Brazilian Journal of Probability and Statistics
2014, Vol. 28, No. 1, 140–149
DOI: 10.1214/12-BJPS199

Hypergeometric functions where two arguments differ by an integer

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Abstract. If $\alpha_1 - \beta_1$ is an integer, then ${}_pF_q(\boldsymbol{\alpha}; \boldsymbol{\beta} : z)$ can be expressed in terms of ${}_{p-1}F_{q-1}$. This leads to a conjectured generalization of Kummer's transformation from ${}_1F_1$ to ${}_pF_q$. Applications are given for noncentral chi-square and Student's t distributions.

1 Introduction and summary

The generalized hypergeometric function ${}_pF_q$ and its variants are basic tools in many areas of mathematics, see Mathai and Saxena (1973, 1978) and Mathai et al. (2010) for most excellent accounts. Many standard functions can be expressed in terms of the generalized hypergeometric function.

By equation (9.14) of Gradshteyn and Ryzhik (2007), ${}_pF_q(\boldsymbol{\alpha}; \boldsymbol{\beta} : z)$ is defined by

$$ {}_pF_q(\boldsymbol{\alpha}; \boldsymbol{\beta} : z) = \sum_{k=0}^{\infty} \frac{(\boldsymbol{\alpha})_k z^k}{(\boldsymbol{\beta})_k k!} \tag{1.1} $$

for $\boldsymbol{\alpha}$ in $\mathbb{C}^p$, $\boldsymbol{\beta}$ in $\mathbb{C}^q$ and z in $\mathbb{C}$, where $(\boldsymbol{\alpha})_k = (\alpha_1)_k(\alpha_2)_k \cdots (\alpha_p)_k$, $(z)_k = \Gamma(z+k)/\Gamma(z) = z(z+1)\cdots(z+k-1)$ for $k = 0, 1, 2, \ldots,$ and $(\boldsymbol{\beta})_k$ is defined similarly. For convenience, we set $\boldsymbol{\alpha}^* = (\alpha_2, \ldots, \alpha_p)$, $\boldsymbol{\beta}^* = (\beta_2, \ldots, \beta_q)'$, so that $\boldsymbol{\alpha} = (\alpha_1, \boldsymbol{\alpha}^*)$ and $\boldsymbol{\beta} = (\beta_1, \boldsymbol{\beta}^*)$.

The sum in (1.1) is finite if $\alpha_1 = 0, -1, -2, \ldots.$ It is absolutely convergent if $p < q+1$ or if $p = q+1$ and $|z| < 1$. These restrictions can be removed by redefining it in terms of an integral transform: see Mathai and Saxena (1973, 1978) and Mathai et al. (2010).

The aim of this short note is to provide some transformation tools for (1.1) when α_1 and β_1 differ by an integer. In Sections 2 and 3, we express ${}_pF_q$ in terms of ${}_{p-1}F_{q-1}$ when $\alpha_1 - \beta_1$ is a positive and negative integer, respectively. Several examples are given, including applications to noncentral chi-square and Student's t distributions.

The nth derivative of (1.1) is

$$ (d/dz)^n {}_pF_q(\boldsymbol{\alpha}; \boldsymbol{\beta} : z) = (\boldsymbol{\alpha})_n (\boldsymbol{\beta})_n^{-1} {}_pF_q(\boldsymbol{\alpha} + n\mathbf{1}_p; \boldsymbol{\beta} + n\mathbf{1}_q : z), $$

Key words and phrases. Generalized hypergeometric functions, Kummer's transformation.
Received January 2012; accepted May 2012.

where $\mathbf{1}_p$ is the vector of p 1's. So,

$$ {}_pF_q(\boldsymbol{\alpha}+n\mathbf{1}_p;\boldsymbol{\beta}+n\mathbf{1}_q:z)=(\boldsymbol{\alpha})_n^{-1}(\boldsymbol{\beta})_n(d/dz)^n\,{}_pF_q(\boldsymbol{\alpha};\boldsymbol{\beta}:z). \tag{1.2} $$

