A problem in last-passage percolation

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Abstract. Let $\{X(v), v \in \mathbb{Z}^d \times \mathbb{Z}_+\}$ be an i.i.d. family of random variables such that $P\{X(v) = e^b\} = 1 - P\{X(v) = 1\} = p$ for some $b > 0$. We consider paths $\pi \subset \mathbb{Z}^d \times \mathbb{Z}_+$ starting at the origin and with the last coordinate increasing along the path, and of length n. Define for such paths $W(\pi) =$ number of vertices π_i, $1 \le i \le n$, with $X(\pi_i) = e^b$. Finally, let $N_n(\alpha) =$ number of paths π of length n starting at $\pi_0 = \mathbf{0}$ and with $W(\pi) \ge \alpha n$. We establish several properties of $\lim_{n \to \infty} [N_n]^{1/n}$.

1 Statement of the problem

The problem studied in the present paper was suggested by the study of the free energy of a directed polymer in random environment. Here we consider a site version of semi-oriented first-passage percolation. To be more precise, we take for $\mathcal{L}$ the graph $\mathbb{Z}^d \times \mathbb{Z}_+$ with the last coordinate oriented in the standard way. A vertex $v \in \mathbb{Z}_+^d$ has an edge to $v \pm e_i + e_{d+1}$ for $1 \le i \le d$, and there are no other outgoing edges from v. Here and in the sequel e_i stands for the ith coordinate vector. We shall use the symbol $\mathbf{0}$ for the origin in $\mathbb{Z}^d$, as well as for the corresponding vertex of $\mathcal{L}$. For $v = (v_1, \ldots, v_d)$ a vertex of $\mathcal{L}$ or of $\mathbb{Z}^d$, $\|v\|$ will be the ℓ_1-norm of v, that is, $\|v\| = \sum_{i=1}^d |v_i|$. We will call a path on $\mathcal{L}$ *semi-oriented* and we will say that we are dealing with the semi-oriented case.

Our arguments can also be carried out in the fully oriented case in which $\mathcal{L}$ is replaced by the graph $\mathbb{Z}_+^{d+1}$ with an edge from v to $v + e_i$ for $v \in \mathbb{Z}_+^{d+1}$ and $1 \le i \le d + 1$. However, we shall not mention the latter case anymore in these notes.

We assign to each $v \in \mathcal{L}$ a random weight $X(v)$. The $\{X(v) : v \in \mathcal{L}\}$ are taken i.i.d. with the common distribution

$$P\{X(v) = e^b\} = p, \qquad P\{X(v) = 1\} = 1 - p$$

for some $b > 0, 0 < p < 1$. Nothing interesting happens when $p = 0$ or 1, so we exclude these values for p. For an oriented path $\pi = (\pi_0, \pi_1, \ldots, \pi_s)$ on $\mathcal{L}$ of length s we define

$$W(\pi) = \text{ number of vertices } \pi_i, 1 \le i \le s, \text{ with } X(\pi_i) = e^b.$$

[Note that $X(\pi_0)$ does not contribute to $W(\pi)$.] We further define for $0 \leq \alpha \leq 1 - p$

$N_s(\alpha) =$ number of paths π of length s starting at $\pi_0 = \mathbf{0}$ with $W(\pi) \geq \alpha s$.

We are interested in these notes in the behavior of $N_s(\alpha)$ for large s and different α.

Related problems have been studied by Comets, Popov and Vashkovskaya (2008), using different approach.

The first lemma is an exponential bound for $P\{N_s(\alpha) = 0\}$ for certain α, as $s \to \infty$. Basically this comes from Gandolfi and Kesten (1994), but the oriented case considered here is simpler than the unoriented case of Gandolfi and Kesten (1994). See also Cranston, Mountford and Shiga (2005).

Lemma 1. *The limit*

$$M = M(p) := \lim_{s \to \infty} \max_{\pi_0 = \mathbf{0}, |\pi| = s} \frac{1}{|\pi|} W(\pi) \tag{1.1}$$

exists and is constant a.s. If $p > 0$, then also $M > 0$. (Here $|\pi| = s$ in the max means that we take the maximum over all oriented paths of length s which start at $\mathbf{0}$.) Moreover, for any $\varepsilon > 0$ there exist constants $0 < C_i < \infty$ for which

$$P\{N_t(M(p) - \varepsilon) = 0\} = P\left\{ \max_{\pi_0 = \mathbf{0}, |\pi| = t} \frac{W(\pi)}{t} < M(p) - \varepsilon \right\}$$
$$\leq C_1 e^{-C_2 t}, \qquad t \geq 0. \tag{1.2}$$

Proof. In the sequel a path will always mean an oriented path on $\mathcal{L}$. However, a path does not have to start at π_0 at time 0. We will call the sequence $(\pi_j, \ldots, \pi_{j+t})$ a *path starting at π_j at time s and of length t* if $\|\pi_j\| = s$ and there is an oriented edge of $\mathcal{L}$ from π_i to π_{i+1} for $j \leq i < j + t$.

The limit M exists and is a.s. constant by Gandolfi and Kesten (1994). In the oriented case considered here this was proven in an easier way in Cranston, Mountford and Shiga (2005) by an application of Liggett's subadditive ergodic theorem [Liggett (1985)]. We merely outline the proof of Cranston, Mountford and Shiga (2005). Define

$$M_s(x, y) = \max_{\pi_0 = x, |\pi| = s, \pi_s = y} W(\pi),$$
$$M_s(x, *) = \max_y M_s(x, y) = \max_{\pi_0 = x, |\pi| = s} W(\pi).$$

Define further

$y(s) =$ first vertex y in lexicographical order for which $M_s(\mathbf{0}, y) = M_s(\mathbf{0}, *)$.

Then, for $s, t \geq 1$

$$M_{s+t}(\mathbf{0}, *) \geq M_s(\mathbf{0}, *) + M_t(y(s), *) = M_s(\mathbf{0}, y(s)) + M_t(y(s), *). \tag{1.3}$$

Indeed, the left-hand side is a maximum over all paths starting at $\mathbf{0}$ and of length $s + t$, while the right-hand side is just a maximum over paths which start at $\mathbf{0}$ but pass through $y(s)$ at time s and have length $s + t$. If one sets $M_0(x, y) = 0$ for all x, y, then (1.3) remains valid even if $s = 0$ or $t = 0$.

We note further that if all $X(v)$ with $\|v\| \leq s$ are given, then $y(s)$ is also fixed and $M_t(y(s), *)$ is defined in the same way as $M_t(\mathbf{0}, *)$, but with $X(v)$ replaced by $X(v + y(s))$. It follows from this that the conditional distribution of $M_t(y(s), *)$ given all $X(v)$ with $\|v\| \leq s$ is just the same as the unconditional distribution of $M_t(\mathbf{0}, *)$, and hence does not depend on the $X(v)$ with $\|v\| \leq s$. Thus, $M_t(y(s), *)$ is independent of those $X(v)$ and has the distribution of $M_t(\mathbf{0}, *)$. These observations allow us to apply Liggett's theorem [Liggett (1985), Theorem VI.2.6] to the variables $X_{s,t} := M_{t-s}(y(s), u(s, t))$, where

$$u(s + t) = \text{first vertex } u \text{ in lexicographical order}$$

$$\text{for which } M_t(y(s), u) = M_t(y(s), *).$$

This shows that $M(p)$ exists and is almost surely constant. The fact that $M > 0$ is immediate from

$$M \geq \lim_{t \to \infty} \frac{W(\pi^{(t)})}{t},$$

where $\pi^{(t)}$ is the path which moves along the first coordinate axis from $\mathbf{0}$ to $(t, 0, \ldots, 0)$ in t steps. Indeed,

$$\frac{W(\pi^{(t)})}{t} = \frac{1}{t} \sum_{i=1}^{t} I[X(i, 0, \ldots, 0) = e^b]$$

and this tends to p by the strong law of large numbers.

