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Stationary infinitely divisible processes

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Abstract. Several recent strands of work has led to the consideration of various types of continuous time stationary and infinitely divisible processes. A review of these types, with some new results, is presented.

1 Introduction

Recent years have seen a revival of interest in continuous time strictly stationary processes. Much of this interest have arisen out of problems in stochastic modelling, coming in particular from the areas of turbulence and of finance. An important aspect is the fact that strictly stationary processes are mostly not of the semimartingale type, so that the comprehensive and very powerful theory of semimartingales is generally not applicable, at least not in a direct fashion, and instead new tools have to be developed, drawing among other things on Malliavin calculus. Among the stationary processes those that have the additional property of being infinitely divisible are of some special interest, both as regards their theoretical properties and their potential in modelling. For brevity we shall refer to continuous time strictly stationary infinitely divisible processes as *SID* processes.

The present paper reviews these developments and adds some new results, in particular introducing a concept of extended subordination and a new type of ambit processes called trawling processes.

We shall primarily discuss one-dimensional processes $Y = \{Y_t\}_{t\in\mathbb{R}}$ that can be represented on the mixed moving average form

$$Y_t = \int_{\mathbb{R}\times\mathbb{R}^d} G(t-s,\xi)M(\mathrm{d}s\,\mathrm{d}\xi), \tag{1.1}$$

where G is a deterministic function and M denotes a homogeneous random measure on $\mathbb{R}\times\mathbb{R}^d$, whose values are infinitely divisible. In case the random measure M is a homogeneous Lévy basis the formula (1.1) is said to constitute a spectral representation or decomposition of the process Y. Multivariate versions of (1.1) and extensions to tempo-spatial settings with stationarity in space and time simultaneously will also be considered.

More specifically, the focus will be on the cases where the function G has the form

$$G(u,\xi) = 1_A(u,\xi)g(u,\xi)$$

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for some subset A of $(-\infty, 0] \times \mathbb{R}^d$ and a nonnegative damping function g, and where $M(\mathrm{d}s\,\mathrm{d}\xi)$ is a subordinated homogeneous Lévy basis $L(\mathrm{d}s\,\mathrm{d}\xi \curlywedge \tau)$.[1] Then (1.1) can be written as

$$Y_t = \int_{A+(t,0)} g(t-s, \xi) L(\mathrm{d}s\,\mathrm{d}\xi \curlywedge \tau). \tag{1.2}$$

The special case (1.2) offers both analytical tractability and modelling flexibility.

More generally one considers homogeneous tempo-spatial ambit fields $\{Y_t(x)\}_{(t,x)\in\mathbb{R}\times\mathbb{R}^d}$ where

$$Y_t(x) = \int_{\mathbb{R}\times\mathbb{R}^d} G(t-s, x-\xi) L(\mathrm{d}s\,\mathrm{d}\xi \curlywedge \tau)$$

and ambit processes embedded in such fields.

Section 2 provides various background material on infinite divisibility and Lévy bases, and Section 3 introduces a generalisation of the concepts of subordination and time change to Lévy bases and tempo-spatial settings. Section 4 briefly reviews some basic ideas and results from ambit stochastics and draws connection to the concept of extended subordination from Section 3. The question of spectral representability is the subject of Section 5. Section 6 then discusses the null-spatial case, that is, where there is no spatial component in (1.1) and (1.2) or, in other words, where $d = 0$. Section 7 treats tempo-spatial settings, that is, where $d > 0$, and discusses OU related processes, volatility modulation using extended subordination, and a new special type of tractable ambit processes termed *trawlings*.

2 Background

In terms of monographic accounts, the literature on general theory of strictly stationary continuous time processes is quite limited, the monograph by Cramér and Leadbetter (1967) being still a standard reference; but infinite divisibility is not a topic in that book. Another useful reference is Chapter XI of Doob (1990), but again, infinite divisibility is not an issue there.

2.1 Infinite divisibility

The basic theory of infinitely divisible distributions and Lévy processes is treated comprehensible and in detail in the already classic monograph by Ken-iti Sato (1999). Maruyama (1970) was the first to give a systematic account of infinite divisibility of processes. Some extensions and clarifications of his work are discussed in Barndorff-Nielsen, Maejima and Sato (2006b) and in Rosiński (2007).

[1] As defined in Barndorff-Nielsen (2010), Barndorff-Nielsen and Pedersen (2010) and discussed in Section 3 below.

By definition a process $X = \{X_t\}_{t \in \mathbb{T}}$, where $\mathbb{T}$ is an arbitrary set, is said to be infinitely divisible if any finite dimensional marginal law is infinitely divisible in the classical sense. Rosiński (2007) proposes a definition of the Lévy measure of X that is somewhat different from and seems more suitable than that of Maruyama (1970), and in terms of this the corresponding Lévy–Khintchine and Lévy–Ito representations are quite similar to those for the classical finite dimensional case.

Conditions for mixing of infinitely divisible processes, in particular *SID* processes, are discussed in Fuchs and Stelzer (2011).

We shall denote the class of m-dimensional infinitely divisible distributions by $\mathrm{ID}(\mathbb{R}^m)$. The subclass $\mathrm{SD}(\mathbb{R}^m)$ of $\mathrm{ID}(\mathbb{R}^m)$ consisting of the m-dimensional self-decomposable laws is of some special interest in the present context. Recall that an m-dimensional random variable X and its law μ are said to be self-decomposable provided that, for all $c \in (0, 1)$, X can be represented in law as

$$X \overset{\text{law}}{=} cX + X_c, \tag{2.1}$$

where the m-dimensional random variable X_c is independent of X. Any such X is necessarily infinitely divisible, and the same is true of the associated X_c. The property of self-decomposability can be simply characterised in terms of the Lévy measure ν of μ. Thus, in the one-dimensional case the condition on ν for μ to be self-decomposable is that ν is absolutely continuous with a density u having the property that $\bar{u}(x) = |x|u(x)$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

In the context of the present paper the main aspect of self-decomposability is its relation to processes of Ornstein–Uhlenbeck type, or OU processes for short. Such processes will be discussed in Sections 6.1 and 7.1. Suffice it here to mention that the one-dimensional OU processes Y are continuous time analogues of AR(1) time series, satisfying a stochastic differential equation

$$\mathrm{d}Y_t = -\lambda Y_t + \mathrm{d}L_t,$$

where L is a Lévy process and $\lambda > 0$. Such an SDE allows a stationary solution if and only if the Lévy measure ν of L satisfies $\int_{|x|>1} \log(1 + |x|)\nu(\mathrm{d}x) < \infty$. And a one-dimensional *SID* process satisfies an SDE of this type if and only if the marginal law of Y_t is in $\mathrm{SD}(\mathbb{R})$.

2.2 Lévy bases

This section recalls basic definitions and properties of Lévy bases on $\mathbb{R}^k$. For proofs and mathematical details we refer to Rajput and Rosiński (1989) and Pedersen (2003).[2]

Let $\mathcal{B}$ denote the family of Borel sets in $\mathbb{R}^k$ and let $\mathcal{B}_b$ be the subfamily consisting of the bounded elements of $\mathcal{B}$. An independently scattered random measure

[2]These authors present the theory of infinitely divisible random measures on spaces $\mathcal{S}$ more general than $\mathbb{R}^k$. We shall consider some instances of this in Sections 5 and 7.

M on $\mathbb{R}^k$ is a collection $\{M(B) : B \in \mathcal{B}_b\}$ of random variables on some probability space $(\Omega, \mathcal{A}, P)$ such that for every sequence $\{B_n\}$ of mutually disjoint sets in $\mathcal{B}_b$ with $\bigcup_{n=1}^{\infty} B_n \in \mathcal{B}_b$ the random variables $M(B_n)$, $n = 1, 2, \ldots,$ are independent and

$$M\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} M(B_n) \qquad \text{a.s.}$$

Note that, in general, M may take both negative and positive values.

Definition. A Lévy basis L on $\mathbb{R}^k$ is an independently scattered random measure on $\mathbb{R}^k$ such that for all $B \in \mathcal{B}_b$ the random variable $L(B)$ is infinitely divisible.

If L is such a basis then it has a Lévy–Khintchine representation of the form[3]

$$\begin{aligned} \mathrm{C}\{\zeta \ddagger L(B)\} = {} & i a(B)\zeta - \frac{1}{2} m(B)\zeta^2 \\ & + \int_{-\infty}^{\infty} (e^{i\zeta x} - 1 - i\zeta x 1_{[-1,1]}(x)) n(\mathrm{d}x; B), \end{aligned} \tag{2.2}$$

where a and m are measures on $\mathbb{R}$ (a in general signed) and $n(\mathrm{d}x; B)$ is for fixed B a Lévy measure on $\mathbb{R}$ and for fixed $\mathrm{d}x$ a Lévy measure on $\mathbb{R}^k$. For the proof of this fact, see Rajput and Rosiński (1989).

The associated measure

$$c(B) = \|a\|(B) + m(B) + \int_{-\infty}^{\infty} (1 \wedge x^2) n(\mathrm{d}x; B),$$

where $\|a\|$ denotes the total variation of a, is called the control meaure of (2.2). We introduce the Radon–Nikodym derivatives

$$a(s) = \frac{\mathrm{d}a}{\mathrm{d}c}(s),$$

$$m(s) = \frac{\mathrm{d}m}{\mathrm{d}c}(s)$$

and

$$\nu(\mathrm{d}x; s) = \frac{n(\mathrm{d}x; \cdot)}{\mathrm{d}c}(s).$$

Thus, in particular,

$$n(\mathrm{d}x; \mathrm{d}s) = \nu(\mathrm{d}x; s) c(\mathrm{d}s).$$

[3]We denote the cumulant function of a random variable Y by $\mathrm{C}\{\zeta \ddagger Y\}$ and the cumulant function of Y conditional on another random variable X by $\mathrm{C}\{\zeta \ddagger Y|X\}$. Similarly, we write $\mathrm{K}\{z \ddagger Y\}$ and $\mathrm{K}\{z \ddagger Y|X\}$ for the associated log Fourier–Laplace transforms, and $\phi(\zeta \ddagger Y)$ and $\phi(\zeta \ddagger Y|X)$ for the characteristic functions. Thus, for example, $\mathrm{K}\{z \ddagger Y\} = \log \mathrm{E}\{e^{zY}\}$ (where $z = \eta + i\zeta$).