The nth integral of (1.1) is

$$ \left(\int_0^z dz\right)^n {}_pF_q(\boldsymbol{\alpha};\boldsymbol{\beta}:z)=(\boldsymbol{\alpha}-n\mathbf{1}_p)_n^{-1}(\boldsymbol{\beta}-n\mathbf{1}_q)_n\,{}_pF_q^n(\boldsymbol{\alpha}-n\mathbf{1}_p;\boldsymbol{\beta}-n\mathbf{1}_q:z), $$

where ${}_pF_q^n(\boldsymbol{\alpha};\boldsymbol{\beta}:z)$ denotes the series for ${}_pF_q(\boldsymbol{\alpha};\boldsymbol{\beta}:z)$ less its first n terms. Note that

$$ {}_pF_q^n(\boldsymbol{\alpha};\boldsymbol{\beta}:z)=\frac{(\boldsymbol{\alpha})_n z^n}{(\boldsymbol{\beta})_n n!}\,{}_{p+1}F_{q+1}(\boldsymbol{\alpha}+n\mathbf{1}_p,1;\boldsymbol{\beta}+n\mathbf{1}_q,1+n:z), \tag{1.3} $$

which we can rewrite as

$$ \begin{aligned} &{}_pF_q\big(\boldsymbol{\alpha}^*,1;\boldsymbol{\beta}^*,1+n:z\big)\\ &\quad={}_{p-1}F_{q-1}^n\big(\boldsymbol{\alpha}^*-n\mathbf{1}_{p-1};\boldsymbol{\beta}^*-n\mathbf{1}_{q-1}:z\big)\frac{(\boldsymbol{\beta}^*-n\mathbf{1}_{q-1})_n n!}{(\boldsymbol{\alpha}^*-n\mathbf{1}_{p-1})_n z^n}. \end{aligned} $$

Compare this with (3.2) below with $\alpha_1=1$.

Mathai and Saxena (1973, pages 165–166) (see also Mathai and Saxena (1978) and Mathai et al. (2010)) give several examples of ${}_2F_1(\boldsymbol{\alpha};\beta:z)$ with coefficients differing by an integer that are not covered by our results: see equation (5.4.2) for $\alpha_2=\alpha_1+m$, equation (5.4.3) for $\alpha_2=\alpha_1+m$, $\beta=\alpha_2+n$, equations (5.4.4) and (5.4.5) for $\beta=\alpha_1+\alpha_2+m$, equations (5.4.6) and (5.7) for $\beta=\alpha_1+\alpha_2-m$. Analogous results for ${}_pF_q(\boldsymbol{\alpha};\boldsymbol{\beta}:z)$ are presumably deducible from their Theorems 5.5.1, 5.6.1, 5.7.1 using their equation (1.1.9).

If $(\alpha_1,\dots,\alpha_m)=(\alpha_1,\alpha_1+1/m,\dots,\alpha_1+(m-1)/m)$ for some $m=1,2,\dots$, then

$$ (\alpha_1)_k\cdots(\alpha_m)_k=m^{-mk}(m\alpha_1)_{mk}. $$

So, if in (1.1) $(\alpha_1)_k$ is changed to $(\alpha_1)_{mk}$ or if $(\beta_1)_k$ is changed to $(\beta_1)_{mk}$, the series remains a hypergeometric function with p or q, respectively, increased by $m-1$.

2 When arguments differ by a positive integer

If $\alpha_1=\beta_1$, then ${}_pF_q(\boldsymbol{\alpha};\boldsymbol{\beta}:z)={}_{p-1}F_{q-1}(\boldsymbol{\alpha}^*;\boldsymbol{\beta}^*:z)$. In this section, we assume that $m=\alpha_1-\beta_1=0,1,2,\dots$. We state our first main result as follows.