Now, to start on the proof of (1.2) note first that the equality of the first and second member in (1.2) is immediate from the definitions. Indeed, $N_t(\alpha) = 0$ means that for all path π of length t and starting at the origin $W(\pi) \leq \alpha t$. We therefore concentrate on the inequality in (1.2). Observe that by the definition of W

$$\frac{1}{|\pi|} W(\pi) \leq 1 \tag{1.4}$$

so that also

$$\frac{M_s}{s} \text{ is bounded} \quad \text{and} \quad \lim_{s \to \infty} \frac{E M_s}{s} = M. \tag{1.5}$$

Let $\varepsilon > 0$ be given. One can then fix s such that $E M_s / s - \varepsilon/2 \geq M(p) - \varepsilon$. Now define recursively $y_0 = \mathbf{0}$, $y_1 = y(s)$,

$$y_{k+1} = \text{first vertex } y \text{ in lexicographical order for which } M_s(y_k, y) = M_s(y_k, *).$$

Analogously to (1.3) we then have

$$M_{ks+t}(x, *) \geq M_{ks}(x, z) + M_t(z, *).$$

This holds for any z and in particular for any z for which $M_{ks}(x, z) = \max_v M_{ks}(x, v)$. By iteration,

$$M_{ks}(\mathbf{0}, *) \geq M_s(\mathbf{0}, y_1) + M_{(k-1)s}(y_1, *)$$
$$\geq M_s(\mathbf{0}, y_1) + M_s(y_1, y_2) + M_{(k-2)s}(y_2, *)$$
$$\geq \cdots \geq \sum_{j=0}^{k-1} M_s(y_j, y_{j+1}).$$

By the argument given a few lines after (1.3), the random variables $M_s(y_k, y_{k+1})$ are i.i.d. Moreover, the variables $M_s(y_j, y_{j+1})$, $j \geq 0$, are bounded [see (1.4)]. By exponential bounds for the sum of i.i.d. variables or Bernstein's inequality [see Chow and Teicher (1988), Exercise 4.3.14] we have

$$P\{M_{ks}(\mathbf{0}, *) \leq ks[M - \varepsilon]\}$$
$$\leq P\left\{ \sum_{j=0}^{k-1} M_s(y_j, y_{j+1}) \leq k[EM_s(\mathbf{0}, y_1) - \varepsilon s/2] \right\} \leq C_1 e^{-C_2 k}. \tag{1.6}$$

This proves (1.2) for t a multiple of s. The extension to arbitrary positive integers t is an easy monotonicity argument. If $ks \leq t < (k+1)s$ and π is a path of length t, let π' be the initial piece of length ks of π. Then $N_t(M - 2\varepsilon) = 0$ implies $W(\pi') \leq W(\pi) \leq t(M - 2\varepsilon) \leq ks(M - \varepsilon)$ for large k and this happens only on a set of probability at most $C_1 \exp[-C_2 ks]$. $\qquad \square$

The next lemma will help us to formulate a concrete problem.

Lemma 2. *For* $0 \leq \alpha \neq M$

$$\lambda(\alpha) = \lambda(\alpha, p) := \lim_{t \to \infty} [N_t(\alpha)]^{1/t} \qquad \textit{exists and is constant a.s.} \tag{1.7}$$

Proof. This proof uses standard arguments for superconvolutive sequences. However the assumptions here seem to differ from the usual ones and we see no way to appeal to a standard theorem such as Hammersley (1974) for the lemma. We therefore go into some details. We break the proof into three steps.

Step 1. To begin with, if $\alpha > M$, then by the fact that the limit in (1.1) exists we have $\max_{\pi_0 = \mathbf{0}, |\pi| = s} W(\pi)/s < \alpha$ eventually. But this says that $N_s(\alpha) = 0$ for all large s, a.s. Thus (1.7) with $\lambda(\alpha) = 0$ is obvious when $\alpha > M$.

Next, fix an α with $\alpha < M$. We shall suppress α in our notation for the rest of this proof. In the rest of this step we define N_t and related quantities and show that they are almost superconvolutive. Define

$$N_t(x) = N_t(x; \alpha)$$

$\qquad$ = number of paths π of length t which start at x and have $W(\pi) \geq \alpha t$,

$$N_t(x, y) = N_t(x, y; \alpha) = \text{number of paths } \pi \text{ of length } t \text{ which start at } x$$
$$\text{and end at } y \text{ and have } W(\pi) \geq \alpha t,$$

and

$$N_t(x, *) = \max_y N_t(x, y).$$

Note that

$$N_t(x) = \sum_y N_t(x, y) \quad \text{and} \quad N_t = N_t(\mathbf{0}).$$

Accordingly, we set

$$N_t(*) = N_t(\mathbf{0}, *).$$

Note also that $N_t(x, y)$ can be nonzero only if $y = x + v$ for some $v \in \mathcal{L}$ with $\|v\| = t$. There are at most $(t + 1)^d$ possible values for v. Thus the max here is really a maximum over at most $(t + 1)^d$ values of y. Consequently,

$$(t + 1)^{-d} N_t(x) \leq N_t(x, *) \leq N_t(x). \tag{1.8}$$

It follows from this that it suffices for (1.7) to prove that

$$\lim_{t \to \infty} [N_t(*)]^{1/t} \text{ exists and is constant a.s.} \tag{1.9}$$

The advantage of $N_t(*)$ is that it is almost superconvolutive. To make this precise, we order the vertices of $\mathcal{L}$ lexicographically. If $N_t > 0$, then also $N_t(*) > 0$. In this case we define

$$z(t) = \text{ first site } z \text{ in the lexicographical ordering for which } N_t(\mathbf{0}, z) = N_t(*).$$

If $N_t = 0$, then also $N_t(*) = 0$. In this case we take for $z(t)$ any fixed vertex z of $\mathcal{L}$ with $\|z\| = t$. For the sake of definiteness we shall take $z(t) = (t, 0, \ldots, 0)$. With these definitions we have for $s, t \geq 1$

$$N_{s+t}(*) \geq N_s(*) \cdot N_t(z(s), *). \tag{1.10}$$

This is trivial if $N_s = 0$, for then also $N_s(*) = 0$. If $N_s > 0$, and hence also $N_s(*) > 0$, then (1.10) follows from the fact that $N_{s+t}(*)$ is no smaller than [number of paths $\pi = (\pi_0, \ldots, \pi_s)$ of length s with $\pi_0 = \mathbf{0}, \pi_s = z(s)$ and $W(\pi) \geq \alpha s$] times [number of paths $\tilde{\pi}$ which start at time s at $\pi_s = z(s)$ and are at time $s + t$ at any fixed vertex z, and have $W(\tilde{\pi}) \geq \alpha t$]. The maximum over all z of the second factor is just $N_t(z(s), *)$.

