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Optimal controllability of manpower system with linear quadratic performance index

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Abstract. In classical manpower systems analysis, control of the system usually results in a set of admissible controls. This forms the basis for the use of the concepts of optimal control to distinguish this set of admissible controls for optimality. In this paper, the concepts of classical deterministic optimal control are extended to examine the optimal controllability of manpower system modeled by stochastic differential equations in terms of the differential flow matrices for both time varying and time invariant manpower systems. Necessary and sufficient conditions for controllability are given. The Hamilton–Jacobi–Bellman (HJB) equation is used to obtain an algebraic Riccati equation for an optimal tracking linear quadratic problem in a finite time horizon. A 2-norm optimality criterion which is equivalent to a minimum effort criterion is used to obtain a 2-norm optimal control for the system. An optimal time control is also obtained.

1 Introduction

Control theory is an aspect of optimization theory concerned with the process of optimizing (minimizing cost of or maximizing reward of) a control process. The control mechanism is the value of the controlled variable which is chosen to influence the trajectory of the system and is obtained as solution of the differential equation representing the system under consideration. Each trajectory of the control has associated with it a cost functional or performance index and optimal control is to optimize this cost functional over all choices of the control variable. There is an extensive literature on optimal control both in theories and applications. Notable references in this area are Sung (2006), Deshmukh et al. (2006), Siska (2007), Ohsawa et al. (2010). Stochastic control is an extension of optimal control to systems whose dynamics is not deterministic but random. In this case, we can no longer minimize the performance index but can only hope to be able to minimize its expected value over all possible future realizations of the random process. Some references on stochastic control are Kushner and Runggaldier (1987), Yao et al. (2001), Mahmudov (2003), Klamka (2008). Klamka (2007) considers a finite-dimensional stationary dynamical control systems described by linear stochastic ordinary differential state equations with single point delay in the control. As can

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be seen in Borkar (2005), researches on stochastic control took root in the sixties as a natural sequel to the developments in the deterministic optimal control on one hand and Markov decision process on the other hand. A good example of stochastic system is a manpower system.

A manpower system is any group of people working together for the purpose of achieving the common goal of an organization. Manpower system is usually made of stocks and flows. Stocks refer to the number in the various categories of the system and flows refer to number moving within the system from one category to another (promotions or internal transfers), from the system to the outside (wastages), and from outside into the system (recruitment). Because manpower system is made of human beings, the dynamics of the stocks and flows in the system can best be described in stochastic terms (Bartholomew et al. (1991)). Manpower system analyses generally consist of two parts, namely quantitative and qualitative analyses. In quantitative analysis, interest is on the exact response of the system to certain input and initial condition, while in qualitative analysis interest is on the manipulation of the system for specific responses; controllability. Controllability of manpower system has two aspects: reachability which has to do with the process of reaching or attaining a desired structure and maintainability which deals on maintaining a desired structure once it has been reached. In the literature there are various approaches to manpower modeling and control using Markov, semi-Markov and renewal theory concepts, see for example (Glen and Yang (1996), Udom (2009)).

Modelling and control of manpower system within the Markovian framework is well developed in (Bartholomew (1982) and Uche (1984)). The Markovian manpower model is briefly reviewed here. Consider a manpower system whose members are divided into k categories, let $\bar{n}_i(t)$ denotes the expected number of people in category i at time t $(t = 0, 1, 2, \ldots)$ and $N(t) = \sum_i \bar{n}_i(t)$ be the total number expected in the system at time t. Let $\bar{R}(t)$ the expected number of recruits to the system at time t be distributed to the k-categories according to the proportion r_i with $\sum_i r_i = 1$. A member of category i moves to category j with probability P_{ij} where $\sum_j P_{ij} < 1$. Because transition of members out of the system is allowed; we denote w_i to be the probability of member in category i moving out of the system such that $\sum_j P_{ij} + w_i = 1$ and $w(t)$ the total number that left the system in t.

Using the above notations, the system can be described with the following recursive relation.

$$\bar{n}_i(t+1) = \sum_j P_{ij}\bar{n}_i(t) + \bar{R}(t+1)r_i. \tag{1.1}$$

In controlling the Markovian system, the terms recruitment control and promotion control respectively are used to describe the problems of choosing $\underline{r}$, the recruitment vector and P, the promotion matrix to control a manpower system. Let n^* be the manpower structure to be controlled, then there must exist P, $\underline{r}$ and $\underline{w}$ such that $n^* = n^*P + n^*\underline{w}'\underline{r}$. If the recruitment flow is the only flow subject to control then it means that P and $\underline{w}$ are assumed fixed and $\underline{r}$ is to be determined. We note here that it is not every structure that can be controlled; therefore an important point in manpower control is to identify those structures which can be controlled. For instance, if our interest is in the relative sizes of the categories it is important to express the system in terms of $q(t) = \frac{n_i(t)}{N(t)}$. Thus, we have

$$q(t+1) = q(t)P + q(t)\underline{w}'\underline{r}. \tag{1.2}$$

Since interest is in recruitment control, we have to find $\underline{r}$ satisfying the control equation. This $\underline{r}$ is $\underline{r} = q(t)(1-P)/q(t)\underline{w}'\underline{r}$. For the system to be controllable, the entries of $\underline{r}$ must be all positive and add up to 1, otherwise the system is not controllable.