Theorem 2.1. *For* $m=\alpha_1-\beta_1=0,1,\dots,$

$$ \begin{aligned} &{}_pF_q(\boldsymbol{\alpha};\boldsymbol{\beta}:z)\\ &\quad=\sum_{j=0}^m\binom{m}{j}\,{}_{p-1}F_{q-1}\big(\boldsymbol{\alpha}^*+j\mathbf{1}_{p-1};\boldsymbol{\beta}^*+j\mathbf{1}_{q-1}:z\big)\big(\boldsymbol{\alpha}^*\big)_j z^j/(\boldsymbol{\beta})_j. \end{aligned} \tag{2.1} $$

Proof. Note that $(\alpha_1)_i/(\beta_1)_i = (\beta_1+m)_i/(\beta_1)_i = (\beta_1+i)_m/(\beta_1)_m$ and

$$(\beta_1+i)_m = \sum_{j=0}^{m} d_{mj}(\alpha_1)[i]_j, \tag{2.2}$$

where $[z]_j = z(z-1)\cdots(z-j+1) = \Gamma(z+1)/\Gamma(z+1-j) = (-1)^j(-z)_j$ and $d_{mj}(\alpha_1) = \binom{m}{j}[\alpha_1-1]_{m-j}$ ((2.2) follows from the last displayed equation in "Relation to umbral calculus" section of http://en.wikipedia.org/wiki/Pochhammer_symbol). Also $[i]_j/i! = 1/(i-j)!$. Setting $k = i-j$ gives

$$_pF_q(\boldsymbol{\alpha};\boldsymbol{\beta}:z) = (\beta_1)_m^{-1}\sum_{j=0}^{m} d_{mj}(\alpha_1)z^j \sum_{k=0}^{\infty}\frac{(\boldsymbol{\alpha}^*)_{k+j}z^k}{(\boldsymbol{\beta}^*)_{k+j}k!}.$$

But $(\boldsymbol{\alpha}^*)_{k+j} = (\boldsymbol{\alpha}^*)_j(\boldsymbol{\alpha}^* + j\mathbf{1}_{p-1})_k$. The theorem now follows using $[\beta_1+m-1]_{m-j}/(\beta_1)_m = 1/(\beta_1)_j$. □

Theorem 2.1 leads us to conjecture that whether or not m is an integer,

$$\begin{aligned}&{}_pF_q(\beta_1+m,\boldsymbol{\alpha}^*;\boldsymbol{\beta}:z)\\ &\quad= \sum_{j=0}^{\infty} {}_{p-1}F_{q-1}\big(\boldsymbol{\alpha}^*+j\mathbf{1}_{p-1};\boldsymbol{\beta}^*+j\mathbf{1}_{q-1}:z\big)\frac{(\boldsymbol{\alpha}^*)_j(-m)_j(-z)^j}{(\boldsymbol{\beta})_j j!}.\end{aligned}$$

The next two examples show that this is true for $(p,q) = (1,\ 1)$ and $(2,\ 1)$.

Example 2.1. By (2.1), for $m = 0, 1, \dots$

$$\begin{aligned}{}_1F_1(\beta+m;\beta:z) &= \sum_{j=0}^{\infty}(\beta+m)_j z^j/\{(\beta)_j j!\}\\ &= \exp(z)\sum_{j=0}^{m}\binom{m}{j}z^j/(\beta)_j = \exp(z)\,{}_1F_1(-m;\beta:-z).\end{aligned}$$

In fact, this holds for any m in $\mathbb{C}$ by Kummer's transformation, equation (13.1.27) of Abramowitz and Stegun (1964).

If $p=q$, one can apply the previous example iteratively. For example, if $p=q$ and $\boldsymbol{\alpha}-\boldsymbol{\beta}$ is a positive vector integer it follows that $\exp(-z)\,{}_pF_q(\boldsymbol{\alpha};\boldsymbol{\beta}:z)$ is a polynomial in z of degree $(\alpha_1-\beta_1)+\cdots+(\alpha_p-\beta_p)$.

Example 2.2. By (2.1),

$$\begin{aligned}{}_2F_1(\beta+m,\alpha;\beta:z) &= \sum_{j=0}^{m}\binom{m}{j}z^j(1-z)^{-\alpha-j}(\alpha)_j/(\beta)_j\\ &= (1-z)^{-\alpha}\,{}_2F_1\big(\alpha,-m;\beta:z/(z-1)\big).\end{aligned}$$

In fact, this holds for any m in $\mathbb{C}$ by equation (15.3.4) of Abramowitz and Stegun (1964). This covers the cases of equation (9.121.23) of Gradshteyn and Ryzhik (2007), and give alternative forms of equations (9.121.16), (9.121.21), (9.121.29), (9.121.32) with n even, and of equations (9.121.17), (9.121.20), (9.121.30), (9.121.31) with n odd.