Step 2. In this step we show that

$$\lim_{t \to \infty} \frac{1}{t} E\{\log N_t(*)\} \text{ exists and lies in } [0, \log(2d)]. \tag{1.11}$$

We set

$$Y_s = Y_s(\alpha) = [\log N_s(*)]^+, \qquad Y_{s,t} = [\log N_t(z(s), *)]^+$$

and

$$Z_s = Z_s(\alpha) = s \log(2d) - Y_s, \qquad Z_{s,t} = t \log(2d) - Y_{s,t}.$$

Note that Y_s is at most equal to the logarithm of the number of paths π of length s with $\pi_0 = \mathbf{0}$, that is, $Y_s \leq s \log(2d)$. Consequently,

$$0 \leq Z_s \leq s \log(2d). \tag{1.12}$$

Similarly,

$$0 \leq Z_{s,t} \leq t \log(2d). \tag{1.13}$$

On the event $A(s,t) := \{N_s > 0, N_t(z(s)) > 0\}$ inequalities $N_s(*) \geq 1$ and $N_t(z(s), *) \geq 1$ hold, so that $Z_s = s \log(2d) - \log N_s(*)$ and $Z_{s,t} = t \log(2d) - \log N_t(z(s), *)$. The relation (1.10) therefore shows that on the event $A(s,t)$ we have

$$Z_{s+t} \leq Z_s + Z_{s,t}. \tag{1.14}$$

Off the event $A_{s,t}$ we need to introduce a correction term. We define

$$\Psi(s,t) = \Psi(s,t,\alpha) := I[N_s = 0]Y_{s,t} + I[N_t(z(s)) = 0]Y_s. \tag{1.15}$$

It is now easy to see that we always have

$$Z_{s+t} \leq Z_s + Z_{s,t} + \Psi(s,t); \tag{1.16}$$

in fact, if $N_s = 0$, then $Y_s = 0$ and the right-hand side equals $(s+t)\log(2d)$. Similarly if $N_t(z(s)) = 0$.

We claim that $N_t(z(s))$ is independent of all $X(v)$ with $\|v\| \leq s$ and has the same distribution as N_t. In fact, if we fix all $X(v)$ with $\|v\| \leq s$, then also $z(s)$ is determined, and $N_t(z(s))$ is defined in the same way as $N_t(\mathbf{0}) = N_t$, but with $X(v)$ replaced by $X(z(s) + v)$. This shows that the conditional distribution of $N_t(z(s))$, given all $X(v)$ with $\|v\| \leq s$ is the same as the unconditional distribution of N_t, which proves our claim. Taking expectations in (1.16) therefore gives

$$\begin{aligned} EZ_{s+t} &\leq EZ_s + EZ_{s,t} + E\Psi(s,t) \\ &\leq EZ_s + EZ_t + P\{N_s = 0\}t \log(2d) + P\{N_t = 0\}s \log(2d). \end{aligned} \tag{1.17}$$

Note that all these expectations are finite by virtue of (1.12) and (1.13). In particular, if K is any positive integer, and $s = t = K2^j$, then

$$\frac{1}{K2^{j+1}}EZ_{K2^{j+1}} \leq \frac{2}{K2^{j+1}}EZ_{K2^j} + \frac{K2^{j+1}\log(2d)}{K2^{j+1}}P\{N_{K2^j} = 0\}.$$

However, if we take $\varepsilon = M - \alpha$, then we see from Lemma 1 that

$$P\{N_t = 0\} \leq C_1 \exp[-C_2 t], \tag{1.18}$$

whence

$$\frac{1}{K2^{j+1}}EZ_{K2^{j+1}} \leq \frac{1}{K2^j}EZ_{K2^j} + \log(2d)C_1\exp[-C_2K2^j].$$

This easily implies

$$\limsup_{j\to\infty}\frac{1}{K2^j}EZ_{K2^j} \leq \liminf_{j\to\infty}\frac{1}{K2^j}EZ_{K2^j},$$

so that

$$\gamma(K) := \lim_{j\to\infty}\frac{1}{K2^j}EZ_{K2^j} \qquad \text{exists and lies in } [0,\log(2d)] \qquad (1.19)$$

[see (1.12) for the bounds on γ]. Next we will prove that $\gamma(K)$ is independent of K. Let $K, L \geq 1$ be integers and let the dyadic expansion of L/K be

$$\frac{L}{K} = \sum_{j=-\infty}^{n} 2^{k_j}, \qquad (1.20)$$

where k_j is increasing in j, $\text{sign}(k_j) = \text{sign}(j)$ and n some finite nonnegative integer. The sum over negative j may actually be finite, but in order to avoid further notation we write sum over the negative j as starting at $-\infty$.

The expansion (1.20) can also be written as

$$L2^\ell = K\sum_{j=-\infty}^{n} 2^{k_j+\ell}$$

for any integer $\ell \geq 0$. We shall let $\ell \to \infty$ later on, but for the moment leave it unspecified. Since we shall use Z_m for some complicated expressions of m, we shall write $Z(m)$ instead of Z_m in the calculations below. Start with an application of (1.17) with $s + t = L2^\ell$, $s = K2^{k_n+\ell}$. Thus we take

$$t = L2^\ell - K2^{k_n+\ell} = K\sum_{j=-\infty}^{n-1} 2^{k_j+\ell}. \qquad (1.21)$$

Since the right-hand side is positive, t is a positive integer. Taking into account that

$$t = \text{right-hand side of } (1.21) \leq K2^{k_n+\ell} = s,$$

we obtain

$$\begin{aligned} EZ(L2^\ell) &\leq EZ(K2^{k_n+\ell}) + EZ(t) + P\{N_s = 0\}t\log(2d) \\ &\quad + P\{N_t = 0\}s\log(2d) \\ &\leq EZ(K2^{k_n+\ell}) + EZ(t) + C_1t\log(2d)\exp[-C_2s] \qquad (1.22) \\ &\quad + C_1s\log(2d)\exp[-C_2t] \qquad [\text{by } (1.18)] \\ &\leq EZ(K2^{k_n+\ell}) + EZ(t) + L2^\ell C_1\exp[-C_2t]\log(2d). \end{aligned}$$

Divide both sides of the inequality by $L2^\ell$ and let $\ell \to \infty$, and note that $t \to \infty$ as $\ell \to \infty$ [see (1.21)]. (1.19) then shows that

$$\gamma(L) \le \gamma(K)\frac{K}{L}2^{k_n} + \limsup_{\ell \to \infty} \frac{EZ(t)}{L2^\ell}.$$

We repeat this argument in the following way. Set

$$t_r = K \sum_{j=-\infty}^{n-r} 2^{k_j+\ell}, \qquad r \ge 0,$$

and apply (1.17) and (1.18) with t_r for $s + t$ and $K2^{k_{n-r}+\ell}$ for s, and consequently t_{r+1} for t. Taking into account that $t_{r+1} \le K2^{k_{n-r}+\ell}$ we obtain

$$
\begin{aligned}
EZ(t_r) &\le EZ(K2^{k_{n-r}+\ell}) + EZ(t_{r+1}) \\
&\quad + C_1 t_r \log(2d) \exp[-C_2 t_{r+1}], \qquad r \ge 0.
\end{aligned}
\tag{1.23}
$$

For $r = 0$ this is just (1.22) with $L2^\ell$ for t_0. This time we successively use (1.23) for $r = 0, 1, \ldots, R-1$ before we divide by $L2^\ell$, where R is determined as follows: (i) if the expansion in (1.20) has only finitely many terms, then we take R such that 2^{k_n-R} is the smallest power of 2 appearing in the right-hand side of (1.20) (so that $t_{R+1} = 0$); (ii) if the expansion in (1.20) has infinitely many terms, then we fix a small number $\eta > 0$ and let $R = R(\eta)$ be the smallest nonnegative integer such that

$$K \sum_{j=-\infty}^{n-R} 2^{k_j} \le \eta. \tag{1.24}$$

Note that R does not depend on ℓ. We get

$$
\begin{aligned}
EZ(L2^\ell) &= EZ(t_0) \\
&\le EZ(K2^{k_n+\ell}) + EZ(t_1) + C_1 t_0 \log(2d) \exp[-C_2 t_1] \\
&\le EZ(K2^{k_n+\ell}) + EZ(K2^{k_{n-1}+\ell}) + EZ(t_2) \\
&\quad + C_1 t_0 \log(2d) \exp[-C_2 t_1] + C_1 t_1 \log(2d) \exp[-C_2 t_2] \\
&\le \cdots \\
&\le \sum_{r=0}^{R-1} EZ(K2^{k_{n-r}+\ell}) + EZ(t_R) + C_1 \log(2d) \sum_{r=0}^{R-1} t_r \exp[-C_2 t_{r+1}].
\end{aligned}
\tag{1.25}
$$