Another way of exercising control over a manpower system is by promotion control, in which case the recruitment vector $\underline{r}$ and the wastage probabilities vector $\underline{w}$ are assumed fixed. For a fixed size organization and under stationarity condition, the problem is to find P satisfying $q = qP + q\underline{w}'\underline{r}$. Such a P must have non-negative elements with the row summing to $1 - w_i$. Uche (1984) has shown that the condition $q \geq q\underline{w}'\underline{r}$ is necessary and sufficient for the system to be controllable and for a hierarchical systems P is usually upper triangular. The problem arising here is that the P satisfying the control equation is not unique. Ossai (2008) also identified this problem in a departmentalized manpower control model. This problem of non-uniqueness of the control matrix forms the basis for the use of the concepts of optimal control to distinguish this set of admissible controls for optimality.

However, the Markovian model assumes an underlying geometric or exponential duration distribution for the completed length of service (CLS) and that transition occur at a constant rate. The renewal model is often based on an assumption of constant category size. In practice, the probability of moving from one category to another, or leaving is usually highly dependent on duration in the category and may exhibit a 'cumulative inertia' effect and the category size may not be constant (McGinnis (1968)). To overcome these, a semi-Markovian model which provides a means of taking into account variations in transition probability with duration of stay before transition to the next category has been proposed. Consider a system with categories $S_1, S_2, \ldots, S_k$ and semi-Markovian transition probabilities α_{ij}, $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, k+1$ of moving from S_i to S_j. S_{k+1} is the category of those who have left the system. Let $f(t)$, $F(t)$ and $G(t)$ be the probability density function, distribution function and survivor function, respectively of the length of time spent in category S_i before making a transition to S_j. Then $p_{ij}(t) = \Pr\{\text{in } S_j \text{ at time } t \mid \text{in } S_i \text{ at time } t = 0\}$, $P(t) = \{p_{ij}(t)\}$. These probabilities can be obtained from their Laplace transform $P^*(s) = (I - g^*(s))^{-1} H^*(s)$ where $g(t) = \{\alpha_{ij} f_{ij}(t)\}$ and $H(t) = \text{Diag}\{\sum_{r=1}^{k+1} \alpha_{ir}(1 - F_{ir}(t))\}$.

For the semi-Markovian model with Poisson recruitment at a rate λ_i to category S_i and $n_i(t)$ the number of staff in category S_i at time t, θ_i the initial number in category i, the joint probability generating function of the $n_i(t)$'s at time t is given by

$$G(z;t) = \prod_{i=1}^{k} (1 + R_i P(t) Z)\theta_i \exp\{\Lambda R(t) Z\},$$

where R_i is a vector with 1 in the ith position and 0's elsewhere,

$$Z = \{Z_i - 1\}, \qquad \Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k), \qquad r(t) = \int_0^t P(u)\,du.$$

This expression is the convolution of multinomial distribution corresponding to the current category of the initial stock and a Poisson distribution representing the current categories of recruits who have joined in $(0, t)$. For more on semi-Markovian manpower model including estimation procedure, readers are referred to McClean (1991) and the references there-in.

Very few researches can be found on optimal control of manpower system modeled by stochastic differential equation, for example, Udom and Uche (2009) uses time as an optimality performance criterion, via the Pontryagin minimum principle, to obtain an optimal recruitment control vector for a manpower system modeled by a stochastic differential equation and it was shown that this recruitment vector minimizes the control time globally. In Mouza (2010), a comparative simple dynamic system (plant) with analytical presentation of stocks and flows is adapted to the formulation of an optimal manpower control problem aiming to achieve in the most satisfactory way, some pre-assigned manpower targets. The work presented a method of solution of the formulated manpower control problem based on the use of the generalized inverse. Other references on optimal control are: (Lin and Wang (2011), Poggiolini and Spadini (2011), Federico et al. (2010), Hermant (2009)).

In this paper, the concepts in classical optimal deterministic control are extended to examine the optimal controllability of manpower system modeled by stochastic differential equations, in terms of differential flow matrices for both time varying and time invariant manpower systems. An infinitesimal version of dynamic programming principle, the Hamilton–Jacobi–Bellman (HJB) equation is used to obtain an algebraic Riccati equation for an optimal tracking linear quadratic problem in a finite time horizon. A 2-norm optimality criterion which is equivalent to a minimum effort criterion is used to obtain a 2-norm optimal control for the system. The major advantage of modeling manpower using stochastic differential equation over the Markovian, semi-Markovian and renewal models is that control parameter is time dependent hence, optimization of control can be with respect to time, whereas this is not possible in the case of Markovian, semi-Markovian and renewal models.

