Brazilian Journal of Probability and Statistics
2014, Vol. 28, No. 2, 209–222
DOI: 10.1214/12-BJPS202

Characterizations of the Weibull and uniform distributions using record values

Saralees Nadarajah[a], **Mahdi Teimouri**[b,c] **and Shou Hsing Shih**[d]

[a]*University of Manchester*
[b]*Amirkabir University of Technology*
[c]*Gonbad Kavous University*
[d]*American University of Sharjah*

Abstract. The Weibull distribution is characterized via the ratio of two upper record statistics. Also the uniform distribution is characterized via the ratio of two lower record statistics. As an application, an improved confidence interval for the Weibull shape parameter is presented.

1 Introduction

The record values were introduced by Chandler (1952). Suppose that $\{X_i\}_{i\geq 1}$ is a sequence of independent and identically distributed random variables with common distribution function $F(x)$ and common probability density function $f(x)$. Set $Y_n = \min(\max)\{X_1, X_2, \ldots, X_n\}$ for $n \geq 1$. We say X_j is a lower (upper) record value of this sequence if $Y_j < (>)\ Y_{j-1}$ for $j > 1$. By definition, X_1 is a lower as well as an upper record value. Let $L_n = \min\{j \mid j > L_{n-1}, Y_j < Y_{L_{n-1}}, n \geq 2\}$ with $L_1 = 1$ denote the times of lower record values. Similarly, let $U_n = \min\{j \mid j > U_{n-1}, Y_j > Y_{L_{n-1}}, n \geq 2\}$ with $U_1 = 1$ denote the times of upper record values. For comprehensive accounts of the theory and applications of record values, we refer the readers to Ahsanullah (1995), Arnold et al. (1998), Ahsanullah (2004), and Ahsanullah and Raqab (2006).

Characterizing distributions via their record statistics has a long history. Some recently published examples include: characterization of generalized extreme value, power function, generalized Pareto and classical Pareto distributions (Wu and Lee, 2001); characterizations of the uniform distribution (Arslan et al., 2005a, 2005b); characterizations of the exponential distribution (Iwinska, 2005; Ahsanullah and Aliev, 2008; Oncel, 2009; Yanev and Ahsanullah, 2009); characterizations based on regression on pairs of record values (Bairamov et al., 2005); characterization based on Fisher information in minima and upper record values (Hofmann et al., 2005); characterizations based on sums of squares of spacings (Kirmani and Wesolowski, 2005); characterizations of general classes of doubly truncated distributions (Sultan and Abd El-Mougod, 2006); characterization based on the conditional expectation of truncated record values (Gupta and Ahsanullah, 2006); characterizations of the generalized Pareto distribution (Tavangar and Asadi, 2007);

Key words and phrases. Record statistics, Weibull distribution, uniform distribution.
Received March 2012; accepted June 2012.

characterizations of the doubly truncated Burr type XII distribution (El-Baset and Ahmad, 2007); characterizations based on regression of adjacent generalized order statistics (Bieniek, 2007); characterizations based on the difference between the counting process of upper records (Gouet et al., 2007); characterizations based on entropy of record values (Baratpour et al., 2007); characterizations based on conditional expectations of functions of generalized order statistics (Beg and Ahsanullah, 2007); characterizations of the exponentiated Pareto distribution (Shawky and Abu-Zinadah, 2008a; Khan and Kumar, 2010); characterizations of the exponentiated gamma distribution (Shawky and Bakoban, 2008); characterization based on bivariate regression of record values (Yanev et al., 2008); characterizations based on the conditional expectation of the kth lower record values (Malinowska and Szynal, 2008); characterization based on entropies of records (Ahmadi, 2009; Ahmadi and Fashandi, 2009); characterizations based on Rényi entropy of record values (Baratpour et al., 2008); characterization through expectation of functions of generalized order statistics (Haque et al., 2009); characterization conditioned on a pair of nonadjacent records (Khan and Khan, 2009); characterizations based on conditional expectation of functions of dual generalized order statistics (Khan et al., 2010a); and, characterizations based on the difference of two conditional expectations, conditioned on a nonadjacent record statistic (Khan et al., 2010b).

For other recent examples, we refer the readers to Shawky and Abu-Zinadah (2006), Akhundov and Nevzorov (2007), Sultan and Abd El-Mougod (2007), Shawky and Abu-Zinadah (2008b), Su et al. (2008), and Shawky and Bakoban (2009).

The aim of this short note is to provide characterizations of the Weibull and uniform distributions via the ratio of two record statistics. The main results are given in Section 2. An application of these results for confidence interval estimation is presented in Section 3. A real data application of the proposed confidence interval in Section 3 is described in Section 4. Finally, some future work are noted in Section 5.