There is no loss of generality in assuming that $\nu(\mathrm{d}x; s)$ is a Lévy measure for each fixed s and we do so.

In other words, any Lévy basis on $\mathbb{R}^k$ determines a quadruplet $(a, m, \nu(\mathrm{d}x; \cdot), c) = (a(s), m(s), \nu(\mathrm{d}x; s), c(\mathrm{d}s))_{s\in\mathbb{R}^k}$ where a and m are functions on $\mathbb{R}^k$, with m nonnegative, $\nu(\mathrm{d}x; s)$ denotes for fixed s a Lévy measure on $\mathbb{R}$ and is for fixed $\mathrm{d}x$ a measurable function on $\mathbb{R}^k$, and c is a measure on $(\mathbb{R}^k, \mathcal{B}_b)$, such that the integral

$$\int_B a(s)c(\mathrm{d}s)$$

determines a (possibly signed) measure on $(\mathbb{R}^k, \mathcal{B}_b)$ and

$$\int_B \nu(\mathrm{d}x; s)c(\mathrm{d}s)$$

is a Lévy measure on $\mathbb{R}$ for each fixed $B \in \mathcal{B}_b$. Conversely, any such quadruplet determines, in law, a Lévy basis on $\mathbb{R}^k$. We refer to $(a, m, \nu(\mathrm{d}x; \cdot), c)$, or $(a(s), m(s), \nu(\mathrm{d}x; s), c(\mathrm{d}s))$, as the *characteristic quadruplet* of the Lévy basis. Given such a quadruplet, we denote $\nu(\mathrm{d}x; s)c(\mathrm{d}s)$ by $n(\mathrm{d}x; \mathrm{d}s)$ and we define $N(\mathrm{d}x; \mathrm{d}s)$ as the Poisson measure on $\mathbb{R}^k$ having compensator n. Generally, n with or without a suffix will stand for a compensator of this type and N with the same suffix denotes the corresponding Poisson measure.

As is the case for Lévy processes, any dispersive[4] Lévy basis has a Lévy–Ito representation

$$L(B) = a(B) + G(B) + \int_{|x|>1} xN(\mathrm{d}x; B) + \int_{|x|\le 1} x(N-n)(\mathrm{d}x; B), \tag{2.3}$$

where a is a, possibly signed measure, $G(B)$ is a Gaussian independently scattered random measure with $G(B) \sim N(0, m(B))$, N is a Poisson measure, independent of G and with compensator $n(\mathrm{d}x; \mathrm{d}s) = \mathrm{E}\{N(\mathrm{d}x; \mathrm{d}s)\}$. This result is due to Pedersen (2003). The notation used here is consistent with that of the Lévy–Khintchine representation (2.2).

Remark 1. The representation (2.3) may conveniently be expressed in infinitesimal form

$$L(\mathrm{d}s) = a(\mathrm{d}s) + G(\mathrm{d}s) + \int_{|x|>1} xN(\mathrm{d}x; \mathrm{d}s) + \int_{|x|\le 1} x(N-n)(\mathrm{d}x; \mathrm{d}s). \tag{2.4}$$

Correspondingly we may write the Lévy–Khintchine representation (2.2) infinitesimally as

$$\begin{aligned} \mathrm{C}\{\zeta \ddagger L(\mathrm{d}s)\} &= ia(\mathrm{d}s)\zeta - \frac{1}{2}m(\mathrm{d}s)\zeta^2 + \int_{-\infty}^{\infty} \left(e^{i\zeta x} - 1 - i\zeta x 1_{[-1,1]}(x)\right)n(\mathrm{d}x; \mathrm{d}s) \\ &= ia(\mathrm{d}s)\zeta - \frac{1}{2}m(\mathrm{d}s)\zeta^2 + \mathrm{C}\{\zeta \ddagger L'(s)\}c(\mathrm{d}s), \end{aligned}$$

[4] A Lévy basis is said to be dispersive if its control measure c is such that $c(\{s\}) = 0$ for all $s \in \mathbb{R}^k$.

where to each $s \in \mathbb{R}$ we have now associated an infinitely divisible random variable $L'(s)$ with Lévy–Khintchine representation

$$\mathrm{C}\{\zeta \ddagger L'(s)\} = \int_{-\infty}^{\infty} \left(e^{i\zeta x} - 1 - i\zeta x 1_{[-1,1]}(x)\right)\nu(\mathrm{d}x; s).$$

We refer to $L'(s)$ as the *Lévy seed* of L at s. By $\{L'_t(s)\}$ we denote the Lévy process generated by $L'(s)$, that is, the Lévy process for which the law of $L'_1(s)$ equals that of $L'(s)$. In case $\nu(\mathrm{d}x; s)$ does not depend on s we identify the Lévy seeds, writing L' for $L'(s)$.

When $\nu(\mathrm{d}x; s)$ does not depend on s, the Lévy basis is said to be *factorisable* and if, moreover, c is proportional to Lebesgue measure and $a(s)$ and $m(s)$ do not depend on s then L is *homogeneous*. Note that to any infinitely divisible distribution there exists a homogeneous Lévy basis on $\mathbb{R}^k$.

Example 1 (Inverse Gaussian basis). We recall that the inverse Gaussian law, denoted $\mathrm{IG}(\delta, \gamma)$, is infinitely divisible with probability density function

$$\frac{\delta}{\sqrt{2\pi}} e^{-\delta\gamma} x^{-3/2} \exp\left\{-\frac{1}{2}(\delta^2 x^{-1} + \gamma^2 x)\right\}, \tag{2.5}$$

where $x > 0$ and the parameters satisfy $\delta > 0$ and $\gamma \geq 0$. This has Lévy density

$$\frac{1}{\sqrt{2\pi}} x^{-3/2} \exp\left\{-\frac{1}{2}\gamma^2 x\right\} \tag{2.6}$$

and cumulant function

$$\mathrm{C}\{\zeta\} = -\delta\gamma + \delta(\gamma^2 - 2i\zeta)^{1/2},$$

and a sum of independent observations from this law must consequently follow the $\mathrm{IG}(n\delta, \gamma)$ distribution. Note also that, by the criterion for self-decomposability mentioned in Section 2.1, it is immediate that $\mathrm{IG}(\delta, \gamma)$ is, in fact, self-decomposable.

An inverse Gaussian homogeneous Lévy basis L may now be specified by taking $L(B)$ to have the $\mathrm{IG}(|B|\delta, \gamma)$law, where $|B|$ is the Lebesgue measure of B. More generally, a non-Gaussian Lévy basis whose seeds are of the form

$$\nu(\mathrm{d}x; s) = \frac{\delta(s)}{\sqrt{2\pi}} x^{-3/2} \exp\left\{-\frac{1}{2}\gamma(s)^2 x\right\} \mathrm{d}x$$

will be referred to as an inverse Gaussian basis.

Example 2 (Normal inverse Gaussian basis). The normal inverse Gaussian distribution $\mathrm{NIG}(\alpha, \beta, \mu, \delta)$ [Barndorff-Nielsen (1998)] equals the law at time 1 of the process obtained by subordinating a Brownian motion of mean μ and drift β

to the inverse Gaussian subordinator with law $\mathrm{IG}(\delta, \gamma)$ at time 1. It is the distribution on $\mathbb{R}$ having probability density function

$$p(x; \alpha, \beta, \mu, \delta) = a(\alpha, \beta, \mu, \delta) q\left(\frac{x-\mu}{\delta}\right)^{-1} K_1\left\{\delta\alpha q\left(\frac{x-\mu}{\delta}\right)\right\} e^{\beta x} \tag{2.7}$$

where $q(x) = \sqrt{1+x^2}$ and

$$a(\alpha, \beta, \mu, \delta) = \pi^{-1}\alpha \exp\{\delta\sqrt{\alpha^2 - \beta^2} - \beta\mu\} \tag{2.8}$$

and where K_1 is the modified Bessel function of the third kind and index 1. The domain of variation of the parameters is given by $\mu \in \mathbb{R}$, $\delta \in \mathbb{R}_+$, and $0 \leq \beta < \alpha$. The Lévy density is

$$\frac{\delta\alpha}{\pi}|x|^{-1} K_1(\alpha|x|) e^{\beta x} \tag{2.9}$$

and the cumulant function has the form

$$\mathrm{C}\{\zeta\} = \delta\{\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + i\zeta)^2}\} + i\mu\zeta. \tag{2.10}$$

A non-Gaussian Lévy basis is then determined by having $L(B) \sim \mathrm{NIG}(\alpha, \beta, |B|\mu, |B|\delta)$. This is the homogeneous normal inverse Gaussian basis, the general form of normal inverse Gaussian bases having Lévy seeds

$$\nu(\mathrm{d}x; s) = \frac{\delta(s)\alpha(s)}{\pi}|x|^{-1} K_1(\alpha(s)|x|) e^{\beta(s)x}\,\mathrm{d}x. \tag{2.11}$$

