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Characterizations and time-dependent association measures for bivariate Schur-constant distributions

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Abstract. Bivariate Schur-constant distributions have an important role in Bayesian analysis of lifetime data, as models possessing no-ageing property. In the present work, we obtain characterizations of bivariate Schur-constant distributions by properties of functions of random variables and reliability concepts. Various time-dependent measures are analysed and shown to be characterized by the ageing property of the marginal distribution.

1 Introduction

Let (X, Y) be a vector of nonnegative random variables with absolutely continuous survival function $\bar{F}(x, y)$ and probability density function $f(x, y)$. Then we say that (X, Y) has a Schur-constant distribution if its survival function can be written as

$$\bar{F}(x, y) = S(x + y), \qquad x, y > 0, \tag{1.1}$$

where $S(\cdot)$ is convex. Obviously, the marginal distributions of X and Y are specified by the survival functions $S(x)$ and $S(y)$, respectively. An important area in which Schur-constant models arise is Bayesian reliability analysis. Barlow and Mendel (1992) have characterized (1.1) in terms of the bivariate no-aging property

$$P(X > x + t|X > x, Y > y) = P(Y > y + t|X > x, Y > y) \tag{1.2}$$

which means that the residual lifetimes of younger and older components with the same survival history, have the same distribution. In Bayesian terminology, regardless of the ages of the components one would bet the same amount on the next increment in life of either component. The concept of Schur-constancy can also be explained in terms of majorization order. For two vectors (x_1, y_1) and (x_2, y_2) it is said that the former is less than the latter in majorization order, written as $(x_1, y_1) \preceq (x_2, y_2)$, if $x_1 + y_1 = x_2 + y_2$ and $\max(x_1, y_1) \leq \max(x_2, y_2)$. A function $p(x, y)$ is Schur-convex (concave) if for all (x_1, y_1) and (x_2, y_2),

$$(x_1, y_1) \preceq (x_2, y_2) \quad \Rightarrow \quad p(x_1, y_1) \leq (\geq) p(x_2, y_2).$$

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When $p(x, y)$ is both Schur-concave and Schur-convex, it is called a Schur-constant function. This definition also justifies the representation (1.1). Finally, (1.1) possesses Archimedean survival copula with generator as the inverse of $S(x)$ and therefore enjoys many nice properties enjoyed by members of this class of copulas. The properties of Schur-constant laws have been studied by several authors like Barlow and Mendel (1993), Caramellino and Spizzichino (1994, 1996), Bassan and Spizzichino (2005), Nelsen (2005) and Chi et al. (2009).

In the present work, we propose to study some new properties of the model (1.1). Characterization problems form an established topic in distribution theory that provide properties unique to a distribution. They are often effective tools in identifying the appropriate model and in estimation and testing. Therefore, we examine whether certain properties of Schur-constant distributions observed by Nelsen (2005) represent characterizations. While modelling data, using bivariate distributions, an important criterion to choose the relevant model is to examine whether the model nearly matches the dependence relation between X and Y. Generally scalar measures of association, time-dependent measures and dependence concepts are used for this purpose. Of these, scalar measures like Pearson's correlation coefficient, Kendall's tau etc. for Schur-constant models have been discussed by various authors like Nelsen (2005) and Chi et al. (2009). Exploiting the fact that Schur-constant bivariate distributions are represented in terms of univariate survival functions, Caramellino and Spizzichino (1994) and Spizzichino (2001) have shown that various dependence concepts like positive quadrant dependence, stochastic increase etc. can be translated in terms of the ageing properties of X. The results of similar nature in Bassan and Spizzichino (1999) and Averous and Dortet-Bernadit (2005) and Spizzichino (2010) are also valid for Schur-constant laws. In various studies on bivariate survival times, measures of association indexed by the ages of the organisms or system provide means of assessing the influence of certain important factors on their lives. See Anderson et al. (1992) for examples and a detailed discussion. Measures that are functions of the times x and y associated with a lifetime random vector (X, Y) are therefore important in survival analysis and are called time dependent measures of association. Such measures are more informative than the scalar measures. As in the case of dependence concepts, there is the possibility that the time-dependent association measures may also be related to the univariate ageing properties in view of the nature of Schur-constant laws. It appears that this aspect of Schur-constant models and the implications between time-dependent association and dependence concepts have not been investigated. In Section 3, we discuss this topic and show that monotonicity of the hazard rate or mean residual life is a necessary and sufficient condition that ensures positive or negative time-dependent association.