In the following, we give some preliminaries. Let $X_{U(m)}$ and $X_{U(n)}$ for $m < n$ denote the upper record statistics from a given family. Let $X_{L(m)}$ and $X_{L(n)}$ for $m < n$ denote the corresponding lower record values. The joint probability density function of $X_{U(m)}$ and $X_{U(n)}$ is given by (see Ahsanullah, 1995; Arnold et al., 1998; Ahsanullah, 2004 and Ahsanullah and Raqab, 2006):

$$
\begin{aligned}
& f_{X_{U(m)},X_{U(n)}}(x, y) \\
&\quad = \frac{[\ln(1 - F(x)) - \ln(1 - F(y))]^{n-m-1}}{\Gamma(m)\Gamma(n - m)} \\
&\qquad \times \left[-\ln\left(1 - F(x)\right)\right]^{m-1} \frac{f(x)f(y)}{1 - F(x)},
\end{aligned}
\tag{1.1}
$$

where $0 < x < y < \infty$, $1 \le m < n$ and $\Gamma(\cdot)$ denotes the gamma function defined by

$$\Gamma(a) = \int_0^\infty t^{a-1} \exp(-t)\, dt.$$

The joint probability density function of $X_{L(m)}$ and $X_{L(n)}$ is given by

$$\begin{aligned} & f_{X_{L(m)}, X_{L(n)}}(x, y) \\ & \quad = \frac{1}{\Gamma(m)\Gamma(n-m)} [-\ln F(x)]^{m-1} \\ & \qquad \times \left[\ln F(x) - \ln F(y)\right]^{n-m-1} \frac{f(x) f(y)}{F(x)}, \end{aligned} \tag{1.2}$$

where $0 < y < x < \infty$ and $1 \le m < n$.

A continuous random variable X is said to have the gamma distribution with shape parameter $a > 0$ if its probability density function is

$$f_X(x) = \frac{x^{a-1} \exp(-x)}{\Gamma(a)},$$

where $x > 0$. A continuous random variable X is said to have the beta distribution with shape parameters $a > 0$ and $b > 0$ if its probability density function is

$$f_X(x) = \frac{x^{a-1} (1-x)^{b-1}}{B(a,b)},$$

where $0 < x < 1$ and $B(\cdot, \cdot)$ denotes the beta function defined by

$$B(a,b) = \int_0^1 t^{a-1} (1-t)^{b-1}\, dt.$$

Finally, if a continuous random variable X has the Weibull distribution with shape parameter $\alpha > 0$ then its distribution function and probability density function are given by

$$F_X(x) = 1 - \exp\left[-\left(\frac{x}{\beta} \right)^\alpha \right] \tag{1.3}$$

and

$$f_X(x) = \frac{\alpha}{\beta} \left(\frac{x}{\beta} \right)^{\alpha-1} \exp\left[-\left(\frac{x}{\beta} \right)^\alpha \right], \tag{1.4}$$

respectively, where $x > 0$ and $\beta > 0$.

There have not been many papers giving characterizations of the Weibull distribution based on records. The only one we are aware of is Chang (2008). It shows that X is a Weibull random variable with shape parameter α if and only if $X_{U(n+1)} / \{X_{U(n+1)} - X_{U(n)}\}$ and $X_{U(n+1)}$ for $n \ge 1$ are independent or equivalently $X_{U(n)} / \{X_{U(n+1)} - X_{U(n)}\}$ and $X_{U(n+1)}$ for $n \ge 1$ are independent or equivalently $\{X_{U(n+1)} + X_{U(n)}\} / \{X_{U(n+1)} - X_{U(n)}\}$ and $X_{U(n+1)}$ for $n \ge 1$ are independent. But Chang (2008) discusses no practical uses of these characterizations. We cannot see any practical use either. Since the if and only if conditions are independent of the shape parameter they cannot be used for estimation for instance.

There are many situations in which only records are observed. Examples include destructive stress testing, meteorology, hydrology, seismology, mining, and athletic events. For a more specific example, consider the situation of testing the breaking strength of wooden beams as described in Glick (1978). Hence, it is important that one has accurate estimation procedures based only on records. We describe such a procedure later in Section 3 and show its superior performance versus known estimation procedures using both simulated and real data.

The Weibull distribution is the most popular model in many areas, including destructive stress testing. Hence, it is even more important that accurate estimation procedures are developed for the Weibull distribution based only on records.

2 Main results

Our main results are Theorems 2.1 and 2.2. Theorem 2.1 provides a characterization of the Weibull distribution based on the ratio of two upper record statistics. Theorem 2.2 provides a characterization of the uniform distribution based on the ratio of two lower record statistics.

Theorem 2.1. *Suppose $X_{U(m)}$ and $X_{U(n)}$ for $m < n$ are the upper record statistics from a given family. Suppose without loss of generality that the distribution function has the form $F(x) = 1 - \exp\{-h(x)/c\}$, where $c > 0$. Suppose further $h(xy) = h(x)h(y)$ for all x, y. Then the family is Weibull with shape parameter α if and only if the distribution of $(X_{U(m)}/X_{U(n)})^\alpha$ is beta with shape parameters $n-m$ and m.*

Proof. Suppose X is a Weibull random variable with shape parameter α. Consider the transformations $R = X_{U(m)}/X_{U(n)}$ and $Q = X_{U(n)}$. The modulus of the determinant of the Jacobian matrix for these transformations is q. It follows from (1.1) that

$$\begin{aligned}
f_{Q,R}(q,r) &= q f_{X_{U(m)},X_{U(m)}}(qr,q) \\
&= \frac{q[\ln(1-F(qr)) - \ln(1-F(q))]^{n-m-1}}{\Gamma(m)\Gamma(n-m)} \\
&\quad \times \left[-\ln\left(1-F(qr)\right)\right]^{m-1} \frac{f(qr)f(q)}{1-F(qr)} \\
&= \frac{\alpha^2 q[(q/\beta)^\alpha - (qr/\beta)^\alpha]^{n-m-1}(qr/\beta)^{\alpha(m-1)+\alpha-1}(q/\beta)^{\alpha-1}}{\beta^2\Gamma(m)\Gamma(n-m)\exp\{(q/\beta)^\alpha\}} \\
&= \frac{\alpha^2 q(q/\beta)^{n\alpha-2}[1-r^\alpha]^{n-m-1} r^{\alpha(m-1)+\alpha-1}\exp\{-(q/\beta)^\alpha\}}{\beta^2\Gamma(m)\Gamma(n-m)}.
\end{aligned}$$