Integration of deterministic functions f with respect to an arbitrary Lévy basis L is defined in Rajput and Rosiński (1989), where criteria for existence of the integral are also given. We denote such an integral by $f \bullet L$. The resulting integral is infinitely divisible with Lévy–Khintchine representation provided by Proposition 2.6 in Rajput and Rosiński (1989). In the above notation this can be written as

$$\mathrm{C}\{\zeta \ddagger f \bullet L\} = -\frac{1}{2}\int_{\mathbb{R}^d} f^2(s) m(s) c(\mathrm{d}s) + \int_{\mathbb{R}^d} \mathrm{C}\{\zeta f(s) \ddagger a(s) + L'(s)\} c(\mathrm{d}s). \tag{2.12}$$

In the special case where $a = m = 0$ this reduces to

$$\mathrm{C}\{\zeta \ddagger f \bullet L\} = \int_{\mathbb{R}^d} \mathrm{C}\{f(s)\zeta \ddagger L'(s)\} c(\mathrm{d}s) \tag{2.13}$$

and the criteria for integrability of a (measurable) function f on $\mathbb{R}^k$ becomes

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}} \min\{1, |xf(s)|^2\} \nu(\mathrm{d}x; s) c(\mathrm{d}s) < \infty \tag{2.14}$$

and

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}} \left|\left(1_{[-1,1]}(xf(s)) - 1_{[-1,1]}(x)\right) f(s)\right| \nu(\mathrm{d}x; s) c(\mathrm{d}s) < \infty. \tag{2.15}$$

In particular, when

$$f(s) = 1_B(s)$$

for some Borel subset B of $\mathbb{R}^k$, condition (2.14) reduces to

$$\int_{\mathbb{R}} \min\{1, |x|^2\} n(\mathrm{d}x; B) < \infty \tag{2.16}$$

while condition (2.15) is vacuous. The domain of the Lévy basis can be extended from $\mathcal{B}_b$ to the class $\mathcal{B}_{ext}$ of Borel sets B in $\mathbb{R}^k$ for which (2.16) holds, by defining $L(B)$ for any $B \in \mathcal{B}_{ext} \backslash \mathcal{B}_b$ through the Lévy–Ito representation (2.3). [The extended class of sets is generally not a σ-algebra; but both $\mathcal{B}_b$ and $\mathcal{B}_{ext}$ are δ-rings. Another possibility for extending the domain arises when L is square integrable, see Barndorff-Nielsen, Benth and Veraart (2010a).]

The Rajput–Rosinski integration concept is restricted to deterministic functions f, and it is desirable to have an integral that is more general and dynamic, in the spirit of the Ito calculus and allowing integration of stochastic processes with respect to random measures. An integration approach of this kind is under development; see Basse-O'Connor, Graversen and Pedersen (2010) for a step in this direction.

The Lévy basis L is said to be non-Gaussian if $G = 0$. In general we will treat the Gaussian case, where $L = G$, and the non-Gaussian case separately, as these two cases are somewhat different in nature.

In the non-Gaussian case

$$L(B) = a(B) + \int_{|x|>1} x N(\mathrm{d}x; B) + \int_{|x|\le 1} x(N - n)(\mathrm{d}x; B), \tag{2.17}$$

and so

$$\mathrm{C}\{\zeta \ddagger L(B)\} = i a(B)\zeta + \int_{-\infty}^{\infty} \big(e^{i\zeta x} - 1 - i\zeta x 1_{(-1,1)}(x)\big) n(\mathrm{d}x; B). \tag{2.18}$$

When L is homogeneous this becomes

$$\mathrm{C}\{\zeta \ddagger L(B)\} = i a(B)\zeta + \mathrm{C}\{\zeta \ddagger L'\} c(B) \tag{2.19}$$

with

$$\mathrm{C}\{\zeta \ddagger L'\} = \int_{-\infty}^{\infty} \big(e^{i\zeta x} - 1 - i\zeta x 1_{(-1,1)}(x)\big) \nu(\mathrm{d}x).$$

For later reference we note that if, in (2.17), $\int_{-\infty}^{\infty} (1 \wedge |x|)\nu(\mathrm{d}x) < \infty$ and

$$a(B) = \int_{-\infty}^{\infty} x 1_{(-1,1)}(x) n(\mathrm{d}x; B)$$

then L can in fact be expressed more simply as

$$L(B) = \int_{-\infty}^{\infty} x N(\mathrm{d}x; B). \tag{2.20}$$

With probability 1, an arbitrary realisation of a Lévy basis of this type is in fact an ordinary (in general signed) measure on $\mathbb{R}^k$. (This property does not hold generally for independently scattered random measures.)

Up till now we have implicitly assumed that the values $L(B)$ of the Lévy basis L are one-dimensional, belonging to $\mathrm{ID}(\mathbb{R})$. However the above theory extends rather straightforwardly to m-dimensional Lévy bases L, that is where L takes values in $\mathrm{ID}(\mathbb{R}^m)$, the class of m-dimensional infinitely divisible distributions [cf., in particular, Pedersen (2003)].

3 Extended subordination and meta-times

Random time change of stochastic processes is a procedure of considerable interest, both theoretically and in various applications; see Barndorff-Nielsen and Shiryaev (2010). Of some special theoretical interest is the concept of subordination [Sato (1999), Bertoin (1999)]. As regards modelling and inference, mathematical finance and financial econometrics provide important cases in point; see Barndorff-Nielsen and Shiryaev (2010), Barndorff-Nielsen and Shephard (2011).

Let X be a d-dimensional Lévy process and let T be a subordinator, that is, a nonnegative Lévy process on $\mathbb{R}_+$. The subordination of X by T, denoted $Y = X \circ T$, is obtained by timewise composition of X by T, that is $Y_t = X_{T_t}$. This concept can be extended to subordination of Lévy bases by positive Lévy bases, and more generally to what may be called “meta-time change" of Lévy bases by independent positive random measures. This type of construction is introduced in Barndorff-Nielsen (2010) and Barndorff-Nielsen and Pedersen (2010), where more detailed discussion can be found. Here we shall just indicate this approach.

So, again, let L be a Lévy basis on $\mathbb{R}^k$ with characteristic quadruplet $(a, m, \nu(\mathrm{d}x;\cdot), c)$ and Lévy–Ito representation (2.17). Let T be a nonnegative random measure on $(\mathbb{R}^k, \mathcal{B})$ that is independent of L, and let $\hat{L}$ be the random measure that conditionally on T is a Lévy basis with characteristic quadruplet $(a, m, \nu(\mathrm{d}x;\cdot), T)$. (Existence of $\hat{L}$ requires a mild regularity assumption. Note also that $\hat{L}$ may take both positive and negative values.) Then, conditionally on T the cumulant functional of $\hat{L}$ is determined by

$$\mathrm{C}\{\zeta \ddagger f \bullet \hat{L}|T\} = \int_{\mathbb{R}^d} \left\{-\frac{1}{2} f^2(s) m(s) + \mathrm{C}\{\zeta f(s) \ddagger a(s) + L'(s)\}\right\} T(\mathrm{d}s) \tag{3.1}$$

while unconditionally

$$\begin{aligned}\mathrm{C}\{\zeta \ddagger f \bullet \hat{L}\} = \log \mathrm{E}\Big\{\exp\Big[-\frac{1}{2}\int_{\mathbb{R}^d} f^2(s) m(s) T(\mathrm{d}s) \\ + \int_{\mathbb{R}^d} \mathrm{C}\{\zeta f(s) \ddagger a(s) + L'(s)\} T(\mathrm{d}s)\Big]\Big\}.\end{aligned} \tag{3.2}$$

Provided the random measure T is infinitely divisible the same is true of $\hat{L}$, and if T is homogeneous the same is true of $\hat{L}$. We say that $\hat{L}$ is *the subordination the*

Lévy basis L by the meta-time T and, for explicitness, we also use the notation $L(\mathrm{d}s \curlywedge T)$ for $\hat{L}(\mathrm{d}s)$. This concept of meta-time change generalises, in particular, the usual concept of subordination of Lévy processes, as shown in Barndorff-Nielsen (2010, Section 2.2).

We note that when $a = m = 0$ then, in terms of the log Fourier–Laplace transform, formula (3.2) becomes

$$\mathrm{K}\{z \ddagger f \bullet \hat{L}\} = \mathrm{K}\big\{1 \ddagger \mathrm{K}\{zf(\cdot) \ddagger L'(\cdot)\} \bullet T\big\}. \tag{3.3}$$

This formula constitutes the generalisation to Lévy bases of the well-known composition relation [cf. for instance Bertoin (1999, Proposition 8.6)] of the Laplace exponents for subordination of Lévy processes. Note also that the corresponding conditional expression is

$$\mathrm{K}\{z \ddagger f \bullet \hat{L}|T\} = \mathrm{K}\{zf(\cdot) \ddagger L'(\cdot)\} \bullet T = \int_{\mathbb{R}^d} \mathrm{K}\{zf(\cdot) \ddagger L'(s)\} T(\mathrm{d}s). \tag{3.4}$$

It follows, in particular, that if L is homogeneous then, for any $B \in \mathcal{B}_b(\mathbb{R}^k)$, we have

$$\hat{L}(B)|T \sim L'_{T(B)}. \tag{3.5}$$

Now suppose that $T = L_0$ where L_0 is a nonnegative and dispersive Lévy basis L_0 on $\mathbb{R}_+^k$ with characteristic quadruplet $(0, 0, \nu_0(\mathrm{d}x; s), c_0(\mathrm{d}s))$. Then L_0 has Lévy–Ito representation

$$L_0(\mathrm{d}s) = \int_0^\infty x N_0(\mathrm{d}x; \mathrm{d}s) \tag{3.6}$$

with compensator

$$n_0(\mathrm{d}x; \mathrm{d}s) = \nu_0(\mathrm{d}x; s) c_0(\mathrm{d}s). \tag{3.7}$$

The realisations of L_0 are almost surely genuine measures on $\mathbb{R}_+^k$.