2 Characterizations

With reference to (X, Y) defined in the previous section, the bivariate hazard rate is given by

$$\big(h_1(x, y), h_2(x, y)\big) = -\nabla \log \bar{F}(x, y)$$

where $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ is the gradient operator Further, the mean residual life is $(r_1(x, y), r_2(x, y))$, where

$$r_1(x, y) = \frac{1}{\bar{F}(x, y)} \int_x^\infty \bar{F}(t, y)\, dt$$

and

$$r_2(x, y) = \frac{1}{\bar{F}(x, y)} \int_y^\infty \bar{F}(x, t)\, dt.$$

The corresponding concepts for the random variable X are its hazard rate $h(x) = -\frac{d \log \bar{F}(x)}{dx}$ and the mean residual life $r(x) = \frac{1}{\bar{F}(x)} \int_x^\infty \bar{F}(t)\, dt$. We also note the identities

$$h(x) = \frac{1 + r'(x)}{r(x)}$$

and

$$h_i(x, y) = \left(1 + \frac{\partial}{\partial x} r_i(x, y)\right) \Big/ r_i(x, y), \qquad i = 1, 2. \tag{2.1}$$

Since $\bar{F}(x, y) = S(x + y)$ in the Schur-constant case, it follows that

$$h_1(x, y) = -\frac{\partial}{\partial x} \log S(x + y) = h(x + y) \tag{2.2}$$

and similarly $h_2(x, y) = h(x + y)$. Also

$$r_1(x, y) = \frac{1}{S(x + y)} \int_x^\infty S(t + y)\, dt = \frac{1}{S(x + y)} \int_{x+y}^\infty S(t)\, dt = r(x + y). \tag{2.3}$$

Similarly $r_2(x, y) = r(x + y)$.

Proposition 2.1. *When (X, Y) follows Schur-constant distribution,*

$$V(Y|X = x) = \frac{2r(x)h(x) - 1}{h^2(x)}. \tag{2.4}$$

Proof.

$$\begin{aligned} E\big(Y^2|X = x\big) &= \int_0^x y^2 \frac{\partial^2 S(x + y)}{\partial\, \partial y}\, dy \\ &= -\frac{2}{f(x)} \frac{\partial}{\partial x} \int_0^\infty \int_y^\infty S(x + t)\, dt \end{aligned}$$

where $f(x)$ is the density function of X. Thus

$$\begin{aligned} E(Y^2|X=x) &= -\frac{2}{f(x)}\frac{\partial}{\partial x}\int_0^\infty S(x+y)r(x+y)\,dy \\ &= -\frac{2}{f(x)}\frac{\partial}{\partial x}\int_x^\infty S(y)r(y)\,dy \\ &= \frac{2r(x)}{h(x)}. \end{aligned} \tag{2.5}$$

From Nelsen (2005),

$$E(Y|X=x) = \frac{1}{h(x)} \tag{2.6}$$

and hence (2.4) follows from (2.5) and (2.6). □

Nelsen (2005) observed that Schur-constant distributions of (X, Y) have the following properties

(i) $U = X + Y$ and $V = \frac{X}{X+Y}$ are independent and V is uniform over (0, 1)

(ii) U is independent of $W = \frac{Y}{X}$ and W has Pareto II distribution with density function $g(w) = (1 + w)^{-2}$, $w > 0$

(iii) the joint distribution of $M = \max(X, Y)$ and $Z = \min(X, Y)$ has survival function

$$\bar{G}(m, z) = 2S(m + z) - S(2m)$$

(iv) $E(Y|X = x) = \frac{1}{h(x)}$ and $E(X|Y = y) = \frac{1}{h(y)}$.