So, the probability density function of R is:

$$f_R(r) = \frac{\alpha^2[1-r^\alpha]^{n-m-1}r^{\alpha m-1}}{\beta^2\Gamma(m)\Gamma(n-m)}\int_0^\infty q(q/\beta)^{n\alpha-2}\exp\{-(q/\beta)^\alpha\}\,dq$$
$$= \frac{\alpha\Gamma(n)}{\Gamma(m)\Gamma(n-m)}\left[1-r^\alpha\right]^{n-m-1}r^{\alpha m-1},$$

where $\alpha > 0$ and $r < 1$. So, the distribution function of $W = R^\alpha$ is:

$$F_W(w) = P\left(R^\alpha \le w\right) = P\left(R \le w^{1/\alpha}\right) = F_R\left(w^{1/\alpha}\right),$$

and hence the probability density function of W is

$$f_W(w) = \frac{(1-w)^{n-m-1}w^{m-1}}{B(n-m,m)},$$

where $0 < w < 1$.

Now consider the converse. The joint probability density function of (Q, R) is

$$f_{Q,R}(q,r) = \frac{1}{c^n\Gamma(m)\Gamma(n-m)}q\left[h(q)-h(qr)\right]^{n-m-1}\left[h(qr)\right]^{m-1} \times h'(qr)h'(q)\exp\{-h(q)/c\}.$$

So, the probability density function of R can be expressed as:

$$f_R(r) = \frac{1}{c^n\Gamma(m)\Gamma(n-m)} \times \int_0^\infty q\left[h(q)-h(qr)\right]^{n-m-1}\left[h(qr)\right]^{m-1} \times h'(qr)h'(q)\exp\{-h(q)/c\}\,dq. \tag{2.1}$$

Note that we must have $h'(\cdot) > 0$ since the integrand in (2.1) must be positive. Also we must have $h(\cdot) > 0$ since the $\exp\{h(q)\}$ term must be finite. Since $h(qr) = h(q)h(r)$ (and so $h'(qr) = h(q)h'(r)/q$), (2.1) can be reduced to

$$f_R(r) = \frac{1}{c^n\Gamma(m)\Gamma(n-m)}\int_0^\infty h(q)^{n-m-1}\left[1-h(r)\right]^{n-m-1}h(q)^{m-1}h(r)^{m-1} \times h(q)h'(q)h'(r)\exp\{-h(q)/c\}\,dq$$
$$= \frac{h'(r)[1-h(r)]^{n-m-1}h(r)^{m-1}}{c^n\Gamma(m)\Gamma(n-m)}\int_0^\infty h(q)^{n-1}h'(q)\exp\{-h(q)/c\}\,dq$$
$$\propto \frac{h'(r)[1-h(r)]^{n-m-1}h(r)^{m-1}}{\Gamma(m)\Gamma(n-m)},$$

where $0 < h(r) < 1$. So, the distribution function and the probability density function of $W = R^\alpha$ are:

$$F_W(w) = P\left(R^\alpha \le w\right) = P\left(R \le w^{1/\alpha}\right) = F_R\left(w^{1/\alpha}\right)$$

and

$$f_W(w) \propto \frac{h'(w^{1/\alpha}) w^{1/\alpha - 1}}{\Gamma(m)\Gamma(n-m)} \left[1 - h(w^{1/\alpha})\right]^{n-m-1} \left[h(w^{1/\alpha})\right]^{m-1},$$

respectively, where $0 < w < 1$. So, the only choice possible for $h(\cdot)$ is $h(q) = q^{\alpha}$. The result follows. □

Theorem 2.2. *Let $X_{L(m)}$ and $X_{L(n)}$ for $m < n$ denote the lower record values from an absolutely continuous distribution function $F(\cdot)$. Then the ratio $W = -\ln(X_{L(n)}/X_{L(m)})$ has the gamma distribution with shape parameter $n - m$ if and only if X is distributed uniformly over the interval* (0, 1).

Proof. Suppose X is a uniform random variable over the interval (0, 1). Consider the transformations $R = X_{L(n)}/X_{L(m)}$ and $Q = X_{L(m)}$. The modulus of the determinant of the Jacobian matrix for these transformations is q. Its follows from (1.2) that the joint probability density function of Q and R is:

$$\begin{aligned} f_{Q,R}(q,r) &= q f_{X_{L(m)},X_{L(n)}}(q, qr) \\ &= \frac{q}{\Gamma(m)\Gamma(n-m)} \left[-\ln F(q)\right]^{m-1} \\ &\quad \times \left[\ln F(q) - \ln F(qr)\right]^{n-m-1} \frac{f(q) f(qr)}{F(q)}. \end{aligned}$$

So, the probability density function of R is:

$$\begin{aligned} f_R(r) &= \int_{-\infty}^{\infty} \frac{q}{\Gamma(m)\Gamma(n-m)} \left[-\ln F(q)\right]^{m-1} \\ &\quad \times \left[\ln F(q) - \ln F(qr)\right]^{n-m-1} \frac{f(q) f(qr)}{F(q)}\, dq \\ &= \int_0^{\infty} \frac{F^{-1}(\exp(-u))}{\Gamma(m)\Gamma(n-m)} u^{m-1} \left[-u - \ln F\left(F^{-1}(\exp(-u)) r\right)\right]^{n-m-1} \\ &\quad \times f\left(r F^{-1}(\exp(-u))\right) du. \end{aligned}$$