Theorem. *Assume that L is purely non-Gaussian (i.e., $a = m = 0$) and that L_0 is as given by (3.6). Then the random measure $\hat{L}$ is a Lévy basis with characteristic quadruplet $(\tilde{a}, 0, \hat{\nu}(\mathrm{d}x; s), c_0(\mathrm{d}s))$ where*

$$\tilde{a}(s) = \int_0^\infty \int_{-1}^1 v P\{L'_x(s) \in \mathrm{d}v\} \nu_0(\mathrm{d}x; s),$$

$$\hat{\nu}(\mathrm{d}v; s) = \int_0^\infty P\{L'_x(s) \in \mathrm{d}v\} \nu_0(\mathrm{d}x; s)$$

and $\{L'_x(s)\}_{x \in \mathbb{R}_+}$ denotes the Lévy process generated from the Lévy seed $L'(s)$ of L at s.

Proof. By formula (3.3) we find

$$
\begin{aligned}
\mathrm{K}\{z \ddagger f \bullet \hat{L}\} &= \mathrm{K}\big\{1 \ddagger \mathrm{K}\{zf(\cdot) \ddagger L'(\cdot)\} \bullet T\big\} \\
&= \int_{\mathbb{R}^d} \mathrm{K}\big\{\mathrm{K}\{zf(s) \ddagger L'(s)\} \ddagger L_0'(s)\big\} c_0(\mathrm{d}s) \\
&= \int_{\mathbb{R}^d} \mathrm{K}\big\{\mathrm{K}\{zf(s) \ddagger L'(s)\} \ddagger L_0'(s)\big\} c_0(\mathrm{d}s) \\
&= \int_{\mathbb{R}^d} \int_0^\infty \big(e^{\mathrm{K}\{zf(s)\ddagger L'(s)\}x} - 1\big) \nu_0(\mathrm{d}x; s) c_0(\mathrm{d}s) \\
&= \int_{\mathbb{R}^d} \int_0^\infty \big(\mathrm{M}\big(zf(s) \ddagger L'(s)\big)^x - 1\big) \nu_0(\mathrm{d}x; s) c_0(\mathrm{d}s),
\end{aligned}
$$

where $\mathrm{M} = e^{\mathrm{K}}$ denotes the Fourier–Laplace transform. It follows that

$$
\begin{aligned}
\mathrm{K}\{z \ddagger f \bullet \hat{L}\} &= \int_{\mathbb{R}^d} \int_0^\infty \big(\mathrm{M}\big(zf(s) \ddagger L_x'(s)\big) - 1\big) \nu_0(\mathrm{d}x; s) c_0(\mathrm{d}s) \\
&= \int_{\mathbb{R}^d} \int_0^\infty \int_{-\infty}^\infty \big(e^{zf(s)v} - 1 - zf(s)v 1_{[-1,1]}(v)\big) \\
&\qquad \times P\{L_x'(s) \in \mathrm{d}v\} \nu_0(\mathrm{d}x; s) c_0(\mathrm{d}s) \\
&\quad + z \int_{\mathbb{R}^d} f(s) \int_0^\infty \int_{-1}^1 v P\{L_x'(s) \in \mathrm{d}v\} \nu_0(\mathrm{d}x; s) c_0(\mathrm{d}s) \\
&= \int_{\mathbb{R}^d} \int_0^\infty \int_0^\infty \big(e^{i\zeta f(s)v} - 1 - i\zeta f(s)v 1_{[0,1]}(v)\big) \hat{\nu}(\mathrm{d}v; s) \\
&\quad + z \int_{\mathbb{R}^d} f(s) \tilde{a}(s) c_0(\mathrm{d}s),
\end{aligned}
$$

where

$$
\tilde{a}(s) = \int_0^\infty \int_{-1}^1 v P\{L_x'(s) \in \mathrm{d}v\} \nu_0(\mathrm{d}x; s)
$$

and

$$
\hat{\nu}(\mathrm{d}x; s) = \int_0^\infty P\{L_x'(s) \in \mathrm{d}x\} \nu_0(\mathrm{d}x; s),
$$

thus verifying the theorem. □

Remark 2. It follows from formula (3.5) [or by comparing (3.8) to Huff (1969); cf. also Theorem 30.1 of Sato (1999)] that the Lévy seeds of L, L_0 and $\hat{L}$ are related by $\hat{L}' = L' \circ L_0'$ where the latter formula is to be understood as saying that for (almost all) $s \in \mathbb{R}^k$ the Lévy process generated by $\hat{L}'(s)$ is equal (in law) to the subordination of the Lévy process generated from $L'(s)$ by the subordinator generated by $L_0'(s)$.

Remark 3. Theorem 1 concerns the case of L being non-Gaussian. On the other hand, if L is Gaussian white noise (i.e., $a=0$, $m=1$ and $\nu=0$), then $\hat{L}$ is non-Gaussian with compensator $\hat{\nu}(\mathrm{d}x;s)c_0(\mathrm{d}s)$ where

$$\hat{\nu}(\mathrm{d}x;s)=\int_0^\infty \varphi(x;u)\nu_0(\mathrm{d}u;s)\,\mathrm{d}x,$$

$\varphi(x;u)$ denoting the density of the Gaussian law of mean 0 and variance u. As an example, suppose that L_0 is the homogeneous $\mathrm{IG}(\delta,\gamma)$ basis. Then $\hat{L}$ is, in law, equal to the homogeneous $\mathrm{NIG}(\delta,0,\gamma,0)$ basis.

Now suppose that τ is an infinitely divisible nonnegative random field on $\mathbb{R}^k$ and let $T(B)=\int_B \tau(s)\,\mathrm{d}s$. If L is a Lévy basis on $(\mathbb{R}^k,\mathcal{B}_b(\mathbb{R}^k))$ having characteristic quadruplet $(0,0,\nu(\mathrm{d}x;s),\mathrm{d}s)$ then $L(\mathrm{d}s\,\mathrm{d}\xi \curlywedge T)$ is an infinitely divisible random measure on $(\mathbb{R}^k,\mathcal{B}_b(\mathbb{R}^k))$ provided

$$\hat{n}(\mathrm{d}x;B)=\int_B \nu(\mathrm{d}x;s)\tau(s)\,\mathrm{d}s \tag{3.8}$$

is a Lévy measure for all $B\in\mathcal{B}_b(\mathbb{R}^k)$.

Assume in particular that τ is of the form

$$\tau(s)=\int_{\mathbb{R}^q} J(u;s)\Lambda(\mathrm{d}u)$$

for some Lévy basis Λ on $\mathbb{R}^q$ with characteristic quadruplet $(0,0,\kappa(\mathrm{d}x;u),\mathrm{d}u)$. Then

$$\begin{aligned}\mathrm{K}\{zf(\cdot)\ddagger L'(\cdot)\}\bullet T&=\int_{\mathbb{R}^k}\mathrm{K}\{zf(s)\ddagger L'(s)\}\int_{\mathbb{R}^q}J(u;s)\Lambda(\mathrm{d}u)\,\mathrm{d}s\\&=\int_{\mathbb{R}^q}H_f(u)\Lambda(\mathrm{d}u),\end{aligned}$$

where

$$H_f(u)=\int_{\mathbb{R}^k}\mathrm{K}\{zf(s)\ddagger L'(s)\}J(u;s)\,\mathrm{d}s. \tag{3.9}$$

Hence we have [cf. (3.4)]

$$\mathrm{K}\{z\ddagger f\bullet\hat{L}\}=\int_{\mathbb{R}^q}\mathrm{K}\{H_f(u)\ddagger\Lambda'(u)\}\,\mathrm{d}u. \tag{3.10}$$

We proceed to illustrate formula (3.10) by two further examples. More advanced applications of extended subordination are discussed in Section 7.2.

Example 3. Let $C\in\mathcal{B}(\mathbb{R}^k)$ and let

$$\tau(s)=\exp\bigl(G(C+s)\bigr),$$

where G is the homogeneous Gaussian Lévy basis on $\mathbb{R}^k$ with mean and variance parameters κ_1 and κ_2. This type of intermittency/volatility field is of special interest in the context of turbulence; cf. Barndorff-Nielsen and Schmiegel

(2004), as—by suitable choice of B—it can be seen as an embodiment of the vortex cascade picture of turbulence and because it allows explicit calculations of tempo-spatial correlators. The field τ is not only infinitely divisible but in fact self-decomposable, due to the fact that the log normal distribution is self-decomposable, as proved by Thorin via the important introduction of the concept of generalised gamma convolutions; see Bondesson (1992).