We show that under certain conditions the converses of (i) through (iv) are also true so that these are characteristic properties of Schur-constant laws.

Proposition 2.2. *Let (X, Y) be a nonnegative random vector with continuous survival function $\bar{F}(x, y)$ and convex survival function $S(x)$ of X. If any one of the properties* (i), (ii) (*on condition that $\bar{F}$ is absolutely continuous*), (iii) *on condition that (X, Y) is exchangeable and* (iv) *on condition that (X, Y) is exchangeable, $\bar{F}$ is absolutely continuous and $\bar{F}(x, y) - S(x + y)$ has the same sign for all $x, y > 0$, is satisfied then $\bar{F}(x, y)$ is Schur-constant.*

Sketch of proof. Result (i) ((ii)) is proved by taking the transformations $X = UV$ and $Y = U(1 - V)$ ($X = \frac{U}{1+W}$, $Y = \frac{UW}{1+W}$) in the joint density function of U and V (U and W). Assuming (iii), we have

$$\bar{F}(m, z) + \bar{F}(z, m) - \bar{F}(m, m) = 2S(m + z) - S(2m).$$

Applying exchangeability of (X, Y), $\bar{F}(u, v) = S(u + v)$. When (iv) holds,

$$\int_y^\infty S(x)\,dx = \int_0^\infty \bar{F}(x, y)\,dx$$

so that from the additional condition on (iv), $\bar{F}(x, y) = S(x + y)$. □

Remark 2.3.

1. It is well known that when X and Y are independent, the independence of U and V with uniform distribution for V characterizes the univariate exponential law. Our result extends this result to Schur-constant models where X and Y are exchangeable. A similar interpretation holds for (ii) also.
2. $Z(M)$ represents the lifetime of a parallel (series) system with identical components with survival function $2S(m) - S(2m)$ $(2S(z))$. The bivariate reliability characteristics of the Schur-constant models that represent the two systems can be evaluated in terms of those of the component lives alone.
3. Characterization of bivariate distributions by regression functions is of particular interest in distribution theory (see, e.g., Kagan et al. (1973)). The hazard rate of the generalized Pareto distribution

$$\bar{F}(x) = \left(1 + \frac{ax}{b}\right)^{-(a+1)/a}, \qquad x > 0, b > 0, a > -1$$

is $h(x) = \frac{a+1}{ax+b}$. Accordingly the bivariate distribution

$$\bar{F}(x, y) = \left(1 + \frac{a}{b}(x + y)\right)^{-(a+1)/a} \tag{2.7}$$

is characterized by a linear regression function. Note that the result in (2.7) provides new characterizations of the bivariate distribution with independent exponential marginals when $a \to 0$, the bivariate Pareto with $\alpha = \frac{a}{b}$, $\beta = \frac{a+1}{a}$ when $a > 0$ and the bivariate beta with $m = -\frac{a}{b}$, $n = -\frac{a+1}{a}$ when $a < 0$.

Proposition 2.4. *The random vector (X, Y) has Schur-constant distribution if and only if any one of the following equivalent conditions are satisfied for all $x, y > 0$, provided $S(x)$ is convex.*

$$\text{(a)} \quad \big(h_1(x, y), h_2(x, y)\big) = \big(h(x + y), h(x + y)\big),$$

$$\text{(b)} \quad \big(r_1(x, y), r_2(x, y)\big) = \big(r(x + y), r(x + y)\big).$$

Proof. When (X, Y) is Schur-constant from (2.2) and (2.3) we have (a) and (b). Conversely, from the representation

$$\bar{F}(x, y) = \exp\left[-\int_0^x h_1(t_1, 0)\, dt - \int_0^y h_2(x, t)\, dt\right]$$

we have $\bar{F}(x, y) = S(x + y)$. Further, the proof of (b) follows from the identity (2.1) and (a). □

Remark 2.5.