2 System description and controllability

As a result of the shortcomings of the Markovian, semi-Markovian and renewal models in manpower control, we propose an n-grade manpower system whose structural form can be represented by the following stochastic differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + \varphi(t), \tag{2.1}$$

where $x(t)$ is an $n \times 1$ structural vector, $u(t)$ is an $n \times 1$ control probability vector and $A(t)$ and $B(t)$ are $n \times n$ differential flow matrices with integrable elements and $\varphi(t)$ is an $n \times 1$ random error vector with expectation equal to zero and variance $\sigma_{\varphi(t)}$. Let Ω and U be the structural and control spaces, respectively being subsets of R^n and let (Ω, F, P) be a complete probability space with probability measure P on Ω and a filtration $\{F_t | t \in (t_0, T)\}$ generated by n-dimensional random process $\{\varphi(t) : t_0 \leq t \leq t_1\}$ defined on the probability space (Ω, F, P).

It can be shown, using method of variation of parameters that the expected value of $x(t)$ in (2.1) can be expressed in an integral convolution form as

$$\begin{aligned} E\bigl(x(t)\bigr) &= \phi(t; x_0, u) = \Phi(t, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t, \tau)B(\tau)u(\tau)\,d\tau \\ &= \Phi(t, t_0)\left[x_0 + \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)u(\tau)\,d\tau\right], \end{aligned} \tag{2.2}$$

where $\Phi(t, t_0)$ is a nonsingular state transition matrix of $\dot{x} = A(t)x$, defined by $\Phi(t, t_0) = \exp[A(t - t_0)]$ and having the following properties:

$$\begin{aligned} \frac{d}{dt}\Phi(t, t_0) &= A\Phi(t, t_0), \\ \Phi(t, t) &= I, \\ \Phi(t, t_0) &= \Phi^{-1}(t_0, t), \\ \Phi(t_0, t) &= \Phi(t_0, t_1)\Phi(t_1, t). \end{aligned}$$

$\Phi(t, t_0)$ is known as the fundamental matrix.

Evaluation of $\exp(At)$ when all the eigenvalues of A are distinct can be achieved by Sylvester's formula (Barnett (1975)).

Definition 1 (Complete controllability). The system is said to be completely controllable at time t_0, if there exist a finite time $t_1 > t_0$ such that for any $x(t_0)$ and a desired structure $x^*(t_1)$ in the structural space Ω, there exist a control input $u_{(t_0,t_1)} \in U$ that will transfer the system from $x(t_0)$ to $x^*(t_1)$. Otherwise, the system is not controllable.

This definition requires that the control input $u_{(t_0,t_1)} \in U$ be capable of moving the system from any point in the space to the desired point in a finite time t_1, which trajectory it should take is not specified.

Definition 2 (Differential controllability). The system is said to be differentially controllable at time t_0, if there exist a finite time $t_1 > t_0$ such that for any $x(t_0)$ and a desired structure $x^*(t_1)$ in the structural space Ω, there exist a control input $u_{(t_0,t_1)} \in U$ that will transfer the system from $x(t_0)$ to $x^*(t_1)$ in an arbitrary infinitesimally small interval of time.

Clearly, differential controllability implies complete controllability.

Definition 3 (Uniform controllability). A system is said to be uniformly controllable if and only if there exist a positive value σ_u and α_i that depends on σ_u such that

$$0 < \alpha_1(\sigma_u)I \le W(t, t+\sigma_u) \le \alpha_2(\sigma_u)I$$

and

$$0 < \alpha_3(\sigma_u)I \le \Phi(t+\sigma_u, t)W(t, t+\sigma_u)\Phi^*(t+\sigma_u, t) \le \alpha_4(\sigma_u)I$$

for all t in $(-\infty, \infty)$, where Φ is the state transition matrix and W is defined as

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)B^*(\tau)\Phi^*(t, \tau)\,d\tau.$$

Proposition 1. *The system* (2.1) *is completely controllable at* t_0 *if and only if there exist a finite* $t_1 > t_0$ *such that the* n *rows of the* $n \times n$ *matrix function* $\Phi(t_0, t_1)B(t_1)$ *are linearly independent on* (t_0, t_1).

Proof.

Sufficiency: If the rows of $\Phi(t_0, t_1)B(t_1)$ are linearly independent on, (t_0, t_1) then $n \times n$ constant matrix $W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)B^*(\tau)\Phi^*(t, \tau)\,d\tau$ is nonsingular. Given any $x(t_0) = x_0$ and any x_1^*, we claim that the control input

$$u_{(t_0,t_1)} = -B^*(t)\Phi^*(t_0, t_1)W^{-1}(t_0, t_1)\big[x_0 - \Phi(t_0, t_1)x_1\big] \tag{2.3}$$

will transfer $x(t_0) = x_0$ to x_1^* at time t_1. Indeed, by substituting (2.3) into (2.2), we have

$$\begin{aligned} E\big(x(t_1)\big) &= \Phi(t_1, t_0)\bigg[x_0 - \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)B^*(t)\Phi^*(t_0, \tau) \\ &\qquad \times W^{-1}(t_0, t_1)\big[x_0 - \Phi(t_0, t_1)x_1\big]\,d\tau\bigg] \\ &= \Phi(t_1, t_0)\big[x_0 - W(t_0, t_1)W^{-1}(t_0, t_1)\big[x_0 - \Phi(t_0, t_1)x_1\big]\big] \\ &= \Phi(t_1, t_0)\Phi(t_0, t_1)x_1 \\ &= x_1 \end{aligned}$$

which implies that the system is controllable.