Now we divide by $L2^\ell$ and let $\ell \to \infty$. Consider first case (i) when the expansion in (1.20) is finite. Now recall

$$\frac{t_r}{L2^\ell} = \frac{K}{L} \sum_{j=-\infty}^{n-r} 2^{k_j} \le 1.$$

On the other hand, $t_{r+1} \to \infty$ as $\ell \to \infty$ for each $r \le R - 1$. The inequality (1.25) therefore implies

$$\frac{1}{L2^{\ell}} C_1 \log(2d) \sum_{r=0}^{R-1} t_r \exp[-C_2 t_{r+1}] \to 0$$

and, by virtue of (1.19) and $t_R = K2^{k_n-R+\ell}$,

$$\gamma(L) = \lim_{\ell \to \infty} \frac{EZ(L2^{\ell})}{L2^{\ell}} \le \sum_{j=0}^{R} \gamma(K) \frac{K}{L} 2^{k_n-j} = \gamma(K). \qquad (1.26)$$

Next, in case (ii) we obtain similarly

$$\gamma(L) \le \sum_{j=0}^{R-1} \gamma(K) \frac{K}{L} 2^{k_n-j} + \limsup_{\ell \to \infty} \frac{EZ(t_R)}{L2^{\ell}} + \limsup_{\ell \to \infty} C_1 \log(2d) \exp[-C_2 t_R].$$

This time we use that

$$\frac{1}{L2^{\ell}} EZ(t_R) \overset{(1.12)}{\le} \frac{1}{L2^{\ell}} t_R \log(2d) \overset{(1.24)}{\le} \frac{\eta \log(2d)}{L}.$$

Finally,

$$t_R \ge K2^{k_n-R-1} 2^{\ell} \to \infty \qquad \text{as } \ell \to \infty,$$

because the term 2^{k_n-R-1} is actually present in (1.20) in case (ii). Thus in case (ii)

$$\gamma(L) \le \sum_{j=0}^{R-1} \gamma(K) \frac{K}{L} 2^{k_n-j} + \frac{\eta \log(2d)}{L} \le \gamma(K) + \frac{\eta \log(2d)}{L}.$$

Since this holds for any $\eta > 0$ we obtain in both cases that $\gamma(L) \le \gamma(K)$. By interchanging the roles of K and L we finally prove that $\gamma(K)$ does not depend on K, as claimed. We shall write γ for the common value of the $\gamma(K)$.

   Step 3. In this step we deduce the almost sure convergence of $(1/t) \log N_t(*)$. As pointed out after (1.8), this will prove (1.7).

   We first show that $[K2^j]^{-1} Z_{K2^j}$ converges almost surely as $j \to \infty$ for any fixed positive integer K. The limit turns out to be independent of K. Recall that $N_t(z(s))$ is independent of all $X(v)$ with $\|v\| \le s$ and has the same distribution as N_t. We now follow the second moment calculations of Hammersley (1974) or Smythe and Wierman (1985). We obtain from (1.16)

$$\frac{EZ^2_{K2^{j+1}}}{[K2^{j+1}]^2} \le \frac{1}{2} \frac{EZ^2_{K2^j}}{[K2^j]^2} + \frac{1}{2} \frac{[EZ_{K2^j}]^2}{[K2^j]^2} + 4 \frac{\sqrt{EZ^2_{K2^j}} \sqrt{E\Psi^2(s,t)}}{[K2^{j+1}]^2}$$
$$+ \frac{E\Psi^2(K2^j, K2^j)}{[K2^{j+1}]^2}. \qquad (1.27)$$

It follows from (1.15), (1.12) and (1.18) that

$$E\Psi^2(K2^j, K2^j) \le 2[K2^j \log(2d)]^2 C_1 \exp[-C_2 K2^j]. \qquad (1.28)$$

By subtracting $[K2^{j+1}]^{-2}[EZ_{K2^{j+1}}]^2$ from both sides of (1.27) and using the bound in (1.28) we now obtain for a suitable constant $C_3 < \infty$

$$\mathrm{Var}\left[\frac{Z_{K2^{j+1}}}{K2^{j+1}}\right] \le \frac{1}{2}\,\mathrm{Var}\left[\frac{Z_{K2^j}}{K2^j}\right] + \frac{[EZ_{K2^j}]^2}{[K2^j]^2} - \frac{[EZ_{K2^{j+1}}]^2}{[K2^{j+1}]^2}$$
$$+ C_3 \exp[-C_2 K2^{j-1}]. \qquad (1.29)$$

Finally, summation of (1.29) from $j = 0$ to $j = J$ and simple algebraic manipulations yield

$$\frac{1}{2}\sum_{j=0}^{J}\mathrm{Var}\left[\frac{Z_{K2^j}}{K2^j}\right] \le \mathrm{Var}\left[\frac{Z_K}{K}\right] + \frac{[EZ_K]^2}{K^2} + C_3\sum_{j=0}^{J}C_3\exp[-C_2 K2^{j-1}].$$

Since this holds for any $J < \infty$, it follows

$$\sum_{j=0}^{\infty}\mathrm{Var}\left[\frac{Z_{K2^j}}{K2^j}\right] < \infty,$$

and then by Chebychev's inequality and Borel–Cantelli

$$\frac{Z_{K2^j} - EZ_{K2^j}}{K2^j} \to 0 \qquad (j \to \infty) \text{ a.s.}$$

Combined with (1.19) and the independence of γ of K, this gives

$$\frac{Z_{K2^j}}{K2^j} \to \gamma \qquad (j \to \infty) \text{ a.s.} \qquad (1.30)$$

It remains to improve the convergence in (1.30) to convergence along all positive integers. To this end we fix a $0 < \varepsilon < 1$ and note that (1.30) implies

$$\frac{Z(\lfloor (1+\varepsilon)^r \rfloor 2^j)}{\lfloor (1+\varepsilon)^r \rfloor 2^j} \to \gamma \qquad \text{for all integers } r \ge 0 \text{ a.s.}$$

Now, for small ε and for all large n we can find $1 \le r \le \frac{2\log 2}{\log(1+\varepsilon)}$ and a j such that

$$\lfloor (1+\varepsilon)^r \rfloor 2^j \le n \le \lfloor (1+\varepsilon)^{r+1} \rfloor 2^j.$$

For such r and j we can apply (1.16) with $s + t = n$, $s = \lfloor (1+\varepsilon)^{r-1} \rfloor 2^j$ and

$$\frac{\varepsilon}{3}\lfloor (1+\varepsilon)^{r-1} \rfloor 2^j \le \frac{\varepsilon}{2}(1+\varepsilon)^{-3}n \le t = n - s \le 3\varepsilon(1+\varepsilon)^{r-1}2^j \le 4\varepsilon(1+\varepsilon)^{-1}n.$$

By (1.17) and (1.18) we then have outside a set of probability

$$P\{N_s = 0\} + P\{N_t = 0\} \le 2C_1 \exp\left[-C_2\frac{\varepsilon}{2}(1+\varepsilon)^{-3}n\right] \qquad (1.31)$$

that

$$Z_n \leq Z\big(\lfloor(1+\varepsilon)^{r-1}\rfloor 2^j\big) + Z(s,t)$$
$$\leq Z\big(\lfloor(1+\varepsilon)^{r-1}\rfloor 2^j\big) + t\log(2d) \qquad [\text{see (1.13)}] \qquad (1.32)$$
$$\leq Z\big(\lfloor(1+\varepsilon)^{r-1}\rfloor 2^j\big) + 4\log(2d)\varepsilon(1+\varepsilon)^{-1}n,$$

and consequently also

$$\frac{Z_n}{n} \leq \frac{Z(\lfloor(1+\varepsilon)^{r-1}\rfloor 2^j)}{\lfloor(1+\varepsilon)^r\rfloor 2^j} + 4\log(2d)\varepsilon(1+\varepsilon)^{-1}. \qquad (1.33)$$