Now consider the transformation $W = -\ln R$. We have:

$$\begin{aligned} f_W(w) &= \exp(-w) f_R\left(\exp(-w)\right) \\ &= \exp(-w) \int_0^{\infty} \frac{F^{-1}(\exp(-u))}{\Gamma(m)\Gamma(n-m)} u^{m-1} \\ &\quad \times \left[-u - \ln F\left(\exp(-w) F^{-1}(\exp(-u))\right)\right]^{n-m-1} \\ &\quad \times f\left(\exp(-w) F^{-1}(\exp(-u))\right) du. \end{aligned} \tag{2.2}$$

If $F^{-1}(z) = z$ for $0 < z < 1$, then

$$f_W(w) = \frac{w^{n-m-1}\exp(-w)}{\Gamma(m)\Gamma(n-m)} \int_0^\infty \exp(-u) u^{m-1}\, du = \frac{w^{n-m-1}\exp(-w)}{\Gamma(n-m)}.$$

So, W is a gamma random variable with shape parameter $n - m$.

Now consider the converse. To obtain a gamma probability density function with shape parameter $n - m$, the integral in (2.2) must evaluate to something of the form $w^{n-m-1}/\Gamma(n-m)$. So, we must have $F^{-1}(\exp(-u)) = \exp(-u)$ and $\ln F(\exp(-w)F^{-1}(\exp(-u))) \propto -(u+w)$. These two requirements result in $F^{-1}(q) = q$ for $0 < q < 1$. The proof is complete. □

3 A simulation study

Here, we use Theorem 2.1 to derive an improved confidence interval for the Weibull shape parameter.

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from (1.3)–(1.4) with shape parameter α. The traditional confidence interval for α is based on normal approximation and the method of maximum likelihood. The maximum likelihood estimator of α, say $\widehat{\alpha}$, is the root of the equation

$$\frac{n}{\alpha} + \sum_{i=1}^n \ln\left(\frac{X_i}{\beta}\right) = \sum_{i=1}^n \left(\frac{X_i}{\beta}\right)^\alpha \ln\left(\frac{X_i}{\beta}\right), \tag{3.1}$$

where $\beta = n^{-1/\alpha}(\sum_{i=1}^n X_i^\alpha)^{1/\alpha}$. By asymptotic normality, an approximate $100(1-\gamma)\%$ confidence interval for α is:

$$\left[\widehat{\alpha} - z_{\gamma/2}\sqrt{\frac{I_{22}}{I_{11}I_{22} - I_{12}^2}}, \widehat{\alpha} + z_{\gamma/2}\sqrt{\frac{I_{22}}{I_{11}I_{22} - I_{12}^2}}\right], \tag{3.2}$$

where $z_{\gamma/2}$ is the $100(1-\gamma/2)\%$ percentile of the standard normal distribution and

$$I_{11} = E\left\{\frac{n}{\widehat{\alpha}^2} + \sum_{i=1}^n \left(\frac{X_i}{\widehat{\beta}}\right)^{\widehat{\alpha}} \left[\ln\left(\frac{X_i}{\widehat{\beta}}\right)\right]^2\right\}, \tag{3.3}$$

$$I_{22} = E\left\{-\frac{n\widehat{\alpha}}{\widehat{\beta}^2} + \frac{\widehat{\alpha}(\widehat{\alpha}+1)}{\widehat{\beta}^{\widehat{\alpha}+2}} \sum_{i=1}^n X_i^{\widehat{\alpha}}\right\}, \tag{3.4}$$

$$I_{12} = E\left\{\frac{n}{\widehat{\beta}} - \frac{1}{\widehat{\beta}^{\widehat{\alpha}+1}} \sum_{i=1}^n X_i^{\widehat{\alpha}} - \frac{\widehat{\alpha}}{\widehat{\beta}^{\widehat{\alpha}+1}} \sum_{i=1}^n X_i^{\widehat{\alpha}} \ln\left(\frac{X_i}{\widehat{\beta}}\right)\right\}, \tag{3.5}$$

where $\widehat{\beta} = n^{-1/\widehat{\alpha}}(\sum_{i=1}^n X_i^{\widehat{\alpha}})^{1/\widehat{\alpha}}$.

By Theorem 2.1, a $100(1-\gamma)\%$ confidence interval for α is:

$$\left[\frac{\ln\{I_\gamma(m, n-m)\}}{\ln(X_{U(n)}/X_{U(m)})} < \alpha < \frac{\ln\{I_{1-\gamma}(m, n-m)\}}{\ln(X_{U(n)}/X_{U(m)})}\right], \tag{3.6}$$

where $1 \le m < n$ and $I_\gamma(a, b)$ denotes the incomplete beta function ratio defined by

$$I_\gamma(a, b) = \frac{1}{B(a, b)} \int_0^\gamma t^{a-1}(1-t)^{b-1}\, dt.$$

An immediate advantage of (3.6) over (3.2) is that the former does not depend on β. For the confidence interval, (3.2), both the maximum likelihood equation, (3.1), and the three expectations, (3.3)–(3.5), are heavily dependent on β.

Another immediate advantage is that (3.6) is computationally less expensive. One only needs to compute the incomplete beta function ratio and in-built routines for this are widely available. The confidence interval, (3.2), requires finding roots of (3.1) as well as computing the three expectations (3.3)–(3.5).