Necessity: Suppose the system is controllable at t_0, but the rows of $\Phi(t_0, t)B(t)$ are linearly dependent on (t_0, t_1) for $t_1 > t_0$. Then there exist a nonzero constant vector ν such that

$$\nu\Phi(t_0, t)B(t) = 0 \qquad \text{for all } t \text{ in } (t_0, t_1). \tag{2.4}$$

Let $x(t_0) = x_0 = \nu^*$ then we have

$$\Phi(t_0, t_1)x(t_1) = \nu^* + \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)u(\tau)\,d\tau. \tag{2.5}$$

Premultiplying both sides of (2.5) by ν, we have

$$\nu\Phi(t_0, t_1)x(t_1) = \nu\nu^* + \int_{t_0}^{t_1} \nu\Phi(t_0, \tau)B(\tau)u(\tau)\,d\tau. \tag{2.6}$$

But the system is controllable at t_0; hence for any point in the structural space, there exist a $u_{(t_0,t_1)}$ such that $x(t_1) = 0$ since $\Phi(t_0, t_1)B(t) = 0$ for all t in (t_0, t_1) (2.6) reduces to $\nu\nu^* = 0 \Rightarrow \nu = 0$. This is a contradiction. □

Notice that the usefulness of this proposition depends on the fundamental matrix Φ or the state transition matrix $\Phi(t, \tau)$ of $\dot{x} = A(t)x$. This is generally not easy. Hence in the following proposition, we give a controllability criterion in terms of the system flow matrices $A(t)$ and $B(t)$ with the additional assumption that $A(t)$ and $B(t)$ are $n - 1$ times differentiable.

Proposition 2. *Assume that the system flow matrices $A(t)$ and $B(t)$ are $n - 1$ times differentiable, then the system is controllable if there exist a finite $t_1 > t_0$ such that*

$$\text{Rank}\big[M_0(t_1) \vdots M_1(t_1) \vdots \cdots \vdots M_{n-1}(t_1)\big] = n,$$

where the sequence of $n \times n$ matrices $M_0(t_1), M_1(t_1), \ldots, M_{n-1}(t_1)$ is defined by

$$M_{r+1}(t) = -A(t)M_r(t) + \frac{d}{dt}M_r(t), \qquad r = 0, 1, 2, \ldots, n-1$$

with $M_0(t) = B(t)$,

$$\Phi(t_0, t)B(t) = \Phi(t_0, t)M_0(t),$$
$$\frac{d}{dt}\Phi(t_0, t)B(t) = \Phi(t_0, t)M_1(t).$$

Generally, $\frac{d^r}{dt^r}\Phi(t_0, t)B(t) = \Phi(t_0, t)M_r(t)$.

Proof. Let

$$\frac{\partial}{\partial t}\Phi(t_0, t)B(t)\bigg|_{t=t_1} = \frac{\partial}{\partial t}\Phi(t_0, t_1)B(t_1).$$

Then

$$\left[\Phi(t_0,t_1)B(t_1)\vdots\frac{\partial}{\partial t_1}\Phi(t_0,t_1)B(t_1)\vdots\cdots\vdots\frac{\partial^{n-1}}{\partial t_1^{n-1}}\Phi(t_0,t_1)B(t_1)\right]$$

$$=\Phi(t_0,t_1)\big[M_0(t_1)\vdots M_1(t_1)\vdots\cdots\vdots M_{n-1}(t_1)\big].$$

Since we know that $\Phi(t_0,t_1)$ is nonsingular, the claim that $\mathrm{Rank}[M_0(t_1)\vdots M_1(t_1)\vdots \cdots\vdots M_{n-1}(t_1)]=n$ implies that

$$\mathrm{Rank}\left[\Phi(t_0,t_1)B(t_1)\vdots\frac{\partial}{\partial t_1}\Phi(t_0,t_1)B(t_1)\vdots\cdots\vdots\frac{\partial^{n-1}}{\partial t_1^{n-1}}\Phi(t_0,t_1)B(t_1)\right]=n.$$

It follows from Proposition 1 that the rows of $\Phi(t_0,t_1)B(t_1)$ are linearly independent on (t_0,t_1) for any $t_1>t_0$. Therefore, the system is controllable. □

Remark. Following the presentation in Chen (1984), it can be proved that for a time invariant manpower system, $\dot{x}=Ax(t)+Bu(t)+\varphi$, controllability can be established if and only if any of the following equivalent statements are satisfied:

1. All rows of $\exp(-At)B$ (and consequently $\exp(At)B$) are linearly independent on $[0,\infty)$.
2. The $n\times nn$ controllability matrix $U=[B\vdots AB\vdots A^2B\vdots\cdots\vdots A^{n-1}B]$ has rank n.
3. For every eigenvalue λ_i of A, the $n\times 2n$ complex matrix $[\lambda_i I-A\vdots B]$ has rank n.
4. The controllability Grammian $W_g=\int_0^t \exp(A\tau)BB^*\exp(A^*\tau)\,d\tau$ is positive definite.