Since the sum over n of the probabilities in (1.31) converges, (1.33) will be almost surely valid for all large n. By taking first the limsup as $n \to \infty$ and then as $\varepsilon \downarrow 0$ we find that

$$\limsup_{n\to\infty} \frac{Z_n}{n} \leq \lim_{\varepsilon\downarrow 0} \lim_{j\to\infty} \sup_{1\leq r\leq 2\log 2/\log(1+\varepsilon)} \frac{Z(\lfloor(1+\varepsilon)^{r-1}\rfloor 2^j)}{\lfloor(1+\varepsilon)^r\rfloor 2^j} = \gamma \qquad \text{a.s.}$$

In almost the same way one can show that outside a set of negligible probability

$$\frac{Z(\lfloor(1+\varepsilon)^{r+1}\rfloor 2^j)}{(1+\varepsilon)^{r+1}2^j} \leq (1+\varepsilon)^2 \frac{Z_n}{n}$$

and obtain $\liminf_{n\to\infty} Z_n/n \geq \gamma$.

We therefore proved that $\lim_{n\to\infty} Z_n/n = \gamma$ almost surely, and (1.7) with $\lambda = 2de^{-\gamma}$ is then immediate from the definition of Z. $\qquad \square$

The main problem in these notes is to find information about $N_n(\alpha)$ as a function of α. In particular, we want to compare $\lambda(\alpha)$ to $\phi = \phi(\alpha) := \lim_{n\to\infty}[EN_n(\alpha)]^{1/n}$. Note that ϕ is easy to evaluate. Indeed, there are $(2d)^n$ oriented paths of length n. A given path π of length n contributes to N_n if and only if $W(\pi) \geq \alpha n$. But, for any given π of length n, $W(\pi)$ has the a binomial distribution with n trials and success probability p. Therefore

$$EN_n(\alpha) = (2d)^n \sum_{k\geq\alpha n} \binom{n}{k} p^k (1-p)^{n-k}, \qquad (1.34)$$

and if $\alpha \geq p$, then

$$\phi = 2d\left(\frac{p}{\alpha}\right)^\alpha \left(\frac{1-p}{1-\alpha}\right)^{1-\alpha}. \qquad (1.35)$$

In the next section we shall prove a few facts concerning λ and ϕ; see also Figure 1.
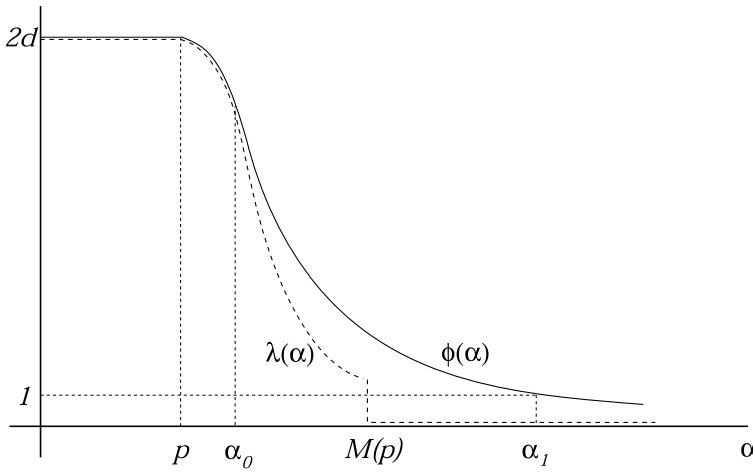
Figure 1 *Illustration of the graph of $\phi(\alpha)$ (the solid curve) and of $\lambda(\alpha)$ (the dotted curve). The figure is not drawn to scale. The points α_0, α_1 and $M(p)$ are explained in Proposition 4, display (3.1) and Lemma 1, respectively.*

2 Properties of λ

Let us first take care of the trivial region when $\alpha \leq p$. Then $P\{W(\pi) \geq \alpha n\}$ is of order 1 as $n \to \infty$ and $\phi(\alpha) = 2d$. So we expect that also $\lambda(\alpha) = 2d$. The following lemma confirms this if $\alpha < p$ or if $d \geq 4$ and $\alpha = p$.

Lemma 3. *For $\alpha < p$, or $d \geq 4$ and $\alpha = p$*

$$\lambda(\alpha) = \phi(\alpha) = 2d. \tag{2.1}$$

Proof. The case $d \geq 4, \alpha = p$, will be included in Proposition 4. We therefore assume throughout this proof that $\alpha < p$. It is evident from the strong law of large numbers that $M(p) \geq p$, since

$$\lim_{n \to \infty} \frac{1}{n} W(\pi^{(n)}) = p \qquad \text{a.s.}$$

if $\pi^{(n)}$ is the path which moves along the first coordinate axis, that is, with $\pi_j^{(n)} = (j e_1, j), 0 \leq j \leq n$. We therefore may assume for the rest of this proof that $\alpha < M(p)$.

$\phi(\alpha) = 2d$ for $\alpha \leq p$ is immediate from (1.34) and the weak law of large numbers, so we concentrate on proving $\lambda(\alpha) = 2d$. Let

$R_n = ($number of paths π of length n starting at $\mathbf{0}$ and with $W(\pi) < \alpha n)$.

Then $E R_n = (2d)^n P\{W(\pi) < \alpha n\}$ for any π of length n and starting at $\mathbf{0}$. Since $W(\pi)$ has a binomial distribution with parameters n, p, and $p > \alpha$, Bernstein's

inequality [Chow and Teicher (1988), Exercise 4.3.14] shows that $P\{W(\pi) \leq \alpha n\} \leq C_1 \exp[-C_2 n]$ for some constants C_1, C_2 (depending on p and α, but not on n). Consequently, ER_n is exponentially small with respect to $(2d)^n$. Thus by Markov's inequality

$$P\left\{R_n \geq \frac{1}{2}(2d)^n\right\} \leq \frac{ER_n}{(1/2)(2d)^n}$$

is also exponentially small. Hence by Borel–Cantelli, almost surely $N_n = (2d)^n - R_n \geq \frac{1}{2}(2d)^n$ eventually. This, together with Lemma 2 proves $\lambda(\alpha) = 2d$. $\square$

The following proposition shows that the equality $\lambda(\alpha) = \phi(\alpha)$ extends to α some distance beyond p. This is much more difficult to prove than the preceding lemma.

Proposition 4. *If $d \geq 4$, then there exists some constant $\alpha_0 = \alpha_0(p) > p$ such that*

$$\lambda(\alpha) = \phi(\alpha) \tag{2.2}$$

for $\alpha < \alpha_0$. In particular $M(p) \geq \alpha_0$ and the limit $\lambda(\alpha)$ in (1.7) exists for all $\alpha < \alpha_0$.