Example 4. Suppose $q = k$ and

$$\tau(s) = \Lambda(s - C),$$

where C is an arbitrary element of $\mathcal{B}_b(\mathbb{R}^k)$. This corresponds to having $J(u; s) = 1_C(u - s)$ and hence

$$H_f(u) = \int_{u+C} \mathrm{K}\{zf(s) \ddagger L'(s)\}\,\mathrm{d}s$$

and

$$\mathrm{K}\{z \ddagger f \bullet \hat{L}\} = \int_{\mathbb{R}^k} \mathrm{K}\left\{\int_{u+C} \mathrm{K}\{zf(s) \ddagger L'(s)\}\,\mathrm{d}s \ddagger \Lambda'(u)\right\}\mathrm{d}u.$$

In case both L and Λ are homogenous this reduces to

$$\mathrm{K}\{z \ddagger f \bullet \hat{L}\} = \int_{\mathbb{R}^k} \mathrm{K}\left\{\int_{u+C} \mathrm{K}\{zf(s) \ddagger L'\}\,\mathrm{d}s \ddagger \Lambda'(u)\right\}\mathrm{d}u.$$

Taking $f(s) = 1_B(s)$ for some $B \in \mathcal{B}_b(\mathbb{R}^k)$ yields $\mathrm{K}\{z \ddagger f \bullet \hat{L}\} = \mathrm{K}\{z \ddagger \hat{L}(B)\}$ and

$$\int_{u+C} \mathrm{K}\{zf(s) \ddagger L'\}\,\mathrm{d}s = \mathrm{K}\{z \ddagger L'\}|(u + C) \cap B|$$

and hence

$$\mathrm{K}\{z \ddagger \hat{L}(B)\} = \int_{\mathbb{R}^k} \mathrm{K}\{\mathrm{K}\{z \ddagger L'\}|(u + C) \cap B| \ddagger \Lambda'(u)\}\,\mathrm{d}u.$$

The above discussion is at the distributional level. A constructive procedure, that introduces a concept of meta-time change and allows dynamic process formulations is presented in Barndorff-Nielsen and Pedersen (2010). More specifically, given a nonnegative Lévy basis T of the form (3.6) we define, constructively from T, a mapping $\mathbf{T}$ from $\mathbb{R}^k_+$ to $\mathbb{R}^k$ in terms of which $\hat{L}$ may be expressed as

$$\hat{L}(B) = L(\mathbf{T}(B)).$$

4 Ambit stochastics

This section first, in brief, reviews the ideas of ambit fields and processes, introduced in Barndorff-Nielsen and Schmiegel (2004, 2007) and further discussed in

Barndorff-Nielsen and Schmiegel (2009), Barndorff-Nielsen, Benth and Veraart (2011, 2010b). This is followed by a discussion of the relevance of the concept of extended subordination, from Section 3, to ambit stochastics. Connections to modelling by stochastic partial differential equations (SPDEs) and to mixed moving averages are also indicated. Finally, questions of inference for ambit processes are discussed briefly. Most of the processes considered in this paper can be viewed as special ambit types.

The general background setting for the concept of ambit processes consists of a stochastic field $Y = \{Y_t(x)\}$ in space–time $\mathbb{R}\times\mathcal{X}$, a curve $\tau(\theta)$ in $\mathbb{R}\times\mathcal{X}$, and the values X_θ of the field along a parametrised curve, the focus being on the dynamic properties of the stochastic process $X = \{X_\theta\}$. Here the space $\mathcal{X}$ is often, but not necessarily, taken as $\mathbb{R}^d$ for $d = 1, 2$ or 3. The stochastic field is supposed to be generated by innovations in space–time and the values $Y_t(x)$ are assumed to depend only on innovations that occur prior to or at time t. More precisely, at each point (t, x) only the innovations in some subset $A_t(x)$ of $\mathbb{R}_t\times\mathcal{X}$ (where $\mathbb{R}_t = (-\infty, t])$ are influencing the value of $Y_t(x)$, and we refer to $A_t(x)$ as the *ambit set*, associated to (t, x), and to Y and X as an *ambit field* and an *ambit process*, respectively.

Specifically, with $\mathcal{X} = \mathbb{R}^d$, an ambit field is defined, up to an additive constant, by

$$Y_t(x) = \int_{A_t(x)} g(\xi, s; x, t)\sigma_s(\xi)L(\mathrm{d}s, \mathrm{d}\xi) + \int_{D_t(x)} q(\xi, s; x, t)\tau_s(\xi)\,\mathrm{d}s\,\mathrm{d}\xi, \quad (4.1)$$

where the *ambit sets* $A_t(x)$, and $D_t(x)$ are subsets of $(-\infty, t]\times\mathbb{R}^d$, g and q are deterministic damping functions, $\sigma \geq 0$ is a stochastic field referred to as the *volatility* or *intermittency*, and L is a Lévy basis. Furthermore, τ is another stochastic field, often chosen as $\tau = \sigma^2$ when L is the Gaussian white noise.

An ambit process is then the realisation of Y along a curve $\gamma(\theta) = (t(\theta), x(\theta))$ in $\mathbb{R} \times \mathbb{R}^d$, with $t(\theta)$ increasing in θ, from minus infinity to plus infinity. Note that, in general, ambit processes are not semimartingales [cf. Barndorff-Nielsen and Schmiegel (2009)]. Many of the standard tools from semimartingale theory are therefore not applicable in the study of ambit stochastics and alternative methods are required.

At the present level of generality we take the integrals in (4.1) to be defined in the sense of independently scattered random measures [cf. Rajput and Rosiński (1989)], assuming that g, σ, q and τ are sufficiently regular for the integrals to exist. However, in more concrete cases it is often of interest to establish whether the definition of the integrals can be sharpened to a more dynamical version, for instance in the spirit of Itô-type integrals, and allowing integrands that are not necessarily independent of the random integrator measure (here L). The paper of Basse-O'Connor, Graversen and Pedersen (2010) represents a step towards such an integration theory.

Of particular interest are ambit processes that are stationary in time and nonanticipative. More specifically, they may be derived from ambit fields Y of the form

$$Y_t(x) = \int_{A_t(x)} g(\xi, t-s; x)\sigma_s(\xi)L(\mathrm{d}s, \mathrm{d}\xi) + \int_{D_t(x)} q(\xi, t-s; x)\tau_s(\xi)\,\mathrm{d}s\,\mathrm{d}\xi. \tag{4.2}$$

Here the ambit sets $A_t(x)$, and $D_t(x)$ are taken to be *homogeneous* and *nonanticipative*, that is, $A_t(x)$ is of the form $A_t(x) = A + (x, t)$ where A only involves nonpositive time coordinates, and similarly for $D_t(x)$. Note that the field (4.2) will also be stationary in the space variable x if the fields σ and τ are stationary and if the kernel g is in fact of the form $g(t-s, x-\xi)$ and the analogous holds for q.

The class of ambit processes also encompasses that of mixed moving averages which are stationary processes of the form

$$Y_t = \int_{\mathcal{X}\times\mathbb{R}} f(t-s, \xi)L(\mathrm{d}s\,\mathrm{d}\xi). \tag{4.3}$$

For a discussion of such processes see the following Sections 5–7.

Now, while the multiplicative position of σ to the basis L in (4.2) goes naturally together with L when L is Gaussian or more generally stable, this is less so in general, and there are advantages in interpretation and calculation in—instead of having σL as the integrator—using $\hat{L}$ obtained by subordinating L to a random meta-time T, in the sense defined in Section 3 and such that T is absolutely continuous with density $\tau = \sigma^2$. In particular, the dependence structure in Y is then relatively simple to describe. Note that in the Gaussian and stable cases the result of the multiplicative approach σL can equally be achieved by the extended subordination. We develop this aspect in Section 7.2.

Many prominent tempo-spatial models are constructed from an ordinary, partial or fractional differential equation by adding a noise term, for instance in the form of white noise, to the equation. The solution to the equation then being often representable as an integral with respect to the noise of the Green's function of the original deterministic differential equation. Thus the solution is taking the form of an ambit process. For some examples with discussion, see Barndorff-Nielsen, Benth and Veraart (2011). Here is a further example.

Example 5. The stochastic heat equation

$$\partial_t u = \frac{1}{2}\partial_x^2 u + \dot{W}(x, t),$$

where $\dot{W}(x, t)$ is white noise, has a solution

$$u(t, x) = \int_{[0,t]\times\mathbb{R}} \varphi(x-\xi; t-s)W(\mathrm{d}s\,\mathrm{d}\xi),$$

where φ is the normal density of mean 0 and variance t and $W$is the white Lévy basis on $\mathbb{R}\times\mathbb{R}$.

As discussed in Burdzy and Swanson (2010), for fixed x the stationary process $\{u(t,x)\}_{t\in[0,\infty)}$ has a nontrivial quartic variation and hence is not a semimartingale. Consequently a stochastic integral with respect to this process cannot be defined in the classical Ito sense. However, the authors define another type of stochastic integral for this case, which exists as a limit of a certain type of Riemann sums and satisfies a change of variable formula with a correction term that is an ordinary Ito integral with respect to a Brownian motion which is independent of the process u itself.

Note that, in contrast to modelling by SPDEs, the ambit approach defines the system dynamics directly. It also differs in specifically including the ideas of ambit sets and stochastic volatility/intermittency fields. Such fields play a key role in many areas of science, particularly in the contexts of turbulence and finance.

Let $Y = \{Y_t(x)\}$ be an ambit field as given by (4.2). Realisations of such a field are rarely directly observable, but time series observations from one or more ambit processes embedded in Y may be available. The question is then what type of inference can be drawn on the elements in Y from observations of this kind, in particular about the damping function g and the volatility/intermittency σ. A key tool for this is the theory of realised quadratic and multipower variations, as developed, from specific problems of finance and turbulence. As mentioned earlier, ambit processes are generally not of the semimartingale type. The theory of multipower variations provides probabilistic and distributional limit laws in both semimartingale and nonsemimartingale settings, but the mathematical theory is quite different in the two cases, in particular drawing heavily on recent results in Malliavin calculus under nonsemimartingale specifications. See Barndorff-Nielsen and Shephard (2002, 2003, 2004), Barndorff-Nielsen et al. (2006a, 2006b), Jacod (2008a, 2008b), Barndorff-Nielsen, Corcuera and Podolskij (2009, 2010), Barndorff-Nielsen and Graversen (2010).