1. The correspondence between univariate hazard rate (mean residual life) and the bivariate counter part in Proposition 2.4 will be used to establish the implications of the ageing properties of X on the measures of association between X and Y in the next section.

2. Proposition 2.4 enables the comparison of Schur-constant distributions in terms of stochastic orders relating to the marginals. Let (X_1, Y_1) and (X_2, Y_2) have Schur-constant distributions with hazard rates $(\bar{h}_1(x, y), \bar{h}_2(x, y))$ and $(\bar{k}_1(x, y), \bar{k}_2(x, y))$ respectively with marginal hazard rates of X_1 and Y_1 as $\bar{h}(x)$ and $\bar{k}(y)$. Recall that X_1 is smaller than X_2 in hazard rate order $X_1 \leq_{\text{hr}} Y_1$, iff $\bar{h}(x) \geq \bar{k}(x)$ for all $x > 0$. Likewise from Hu et al. (2003) (X_1, Y_1) is smaller than (X_2, Y_2) in weak hazard rate order, $(X_1, Y_1) \leq_{\text{whr}} (X_2, Y_2)$, iff $\bar{h}_i(x, y) \geq \bar{k}_i(x, y)$, $i = 1, 2$ for all $x, y > 0$. It now follows that

$$X_1 \leq_{\text{hr}} Y_1 \quad \Leftrightarrow \quad (X_1, Y_1) \leq_{\text{whr}} (X_2, Y_2).$$

3. A similar result exists with regard to mean residual life functions also. Let $\bar{r}(x)$ and $\bar{\mu}(x)$ denote mean residual lives of X_1 and Y_1 respectively and $(\bar{r}_1(x, y), \bar{r}_2(x, y))$ and $(\bar{\mu}_1(x, y), \bar{\mu}_2(x, y))$ the bivariate mean residual lives of (X_1, Y_1) and (X_2, Y_2). Then we say that X_1 is smaller than Y_1 in mean residual life denoted by $X_1 \leq_{\text{mrl}} Y_1$ iff $\bar{r}(x) \leq \bar{\mu}(x)$ for all x. Similarly (X_1, Y_1) is smaller than (X_2, Y_2) in bivariate mean residual life or $(X_1, Y_1) \leq_{\text{bmrl}} (X_2, Y_2)$, iff $\bar{r}_i(x, y) \leq \bar{\mu}_i(x, y)$, $i = 1, 2$. Then

$$X_1 \leq_{\text{mrl}} Y_1 \quad \Leftrightarrow \quad (X_1, Y_1) \leq_{\text{bmrl}} (X_2, Y_2).$$

Further a sufficient condition for $(X_1, Y_1) \leq_{\text{bmrl}} (X_2, Y_2)$ is that $X_1 \geq_{\text{hr}} X_2$.

3 Measures of association and dependence

In this section, we examine the relationships between some time-dependent measures of association have with the ageing properties of the marginal distribution of X. In fact, it will be shown that in the case of Schur-constant models, positive (negative) association between X and Y corresponds to the positive (negative) ageing property of X, Time-dependent measures are of importance in survival analysis, where identification of the age at which association is maximum is of special interest. They allow comparisons of variations in association overtime and also help in identification of models.

Clayton (1978) proposed

$$\theta(x, y) = \bar{F}(x, y) \frac{\partial^2 \bar{F}}{\partial x \, \partial y} \Big/ \left(\frac{\partial \bar{F}}{\partial x} \frac{\partial \bar{F}}{\partial y} \right) \tag{3.1}$$

as a measure of association between X and Y. For a detailed study of the interpretations, properties and applications of (3.1) we refer to Oakes (1989), Anderson et al. (1992) and Gupta (2003). When X and Y are positively (negatively) associated, $\theta(x, y) > (<) 1$ and $\theta = 1$ implies independence of X and Y. An alternative representing of (3.1) in terms of bivariate hazard rates is