We now perform a simulation study to see how the empirical coverage lengths and empirical coverage probabilities of (3.2) and (3.6) compare. We simulate ten thousand samples of size n from (1.3)–(1.4) for $n = 2, 3, \ldots, 100$, $\alpha = 0.2, 0.5, 1, 2, 3, 5$ and $\beta = 1$. For each sample, we calculate the limits of the confidence intervals, (3.2) and (3.6), for $\gamma = 0.05$. For given n, α and β, we calculate empirical coverage length as the average of the empirical coverage lengths over the ten thousand samples. We calculate empirical coverage probability as the proportion of the ten thousand confidence intervals containing the true value of α.

The empirical coverage lengths and empirical coverage probabilities versus $n = 2, 3, \ldots, 100$ and for $\alpha = 0.2, 0.5, 1, 2, 3, 5$ are shown in Figures 1 and 2. The horizontal line in Figure 2 corresponds to the empirical coverage probability being equal to 0.95. The actual values plotted are the lowess (Cleveland, 1979, 1981) smoothed versions versus n for $n = 2, 3, \ldots, 100$.

It is evident from the figures that the empirical coverage probability for (3.6) is generally closer to the nominal level for all n. The empirical coverage length for (3.6) appears smaller for small to moderate n. However, for $n \ge 80$, the empirical coverage lengths for (3.2) and (3.6) appear indistinguishable. It is also worth noting that empirical coverage lengths generally increase with increasing α and decrease with increasing n.

In summary, the confidence interval given by (3.6) does not depend on β, is easy to compute, and has better empirical coverage probabilities and better empirical coverage lengths. The empirical coverage lengths for (3.2) and (3.6) appear indistinguishable for all sufficiently large n.

4 A real data application

Here, we compare the performance of the confidence intervals, (3.2) and (3.6), for a real data set. The data set is on the times in days between successive earthquakes of magnitudes greater or equal to 6.5 in Iran for the years from 1989 to 2008. The data set given in Table 1 is extracted from the International Institute of Earthquake Engineering and Seismology (IIEES) web-site, http://www.iiees.ac.ir.

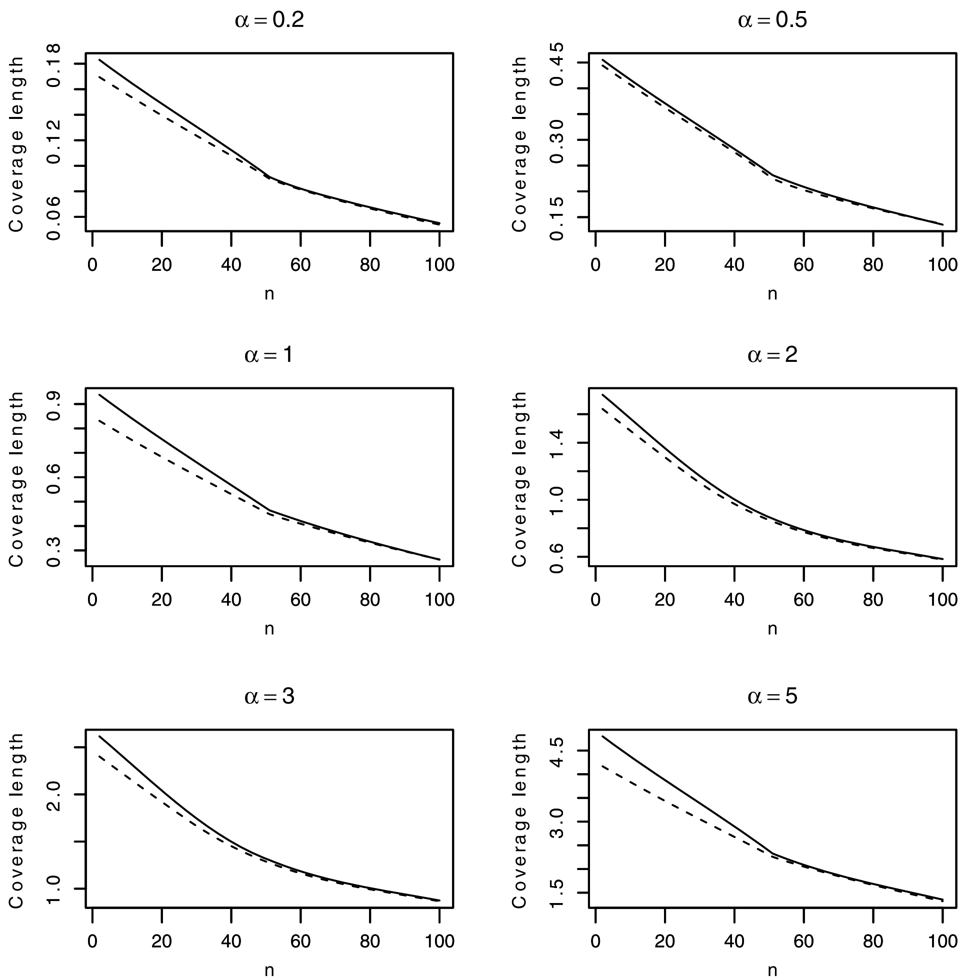

Figure 1 *Empirical coverage lengths of* (3.2) (*solid*) *and* (3.6) (*broken*) *for samples of size n from* (1.3)–(1.4) *each replicated ten thousand times.*

We computed $100(1-\gamma)\%$ confidence intervals for the shape parameter, α, using both (3.2) and (3.6). The results are shown in Table 2. We see that the proposed confidence interval is narrower for each γ. This supports are our findings from the simulation study.