3 Optimal control of the system

For a system that is controllable, there are generally many different control inputs $u_{(t_0,t_1)}\in U$ that can transfer the system from $x(t_0)$ to $x^*(t_1)$, since the trajectory between $x(t_0)$ and $x^*(t_1)$ is not specified. This is also the case even when the trajectory is specified. Among these possible admissible control inputs that may achieve the same mission, interest may be on which control input is optimal according to some priori criteria, because what is optimal depends on the optimality criterion used. Our main purpose here therefore, is to obtain optimal control base on some apriori optimality criteria to be defined.

3.1 2-norm optimality

Here, the desired structure is to be reached with minimum total expenditure of control effort. This is equivalent to minimum effort control (Klamka (2007)). The 2-norm optimality criterion has the following form $J(u)=\int_{t_0}^{t_1}\|u(t)\|^2\,dt$.

Proposition 3. *Let $u^*(t)$ be a control input defined by*

$$u^*(t) = \big(\Phi(t_0, t)B(t)\big)^* W^{-1}(t_0, t)\big[\Phi(t_0, t)x_1 - x_0\big] \quad \text{for all } t \text{ in } (t_0, t_1). \tag{3.1}$$

And let $u(t)$ be any other control input capable of transferring the system from $x(t_0)$ to $x^*(t_1)$, then $\int_{t_0}^{t_1} \|u(t)\|^2\, dt \geq \int_{t_0}^{t_1} \|u^*(t)\|^2\, dt$.

Proof. The expectation of the system structural equation at t_1 is

$$E\big(x(t_1)\big) = \Phi(t_1, t_0)\left[x_0 + \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)u(\tau)\, d\tau\right]. \tag{3.2}$$

Define $\bar{x} = \Phi^{-1}(t_1, t_0)x(t_1) - x(t_0) = \Phi(t_0, t_1)x(t_1) - x(t_0)$.

Then the assumptions that $u^*(t)$ and $u(t)$ transfer $x(t_0)$ to $x(t_1)$ in finite time $t_1 > t_0$ imply that

$$\begin{aligned}
\bar{x} &= \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)u^*(\tau)\, d\tau = \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)u(\tau)\, d\tau \\
\Rightarrow &\quad \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)\big[u(\tau) - u^*(\tau)\big]\, d\tau = 0 \\
\Rightarrow &\quad \left\langle \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)\big[u(\tau) - u^*(\tau)\big]\, d\tau, W^{-1}(t_0, t_1)\bar{x} \right\rangle = 0.
\end{aligned}$$

Which yields

$$\int_{t_0}^{t_1} \big\langle \big[u(\tau) - u^*(\tau)\big], \big(\Phi(t_0, \tau)B(\tau)\big)^* W^{-1}(t_0, t_1)\bar{x} \big\rangle\, d\tau = 0. \tag{3.3}$$

With the use of equation (3.1), (3.3) becomes

$$\int_{t_0}^{t_1} \big\langle \big[u(\tau) - u^*(\tau)\big], u^*(\tau) \big\rangle\, d\tau = 0. \tag{3.4}$$

It is easy to see that

$$\begin{aligned}
\int_{t_0}^{t_1} \|u(\tau)\|^2\, d\tau &= \int_{t_0}^{t_1} \|u(t) - u^*(t) + u^*(t)\|^2\, d\tau \\
&= \int_{t_0}^{t_1} \|u(t) - u^*(t)\|^2\, d\tau + \int_{t_0}^{t_1} \|u^*(t)\|^2\, d\tau \\
&\quad + 2\int_{t_0}^{t_1} \big\langle u(t) - u^*(t), u^*(t) \big\rangle\, d\tau \\
&= \int_{t_0}^{t_1} \|u(t) - u^*(t)\|^2\, d\tau + \int_{t_0}^{t_1} \|u^*(t)\|^2\, d\tau.
\end{aligned}$$