$$\theta(x, y) = \frac{\bar{k}(x, y)}{h_1(x, y)h_2(x, y)}, \tag{3.2}$$

where

$$\bar{k}(x, y) = \frac{f(x, y)}{\bar{F}(x, y)}$$

is the scalar bivariate hazard rate of Basu (1971). Since

$$h_1(x, y) = -\frac{1}{\bar{F}}\frac{\partial \bar{F}}{\partial x},$$
$$\frac{\partial h_1}{\partial y} = h_1(x, y)h_2(x, y) - \bar{k}(x, y)$$

and hence

$$\theta(x, y) = 1 - \frac{\partial h_1}{\partial y}\Big/\big(h_1(x, y)h_2(x, y)\big). \tag{3.3}$$

Specializing to the Schur-constant case, (3.3) becomes

$$\theta(x, y) = 1 - \frac{\partial h(x + y)}{\partial x}\Big/\big(h^2(x + y)\big). \tag{3.4}$$

Proposition 3.1. *For Schur-constant distributions,*

(i) $\theta(x, y) > (<) 1$ *or* X *and* Y *are positively (negatively) associated if and only if* X *is strictly DHR (IHR)*
(ii) $\theta(x, y) = 1$ *or* X *and* Y *are independent if and only if* $h(x + y)$ *is a constant or* X *is exponential. In this case*

$$S(x + y) = \exp[-x - y], \qquad x, y > 0. \tag{3.5}$$

Example 1. (a) Let (X, Y) follow the bivariate Schur-constant Pareto law

$$S(x + y) = (1 + \alpha x + \alpha y)^{-\beta}, \qquad x, y > 0, \alpha, \beta > 0. \tag{3.6}$$

Then $S(x) = (1 + \alpha x)^{-\beta}$ and therefore from (3.4)

$$\theta(x, y) = 1 + \frac{1}{\beta} > 1.$$

The Pareto law is DHR and accordingly (X, Y) is positively associated.

(b) For the bivariate beta distribution,

$$S(x+y) = (1 - p_1 x - p_1 y)^{q_1},$$
$$0 < x < \frac{1}{p_1}, 0 < y < \frac{1 - p_1 x}{p_1}, \; p_1 > 0, q_1 > 1, \tag{3.7}$$

the marginal distribution $S(x+y) = (1 - p_1 x)^{q_1}$ is IHR. Also

$$\theta(x, y) = \frac{q_1 - 1}{q_1} < 1$$

showing that there is negative association between X and Y.

(c) Consider the bivariate Weibull distribution

$$S(x+y) = \exp[-(x+y)^{\lambda}], \qquad x, y > 0, 0 < \lambda \le 1.$$

The marginal distributions are DHR and

$$\theta(x, y) = \frac{1 - \lambda + \lambda(x+y)^{\lambda}}{\lambda(x+y)^{\lambda}} \ge 1.$$

In the above examples, the time-dependent measures in the first two cases are independent of x and y. Motivated by this property, as a point of departure, we enumerate the class of bivariate distributions for which $\theta(x, y)$ is a constant.

Proposition 3.2. *Among Schur-constant bivariate models, the only distributions for which* $\theta(x, y) = c$, *a constant, are* (3.5), (3.6) *and* (3.7).

Proof. When $\theta(x, y) = c$, from (3.4),

$$\frac{\partial}{\partial x}\left(\frac{1}{h(x+y)}\right) = 1 - c \tag{3.8}$$

which gives the solution

$$h(x+y) = \left[(1-c)x + a_1(y)\right]^{-1}.$$

By exchangeability

$$h(x+y) = \left[(1-c)y + a_1(x)\right]^{-1}$$

giving

$$(1-c)x + a_1(y) = (1-c)y + a_1(x).$$

Since the left side is linear in x, right side must also be linear in x. Similarly for y. Hence,

$$h(x+y) = \left[(1-c)(x+y)\right]^{-1}.$$

The case $c > 1$ ($c < 1$) leads to the Pareto (beta) distribution. Further $c = 1$ in (3.8) gives the bivariate exponential distributions (3.5). □

Remark 3.3. We can see that $\theta(x, y)$ has the same values for the above three distributions even if the scale parameters of X and Y are different.