5 Future work

In this short note, we have derived characterizations of the Weibull and uniform distributions based on records. As an application, we have proposed a confidence

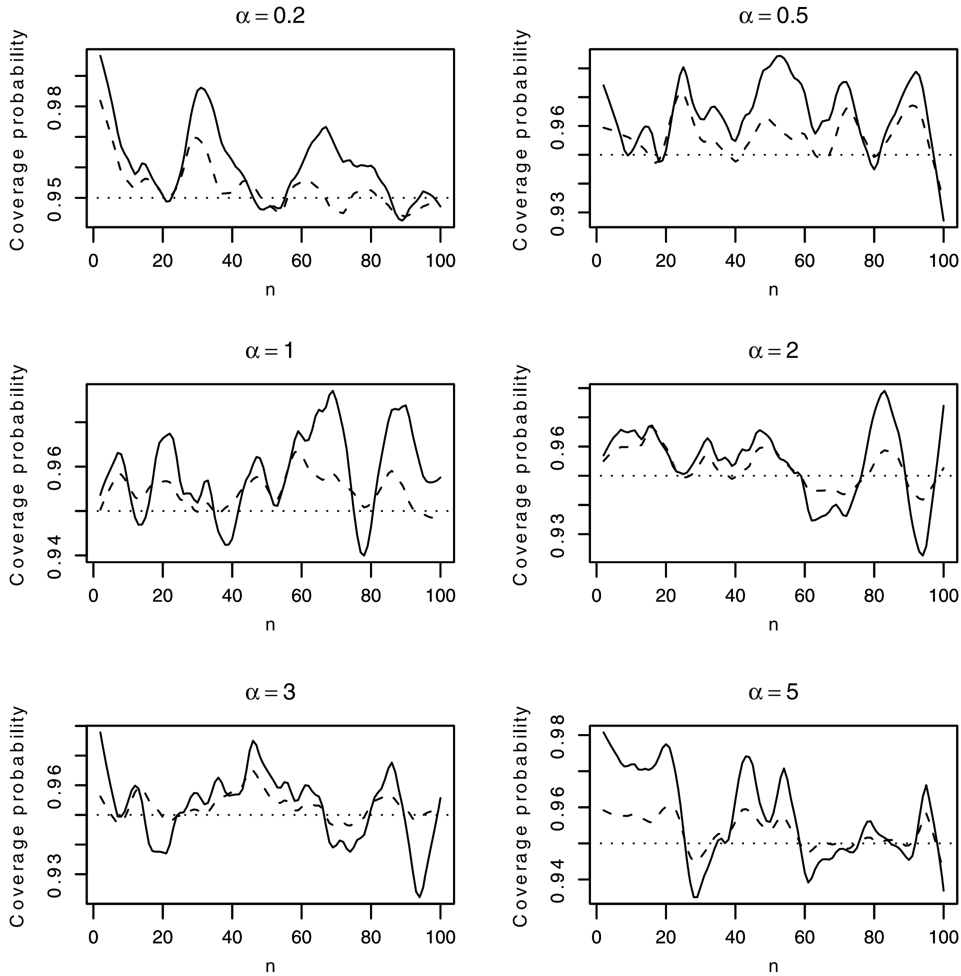

Figure 2 *Empirical coverage lengths of* (3.2) (*solid*) *and* (3.6) (*broken*) *for samples of size n from* (1.3)–(1.4) *each replicated ten thousand times.*

interval for the Weibull shape parameter. We have shown that this confidence interval is superior to the one based on maximum likelihood estimation.

A natural extension is to consider joint confidence regions for the Weibull shape and scale parameters. Asgharzadeh and Abdi (2011) give such confidence regions. But they do not compare their confidence regions with known ones; for example, the one based on maximum likelihood estimation. So, the practical values of their contribution (although quite novel) are not clear.

The characterization in Theorem 2.1 involves the ratio of two record statistics each with the same scale parameter. Therefore, the characterization is independent of the scale parameter. In other words, a joint confidence region of the Weibull

Table 1 *Times (in days) between successive earthquakes in Iran*

284, 246, 139, 2280, 95, 308, 355, 607, 11, 563, 553

Table 2 *Comparison of (3.2) and (3.6) for the real data*

	Confidence intervals	
γ	using (3.2)	using (3.6)
0.1	(0.5896652, 1.280012)	(0.7030108, 1.276851)
0.05	(0.5235391, 1.346138)	(0.5519302, 1.252841)
0.005	(0.3457811, 1.523896)	(0.4212522, 1.460757)
0.0005	(0.2044001, 1.665277)	(0.2231795, 1.500415)

shape and scale parameters cannot be constructed using Theorem 2.1. A completely different characterization will be required. This is a possible future work.