Example 2.3. By (2.1),

$$
\begin{aligned}
{}_1F_2(3/2;\beta_1,1/2:z) &= f_1 + 2zf_3/\beta_1,\\
{}_1F_2(5/2;\beta_1,1/2:z) &= f_1 + (4/3)zf_3/\beta_1 + 4z^2 f_5/(\beta_1)_2,\\
{}_1F_2(7/2;\beta_1,1/2:z) &= f_1 + 6zf_3/\beta_1 + 4z^2 f_5/(\beta_1)_2 + (8/15)z^3 f_7/(\beta_1)_3,
\end{aligned}
$$

and so on, where $f_j = {}_0F_1(j/2:z)$. Set $Z = 2z^{1/2}$, $C = \cosh Z$ and $S = \sinh Z$. Then $f_1 = C$. By (1.2),

$$
\begin{aligned}
f_3 &= S/Z,\\
f_5 &= 3Z^{-2}(C - S/Z),\\
f_7 &= 15Z^{-5}\big(SZ^2 - 3CZ + 3S\big),
\end{aligned}
$$

and so on. In this way one may express ${}_1F_2(j/2;\beta_1,1/2:z)$ as a linear combination of C and S. More generally, by equation (9.6.10) of Abramowitz and Stegun (1964, page 375),

$$
{}_0F_1(\beta:z) = \Gamma(\beta)z^{(1-\beta)/2} I_{\beta-1}(Z), \tag{2.3}
$$

where $Z = 2z^{1/2}$, where $I_\nu(\cdot)$ is the modified Bessel function of the first kind of order ν. So, $f_j = \Gamma(j/2)(2/Z)^{j/2-1} I_{j/2-1}(Z)$, which is given in terms of C, S by equations (10.2.12)–(10.2.14) of Abramowitz and Stegun (1964, page 443).

Example 2.1 allows one to find the moments of the noncentral chi-square random variable $X \sim \chi^2_\nu(\delta)$. For, by equation (3) of Johnson and Kotz (1970, page 132),

$$
\begin{aligned}
\mathbb{E}\big[X^\theta\big] &= \exp(-\delta/2)2^\theta\, {}_1F_1(\nu/2+\theta;\nu/2:\delta/2)\Gamma(\nu/2+\theta) && (2.4)\\
&= \exp(-\delta/2)2^m\Gamma(\nu/2+m)\sum_{j=0}^{\infty}(\nu/2+m)_j(\delta/2)^j/\{(\nu/2)_j\, j!\} && (2.5)
\end{aligned}
$$

if $\theta = m$. Note that $(\beta+j)_{m-j} = (\beta+j)(\beta+j+1)\cdots(\beta+m-1) = \Gamma(\beta+m)/\Gamma(\beta+j)$. This expression (2.5) appears to be new. For comparison, by Johnson and Kotz (1970, page 133), it follows that the mth cumulant of $\chi^2_\nu(\delta)$ is

$$
\kappa_m(X) = 2^{m-1}(m-1)!(\nu+m\delta).
$$

Note that $X^{1/2}$ is known to electrical engineers as the Ricean random variable. By (2.4) its mean requires ${}_1F_1(\beta+1/2;\beta:z)$ for $\beta = \nu/2$. This is one of many cases where it would be nice to have a simplification for ${}_pF_q(\boldsymbol{\alpha};\boldsymbol{\beta}:z)$ when $2(\alpha_1-\beta_1)$ is an integer.