5 Spectral representability

In a variety of situations there are considerable gains in interpretability and tractability if a given stochastic object—a probability distribution, a stochastic process, etc.—can be expressed as some simple combination of more elemental objects. One such type of reformulation consists in representing a stochastic process as a random time change of a basic kind of process, for instance time change of a compound Poisson process or a Brownian motion with drift [for a systematic discussion of this, see Barndorff-Nielsen and Shiryaev (2010)]. Another is that of representing important classes of infinitely divisible probability laws as stochastic integrals with respect to Lévy processes; see Barndorff-Nielsen, Maejima and Sato (2006a), Sato (2007), Maejima, Pérez-Abreu and Sato (2010) and references given there.

The simplest general example of a spectral representation result for continuous time stationary processes is the continuous time Wold–Karhunen decomposition. It says that any second-order stationary stochastic process Z, possibly complex valued, of mean 0 and continuous in quadratic mean can be represented as

$$Z_t = \int_{-\infty}^{t} \phi(t-s)\,\mathrm{d}\Xi_s + V_t, \tag{5.1}$$

where the deterministic function ϕ is an in general complex, deterministic square integrable function, the process Ξ has orthogonal increments with $E\{|\mathrm{d}\Xi_t|^2\} = \varpi\,\mathrm{d}t$ for some constant $\varpi > 0$ and the process V is nonregular (i.e., its future values can be predicted, in the L^2 sense, by linear operations on past values without error). Under the further condition that $\bigcap_{t\in\mathbb{R}} \overline{\mathrm{sp}}\{Z_s : s \le t\} = \{0\}$, the function ϕ is real and uniquely determined up to a real constant of proportionality; and the same is therefore true of Ξ (up to an additive constant). If the spectral measure of the autocorrelation function of Z is absolutely continuous then V in the above representation is 0 and we have a second-order spectral decomposition.

However, explicit expressions for the kernel ϕ are rarely available (but some cases where such expressions have been determined are reviewed in Section 6.1). Furthermore, the Wold–Karhunen representation is in the L^2sense only and a further important restriction is that it does not involve a spatial component in the integral, such as is the case generally in (1.1) and (4.3). In fact, even in the case of square integrability it is generally not possible to represent all properties of a strictly stationary infinitely divisible process by the Wold–Karhunen formula.

The general question of spectral representations of infinitely divisible processes is discussed in a basic and detailed way in the seminal paper by Rajput and Rosiński (1989). These authors set up two criteria that such representations should ideally meet and showed how in principle that may be achieved, establishing as part of the discussion the integration approach outlined above in Section 2.2. In general the representations are, formally written, of the type

$$Y = \int f\,\mathrm{d}L \tag{5.2}$$

with f deterministic and L a Lévy basis. The extent to which such representations are valid not only in law but almost surely is also considered in Rajput and Rosiński (1989). However, like in the case of the Wold–Karhunen representation, explicit expressions of the ingredients f and L are rarely available. In Basse (2009) a detailed study is given of necessary and sufficient conditions on the kernel f in (5.2) for the process Y to be a semimartingale.

When Y is stationary the relevant type of spectral representation is in the form of a *mixed moving average*

$$Y_t = \int_{\mathbb{R}\times\mathcal{X}} f(t-s,\xi)L(\mathrm{d}s\,\mathrm{d}\xi), \tag{5.3}$$

where $\mathcal{X}$ is a nonempty set and L a Lévy basis on $\mathbb{R}\times\mathcal{X}$. In general, to obtain a representation of this type for a given *SID* process Y it is necessary to include the $\mathcal{X}$ component in (5.3).

Using characterisation and properties of flows available from ergodic theory, Rosiński (1995) has shown that not all symmetric α stable stationary processes can be represented as in (5.3), with L being a symmetric α stable Lévy basis, and he has given a criterion [Corollary 4.2 in Rosiński (1995)] for when such a representation is possible.

Finally, in Rosiński (2007) the author shows, using the theory of Upsilon transformations [cf. Barndorff-Nielsen, Rosinski and Thorbjørnsen (2008)], that a broad range of non-Gaussian infinitely divisible stationary processes can be represented on the form

$$Y_t = \int_{\mathcal{S}} f(\phi_t(s))\{M(\mathrm{d}s) - c(f(\phi_t(s)))m(\mathrm{d}s)\},$$

where ϕ_t is a measure preserving flow on a Borel space $(\mathcal{S}, \mathcal{B}(\mathcal{S}), m)$, M is a Lévy basis on $\mathcal{S}$ and c is a specified centering function. Classes of processes whose finite dimensional laws are stable or tempered stable or self-decomposable are among the cases covered by this formula.

Further links to mixed moving averages occur in Sections 6 and 7 below.

6 Null-spatial settings

The present section reviews, in brief, two important types of one-dimensional *SID* processes, both of which are particular cases of the ambit class.

6.1 OU and closely related processes

SID processes that are also Markovian are clearly of special interest. The most important of these are the OU processes.

A one-dimensional *SID* process Y is to said be an OU process if it is representable in law as

$$Y_t = \int_{-\infty}^{t} e^{-\lambda(t-s)}\,\mathrm{d}L_t, \tag{6.1}$$

where L is a Lévy process on $\mathbb{R}$. Recall that a random variable X is said to be self-decomposable if and only if it is representable as in (2.1) or, equivalently, if and only if the characteristic function ϕ of X is such that for every constant $c \in (0, 1)$ there exists a characteristic function $\phi_c$for which $\phi(t) = \phi(ct)\phi_c(t)$.

The basic theory of OU processes is presented in detail in the monograph by Sato (1999). For some of the later developments, including multivariate versions, superpositions of OU processes and doubly stochastic (or generalised) OU processes, see Barndorff-Nielsen and Pérez-Abreu (1999), Barndorff-Nielsen (2001), Rocha-Arteaga and Sato (2003), Barndorff-Nielsen, Maejima and Sato (2006a), Barndorff-Nielsen and Stelzer (2011a), Behme, Lindner and Maller (2011) and references given there. For applications to financial econometrics, see Barndorff-Nielsen and Shephard (2001, 2011), Barndorff-Nielsen and Stelzer (2011b) and Barndorff-Nielsen and Veraart (2011).

OU processes are the continuous time analogue of AR(1) time series. More generally, continuous time versions of ARMA models, called CARMA processes, and some ramifications of these, are introduced and discussed in Brockwell (2001, 2004), Brockwell and Marquardt (2005), Marquardt and Stelzer (2007) and Brockwell and Lindner (2007). Like the OU case, the CARMA processes are Lévy driven, that is, they are representable as integrals of suitable deterministic kernels with respect to a Lévy process.

A one-dimensional OU process (6.1) is the stationary solution to the stochastic differential equation

$$\mathrm{d}Y_t = -\lambda Y_t \,\mathrm{d}t + \mathrm{d}L_t, \tag{6.2}$$

where L is a Lévy process. Positive stationary solutions to this SDE are obtained when L is a subordinator, that is, a process with positive increments, and such solutions are used in particular to model the random fluctuations of the latent variance, that is, the volatility, of financal assets, such as stock prices or exchange rates, see Barndorff-Nielsen and Shephard (2001, 2011) and references there. In recent years it has been realised that an additional layer of volatility, referred to as volatility of volatility, is called for to explain the actual fluctuations of such assets. A natural approach to model this is to introduce a further layer of variation through volatility modulation in (6.2). The most tractable way to do this is by replacing the background driving Lévy process by a time changed version of this where the time change is the integral of another, independent, positive OU process. For a discussion of this and of alternative kinds of modulation see Barndorff-Nielsen and Veraart (2011).

The SDE (6.2) for OU processes is a special case of the generalised Langevin equation

$$\mathrm{d}Y_t = -\lambda Y_t \,\mathrm{d}t + \mathrm{d}N_t, \tag{6.3}$$

where N denotes a process with stationary, in general not independent, increments. When N is not a Lévy process we refer to stationary solutions of (6.3) as *quasi Ornstein–Uhlenbeck* (QOU) processes. Existence and properties of QOU processes is discussed in Barndorff-Nielsen and Basse-O'Connor (2009). In the case of square integrability such a process can in principle be represented by the Wold–Karhunen decomposition but this embodies the L^2 aspects only and moreover an explicit form of the kernel function ϕ in (5.1) is seldomly available; cf. Section 5. A class of instances where an explicit expression can be given is presented in Barndorff-Nielsen and Basse-O'Connor (2009). More specifically, suppose that the driving process N has a pseudo moving average representation in the sense that

$$N_t = \int_{\mathbb{R}} \big(f(t-s) - f(s)\big)\,\mathrm{d}Z_s \tag{6.4}$$

for some deterministic function f and an integrable and centered Lévy process Z. Then there exists a stationary solution Y to the generalised Langevin equation (6.3) and Y has a moving average representation

$$Y_t = \int_{\mathbb{R}} \psi_f(t-s)\,\mathrm{d}Z_s,$$

where the function ψ is given by

$$\psi_f(t) = f(t) - \lambda e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda s} f(s)\,\mathrm{d}s.$$

An example in point is where $f(t) = c_H t^{H-1/\alpha}$, for some constants $c_H > 0$ and $H \in (0,1)$, and where Z is an α-stable Lévy process. Then N is a linear fractional α-stable motion and Y is a so-called fractional OU process.

