 0$. Thus, the conditions of Corollary 3.12(iii) are satisfied for any pair if r.v.s Θ_1 and Θ_2 such that $\Theta_1 \le_{rh} \Theta_2$.

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