3 When arguments differ by a negative integer

In this section, we suppose that $n = \beta_1 - \alpha_1$ is a positive integer. So, $(\alpha_1)_i/(\alpha_1 + n)_i = (\alpha_1)_n/(\alpha_1 + i)_n$. Expanding in partial fractions gives

$$1/(\alpha_1 + i)_n = \sum_{j=0}^{n-1} c_{nj}/(\alpha_1 + i + j), \tag{3.1}$$

where

$$\begin{aligned} c_{nj} &= (z + j)/(z)_n|_{z=-j} = (-1)^j \binom{n-1}{j} \Big/ (n-1)! \\ &= (-1)^j / \{n! B(n - j, j + 1)\}, \end{aligned}$$

where $B(\cdot, \cdot)$ denotes the beta function. That is,

$$\begin{aligned} c_{10} &= 1, \\ (c_{20}, c_{21}) &= (1, -1), \\ (c_{30}, c_{31}, c_{32}) &= (1, -2, 1)/2!, \\ (c_{40}, c_{41}, c_{42}, c_{43}) &= (1, -3, 3, -1)/3!, \end{aligned}$$

and so on. So, using the notation of (1.3) we obtain our second main result.

Theorem 3.1. *For* $n = 1, 2, \ldots,$

$${}_pF_q\big(\boldsymbol{\alpha}; \alpha_1 + n, \boldsymbol{\beta}^* : z\big) = (\alpha_1)_n \sum_{j=0}^{n-1} c_{nj}\, {}_pA_{q-1}\big(\alpha_1 + j, \boldsymbol{\alpha}^*; \boldsymbol{\beta}^* : z\big), \tag{3.2}$$

where

$${}_pA_{q-1}\big(\boldsymbol{\alpha}; \boldsymbol{\beta}^* : z\big) = \sum_{i=0}^{\infty} \frac{(\boldsymbol{\alpha}^*)_i z^i}{(i + \alpha_1)(\boldsymbol{\beta}^*)_i i!}.$$

If $\alpha_1 > 0$, *then*

$${}_pA_{q-1}\big(\boldsymbol{\alpha}; \boldsymbol{\beta}^* : z\big) = z^{-\alpha_1} \int_0^z y^{\alpha_1 - 1}\, {}_{p-1}F_{q-1}\big(\boldsymbol{\alpha}^*; \boldsymbol{\beta}^* : y\big)\, dy.$$

If $\alpha_1 < 0$, *then*

$$\begin{aligned} {}_pA_{q-1}\big(\boldsymbol{\alpha}; \boldsymbol{\beta}^* : z\big) = &\sum_{i=0}^{[-\alpha_1]} \big(\boldsymbol{\alpha}^*\big)_i z^i / \{(i + \alpha_1)\big(\boldsymbol{\beta}^*\big)_i i!\} \\ &+ z^{-\alpha_1} \int_0^z y^{\alpha_1 - 1}\, {}_{p-1}F_{q-1}^{1+[-\alpha_1]}\big(\boldsymbol{\alpha}^*; \boldsymbol{\beta}^* : y\big)\, dy, \end{aligned} \tag{3.3}$$

where $[x]$ *is the integral part of* x.

For example, taking $n = 1, 2$ gives

$$ {}_pF_q\big(\boldsymbol{\alpha}; \alpha_1 + 1, \boldsymbol{\beta}^* : z\big) = \alpha_1\, {}_pA_{q-1}\big(\boldsymbol{\alpha}; \boldsymbol{\beta}^* : z\big), \tag{3.4} $$

$$ \begin{aligned} {}_pF_q\big(\boldsymbol{\alpha}; \alpha_1 + 2, \boldsymbol{\beta}^* : z\big) = (\alpha_1)_2 \big\{ -{}_pA_{q-1}\big(\alpha_1 + 1, \boldsymbol{\alpha}^*; \boldsymbol{\beta}^* : z\big) \\ + {}_pA_{q-1}\big(\boldsymbol{\alpha}; \boldsymbol{\beta}^* : z\big)\big\}. \end{aligned} \tag{3.5} $$

The theorem is useful because it expresses ${}_pF_q$ in terms of ${}_{p-1}F_{q-1}$. As an example of (3.3), if $-1 < \alpha_1 < 0$, then

$$ {}_pA_{q-1}\big(\boldsymbol{\alpha}; \boldsymbol{\beta}^* : z\big) = {\alpha_1}^{-1} + z^{-\alpha_1} \int_0^z y^{\alpha_1 - 1} \big\{ {}_{p-1}F_{q-1}\big(\boldsymbol{\alpha}^*; \boldsymbol{\beta}^* : y\big) - 1 \big\}\, dy. $$

For an application of (3.3) to ${}_2F_1$, see equation (9.121.2) of Gradshteyn and Ryzhik (2007).