It is clear from the very definition (2.1) of self-decomposability that no integer valued infinitely divisible distribution can be self-decomposable. A concept of discrete self-decomposability, somewhat analogous to ordinary self-decomposability, that holds for certain types of discrete elements of ID($\mathbb{R}$), was introduced by Steutel and van Harn and is discussed in considerable detail in Bondesson (1992) and Steutel and van Harn (2004). This concept was generalised by Zhu and Joe (2003) and applied to the construction of a discrete analogue of OU processes. These discrete OU processes are Markovian and infinitely divisible and, like their continuous counterparts, have negative exponential autocorrelation functions provided they are square integrable. Examples include processes with negative binomial marginal law, in particular a linear birth and death process. The discrete analogue of the defining relation (2.1) of self-decomposability expresses the possibility of representing an integer valued random variable X as

$$X \overset{\text{law}}{=} \sum_{j=0}^{X} Z_j(c) + \Xi_c$$

for a continuum of values of the index c, say $c \in (0,1)$, and where, given any such value, the integer valued random variables $Z_j(c)$ are independent and identically distributed and independent of the random variable Ξ_c. This idea is closely related to binomial thinning.

6.2 Brownian and Lévy semistationary processes

Brownian semistationary processes, $\mathcal{BSS}$ processes for short, were defined [Barndorff-Nielsen and Schmiegel (2009)] as processes $Y = \{Y_t\}_{t\in\mathbb{R}}$ that, up to an additive constant, are representable as

$$Y_t = \int_{-\infty}^{t} g(t-s)\sigma_s\,\mathrm{d}B_s + \int_{-\infty}^{t} q(t-s)a_s\,\mathrm{d}s, \tag{6.5}$$

where B is Brownian motion, g and q are nonnegative deterministic functions on $\mathbb{R}$, with $g(t) = q(t) = 0$ for $t \leq 0$, and ω and a are càdlàg processes. The process Y is stationary provided σ and a are stationary, as we henceforth assume. In Barndorff-Nielsen and Schmiegel (2009) this type of process was introduced in the context of modelling turbulence in fluids.

Subject to a regularity condition, the $\mathcal{BSS}$ processes have conditional full support, a property of importance in mathematical finance; see Pakkanen (2010).

Substituting the Brownian motion in (6.5) by a general Lévy motion gives rise to the class $\mathcal{LSS}$ of Lévy semistationary processes, that is, processes of the form

$$Y_t = \int_{-\infty}^{t} g(t-s)\sigma_s \,\mathrm{d}L_s + \int_{-\infty}^{t} q(t-s)a_s \,\mathrm{d}s \tag{6.6}$$

or

$$Y_t = \int_{-\infty}^{t} g(t-s) \,\mathrm{d}L_{\tau_s} + \int_{-\infty}^{t} q(t-s)a_s \,\mathrm{d}s, \tag{6.7}$$

where τ denotes an increasing càdlàg process with stationary increments. Such processes are used in Barndorff-Nielsen, Benth and Veraart (2010a) to model spot prices in finance.

Note that (6.6) can be rewritten in the form (6.7) provided L is a stable motion, in particular Brownian motion, but in general not otherwise.

$\mathcal{LSS}$ and in particular $\mathcal{BSS}$ processes are generally not of the semimartingale type. Therefore new tools are being developed to handle them, in regard both to their dynamic properties and questions of inference on the various elements of their structure; see Barndorff-Nielsen, Corcuera and Podolskij (2009, 2010), Basse-O'Connor, Graversen and Pedersen (2010), Barndorff-Nielsen and Graversen (2010). [For a discussion of the canonical decomposition of stationary Gaussian semimartingales, see Basse (2010).]

Here we briefly comment on the nonsemimartingale question. A classical necesssary and sufficient condition, due to Knight (1992), for the process Y to be a semimartingale, valid in the special simple situation where $\sigma = 1$, $a = 0$, and L equals the Brownian motion B, says that $(Y_t)_{t\geq 0}$ is a semimartingale in the Brownian filtration if and only if

$$g(t) = c + \int_0^t b(s) \,\mathrm{d}s \tag{6.8}$$

for some $c \in \mathbb{R}$ and a square integrable function b. More generally, one may ask under what conditions quasi moving average processes of the form

$$X_t = \int_{-\infty}^{\infty} \big(g(t-s) - h(-s)\big) \,\mathrm{d}B_s$$

with g and h deterministic, are semimartingales; specifically, when is $(X_t)_{t\geq 0}$ a $(\mathcal{F}_t^X)_{t\geq 0}$-seminartingale, where $\mathcal{F}_t^X$ is the σ-algebra generated by $\{X_s, s \leq t\}$.

Constructive necessary and sufficient conditions for this are given in a recent paper by Basse; see Basse (2008). At a further level of generalisation, Basse and Pedersen (2009), consider processes X of the general form

$$X_t = \int_{-\infty}^{t} \big(\phi(t-s) - \psi(-s)\big)\,\mathrm{d}L_s,$$

where L is a (two-sided) nondeterministic Lévy process with characteristic triplet (γ, σ^2, ν) and ϕ and ψ are deterministic functions. These authors establish various necessary conditions on (γ, σ^2, ν) and ϕ, ψ in order for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^L)_{t\geq 0}$-semimartingale.

For the case where the driving Lévy process in (6.6) is the Brownian motion and σ and a are stationary a set of sufficient conditions for Y to be a semimartingale are [Barndorff-Nielsen and Schmiegel (2009)]: (i) $g(0+)$ and $q(0+)$ exist and are finite; (ii) g is absolutely continuous with square integrable derivative $\dot{g}$; (iii) the process $\dot{g}(-\cdot)\sigma.$ is square integrable; (iv) the process $\dot{q}(-\cdot)a.$ is integrable. These conditions must be close to necessary as well; cf. the above-mentioned results by Knight and Basse.

7 Tempo-spatial settings

Having discussed the null-spatial case, we now turn to consider a few aspects of genuinely tempo-spatial ambit settings. The section is divided into three subsections: on OU related processes, on volatility modulation, and on a new type of ambit processes called *trawlings*.

7.1 OU and closely related processes

One-dimensional processes of OU and supOU type were discussed in Section 6.1. Multivariate, in particular matrix valued, versions of these, using the concept of matrix subordinators [Barndorff-Nielsen and Pérez-Abreu (2008)], are considered in Barndorff-Nielsen and Stelzer (2011a, 2011b) and Moser and Stelzer (2010), and those papers also discuss the application of such processes to stochastic volatility modelling, extending the OU-based modelling introduced in Barndorff-Nielsen and Shephard (2001). Following the initial approach to the definition of one-dimensional supOU [Barndorff-Nielsen (2001)] the multivariate supOU processes are represented as mixed moving averages [cf. (5.3)]. Specifically, a d-dimensional supOU process is defined as a process of the form

$$Y_t = \int_{M_d^-} \int_{-\infty}^{t} e^{A(t-s)} L(\mathrm{d}s, \mathrm{d}A),$$

where M_d^- denotes the set of $d\times d$ real matrices whose spectrum is contained in the negative complex halfplane and L is an $\mathbb{R}^d$-valued Lévy basis on $\mathbb{R}\times M_d^-$ with generating quadruplet $(a, m, \nu(\mathrm{d}x), \mathrm{d}s \times \pi(\mathrm{d}\cdot))$ such that π is a probability measure on M_d^- and

$$\int_{\|x\|>1} \log(\|x\|)\nu(\mathrm{d}x) < \infty.$$

In Barndorff-Nielsen and Schmiegel (2004) the concept of OU processes (6.1) [see also Barndorff-Nielsen and Schmiegel (2007)] is extended in a tempo-spatial setting to processes denoted $\mathrm{OU}_\wedge$. These are defined through integration backwards in time over wedge shaped regions and with negative exponential weighing, the integration being with respect to nonnegative homogeneous Lévy bases on $\mathbb{R} \times \mathbb{R}^d$. This type of extension is motivated in the aim to model the energy dissipation (or intermittency/volatility) in homogeneous turbulent fluids. The $\mathrm{OU}_\wedge$ processes are Markovian in character and allow for explicit calculations.

A direct multiparameter extension of the one-dimensional OU process (6.1) to

$$Y_{t.} = \int_{s.\leq t.} e^{-(t_+ - s_+)} L(\mathrm{d}s.),$$

where $t. = (t_1, \ldots, t_k)$, $t_+ = t_1 + \cdots + t_k$ and L is a homogeneous Lévy basis on $\mathbb{R}_+^k$ is introduced in Graversen and Pedersen (2010) and related to the Urbanik subclasses of Lévy and OU processes.

7.2 Volatility modulation

In stochastic modelling it is often natural to start by specifying a relatively simple model where the stochastic input only consists of independent and identically distributed innovations, afterwards making the model more realistic by volatility modulation, that is, by introducing additional stochastic variation to reflect the fact that the phenomenon modeled exhibits varying degrees of variation over time and/or space. We have touched upon the role of volatility/intermittency above, particularly in Sections 3 and 4, and we now return to the possibility, indicated there, of volatility modulation by extended subordination.

Consider first a general specification of a stochastic process without volatility element

$$X_t = \int_{\mathbb{R}^k} K_t(s) L(\mathrm{d}s),$$

where K is a deterministic function and L a Lévy basis on $\mathbb{R}^k$ with characteristic quadruplet $(0, 0, \nu(\mathrm{d}x; s), \mathrm{d}s)$; such a process is referred to as a Lévy-driven Volterra process. Then for a general measure μ on $\mathbb{R}$ we have

$$\mu(X.) = \int_{\mathbb{R}^k} \mu(K.(s)) L(\mathrm{d}s) = \mu(K.(\cdot)) \bullet L$$

and the kumulant functional of X (which determines the law of the process X uniquely) is

$$\mathrm{K}\{z \ddagger \mu(X.)\} = \int_{\mathbb{R}^k} \mathrm{K}\{z K.(s) \ddagger L'(s)\}\, \mathrm{d}s = \mathrm{K}\{z \ddagger \mu(K.(\cdot)) \bullet L\}$$

[where $L'(s)$ is the seed of L at s as defined in Section 3].