Another natural extension is to consider the case that X in Theorem 2.1 is an exponentiated Weibull random variable (Mudholkar and Srivastava, 1995) with distribution function and probability density function specified by

$$F_X(x) = \{1 - \exp[-x^c]\}^\alpha$$

and

$$f_X(x) = c\alpha x^{c-1} \exp[-x^c]\{1 - \exp[-x^c]\}^{\alpha-1},$$

respectively, where $x > 0$, $\alpha > 0$ and $c > 0$. In this case, the joint probability density function of (Q, R) becomes

$$\begin{aligned} f_{Q,R}(q,r) = {} & \frac{q\{\ln[1-\{1-\exp[-(qr)^c]\}^\alpha] - \ln[1-\{1-\exp[-q^c]\}^\alpha]\}^{n-m-1}}{\Gamma(m)\Gamma(n-m)} \\ & \times \{-\ln[1-\{1-\exp[-(qr)^c]\}^\alpha]\}^{m-1} \\ & \times c^2\alpha^2 q^{2c-2} r^{c-1} \exp[-(qr)^c - q^c] \\ & \times \frac{\{1-\exp[-q^c]\}^{\alpha-1}\{1-\exp[-(qr)^c]\}^{\alpha-1}}{1-\{1-\exp[-(qr)^c]\}^\alpha}. \end{aligned}$$

So, the marginal probability density function of R is

$$\begin{aligned} f_R(r) = {} & \frac{c^2\alpha^2 r^{c-1}}{\Gamma(m)\Gamma(n-m)} \\ & \times \int_0^\infty \{\ln[1-\{1-\exp[-(qr)^c]\}^\alpha] \\ & \quad - \ln[1-\{1-\exp[-q^c]\}^\alpha]\}^{n-m-1} \end{aligned} \quad (5.1)$$

$$\times \{-\ln[1-\{1-\exp[-(qr)^c]\}^\alpha]\}^{m-1} q^{2c-1} \exp[-(qr)^c + q^c]$$
$$\times \frac{\{1-\exp[-q^c]\}^{\alpha-1}\{1-\exp[-(qr)^c]\}^{\alpha-1}}{1-\{1-\exp[-(qr)^c]\}^\alpha} \, dq.$$

Clearly, (5.1) is not in a recognizable form. Hence, the approach of Theorem 2.1 is unlikely to give a characterization of the exponentiated Weibull distribution. An alternative approach will be required, a possible future work.

Acknowledgments

The authors would like to thank the Editor and the two referees for careful reading and for comments which greatly improved the paper.

References

Ahmadi, J. (2009). Entropy properties of certain record statistics and some characterization results. *Journal of the Iranian Statistical Society* **8**, 1–13. MR2629831

Ahmadi, J. and Fashandi, M. (2009). Some characterization and ordering results based on entropies of current records. *Statistics and Probability Letters* **79**, 2053–2059. MR2571768

Ahsanullah, M. (1995). *Record Statistics*. New York: Nova Science Publishers. MR1443904

Ahsanullah, M. (2004). *Record Values—Theory and Applications*. New York: Univ. Press of America.

Ahsanullah, M. and Aliev, F. (2008). Some characterizations of exponential distribution by record values. *Journal of Statistical Research* **42**, 41–46. MR2751008

Ahsanullah, M. and Raqab, M. Z. (2006). *Bounds and Characterizations of Record Statistics*. Hauppauge, NY: Nova Science Publishers. MR2294882

Akhundov, I. S. and Nevzorov, V. B. (2007). Conditional distributions of record values and characterizations of distributions. *Journal of Mathematical Sciences (New York)* **145**, 4837–4842. MR2355398

Arnold, B. C., Balakrishnan, N. and Nagaraja, H. N. (1998). *Records*. New York: Wiley. MR1628157

Arslan, G., Ahsanullah, M. and Bairamov, I. G. (2005a). On characteristic properties of the uniform distribution. *Sankhyā* **67**, 715–721. MR2283011

Arslan, G., Ahsanullah, M. and Bairamov, I. G. (2005b). On some characteristic properties of the uniform distribution. In *Advances on Models, Characterizations and Applications* 153–160. Boca Raton, FL: Chapman & Hall/CRC. MR2190601

Asgharzadeh, A. and Abdi, M. (2011). Joint confidence regions for the parameters of the Weibull distribution based on records. *ProbStat Forum* **4**, 12–24. MR2788625

Bairamov, I., Ahsanullah, M. and Pakes, A. G. (2005). A characterization of continuous distributions via regression on pairs of record values. *Australian and New Zealand Journal of Statistics* **47**, 543–547. MR2235422

Baratpour, S., Ahmadi, J. and Arghami, N. R. (2007). Some characterizations based on entropy of order statistics and record values. *Communications in Statistics—Theory and Methods* **36**, 47–57. MR2386975

Baratpour, S., Ahmadi, J. and Arghami, N. R. (2008). Characterizations based on Rényi entropy of order statistics and record values. *Journal of Statistical Planning and Inference* **138**, 2544–2551. MR2414262

Beg, M. I. and Ahsanullah, M. (2007). On characterizing distributions by conditional expectations of functions of generalized order statistics. *Journal of Applied Statistical Science* **15**, 229–244. MR2400535

Bieniek, M. (2007). On characterizations of distributions by regression of adjacent generalized order statistics. *Metrika* **66**, 233–242. MR2329035

Chandler, K. N. (1952). The distribution and frequency of record values. *Journal of the Royal Statistical Society, Ser. B* **14**, 220–228. MR0053463

Chang, S. K. (2008). Characterizations of the Weibull distribution by the independence of record values. *Journal of the Chungcheong Mathematical Society* **21**, 279–285.

Cleveland, W. S. (1979). Robust locally weighted regression and smoothing scatterplots. *Journal of the American Statistical Association* **74**, 829–836. MR0556476

Cleveland, W. S. (1981). LOWESS: A program for smoothing scatterplots by robust locally weighted regression. *The American Statistician* **35**, 54.