Since $\int_{t_0}^{t_1} \|u(t) - u^*(t)\|^2\, d\tau$ is always nonnegative, we conclude that

$$\int_{t_0}^{t_1} \|u(t)\|^2\, dt \geq \int_{t_0}^{t_1} \|u^*(t)\|^2\, dt.$$

□

3.2 Minimum time optimality

Here we examine the control strategy that will transfer the system from an initial structure $x(t)$ to a desired structure $x^*(t)$ in the shortest possible time. This will be done by minimizing a time performance index $J[x(t), u(t)]$ within the specified interval $t_0 \leq t \leq t_1$ where $x(t)$ is the structure of the system at time t and $u(t)$ is the control variable at time t. t_1 is the first instant of time at which the desired state $x^*(t)$ is reached. This is achieved by applying the Pontryagin theorem to a class of problem in which the cost functional is a quadratic time performance index in the structural and control variables. Specifically, the problem is to find the control $u(t)$ that can control the system represented by (2.1) during the time interval $t_0 \leq t \leq t_1$ from an initial state $x(t_0)$ such that

$$J(x, u; t) = E\left(\frac{1}{2}x^T(t_1)Sx(t_1) + \frac{1}{2}\int_{t_0}^{t_1}[x^T Px + x^T Qu + u^T Ru]\,dt\right)$$

is minimized,

where the weighting matrices P, Q, R and S are assumed to be positive definite.

The Hamiltonian for this problem is

$$H' = -\tfrac{1}{2}p_{ij}x_i x_j - q_{ik}x_i u_k - \tfrac{1}{2}r_{ks}u_k u_s + \lambda(a_{ij}x_j + b_{ik}u_k).$$

Applying the Pontryagin principle, we have that the λ_i must satisfy

$$\dot{\lambda}_i = -\frac{\partial H'}{\partial x_i} = p_{ij}x_j + q_{ik}u_k - \lambda_i a_{ji}.$$

To maximize H' it is necessary that $\frac{\partial H'}{\partial u_k} = -q_{ik}x_i - r_{ks}u_s + \lambda_i b_{ik} = 0$, since R is positive definite, the critical point is a maxima.

Therefore, we can have

$$\dot{\lambda} = Px + Qu - A^T\lambda. \tag{3.5}$$

Since the Hamiltonian H' is to be maximized, it is required that $u = u^*$, where

$$\begin{aligned} &-Q^T x - Ru^* + B^T\lambda = 0 \\ \Rightarrow\quad &u^* = -R^{-1}Q^T x + R^{-1}B^T\lambda. \end{aligned} \tag{3.6}$$

To solve for the unknown λ, we substituted the value of $u^*(t)$ in the structure and costructure equations and we obtain

$$\begin{aligned} \dot{x} &= [A - BR^{-1}Q^T]x + [BR^{-1}B^T]\lambda, \\ \dot{\lambda} &= [P - QR^{-1}Q^T]x + [QR^{-1}B^T - A^T]\lambda. \end{aligned}$$

This is a two-point boundary value problem with end conditions $x(t_0) = x_0$ and $x(t_1)$, and since t_1 is not specified, $\lambda(t) = -Sx(t_1)$. However, this can be transform so that we have a final-value problem in which the boundary conditions are at $t = t_1$.

This is equivalent to

$$\begin{pmatrix} x \\ \lambda \end{pmatrix} = \Phi(t, t_1) \begin{pmatrix} x(t_1) \\ \lambda(t_1) \end{pmatrix},$$

where

$$\Phi(t, t_1) = \begin{pmatrix} \Phi_{11}(t, t_1) & \Phi_{21}(t, t_1) \\ \Phi_{12}(t, t_1) & \Phi_{22}(t, t_1) \end{pmatrix}$$

is the state transition matrix. Thus,

$$x(t) = \Phi_{11}(t, t_1)x(t_1) + \Phi_{21}(t, t_1)\lambda(t_1),$$
$$\lambda(t) = \Phi_{12}(t, t_1)x(t_1) + \Phi_{22}(t, t_1)\lambda(t_1).$$

Now since the end condition is

$$\lambda(t_1) = -Sx(t_1)$$

we obtain

$$x(t) = (\Phi_{11} - \Phi_{21}S)x(t_1),$$
$$\lambda(t) = (\Phi_{12} - \Phi_{22}S)x(t_1).$$

Which gives $\lambda(t) = (\Phi_{12} - \Phi_{22}S)(\Phi_{11} - \Phi_{21})^{-1}x(t_1)$.

The optimal time control therefore is

$$u^* = -R^{-1}Q^T x(t) + R^{-1}B^T(\Phi_{12} - \Phi_{22}S)(\Phi_{11} - \Phi_{21})^{-1}x(t_1). \quad (3.7)$$

Proposition 4. *Let $J^*[x(t), u(t)]$ be the time performance of the control given by $u^*(t)$ and $\tilde{J}[x(t), u(t)]$ be the time performance of any other control satisfying the conditions of the Pontryagin theorem, then we have the following variational inequality*: $\tilde{J}[x(t), u(t)] - J^*[x(t), u(t)] \geq 0$.