Proof. By the proof of Lemma 3 we only have to prove (2.2) for $p \leq \alpha \leq \alpha_0$ for some $\alpha_0 > p$. For the remainder of this proof a path is tacitly assumed to have length n and to start at **0**. Let

$$I[\pi] = \begin{cases} 1, & \text{if } W(\pi) \geq \alpha n, \\ 0, & \text{if } W(\pi) < \alpha n. \end{cases}$$

Then

$$N_n = \sum_{\pi_0 = \mathbf{0}, |\pi| = n} I[\pi].$$

We shall prove that for suitable $\alpha_0 > p$

$$EN_n^2 \leq C_3 [EN_n]^2 \qquad \text{for } p \leq \alpha \leq \alpha_0 \tag{2.3}$$

for a suitable constant $C_3 < \infty$ (independent of n). By Schwarz' inequality [Durrett (1996)] this will imply

$$P\{N_n \geq EN_n/2\} \geq \frac{1}{4C_3}. \tag{2.4}$$

In particular this will imply

$$M(p) \geq \limsup_{n \to \infty} \frac{1}{n} N_n(\alpha) \geq \limsup_{n \to \infty} \frac{1}{2n} EN_n(\alpha) = \frac{1}{2}\phi(\alpha) > 0 \qquad [\text{see } (1.35)]$$

for $\alpha \leq \alpha_0$. But $\limsup_{n \to \infty} \frac{1}{n} N_n(\alpha') = 0$ for $\alpha' > M(p)$, by definition of $M(p)$, so that $M(p) \geq \alpha_0$. Lemma 2 then shows that $\lambda(\alpha) = \lim_{n \to \infty} [N_n(\alpha)]^{1/n}$ exists

almost surely for all $\alpha < \alpha_0$. Finally, (2.4) will then show that the almost sure limit of $[N_n(\alpha)]^{1/n}$ satisfies

$$\lambda(\alpha) \geq \lim_{n\to\infty} [EN_n(\alpha)]^{1/n} = \phi(\alpha) \qquad \text{for } p \leq \alpha < \alpha_0.$$

In the other direction, Markov's inequality immediately implies that always

$$\lambda \leq \phi. \tag{2.5}$$

Together these inequalities will prove (2.2) and the last statement in the proposition.

We turn now to the proof of (2.3). Obviously

$$
\begin{aligned}
EN_n^2 &= \sum_{\pi'} \sum_{\pi''} E\{I[\pi']I[\pi'']\} \\
&= \sum_{k=1}^{n} \sum_{\pi'} \sum_{\substack{\pi'' \text{ with } |\pi'\cap\pi''|=k}} E\{I[\pi']I[\pi'']\}.
\end{aligned}
\tag{2.6}
$$

Let $\{S'_n\}_{n\geq 0}$ and $\{S''_n\}_{n\geq 0}$ be two independent simple random walks on $\mathcal{L}$, both starting at $\mathbf{0}$, and let T_n be a random variable with a binomial distribution with parameters n and p. Further let

$$\rho = P\{S'_n = S''_n \text{ for some } n \geq 1\}.$$

Then the number of pairs of paths π', π'' which meet at least k times (not including at time 0, when both paths are at $\mathbf{0}$) is at most $(2d)^{2n}\rho^k$, provided $k \leq n$; there are no pairs of paths of length n which meet more than n times. Let J be the collection of vertices which π' and π'' have in common (again excluding $\mathbf{0}$). Then, if J contains exactly k vertices,

$$P\{W(\pi'') \geq \alpha n \,|\, X(v) \text{ for } v \in \pi''\}$$

$$= P\left\{ \sum_{v\in\pi'' \text{ but } v\notin J} I[X(v)=e^b] \geq \alpha n - \sum_{v\in J} I[X(v)=e^b] \,\bigg|\, X(v) \text{ for } v \in J \right\}$$

$$\leq P\left\{ \sum_{v\in\pi'' \text{ but } v\notin J} I[X(v)=e^b] \geq \alpha n - k \right\}$$

$$= P\{T_{n-k} \geq \alpha n - k\}.$$

Consequently, if $|\pi' \cap \pi''| = k$, then

$$
\begin{aligned}
E\{I[\pi']I[\pi'']\} &= P\{W(\pi') \geq \alpha n\}P\{W(\pi'') \geq \alpha n \,|\, W(\pi') \geq \alpha n\} \\
&\leq P\{T_n \geq \alpha n\}P\{T_{n-k} \geq \alpha n - k\} \\
&\leq P\{T_n \geq \alpha n\}.
\end{aligned}
$$

We substitute these bounds in (2.6). We then see that the right-hand side of (2.6) is for any $0 < \beta \leq 1$ at most

$$\sum_{1 \leq k \leq \beta n} (2d)^{2n} \rho^k P\{T_n \geq \alpha n\} P\{T_{n-k} \geq \alpha n - k\} + (2d)^{2n} \rho^{\beta n} P\{T_n \geq \alpha n\}$$

$$\leq [(2d)^n P\{T_n \geq \alpha n\}]^2$$

$$\times \left[\sum_{1 \leq k \leq \beta n} \rho^k \frac{P\{T_{n-k} \geq \alpha n - k\}}{P\{T_n \geq \alpha n\}} + \rho^{\beta n} \frac{1}{P\{T_n \geq \alpha n\}} \right] \tag{2.7}$$

$$= [E N_n]^2 \left[\sum_{1 \leq k \leq \beta n} \rho^k \frac{P\{T_{n-k} \geq \alpha n - k\}}{P\{T_n \geq \alpha n\}} + \rho^{\beta n} \frac{1}{P\{T_n \geq \alpha n\}} \right].$$

Note that ρ depends on p and d only, so it is a constant < 1 for our purposes here. Moreover, by (1.35), for any given $\alpha_0 > p$, it will be the case that for all $p \leq \alpha \leq \alpha_0$

$$\lim_{n \to \infty} [P\{T_n \geq \alpha n\}]^{1/n} \geq \lim_{n \to \infty} [P\{T_n \geq \alpha_0 n\}]^{1/n} = (2d)^{-1} \phi(\alpha_0)$$

$$= \left(\frac{p}{\alpha_0} \right)^{\alpha_0} \left(\frac{1-p}{1-\alpha_0} \right)^{1-\alpha_0}.$$

Therefore, for any $0 \leq \beta \leq 1$, we can choose $\alpha_0 = \alpha_0(\beta) > 0$ so close to p that $\rho^{\beta n} [P\{T_n \geq \alpha n\}]^{-1}$ is exponentially small, uniformly in $p \leq \alpha \leq \alpha_0$. In other words, the second term in the right-hand side of (2.7) can be taken care of by taking $\alpha_0 - p > 0$ small, after we have picked β. Thus, to prove (2.2) it suffices to show that we can pick $\alpha_0 > p$ and $\beta > 0$ so small that

$$\sum_{1 \leq k \leq \beta n} \rho^k \frac{P\{T_{n-k} \geq \alpha n - k\}}{P\{T_n \geq \alpha n\}} \leq C_3 - 1 \tag{2.8}$$

uniformly for $p \leq \alpha \leq \alpha_0$. Without loss of generality we take $\beta < p$ so that $\beta < \alpha$.