For example, the bivariate Pareto model

$$\bar{F}(x, y) = (1 + \alpha_1 x + \alpha_2 y)^{-\beta} \neq S(x + y)$$

is not Schur-constant, but $\theta(x, y) = 1 + \frac{1}{\beta}$.

Bjerve and Doksum (1993) defined a local nonparametric dependence function which measures the strength of association between X and Y as a function of x called the correlation curve. The measure provides the strength of relationships in nonnormal models that reduce to the usual correlation coefficient in the normal case. It is defined as

$$\rho(x) = \frac{\sigma_1 \mu_1'(x)}{[(\sigma_1 \mu_1'(x))^2 + \sigma^2(x)]^{1/2}} \tag{3.9}$$

where $\mu_1(x) = E[Y|X = x]$, $\sigma^2(x) = V(Y|X = x)$, V standing for variance and $\sigma_1^2 = V(X)$. It may be noted $-1 \le \rho(x) \le 1$ and X and Y are independent when $\rho(x) = 0$.

With the aid of (2.2) and (2.3), the expression for $\rho(x)$ reduces to

$$\rho(x) = \sigma_1 \frac{d}{dx}\frac{1}{h(x)} \Big/ \left[\left(\sigma_1 \frac{d}{dx}\frac{1}{h(x)}\right)^2 + \frac{2r(x)h(x) - 1}{h^2(x)}\right]^{1/2},$$

on using $E(Y|X = x) = \frac{1}{h(x)}$ for Schur-constant models. Further simplification leads to

$$\rho(x) = -\sigma_1 \frac{dhx}{dx} \Big/ \left[\sigma_1^2 \left(\frac{dh(x)}{dx}\right)^2 + 2h^3(x)r(x) - h^2(x)\right]^{1/2}. \tag{3.10}$$

Recalling that $\rho(x) > (<)\, 0$ implies positive (negative) dependence, (3.10) provides the following proposition

Proposition 3.4. (a) *When X is DHR (IHR) in the strict sense, $\rho(x) > (<)\, 0$ and conversely*

(b) *In the independent case $\rho(x) = 0$ if and only if $X(Y)$ is exponential*

(c) $\theta(x, y) > 1 \Leftrightarrow$ *X is DHR* $\Leftrightarrow \rho(x) > 0, \theta(x, y) < 1 \Leftrightarrow$ *X is IHR* $\Leftrightarrow \rho(x) < 0$.

Bjerve and Doksum (1993) also suggested the dependence function named conditional correlation curve

$$\xi(x) = \frac{\sigma_1 \mu_1'(x)}{\sigma(x)}. \tag{3.11}$$

In terms of reliability functions,

$$\xi(x) = -\sigma_1 \frac{dh(x)}{dx} \Big/ \left(h(x)[2r(x)h(x) - 1]^{1/2}\right).$$

Thus there is positive (negative) dependence when X is DHR (IHR) and $\xi(x) = 0$ when X is exponential.

Remark 3.5. Unlike $\theta(x, y)$, $\rho(x)$ is not a constant for the distributions (3.6) and (3.7). For example, the Pareto model (3.6) has

$$\rho(x) = \alpha^{-1}\left[\beta^2 + (\beta+1)\beta(\beta-1)^2(\beta-2)(1+\alpha x)^2\right]^{-1/2}, \qquad \beta > 2.$$

Remark 3.6. One of the referees has pointed out that $\rho(x)$ is constant for the bivariate normal distribution and asked whether there are Schur-constant distributions for which $\rho(x)$ is a constant. When $\rho(x) = C$, a constant, equation (3.9) when simplified, reduces to a second order second degree differential equation in $r(x)$, which appears difficult to solve for a general form of $r(x)$ and hence $\bar{F}(x)$. Schur-constant models generated from distributions with tractable expressions for $r(x)$ like exponential, Pareto, half-logistic etc and also the half-normal do not possess this property.