Proof. The function $F = \frac{1}{2}(x^T Px + x^T Qu + u^T Ru)$ is convex in $(x(t), u(t))$ because P, Q and R have been assumed to be positive definite. Let $\tilde{u}$ be some other control vector satisfying the optimality conditions of the Pontryagin theorem, then we have

$$\begin{aligned} &2\big[F\big(\tilde{x}(t), \tilde{u}(t)\big) - F\big(x(t)^*, u^*(t)\big)\big] \\ &\quad \geq \big(\tilde{x} - x^*\big)^T P\big(\tilde{x} - x^*\big) + \big(\tilde{x} - x^*\big)^T Q(\tilde{u} - u^*) + (\tilde{u} - u^*)^T R(\tilde{u} - u^*). \end{aligned}$$

For some positive definite matrix S a similar inequality holds, that is, $S\tilde{x}(t_1) - Sx^*(t_1) \geq (\tilde{x}(t_1) - x^*(t_1))S(\tilde{x}(t_1) - x^*(t_1))$ with these two inequalities, the difference between $\tilde{J}[x(t), u(t)]$ and $J^*[x(t), u(t)]$ satisfies

$$\begin{aligned} &\tilde{J}\big[x(t), u(t)\big] - J^*\big[x(t), u(t)\big] \\ &\quad \geq \Delta x^T(t_1)S\Delta x(t_1) + \int_{t_0}^{t_1} \big[\Delta x^T P\Delta x + \Delta x^T Qu + \Delta u^T R\Delta u\big]\,dt, \end{aligned} \quad (3.8)$$

where $\Delta g(t) = \tilde{g}(t) - g^*(t)$.

Now using equations (3.5) and (3.6) to express $Q\tilde{x}$ and $R\tilde{u}$ in terms of λ in equation (3.8) and integrating by parts, we obtain the following result

$$\begin{aligned}&\tilde{J}\big[x(t),u(t)\big] - J^*\big[x(t),u(t)\big]\\&\quad\geq \Delta x^T(t_1)\big[Sx(t_1)+\lambda(t_1)\big]\\&\qquad + \Delta x^T(t_0)\lambda(t_0) + \int_{t_0}^{t_1}\big[\big(\Delta\dot{x}^T(t)\big) - \Delta x^T(t)A^T - \Delta u^T(t)B^T\big]\lambda\,dt.\end{aligned}$$

The first and second terms of the above equation vanish because $\lambda(t_1) = -Sx(t_1)$. For similar reason, the second and third term vanish because of equation of state (2.1), thus we have

$$\tilde{J}\big[x(t),u(t)\big] - J^*\big[x(t),u(t)\big] \geq 0. \qquad \square$$

3.3 Linear quadratic problem and algebraic Riccati equation

There is one class of optimal control problems for which the optimal control can be given in feedback form. These are problems involving linear system dynamics and Lagrangian quadratic both in state and control variables. Here we consider a close-loop linear quadratic regulator problem of minimizing

$$J(x,u;t) = E\left(\frac{1}{2}\int_{t_0}^{t_1}\big[x^TP(t)x + x^TQ(t)u + u^TR(t)u\big]\,dt\right)$$

$$\text{subject to } \dot{x} = [Ax+Bu] + \sum_{j=1}^{m}[C_jx + D_ju]\,dw_j(t),$$

where the matrices C and D are integrable weighting matrices whose values are chosen to reflect the relative importance associated with the corresponding grade sizes in the system. This is achieved by obtaining and finding a solution to an algebraic Riccati equation using the Hamilton–Jacobi–Bellman equation approach, an infinitesimal version of dynamic programming principle.

Let

$$V\big(x(t_0)\big) = \underset{u}{\text{Min}}\, E_{t_0}\left(\int_{t_0}^{t_1}\big[x^TPx + x^TQ(t)u + u^TRu\big]\,dt + V\big(x(t_1)\big)\big|x(t_0)\right)$$
$$\text{for any finite } t_1 > t_0,$$

where E_{t_0} denote the conditional expectation given $t = t_0$.

Since $V(x(t))$ is a function of function of t, Itȯ's formula can be used to obtain

$$\begin{aligned}&dV\big(x(t)\big)\\&\quad = V_t\,dt + V_x\,dx + \frac{1}{2}dx\,V_{xx}\,dx\end{aligned}$$

$$= \Big(V_t + V_x^T (Ax + Bu) + \frac{1}{2} \sum_j (C_j x + D_j u)^T V_{xx} \sum_j (C_j x + D_j u) \Big) dt$$

$$+ V_x^T \sum_j (C_j x + D_j u)\, dw_j(t).$$

Integrating from t_0 to t_1 and taking conditional expectation at $t = t_0$ and noting that the resultant stochastic integral is a martingale, we obtain

$$\begin{aligned} &V\big(x(t_1)\big) - V\big(x(t)\big) \\ &= E_{t_0}\bigg(\int_{t_0}^{t_1} \Big[V_t + V_x^T (Ax + Bu) \\ &\qquad + \frac{1}{2} \sum_j (C_j x + D_j u)^T V_{xx} \sum_j (C_j x + D_j u) \Big] dt \bigg) \\ &= 0. \end{aligned}$$