To prove (2.8) we start from

$$\frac{P\{T_{n-k} \geq \alpha n - k\}}{P\{T_n \geq \alpha n\}} \leq \frac{P\{T_n \geq \alpha n - k\}}{P\{T_n \geq \alpha n\}} = \prod_{j=1}^{k} \frac{P\{T_n \geq \alpha n - j\}}{P\{T_n \geq \alpha n - j + 1\}}. \tag{2.9}$$

In addition, if for simplicity we write $\alpha n - j$ for $\lceil \alpha n - j \rceil$, we shall use

$$P\{T_n \geq \alpha n - j\} = \sum_{r=\alpha n - j}^{n} \binom{n}{r} p^r (1-p)^{n-r}$$

$$= n \binom{n-1}{\alpha n - j - 1} \int_0^p x^{\alpha n - j - 1} (1-x)^{(1-\alpha)n + j} \, dx$$

and

$$\frac{P\{T_n \geq \alpha n - j\}}{P\{T_n \geq \alpha n - j + 1\}} = \frac{\alpha n - j}{(1-\alpha)n + j} \frac{\int_0^p x^{\alpha n - j - 1}(1-x)^{(1-\alpha)n+j}\,dx}{\int_0^p x^{\alpha n - j}(1-x)^{(1-\alpha)n+j-1}\,dx}. \quad (2.10)$$

We want to show that the ratio here is close to 1 uniformly in $\alpha \in [p, \alpha_0]$ when α_0 is close to p, and $1 \leq j \leq k \leq \beta n$ with β small. We first show that we may replace the integrals over the interval $[0, p]$ here, by integrals over $[p - \varepsilon, 1]$ for any fixed (but sufficiently small) $\varepsilon > 0$, without influence on the asymptotic behavior of the right-hand side in (2.10). To be more precise, set

$$A = \alpha - \frac{j+1}{n}, \qquad B = (1-\alpha) + \frac{j}{n} \quad \text{and} \quad f(x) = f(x; j, n) = x^A(1-x)^B,$$

so that $x^{\alpha n - j - 1}(1-x)^{(1-\alpha)n+j} = f^n(x; j, n)$. Now

$$f'(x; j, n) = \left[\frac{A}{x} - \frac{B}{1-x}\right] f(x).$$

One sees from this that $f(x)$ is strictly increasing in $[0, A/(A+B)]$ and strictly decreasing in $[A/(A+B), 1]$. In particular, if $0 \leq \alpha - p \leq \varepsilon/4$ and $j \leq \beta n$ with $0 \leq \beta \leq (\varepsilon/8) \wedge p$, then $\max_x f(x)$ is achieved at the single point

$$x_0 := \frac{A}{A+B} \in [(\alpha - \beta) - 1/(n-1), \alpha] \subset [p - \varepsilon/2, p + \varepsilon/4]$$

(provided $n \geq 1 + \varepsilon/4$) and consequently

$$\int_0^{p-\varepsilon} f^n(x)\,dx \leq (p - \varepsilon)f^n(p - \varepsilon; n, j) \quad (2.11)$$

while

$$\int_0^p f^n(x)\,dx \geq \int_{p-3\varepsilon/4}^{p-\varepsilon/2} f^n(x)\,dx \geq \frac{\varepsilon}{4} f^n(p - 3\varepsilon/4; n, j). \quad (2.12)$$

Finally,

$$f''(x) = -\frac{A}{x^2} - \frac{B}{(1-x)^2} \leq -(A+B) < 0$$

so that, by Rolle's theorem,

$$f'(x) \geq f'(p - 3\varepsilon/4) \geq f'(p - \varepsilon/2) + (A+B)\frac{\varepsilon}{4} \geq C > 0 \qquad \text{for } x \leq p - 3\varepsilon/4,$$

and some constant $C = C(\varepsilon) > 0$, independent of α and n. Also, again by Rolle's theorem,

$$f(x) \leq f(p - 3\varepsilon/4) - \frac{\varepsilon}{4} f'(p - 3\varepsilon/4) \leq f(p - 3\varepsilon/4) - C\frac{\varepsilon}{4} \qquad \text{for } x \leq p - \varepsilon.$$

We combine this result with (2.11) and (2.12) to obtain that

$$
\frac{\int_0^{p-\varepsilon} x^{\alpha n - j - 1}(1-x)^{(1-\alpha)n+j}\, dx}{\int_0^{p} x^{\alpha n - j - 1}(1-x)^{(1-\alpha)n+j}\, dx}
$$

$$
\leq \frac{4p}{\varepsilon}\left[\frac{f(p-\varepsilon)}{f(p-3\varepsilon/4)}\right]^n
$$

$$
\leq \frac{4p}{\varepsilon}\left[1 - \frac{C\varepsilon}{4f(p-3\varepsilon/4)}\right]^n \to 0 \qquad \text{as } n \to \infty.
$$

In fact, since $f(x) \leq 1$, this convergence is uniform in $\alpha \in [p, \alpha_0]$ for α_0 sufficiently close to p and β sufficiently small. This shows that replacement of the integral over $x \in [0, p]$ in the numerator of the right-hand side of (2.10) by the same integral over $[p - \varepsilon, 1]$, does not change the right-hand side of (2.10) much for large n. On the other hand, the right-hand side of (2.9) can only increase if we replace the integral in the denominator by the integral over $[p - \varepsilon, p]$. It follows that for small $\varepsilon > 0$ and $p \leq \alpha_0 \leq p + \varepsilon/4$, $0 < \beta \leq (\varepsilon/8) \wedge p$, the right-hand side of (2.10) is for $p \leq \alpha \leq \alpha_0$ and all large n at most

$$
(1+\varepsilon)\frac{\alpha}{1-\alpha+\beta}\frac{\int_{p-\varepsilon}^{p} x^{\alpha n - j - 1}(1-x)^{(1-\alpha)n+j}\, dx}{\int_{p-\varepsilon}^{p} x^{\alpha n - j}(1-x)^{(1-\alpha)n+j-1}\, dx}
$$

$$
\leq (1+\varepsilon)\frac{p+\varepsilon/4}{1-p-\varepsilon/4}\cdot\frac{1-p+\varepsilon}{p-\varepsilon}.
$$

Here we used that the integrand in the numerator is at most a factor $(1-x)/x \leq (1-p+\varepsilon)/(p-\varepsilon)$ times the integrand in the denominator for $x \in [p-\varepsilon, p]$. Similar lower bounds hold for (2.10), but we will not need them.

The preceding estimates show that we can choose $\varepsilon_0 > 0$ and $\alpha_0 > p$ such that for $\alpha \in [p, \alpha_0]$ and all large n for all $1 \leq j \leq k \leq \beta n$, ρ times the right-hand side of (2.10) is less than $1 - \varepsilon_0$ (recall that $\rho < 1$). The inequality (2.9) then shows

$$
\rho^k \frac{P\{T_{n-k} \geq \alpha n - k\}}{P\{T_n \geq \alpha n\}} \leq [1-\varepsilon_0]^k \qquad \text{for } k \leq \beta n,
$$

and hence also proves (2.8) with $C_3 = [\varepsilon_0]^{-1} + 1$. $\qquad\square$

Corollary 5. *For $d \geq 4$ and all $0 < p < 1$ it holds*

$$
M(p) > p. \tag{2.13}
$$

Proof. This is immediate from Proposition 4 and the fact that $\alpha_0 > p$ in this proposition. $\qquad\square$

3 Behavior of $\lambda(\alpha)$ for "large" α

The last proposition gives the behavior of λ for "small" α, that is, from $\alpha = 0$ to a little beyond p. In this section we shall look at the behavior of $\lambda(\alpha)$ when $\lambda(\alpha)$ is small, which corresponds to large α.

It is well known that on the regular $(2d)$-ary tree (in which each vertex has degree $2d$) one has

$$M(p) = \sup\{\alpha : \phi(\alpha) > 1\}$$

[see Biggins (1977), Formula (3.4)]. One can also use a branching random walk proof to show that on such a rooted regular tree, oriented away from the root, for α such that $\phi(\alpha) > 1$, it holds $\lambda(\alpha) = \phi(\alpha)$. As we shall demonstrate soon, this is not the case for walks on $\mathcal{L} = \mathbb{Z}^d \times \mathbb{Z}_+$.