Next we introduce stochastic volatility by changing X to $\hat{X}$ defined by

$$\hat{X}_t = \int_{\mathbb{R}^k} K_t(s)\hat{L}(\mathrm{d}s)$$

where $\hat{L}(\mathrm{d}s) = L(\mathrm{d}s \curlywedge T)$ for some meta-time T on $\mathbb{R}^k$. Then, conditionally on T,

$$\mathrm{K}\{z \ddagger \mu(\hat{X}.)|T\} = \int_{\mathbb{R}^k} \mathrm{K}\{zK.(s) \ddagger L'(s)\}T(\mathrm{d}s). \tag{7.1}$$

Suppose in particular that T is absolutely continuous with density τ of the form

$$\tau(s) = \int_{\mathbb{R}^q} J(u;s)\Lambda(\mathrm{d}u) \tag{7.2}$$

for a Lévy basis Λ on $\mathbb{R}^q$ with characteristic quadruplet $(0, 0, \kappa(\mathrm{d}x; u), \mathrm{d}u)$. Then, in view of (7.1), we find that the kumulant functional of $\hat{X}$ can be written as

$$\mathrm{K}\{z \ddagger \mu(\hat{X}.)\} = \int_{\mathbb{R}^q} \mathrm{K}\left\{\int_{\mathbb{R}^k} \mathrm{K}\{z\mu(K.(s)) \ddagger L'(s)\}J(u;s)\,\mathrm{d}s \ddagger \Lambda'(u)\right\}\mathrm{d}u. \tag{7.3}$$

In case L is homogeneous this reduces to

$$\mathrm{K}\{z \ddagger \mu(\hat{X}.)\} = \int_{\mathbb{R}^q} \mathrm{K}\left\{\int_{\mathbb{R}^k} \mathrm{K}\{z\mu(K.(s)) \ddagger L'\}J(u;s)\,\mathrm{d}s \ddagger \Lambda'(u)\right\}\mathrm{d}u \tag{7.4}$$

and if further Λ is homogeneous we obtain the following expression for the kumulant functional of the volatility modulation of X by τ:

$$\mathrm{K}\{z \ddagger \mu(\hat{X}.)\} = \int_{\mathbb{R}^q} \mathrm{K}\left\{\int_{\mathbb{R}^k} \mathrm{K}\{z\mu(K.(s)) \ddagger L'\}J(u;s)\,\mathrm{d}s \ddagger \Lambda'\right\}\mathrm{d}u. \tag{7.5}$$

For illustration, we here apply this to the main element of the ambit setting, that is, where the ambit field is

$$Y_t(x) = \int_{A_t(x)} g(\xi, t-s; x)\sigma_s(\xi)L(\mathrm{d}\xi, \mathrm{d}s). \tag{7.6}$$

But, instead of having the volatility entering as the factor $\sigma_s(\xi)$, we now introduce it as the field

$$\tau(s,\xi) = \sigma_s^2(\xi) \tag{7.7}$$

in the above setup with $\mathbb{R}^k = \mathbb{R}^q = \mathbb{R} \times \mathbb{R}^d$. Thus we modify the ambit field (7.6) into

$$\hat{Y}_t(x) = \int_{A_t(x)} g(t-s, \xi; x)\hat{L}(\mathrm{d}s\,\mathrm{d}\xi), \tag{7.8}$$

where $\hat{L}(\mathrm{d}s\,\mathrm{d}\xi) = L(\mathrm{d}s\,\mathrm{d}\xi \curlywedge T)$ and T has density τ. More specifically, we take $A_t(x) = A + (t,x)$ with $A \subset (-\infty, 0] \times \mathbb{R}^d$, and τ of the form

$$\tau(s,\xi) = \int_{\mathbb{R}\times\mathbb{R}^d} H(s-u, \zeta; \xi)\Lambda(\mathrm{d}u\,\mathrm{d}\zeta)$$

for some deterministic kernel H and with Λ homogeneous. This, in particular, ensures timewise stationarity of $\hat{Y}_t(x)$for any given x. Then, letting

$$G_t(s,\xi;x) = 1_{A_t(x)}(s,\xi)g(t-s,\xi;x),$$

we have

$$\begin{aligned}\mathrm{K}\{z \ddagger \mu(\hat{Y}.(x))\} = \int_{\mathbb{R}\times\mathbb{R}^d} \mathrm{K}\bigg\{\int_{\mathbb{R}\times\mathbb{R}^d} &\mathrm{K}\{z\mu(G.(s,\xi;x)) \ddagger L'\} \\ &\times H(s-u,\zeta;\xi)\,\mathrm{d}s\,\mathrm{d}\xi \ddagger \Lambda'\bigg\}\,\mathrm{d}u\,\mathrm{d}\zeta,\end{aligned} \tag{7.9}$$

the kumulant functional of the modulation of $\{Y_t(x)\}_{t\in\mathbb{R}}$ by τ.

7.3 Trawling

We now define and discuss what we propose to call *trawling processes*. These include particular types of ambit processes and are generally based on homogeneous random fields.

Initially, consider the special case of homogeneous ambit fields $Y = \{Y_t(x)\}$ on $\mathbb{R}\times\mathbb{R}^d$ for which both the kernel function g and the volatility field σ are constant and equal to 1, and note that, with respect to the Lévy basis L of Y, we have that $\mathcal{B}_{ext}(\mathbb{R}\times\mathbb{R}^d) = \{A \in \mathcal{B}(\mathbb{R}\times\mathbb{R}^d) : |A| < \infty\}$. Then, for any set $A \in \mathcal{B}_{ext}(\mathbb{R}\times\mathbb{R}^d)$ and letting $A_t = A + (t,0)$, the specification

$$Y_t = \int_{\mathbb{R}\times\mathbb{R}^d} 1_A(t-s,\xi)L(\mathrm{d}s\,\mathrm{d}\xi) = L(A_t),$$

where L is a homogeneous Lévy basis with values in $\mathrm{ID}(\mathbb{R}^m)$, determines an m-dimensional *SID* process Y. The condition $|A| < \infty$ ensures the existence of the integral in the Rajput–Rosinski sense, and Y is an ambit process provided $A \subset (-\infty,0]\times\mathbb{R}^d$. We shall say that $Y$is a *trawling process* with *trawl* A.

Proposition. *To any infinitely divisible law $q \in \mathrm{ID}(\mathbb{R}^k)$ there exist trawling SID processes having one-dimensional marginal law q.*

Proof. By formula (2.19), the law of Y_t has cumulant function

$$\mathrm{C}\{\zeta \ddagger Y_t\} = |A|\mathrm{C}\{\zeta \ddagger L'\}, \tag{7.10}$$

from which the statement follows directly. □

The dependence structure of a one-dimensional trawling process Y with trawl A is reflected in the *autocorrelator* function

$$r(u) = |A \cap A_u|.$$

Clearly, in case L' has mean 0 and variance 1 then r is the autocorrelation function of Y. A great variety of autocorrelators can be constructed by suitable choice of the trawl A.

Example 6. Suppose that

$$-A = \{(s, \xi) : s > 0, \xi \in (-a(s), a(s))\} \tag{7.11}$$

where a is a positive strictly decreasing function on $(0, \infty)$ such that $|A| < \infty$. For the inverse a^{-1} of a we have $a^{-1}(0) = \infty$ and we let $a^{-1}(\xi) = 0$ in case $\xi \notin (-a(0), a(0))$. Then

$$|A_0 \cap A_u| = 2 \int_u^\infty a(s)\,\mathrm{d}s.$$

In particular, taking $a(s) = \frac{\lambda}{2} e^{-\lambda s}$ we have $r(u) = |A_0 \cap A_u| = e^{-\lambda u}$. Thus, in this case, if L' is square integrable with mean 0 then the trawling process has the same autocorrelation function as an OU process. In contrast to OU processes, the trawling processes are not Markovian; on the other hand, the assumption of self-decomposability is not needed in the definition of trawling processes; cf. the above proposition.

More generally, we may define multidimensional *SID* trawling processes $Y = (Y^{(1)}, \dots, Y^{(m)})$ by arbitrarily choosing points $x_1, \dots, x_m$ from $\mathbb{R}^d$ and trawls $A^{(1)}, \dots, A^{(m)}$ and letting

$$Y_t^{(j)} = \int_{\mathbb{R}\times\mathbb{R}^d} 1_{A^{(i)}}(t - s, \xi - x_j) L(\mathrm{d}s\,\mathrm{d}\xi) = L\big(A_t^{(j)}\big)$$

for $i = 1, \dots, m$ and where $A_t^{(j)} = A^{(j)} + (t, x_j)$. We define the *autocorrelator* between $Y^{(i)}$ and $Y^{(j)}$ by

$$r_{ij}(u) = \big|A_0^{(i)} \cap A_u^{(j)}\big|,$$

where $A_u^{(j)} = A^{(j)} + (u, 0)$.

Extending the above definition, by a *trawling process* we mean a process of the form

$$Y_t = M\big(A + (t, 0)\big)$$

for some $A \in \mathcal{B}_b(\mathbb{R} \times \mathbb{R}^d)$ and where M is a homogeneous infinitely divisible random measure on $(\mathbb{R} \times \mathbb{R}^d, \mathcal{B}_b(\mathbb{R} \times \mathbb{R}^d))$.

This applies in particular to subordinated Lévy bases $M(\mathrm{d}s\,\mathrm{d}\xi) = L(\mathrm{d}s\,\mathrm{d}\xi \curlywedge T)$. Typically, see Section 7.2, such a basis is constructed from a homogeneous Lévy basis L by introducing a stochastic volatility/intermittency in the model, via extended subordination.

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