Example 3.1. Note that ${}_0F_0(y) = \exp(y)$. Set $p = q = 1$. For $\alpha > 0$,

$$ \begin{aligned} {}_1A_0(\alpha : z) &= z^{-\alpha} \int_0^z y^{\alpha-1} \exp(y)\, dy \\ &= \exp(z) \sum_{i=0}^{\infty} (-z)^i / (\alpha)_{i+1} && (3.6) \\ &= \alpha^{-1} \exp(z)\, {}_1F_1(1; \alpha + 1 : -z) && (3.7) \\ &= (-z)^{-\alpha} \Gamma(\alpha) \left\{ 1 - \exp(z) \sum_{i=0}^{\alpha-1} (-z)^i / i! \right\} && (3.8) \end{aligned} $$

for $\alpha = 1, 2, \ldots$. The first series is obtained by integrating by parts to obtain a recurrence relation and then showing the remainder after n terms goes to zero as $n \to \infty$. The last line is equivalent to equation (6.5.13) of Abramowitz and Stegun (1964, page 260). So, by (3.2), for $\alpha > 0$,

$$ \begin{aligned} {}_1F_1(\alpha; \alpha + n : z) &= (\alpha)_n \exp(z) \sum_{j=0}^{n-1} c_{nj} (\alpha + j)^{-1}\, {}_1F_1(1; \alpha + j + 1 : -z) \\ &= (\alpha)_n \exp(z) \sum_{i=0}^{\infty} (-z)^i c_n^i(\alpha), \end{aligned} $$

where

$$ c_n^i(\alpha) = \sum_{j=0}^{n-1} c_{nj} / (\alpha + j)_{i+1} = (n)_i / \{ i!(\alpha)_{n+i} \}, $$

where the last equality follows by (3.1). So, we obtain

$$ {}_1F_1(\alpha; \alpha + n : z) = \exp(z)\, {}_1F_1(n; \alpha + n : -z), \tag{3.9} $$

a form of Kummer's transformation. From (3.2), (3.8) above, for $\alpha = 1, 2, \ldots$,

$$ {}_1F_1(\alpha; \alpha + n : z) = \sum_{j=0}^{n-1} c_{nj} (-z)^{-\alpha-j} \Gamma(\alpha + j) \left\{ 1 - \exp(z) \sum_{i=0}^{\alpha+j-1} (-z)^i / i! \right\}. $$

For example, taking $n = 1$ gives for $\alpha = 1, 2, \ldots$,

$$ {}_1F_1(\alpha; \alpha + 1 : z) = (-z)^{-\alpha} \Gamma(\alpha + 1) \left\{ 1 - \exp(z) \sum_{i=0}^{\alpha-1} (-z)^i / i! \right\}. $$

For $\alpha < 0$,

$$ {}_1A_0(\alpha : z) = \sum_{i=0}^{[-\alpha]} (i + \alpha)^{-1} z^i / i! + z^{-\alpha} \int_0^z y^{\alpha-1} \left\{ \exp(y) - \sum_{i=0}^{[-\alpha]} y^i / i! \right\} dy, $$

which is equal to the right hand side of (3.6) for $\alpha \neq -1, -2, \ldots$.

Example 3.2. For $\mathrm{Re}(y) \leq 1$ and $\mathrm{Re}(\alpha_2) \geq 1$, ${}_1F_0(\alpha_2 : y) = (1 - y)^{-\alpha_2}$. So, for $\alpha_1 > 0$, $(p, q) = (2, 1)$ and $\mathrm{Re}(z) \leq 1$,

$$ \begin{aligned} {}_2A_0(\boldsymbol{\alpha} : z) &= z^{-\alpha_1} \int_0^z y^{\alpha_1-1} (1 - y)^{-\alpha_2} \, dy \\ &= 2z^{-\alpha_1} \int_0^\theta \sin^{2\alpha_1-1}(\theta) \cos^{1-2\alpha_2}(\theta) \, d\theta \end{aligned} \tag{3.10} $$

at $\sin^2(\theta) = z$. This has a closed form if, for example,

(i) $\alpha_1 = 1, 2, 3, \ldots$;
(ii) $\alpha_1 = 1/2, 3/2, 5/2, \ldots$ and $\alpha_2 = 1/2, 0, -1/2, \ldots$;
(iii) $\alpha_1 = 1/2, \alpha_2 = 1$.