El-Baset, A. and Ahmad, A. (2007). Relations for single and product moments of generalized order statistics from doubly truncated Burr type XII distribution. *Journal of the Egyptian Mathematical Society* **15**, 117–128. MR2512793

Glick, N. (1978). Breaking record and breaking boards. *American Mathematical Monthly* **85**, 2–26. MR0461727

Gouet, R., López, F. J. and Sanz, G. (2007). A characteristic martingale related to the counting process of records. *Journal of Theoretical Probability* **20**, 443–455. MR2337136

Gupta, R. C. and Ahsanullah, M. (2006). Some characterization results based on the conditional expectation of truncated order statistics (record values). *Journal of Statistical Theory and Applications* **5**, 391–402. MR2345262

Haque, Z., Athar, H. and Khan, R. U. (2009). Characterization of probability distributions through expectation of function of generalized order statistics. *Journal of Statistical Theory and Applications* **8**, 416–426. MR2681428

Hofmann, G., Balakrishnan, N. and Ahmadi, J. (2005). Characterization of hazard function factorization by Fisher information in minima and upper record values. *Statistics and Probability Letters* **72**, 51–57. MR2126293

Iwinska, M. (2005). On the characterization of the exponential distribution by record values with a random index. *Fasciculi Mathematici* **36**, 33–39. MR2223640

Khan, A. H., Anwar, Z. and Chishti, S. (2010a). Characterization of continuous distributions through conditional expectation of functions of dual generalized order statistics. *Pakistan Journal of Statistics* **26**, 615–628. MR2680985

Khan, A. H., Faizan, M. and Haque, Z. (2010b). Characterization of continuous distributions through record statistics. *Communications of the Korean Mathematical Society* **25**, 485–489. MR2676000

Khan, A. H. and Khan, M. J. S. (2009). On characterization of continuous distributions conditioned on a pair of non-adjacent records. *Journal of Applied Probability and Statistics* **4**, 65–75. MR2668779

Khan, R. U. and Kumar, D. (2010). On moments of lower generalized order statistics from exponentiated Pareto distribution and its characterization. *Applied Mathematical Sciences* **4**, 2711–2722. MR2729567

Kirmani, S. and Wesolowski, J. (2005). Regressions for sums of squares of spacings. *Annals of the Institute of Statistical Mathematics* **57**, 39–47. MR2165606

Malinowska, I. and Szynal, D. (2008). On characterization of certain distributions of kth lower (upper) record values. *Applied Mathematics and Computation* **202**, 338–347. MR2437163

Mudholkar, G. S. and Srivastava, D. K. (1995). The exponentiated Weibull family: A reanalysis of the bus-motor-failure data. *Technometrics* **37**, 436–445.

Oncel, S. Y. (2009). On characterizing the exponential distribution by conditional variance of generalized order statistics. *Journal of Applied Statistical Science* **16**, 281–286. MR2523025

Shawky, A. I. and Abu-Zinadah, H. H. (2006). General recurrence relations and characterizations of certain distributions based on record values. *Journal of Approximation Theory and Applications* **2**, 149–159. MR2475348

Shawky, A. I. and Abu-Zinadah, H. H. (2008a). Characterizations of the exponentiated Pareto distribution based on record values. *Applied Mathematical Sciences* **2**, 1283–1290. MR2443775

Shawky, A. I. and Abu-Zinadah, H. H. (2008b). General recurrence relations and characterizations of certain distributions based on record statistics. *Journal of Statistical Theory and Applications* **7**, 93–117. MR2424584

Shawky, A. I. and Bakoban, R. A. (2008). Characterization from exponentiated gamma distribution based on record values. *Journal of Statistical Theory and Applications* **7**, 263–277. MR2477042

Shawky, A. I. and Bakoban, R. A. (2009). Conditional expectation of certain distributions of record values. *International Journal of Mathematical Analysis* **3**, 829–838. MR2604459

Su, J.-C., Su, N.-C. and Huang, W.-J. (2008). Characterizations based on record values and order statistics. *Journal of Statistical Planning and Inference* **138**, 1358–1367. MR2388016

Sultan, K. S. and Abd El-Mougod, G. A. (2006). Characterizations of general classes of doubly truncated distributions based on record values. *JPSS Journal of Probability and Statistical Sciences* **4**, 65–72. MR2240302

Sultan, K. S. and Abd El-Mougod, G. A. (2007). Characterization of general classes of distributions based on generalized order statistics. *Journal of Statistical Theory and Applications* **6**, 190–205. MR2397321

Tavangar, M. and Asadi, M. (2007). Generalized Pareto distributions characterized by generalized order statistics. *Communications in Statistics—Theory and Methods* **36**, 1333–1341. MR2396539

Wu, J.-W. and Lee, W.-C. (2001). On the characterization of generalized extreme value, power function, generalized Pareto and classical Pareto distributions by conditional expectation of record values. *Statistical Papers* **42**, 225–242. MR1836849

Yanev, G. P. and Ahsanullah, M. (2009). On characterizations based on regression of linear combinations of record values. *Sankhyā, A* **71**, 109–121. MR2579651

Yanev, G. P., Ahsanullah, M. and Beg, M. I. (2008). Characterizations of probability distributions via bivariate regression of record values. *Metrika* **68**, 51–64. MR2407409

S. Nadarajah
School of Mathematics
University of Manchester
Manchester M13 9PL
UK
E-mail: mbbsssn2@manchester.ac.uk

M. Teimouri
Department of Statistics
Faculty of Mathematics and Computer Science
Amirkabir University of Technology
424 Hafez Ave.
Tehran 15914
Iran
and
Department of Statistics
Faculty of Science
Gonbad Kavous University
Gonbad Kavous
Iran

S. H. Shih
Department of Mathematics and Statistics
American University of Sharjah
Sharjah
UAE