Instead of considering the regression function, Anderson et al. (1992) employed the mean residual life in suggesting their measure of association, as

$$\phi(x, y) = \frac{r_1(x, y)}{r(x)}.$$

In the Schur-constant case,

$$\phi(x, y) = \frac{r(x+y)}{r(x)}.$$

Values of $\phi(x, y)$ very different from 1 indicate strong association between X and Y. If X and Y are positively associated as y increases $\phi(x, y)$ also should increase.

$$\phi(x, y) > 1 \quad \Leftrightarrow \quad r(x+y) > r(x) \quad \Leftrightarrow \quad X \text{ is strictly IMRL}$$

and $\phi(x, y) = 1$ iff X and Y are independently distributed as exponentials.

A second measure proposed in Anderson et al. (1992) is based on the ratio of survival functions

$$\psi(x, y) = \frac{P(X > x | Y > y)}{P(X > x)} = \frac{\bar{F}(x, y)}{S(x)S(y)}.$$

Assuming Schur-constancy for (X, Y),

$$\psi(x, y) = \frac{S(x+y)}{S(x)S(y)}$$

so that

$$\log\psi(x, y) = \log S(x+y) - \log S(x) - \log S(y).$$

Differentiating,

$$\frac{1}{\psi(x,y)}\frac{\partial}{\partial y}\psi(x,y)=h(y)-h(x+y). \tag{3.12}$$

When X and Y are independent $\psi(x,y)=1$ and large values of ψ indicates positive association. From the construction of ψ, it is evident that as y increases ψ also should increase. Hence increasing $\psi(x,y)$ is indicative of positive association. However from (3.12), $\psi(x,y)$ increasing in y, if and only if X is DHR.

Nair and Sankaran (2010) suggested the measure

$$\delta(x,y)=\frac{M(x,y)}{r_1(x,y)r_2(x,y)}, \tag{3.13}$$

where

$$M(x,y)=E\big((X-x)(Y-y)|X>x,Y>y\big)=\frac{\int_x^\infty\int_y^\infty \bar{F}(t,s)\,dt\,ds}{\bar{F}(x,y)}$$

is the product moment of residual life. By means of (3.13), the vector (X,Y) is positively (negatively) associated if $\delta(x,y)>(<)\,1$ and X and Y are independent if $\delta(x,y)=1$. Further, if (X_1,Y_1) is a nonnegative random vector specified by the density function

$$f_1(x,y)=\frac{\bar{F}(x,y)}{E(XY)},$$

then

$$\delta(x,y)=\theta_1(x,y)$$

where $\theta_1(x,y)$ is the Clayton measure (3.1) corresponding to (X_1,Y_1). Accordingly it follows from the earlier discussion that if X_1 is DHR (IHR) then (X,Y) is positively (negatively) associated and X and Y are independent if X_1 is exponential whenever the distribution of (X,Y) is Schur-constant. Further, in general, $\delta(0,0)\geq 1$ implies $\mathrm{Cov}(X,Y)\geq 0$ and thus X and Y are positively correlated.

Example 2. Let (X,Y) be distributed as bivariate Pareto (3.6). Then (X_1,Y_1) has density function

$$f_1(x,y)=\alpha^2(\beta-1)(\beta-2)(1+\alpha x+\alpha y)^{-\beta}$$

and survival function

$$\bar{F}_1(x,y)=(1+\alpha x+\alpha y)^{-\beta+2}.$$

Thus, X_1 is DHR and

$$\delta(x,y)=\frac{\beta-1}{\beta-2}=\theta_1(x,y)>1$$

and hence X and Y positively associated.

In a discussion on the properties and interrelationships between association measures, Gupta (2003) has shown that

$$\theta(x,y)>1 \quad\Rightarrow\quad \phi(x,y)>1 \quad \text{and also that}$$
$$\theta(x,y)>1 \quad\Rightarrow\quad \psi(x,y)>1.$$

From the results given above, we have a wider chain of implications for Schur-constant laws, that corresponds to ageing concepts.