Note that (i) covers the cases of equations (9.121.5), (9.121.6), (9.121.7), (9.121.24) of Gradshteyn and Ryzhik (2007), (ii) covers the cases of equations (9.121.13), (9.121.26), (9.121.28) and (iii) covers the cases of equations (9.121.15), (9.121.27). In all of these cases except the first, $n = 1$ so that by (3.4),

$$ {}_2F_1(\boldsymbol{\alpha}; \alpha_1 + 1 : z) = \alpha_1 \, {}_2A_0(\boldsymbol{\alpha} : z) \tag{3.11} $$

of (3.10). Another special case of (3.11) is given by equation (4.5) of Johnson and Kotz (1970, page 96): Student's t distribution with ν degrees of freedom is

$$ \Pr(t_\nu < t) = 1/2 + ct \, {}_2F_1\big(1/2, \alpha_2; 3/2 : -t^2/\nu\big) $$

for $t^2 < \nu$, where $\alpha_2 = (\nu + 1)/2$ and $c = (\pi\nu)^{-1/2} \Gamma(\alpha_2) \Gamma(\nu/2)^{-1}$. By (3.11),

$$ \begin{aligned} {}_2F_1(1/2, \alpha_2; 3/2 : -z) &= 2^{-1} {}_2A_0(1/2, \alpha_2 : -z) \\ &= 2^{-1} (-z)^{-1/2} \int_0^{-z} y^{-1/2} (1 - y)^{-\alpha_2} \, dy \\ &= -2^{-1} z^{-1/2} \int_0^z x^{-1/2} (1 + x)^{-\alpha_2} \, dx. \end{aligned} $$

Corollary 3.1. *For* $\alpha_1 > 0$,

$$
\begin{aligned}
&{}_2F_1(\boldsymbol{\alpha};\alpha_1+n:z) \\
&\quad = (\alpha_1)_n(1-z)^{-\alpha_2}\sum_{j=0}^{n-1} c_{nj}(\alpha_1+j)^{-1}{}_2F_1\big(1,\alpha_2;\alpha_1+1+j:z/(1-z)\big) && (3.12)\\
&\quad = (1-z)^{-\alpha_2}{}_2F_1\big(n,\alpha_2;\alpha_1+n:z/(1-z)\big). && (3.13)
\end{aligned}
$$

Proof. Set

$$
\begin{aligned}
J(\boldsymbol{\alpha}) &= z^{\alpha_1}{}_2A_0(\boldsymbol{\alpha}:z) = \int_0^z y^{\alpha_1-1}(1-y)^{-\alpha_2}\,dy,\\
\mathbf{q} &= (q_1,q_2) = \big(z,1/(1-z)\big),\\
K(\boldsymbol{\alpha}) &= \mathbf{q}^{\boldsymbol{\alpha}} = z^{\alpha_1}(1-z)^{\alpha_2}.
\end{aligned}
$$

Integrating by parts gives for $\boldsymbol{\alpha} > \mathbf{0}$,

$$
\begin{aligned}
J(\boldsymbol{\alpha}) &= \alpha_1^{-1}\big\{K(\boldsymbol{\alpha}) - \alpha_2 J(\boldsymbol{\alpha}+\mathbf{1}_2)\big\}\\
&= K(\boldsymbol{\alpha})/\alpha_1 - \alpha_2K(\boldsymbol{\alpha}+\mathbf{1}_2)/(\alpha_1)_2 + (\alpha_2)_2K(\boldsymbol{\alpha}+2\mathbf{1}_2)/(\alpha_1)_3 - \cdots\\
&= \alpha_1^{-1}\mathbf{q}^{\boldsymbol{\alpha}}{}_2F_1(1,\alpha_2;\alpha_1+1:t),
\end{aligned}
$$