Dividing by $\Delta t = t_1 - t_0$ and taking limit as $\Delta t \to 0$ as we have

$$\begin{aligned} \operatorname*{Min}_u E_{t_0} \Big(& x^T P x + x^T Q u + u^T R u + V_x^T (Ax + Bu) \\ & + \frac{1}{2} \sum_j (C_j x + D_j u)^T V_{xx} (C_j x + D_j u) \Big) = 0. \end{aligned}$$

Since it is assumed that x and u at t_0 are known, it is easy to see that the expectation operator can be dropped so that we have

$$\begin{aligned} \operatorname*{Min}_u \Big(& x^T P x + x^T Q u + u^T R u + V_x^T (Ax + Bu) \\ & + \frac{1}{2} \sum_j (C_j x + D_j u)^T V_{xx} (C_j x + D_j u) \Big) = 0. \end{aligned}$$

Now let $V(t) = x^T M x \Rightarrow V_x = 2Mx$ and $V_{xx} = 2M$. Then

$$\begin{aligned} \operatorname*{Min}_u \Big(& x^T P x + x^T Q u + u^T R u + (2Mx)^T (Ax + Bu) \\ & + \sum_j (C_j x + D_j u)^T M (C_j x + D_j u) \Big) = 0. \end{aligned} \tag{3.9}$$

Taking the derivative with respect to u yields

$$Ru + B^T M x + \sum_j (D_j^T M C_j) x + \sum_j (D_j^T M D_j) u = 0$$

$$\Rightarrow \quad u^* = -\Big[R + \sum_j (D_j^T M D_j) \Big]^{-1} \Big[B^T M + \sum_j (D_j^T M C_j) \Big] x.$$

Feeding this back into (3.9), we have the following algebraic Riccati equation

$$
\begin{aligned}
x^T\Big[&A^TM + MA + \sum_j (C_j^T M C_j) + Q \\
&- \Big(MB + \sum_j (C_j^T M D_j)\Big)\Big(R + \sum_j (D_j^T M D_j)\Big)^{-1} \\
&\times \Big(B^T M + \sum_j (D_j^T M C_j)\Big)\Big]x = 0 \\
\Rightarrow \quad & Q + A^T M + MA + \sum_j (C_j^T M C_j) \\
&- \Big(MB + \sum_j (C_j^T M D_j)\Big)\Big(R + \sum_j (D_j^T M D_j)\Big)^{-1} \\
&\times \Big(B^T M + \sum_j D_j^T M C_j\Big) = 0.
\end{aligned}
$$

A solution to an algebraic Riccati equation can be obtained using Semi-Definite Programming (SDP) (Yao et al. (2001)).

4 Illustrative example

The purpose of this section is to demonstrate, the applicability of the theoretical results presented in this paper by application to real life situation. For the purpose of illustration, we consider the academic staff structure of a University system in Nigeria. There are three categories: (1) Lectureship cadre made of academic staff members in Lecturer I, Lecturer II and Assistant Lecturer positions, (2) Senior lectureship cadre made of Senior Lecturer position and (3) Professorial cadre made of Associate Professor and Professor positions. It is required by the supervising commission, the National Universities Commission (NUC), for the purpose of academic programme accreditation, that the ratio configuration of staff members in the categories be 3:4:3, assuming that the required number in establishment is met. However, the ratio on ground is 5:1:3. The question is: what recruitment control input $u(t)$ do we need so as to reach the desired structure in the shortest possible time, taking into account the recruitment, promotion, and retirement policies of the institution? To answer this question, we use this ratio configuration to form the vector of initial structure $x(t_0) = (0.5 \;\; 0.1 \;\; 0.3)$ and the vector of desired structure $x^*(t) = (0.3 \;\; 0.4 \;\; 0.3)$. Let the parameters of the stochastic differential equation (SDE) model of the system be as follows

$$A(t) = \begin{pmatrix} 0.5 & 0.45 & 0.15 \\ 0 & 0.65 & 0.25 \\ 0 & 0 & 0.9 \end{pmatrix}, \qquad B(t) = \begin{pmatrix} 98 & 0 & 0 \\ 0 & 98 & 0 \\ 0 & 0 & 98 \end{pmatrix}.$$

The weighting matrices P, Q, R and S are taken to be identity matrix, which means that all categories have equal weights. Using the optimal time control we obtain the following control input:

$$u^*(t) = (0.29 \quad 0.39 \quad 0.32). \tag{4.1}$$

Since the interest of management is to attain the desired structure within the shortest possible time, the managerial implication therefore, is that the control specified by $u^*(t)$ must be the choice of management. This means that if the desired structure is to be reached in the smallest possible time, the recruits must be distributed to the categories according to the probabilities given in (4.1).

5 Conclusion

The different modes of controllability and necessary and sufficient conditions under which a manpower system modeled by a stochastic differential equation (SDE) can be controllable have been discussed. A 2-norm optimality criterion and optimal time control have also been obtained. The Hamilton–Jacobi–Bellman (HJB) equation was used to obtain an algebraic Riccati equation for an optimal tracking linear quadratic problem in a finite time horizon. An illustrative example on minimum time control, involving the academic staff structure of a University system in Nigeria is presented.

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