Proposition 3.7. *If* (X,Y) *follows Schur-constant distribution, then*

$$\text{(a)}\quad X \text{ is DHR} \quad\Leftrightarrow\quad \theta(x,y)>1 \quad\Leftrightarrow\quad \rho(x)>0 \quad\Leftrightarrow\quad \xi(x)>0$$
$$\Leftrightarrow\quad \psi(x,y)>1 \quad\Rightarrow\quad \phi(x,y)>1.$$
$$\text{(b)}\quad X \text{ is IMRL} \quad\Leftrightarrow\quad \phi(x,y)>1.$$

Averous and Dortet-Bernadit (2005) and Spizzichino (2010) have studied the correspondence between various dependence concepts of an Archimedian copula and the ageing properties of the corresponding survival functions. Since Schur-constant models possess Archimedian copula whose generator is the inverse of S, their results along with the discussions in the present work, provide us the correspondence between some dependence properties and the various measures of association. We recall that

(i) (X,Y) is right convex set increasing (RCSI) iff $P(X>x, Y>y|X>x', Y>y')$ is nondecreasing in x', y' for all (x,y)
(ii) Y is right tail increasing in X (RTI $(Y|X)$) if $P(Y>y|X>x)$ is increasing in x for all y
(iii) Y is stochastically increasing in X for all y (SI $(Y|X)$) or positively regression dependent (PRD) iff $P(Y>y|X=x)$ is nondecreasing in x for all y
(iv) (X,Y) is positive quadrant dependent (PQD) iff $P(X>x, Y>y) \geq P(X>x)(Y>y)$
(v) Y is left-tail decreasing in X (LTD $(Y|X)$) if $P[Y\leq y|X\leq x]$ is decreasing in x for all y

From Spizzichino (2001), Averous and Dortet-Bernadit (2005) and Spizzichino (2010), we have

$$X \text{ is NWU} \quad\Leftrightarrow\quad (X,Y) \text{ is PQD},$$
$$X \text{ is DHR} \quad\Leftrightarrow\quad (X,Y) \text{ is LTD } (Y|X),$$
$$X \text{ is DHR} \quad\Rightarrow\quad \text{RTI } (X|Y).$$

Comparing these results with correspondence between ageing properties and measures of association we have the following propositions providing more general results.

Proposition 3.8. *If* (X, Y) *follows Schur-constant distribution*, *then*

(a) $\theta(x, y) > 1 \Leftrightarrow \rho(x) > 0 \Leftrightarrow \xi(x) > 0 \Leftrightarrow \psi(x, y) > 1 \Leftrightarrow (X, Y)$ *is RCSI.*
(b) $\rho(x) \geq 0 \Leftrightarrow (X, Y)$ *is PRD* (*SI* $(Y|X)$).
(c) $\psi(x, y) > 1 \Leftrightarrow (X, Y)$ *is PQD.*

Proof. Part (a) follows from Gupta (2003) who proved that $\theta(x, y) > 1 \Leftrightarrow (X, Y)$ is RCSI. If $\rho(X) \geq 0$, (X, Y) is PRD (Lai and Xie, 2006, p. 305). Conversely when (X, Y) is PRD from (2.6), X is DHR and $\rho(x) \geq 0$. The proof of (c) is obvious. □

Remark 3.9. In general among bivariate laws there is no direct relationship between SI and RCSI. Further PQD neither implies RCSI nor SI.

4 Conclusion

In the present work, we have established several characterizations of Schur-constant models that may be useful in identifying the distribution that posses the bivariate ageing property. The analysis of dependent measures make it easier to find the nature of association based on the aging property of the marginal. The correspondence between aging property and association also help the choice of appropriate Schur-constant models. While discussing time-dependent measures we have not considered the measures proposed by Holland and Wang (1987), the local dependence measure proposed in Bairamov and Kotz (2000) and developed in Bairamov et al. (2003). Although these two measures can be expressed in terms of reliability functions, the problem of relating them to aging concepts was found difficult to resolve.

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