If α is such that $\phi(\alpha) < 1$, then, by the definition of ϕ, $EN_n(\alpha)$ tends to 0 exponentially fast, so almost surely $N_n(\alpha) = 0$ eventually. Of course $\lambda(\alpha) = 0$ in this case. If p is small, this case applies for $\alpha > \alpha_1$ with

$$\alpha_1 \sim \frac{\log(2d)}{\log(1/p)}. \tag{3.1}$$

We can do better, though. By definition of $M(p)$, if $\alpha > M(p)$, then $N_n(\alpha) = 0$ for large n. Thus

$$\lambda(\alpha) = 0 \qquad \text{if } \alpha > M(p). \tag{3.2}$$

But it is shown in Lee (1994) that there exist constants $C_1, C_2 \in (0, \infty)$ such that

$$C_1 p^{1/(d+1)} \leq M(p) \leq C_2 p^{1/(d+1)}. \tag{3.3}$$

Thus, by (3.2), for small p it holds

$$\lambda(\alpha) = 0 \qquad \text{if } \alpha > C_2 p^{1/(d+1)}. \tag{3.4}$$

Clearly this improves (3.1) for small p; it shows that $\lambda(\alpha)$ is still zero for smaller values of α than indicated by (3.1). We shall next show that (3.2) is best possible in the following sense.

Proposition 6. *For $d \geq 4$ and each $p \in (0, 1)$ it holds*

$$\lambda(\alpha) > 1 \qquad \text{for all } \alpha < M(p). \tag{3.5}$$

Proof. For $\alpha \leq p$, (2.1) already shows that $\lambda(\alpha) = 2d > 1$. For the remainder of this proof we therefore take $\alpha > p$. As before it is tacitly assumed that all paths in this proof start at $\mathbf{0}$.

Fix $\eta \in (0, 1/4)$ and define $\widetilde{\alpha} = [\alpha + M(p)]/2$, so that $p < \widetilde{\alpha} < M(p)$ by (2.13). By Theorem 2 in Gandolfi and Kesten (1994) there then exists an $M_0 < \infty$ such that with probability at least $(1 - \eta)$ there exists for each $n \geq M_0$ a path $\widetilde{\pi}$ starting

at $\mathbf{0}$ and of length n which has $W(\tilde{\pi}) \geq \tilde{\alpha}n$. Now fix $n \geq M_0$ and let $\tilde{\pi} = (\mathbf{0} = \tilde{\pi}_0, \tilde{\pi}_1, \ldots, \tilde{\pi}_n)$ be a path with the above properties. Assume that for a certain $k \leq n-2$

$$e(i_{k+1}) := \tilde{\pi}_{k+2} - \tilde{\pi}_{k+1} \neq e(i_k) := \tilde{\pi}_{k+1} - \tilde{\pi}_k. \tag{3.6}$$

We can then interchange the two steps $e(i_k)$ and $e(i_{k+1})$ to get the new path

$$\hat{\pi} = \big(\mathbf{0}, \tilde{\pi}_1, \ldots, \tilde{\pi}_k, \tilde{\pi}_k + e(i_{k+1}), \tilde{\pi}_k + e(i_{k+1}) + e(i_k) = \tilde{\pi}_{k+2}, \tilde{\pi}_{k+3}, \ldots, \tilde{\pi}_n\big).$$

This path differs only in its point at time $k+1$ from $\tilde{\pi}$, so that

$$W(\hat{\pi}) - W(\tilde{\pi}) \geq -X(\tilde{\pi}_{k+1}) \geq -1. \tag{3.7}$$

However $\hat{\pi}$ will still be self-avoiding, since $\tilde{\pi}$ does not visit $\hat{\pi}_{k+1}$, because $\|\hat{\pi}_{k+1}\| = \|\tilde{\pi}_{k+1}\| = k+1$ and $\tilde{\pi}$ can visit only one point with ℓ_1-norm $k+1$.

If there are m values of k for which (3.6) holds, then we can interchange two successive steps as described above or not at at least $m/2$ places such that these interchanges do not interfere with each other [say, at any subset of the even k's which satisfy (3.6)]. This yields at least $2^{m/2}$ paths with weight $W \geq \tilde{\alpha}n - m/2$. In other words, $[N_n(\tilde{\alpha} - m/(2n))]^{1/n} \geq 2^{m/(2n)}$ in this case. If we take

$$0 < \liminf_{n \to \infty} m/(2n) \leq \limsup_{n \to \infty} m/(2n) \leq \tilde{\alpha} - \alpha$$

then this method results in

$$\liminf_{n \to \infty} [N_n(\alpha)]^{1/n} \geq \exp\Big[\liminf_{n \to \infty} m/(2n) \log 2\Big] > 1.$$

In view of the preceding paragraph and the fact that $\lim_n [N_n(\alpha)]^{1/n}$ exists, it suffices for the proposition that there is for all large n at least a probability η that there is a path $\tilde{\pi}$ of length n and $W(\tilde{\pi}) \geq \tilde{\alpha}n$ and for which (3.6) holds for at least $C_3 n$ values of k (with $C_3 > 0$ and independent of n and $\tilde{\pi}$). In this case we may take $m = (C_3 \wedge (\tilde{\alpha} - \alpha))n$ in the preceding argument. Let us now make sure that we can find $\tilde{\pi}$ so that $W(\tilde{\pi}) \geq \tilde{\alpha}n$ and such that (3.6) holds for many k. We shall bound the probability that no such path exists. This last probability is, for $n \geq M_0$, bounded by

$$P\{\text{there is no path } \pi \text{ of length } n \text{ with } W(\pi) \geq \tilde{\alpha}n\}$$

$$+ P\{\text{there exists a path } \tilde{\pi} \text{ of length } n \text{ with } W(\tilde{\pi}) \geq \tilde{\alpha}n \tag{3.8}$$

$$\text{but fewer than } C_3 n \text{ values of } k \text{ for which (3.6) holds}\}$$

$$\leq \eta + (\text{number of paths } \tilde{\pi} \text{ for which (3.6) holds for}$$

$$\text{no more than } C_3 n \text{ values of } k) P\left\{\sum_{i=1}^{n} Y_i \geq \tilde{\alpha}n\right\}, \tag{3.9}$$

where the Y_i are i.i.d., each with the distribution $P\{Y_i = 1\} = 1 - P\{Y_i = 0\} = p$. But any path $\tilde{\pi}$ of length n is determined by the values of the k for which (3.6)

holds as well as the values of the corresponding $e(i_{k+1})$, and also $\widetilde{\pi}_1$. Indeed, this gives the places at which the direction of the steps of $\widetilde{\pi}$ changes and the value of this direction immediately after the change (plus the starting direction). The number of paths for which (3.6) holds for no more than $C_3 n$ values of k and the number of choices for the directions right after the k_i and at time 0 is at most

$$2d \sum_{j \leq C_3 n} \binom{n}{j} (2d-1)^j \leq C_4 \exp[nC_3 \log(1/C_3) + nC_3 \log(2d-1)]$$

for small C_3. But $\widetilde{\alpha} > p$, and by simple exponential bounds for the binomial distribution [e.g., Bernstein's inequality in Chow and Teicher (1988), Exercise 4.3.14]

$$P\left\{ \sum_{i=1}^n Y_i \geq \widetilde{\alpha} n \right\} \leq C_5 \exp[-C_6 n]$$

for some constants $0 < C_5(p, \widetilde{\alpha}), C_6(p, \widetilde{\alpha}) < \infty$. Thus the right-hand side of (3.9) is bounded by

$$\eta + C_4 C_5 \exp[nC_3 \log(1/C_3) + nC_3 \log(2d-1) - nC_6].$$

Since C_6 is independent of C_3, we can choose $C_3 > 0$ so small that this expression is at most 2η for large n. The complementary probability is then

$P\{$there exist a path $\widetilde{\pi}$ of length n with $W(\widetilde{\pi}) \geq \widetilde{\alpha} n$,

and all such paths have at least $C_3 n$ values of k for which (3.6) holds$\}$

$\geq 1 - 2\eta > \eta$

(recall $\eta \leq 1/4$). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Acknowledgments

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