where $t = -q_1q_2 = z/(z-1)$. So, by (3.2),

$$
\begin{aligned}
{}_2F_1(\boldsymbol{\alpha};\alpha_1+n:z) &= (\alpha_1)_n q_2^{\alpha_2}\sum_{j=0}^{n-1} c_{nj}(\alpha_1+j)^{-1}{}_2F_1(1,\alpha_2;\alpha_1+j+1:t)\\
&= (\alpha_1)_n q_2^{\alpha_2}\sum_{i=0}^{\infty} t^i(\alpha_2)_i d_n^i(\alpha_1),
\end{aligned}
$$

where

$$
d_n^i(\alpha_1) = \sum_{j=0}^{n-1} c_{nj}/(\alpha_1+j)_{i+1} = (n)_i/\big\{i!(\alpha_1)_{n+i}\big\},
$$

where the last equality follows by (3.1). □

So, just as (3.2) with (3.7) reduces ${}_1F_1(\alpha;\alpha+n:z)$ to a sum of n ${}_1F_1$'s with $\alpha = 1$, and (3.9) reduces ${}_1F_1(\alpha;\alpha+n:z)$ to $\alpha = n$, so does (3.12) reduce ${}_2F_1(\boldsymbol{\alpha};\alpha_1+n:z)$ to a sum of n ${}_2F_1$'s with $\alpha_1 = 1$, and (3.13) reduces ${}_2F_1(\boldsymbol{\alpha};\alpha_1+n:z)$ to $\alpha_1 = n$.

If $\alpha_2 < 1$, one can write (3.10) in the form

$$
{}_2A_0(\alpha_1,\alpha_2:z) = z^{-\alpha_1}B(\alpha_1,1-\alpha_2)I_z(\alpha_1,1-\alpha_2),
$$

where $I_z(\alpha, \beta)$ is the distribution of a beta random variable. This is the notation of Section 26.5 of Abramowitz and Stegun (1964, page 944). Also given in Abramowitz and Stegun (1964) are expansions for the incomplete beta function. The coefficient of the second term in equation (26.5.12) should be a not b. Finite expansions are available when α_1 or $1-\alpha_2$ are positive integers. The incomplete beta is a simple transformation of the F distribution.

Example 3.3. Note that ${}_1F_2(\alpha; \alpha+n, \beta : z)$ is given by (3.2) in terms of ${}_1A_1$. For $\alpha > 0$,

$$ {}_1A_1(\alpha; \beta : z) = z^{-\alpha} \int_0^z y^{\alpha-1} {}_0F_1(\beta : y)\, dy = \Gamma(\beta) z^{-\alpha} \int_0^Z (x/2)^{2\alpha-\beta} I_{\beta-1}(x)\, dx $$

by (2.3), where $Z = 2z^{1/2}$. For example, by (3.4)–(3.5) and equation (10.2.13) of Abramowitz and Stegun (1964, page 443),

$$ \begin{aligned} {}_1F_2(1/2; 3/2, 3/2 : z) &= (1/2) {}_1A_1(1/2; 3/2 : z) \\ &= Z^{-1} \Gamma(1.5) 2^{1/2} \int_0^Z I_{0.5}(x)/x^{0.5}\, dx \\ &= Z^{-1} \int_0^Z x^{-1} \sinh x\, dx, \end{aligned} $$

and

$$ {}_1F_2(1/2; 5/2, 3/2 : z) = (1/2) {}_1A_1(1/2; 5/2 : z), $$

where

$$ \begin{aligned} {}_1A_1(3/2; 3/2 : z) &= z^{-1.5} \Gamma(1.5) \int_0^Z (x/2)^{1.5} I_{0.5}(x)\, dx \\ &= 2Z^{-3} \int_0^Z x \sinh x\, dx \\ &= 2Z^{-3} (Z \cosh Z - \sinh Z), \end{aligned} $$

and

$$ \begin{aligned} {}_1A_1(1/2; 5/2 : z) &= 3\sqrt{2\pi} Z^{-1} \int_0^Z I_{1.5}(x)/x^{1.5}\, dx \\ &= 6Z^{-1} \int_0^Z (\cosh x/x^2 - \sinh x/x^3)\, dx. \end{aligned} $$

Acknowledgments

The authors would like to thank the Editor and the referee for careful reading and for their comments which greatly improved the paper.

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