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# On the optimality of bivariate ranked set sample design for the matched pairs sign test

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**Abstract.** An optimal alternative bivariate ranked set sample designs for the matched pairs sign test are obtained. Our investigation revealed that the optimal bivariate ranked set sample designs for matched pairs sign test are those with quantifying order statistics with labels $\{(\frac{r+1}{2}, \frac{r+1}{2})\}$ when the set size $r$ is odd and $\{(\frac{r}{2}, \frac{r}{2}), (\frac{r}{2}+1, \frac{r}{2}+1)\}$ when the set size $r$ is even. The exact null distributions, asymptotic distributions and Pitman efficiencies of those designs are derived. Numerical analysis of the power of the proposed optimal designs is included. An illustration using real data with a bootstrap algorithm for $P$-value estimation is used.

## 1 Introduction

Hennekens and Buring (1987) argued that matching as a technique for the control of confounding has great intuitive appeal and has been widely used over the years in many epidemiological studies clinical trials. Unlike randomization and restriction, which used to control for confounding in the design stage of a study, matching is a strategy that must include elements of both design and analysis. Examples for matched pairs studies are found in identical twins, before and after and other studies, where subjects are matched based on some confounding factors.

These types of studies produce data consisting of observations in a bivariate random sample $\{(X_i, Y_i), i = 1, 2, \ldots, n\}$, where there are $n$ pairs of observations. As in Hettmansperger (1984), assume that the bivariate cumulative distribution function (CDF) $F$ is absolutely continuous with absolutely continuous marginal CDFs. Thus, within each pair $(X_i, Y_i)$ a comparison is made, and the pair is classified as "+" if $X_i < Y_i$, "−" if $X_i > Y_i$ or "0" if $X_i = Y_i$. Here the measurement scale needs only to be ordinal. Other needed assumptions are: (1) The bivariate variables $(X_i, Y_i), i = 1, 2, \ldots, n$, are mutually independent. (2) The pairs $(X_i, Y_i)$ are internally consistent in that if $P(+) > P(-)$ for one pair $(X_i, Y_i)$, then $P(+) > P(-)$ for all pairs. The same is true for $P(+) < P(-)$ and $P(+) = P(-)$; see Conover (1980) and Samawi et al. (2008).

The types of null hypotheses that can be tested using the matched pairs sign test are:

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(1) $H_0 : P(+) = P(-) = \frac{1}{2}$.
(2) $H_0 : E(X_i) = E(Y_i)$ for all $i$, which is interpreted as $X_i$ and $Y_i$ have the same location parameter.
(3) $H_0$ : The median of $X_i$ equals the median of $Y_i$ for all $i$ (see Conover (1980)).

The matched pairs sign test statistic which denoted by $T_{\text{BVSRS}}$, for testing the above hypotheses equals the number of "+" pairs, that is

$$T_{\text{BVSRS}} = \sum_{i=1}^{n} I(X_i < Y_i),$$

where

$$I(X_i < Y_i) = \begin{cases} 1, & \text{if } X_i < Y_i, \\ 0, & \text{otherwise.} \end{cases}$$

Discarding all tied pairs and let $n$ equal the number of the remaining pairs (see Conover (1980)). Depending on whether the alternative hypothesis is one-tailed or two-tailed, and if $n \le 20$, then one can use the binomial distribution with the values $n$ and $p = 1/2$ for finding the critical region of approximately size $\alpha$. For $n$ larger than 20 and when the null hypothesis is true then $T_{\text{BVSRS}} \sim N(\frac{n}{2}, \frac{n}{4})$. Therefore the critical region can be defined based on the normal distribution. It has been argued that $T_{\text{BVSRS}}$ is an unbiased and a consistent test statistic when testing

$$H_0 : P(+) = P(-).$$

However, for testing

$$H_0 : E(X_i) = E(Y_i)$$

for all $i$, which is interpreted as $X_i$ and $Y_i$ have the same location parameter and

$H_0$ : The median of $X_i$ equals the median of $Y_i$ for all $i$,

$T_{\text{BVSRS}}$ is neither unbiased nor consistent (see Conover (1980)).

In most statistical applications the data used is assumed to consist of a simple random sample (SRS). However, it becomes obvious in some situations that quantification of sampling units with respect to the variable of interest is costly as compared with the physical acquisition of the unit. Cost savings of quantifying sampling units can be achieved by using ranked set sampling (RSS) method which was introduced first by McIntyre (1952) without any mathematical proof, to estimate the population mean and later called RSS by Halls and Dell (1966).

The RSS procedure can be described as follows: randomly sample a group of sampling units from the target population. Then randomly partition the group into disjoint subsets each having a preassigned size $r$. In most practical situations, the size $r$ will be $2, 3$ or 4. Rank the elements in each subset by a suitable method of ranking, such as prior information, visual inspection or by the subject matter experimenter himself, ..., etc. Then the $i$th order statistic from the

$i$th subset, $X_{i(i)}$, $i = 1, \ldots, r$, will be quantified (actual measurement). Therefore, $X_{1(1)}, X_{2(2)}, \ldots, X_{r(r)}$ constitutes the RSS. This represents one cycle. The whole procedure can be repeated $m$-times as needed, to get a RSS of size $n = mr$. For the theoretical aspects of RSS, see Takahasi and Wakimoto (1968) or Dell and Clutter (1972). For more about univariate RSS and its variations, see Kaur et al. (1995) and Patil, Sinha and Tillie (1999).

Recently, ranked set samples were used for quantiles and distribution estimation by Stokes and Sager (1988), Chen (2000), Samawi (2001) and Samawi and Al-Saleh (2004). Optimality of ranked set sample scheme for inference on population quantiles was suggested by Chen (2001). Other authors have used the RSS sampling method to improve parametric and nonparametric statistical inference. For nonparametric methods, RSS was considered by Bohn and Wolfe (1992, 1994), Kvam and Samaniego (1994) and Hettmansperger (1995). Koti and Babu (1996) showed that the RSS sign test provides a more powerful test than the SRS sign test. Barabesi (1998) provided a simpler and faster method for computing the exact distribution of the RSS sign test.

The optimality of the RSS sign test has been established by several researchers in the literature via Pitman asymptotic efficacy. It was shown that the median ranked set sample (MRSS) is the best among all possible sampling schemes in the ranked set sampling environment for the sign test procedure; for example, see Öztürk (1999), Öztürk and Wolfe (2000) and Samawi and Abu-Dayyeh (2003).

Another RSS procedure for estimation of bivariate characteristics using bivariate ranked set sampling (BVRSS) was introduced by Al-Saleh and Zheng (2002). Their procedure can be described as follows:

Suppose $(X, Y)$ is a bivariate random vector with the joint probability density function (PDF) $f(x, y)$.

1. A random sample of size $r^4$ is identified from the population and randomly allocated into $r^2$ pools each of size $r^2$, where each pool is a square matrix with $r$ rows and $r$ columns.
2. In the first pool, identify the minimum value by judgment with respect to the first characteristic $X$, for each of the $r$ rows.
3. For the $r$ minima obtained in step 2, choose the pair that corresponds to the minimum value of the second characteristic $Y$, identified by judgment, for actual quantification. This pair, which resembles the label $(1, 1)$, is the first element of the BVRSS sample.
4. Repeat steps 2 and 3 for the second pool, but in step 3, the pair that corresponds to the second minimum value with respect to the second characteristic, $Y$, is chosen for actual quantification. This pair resembles the label $(1, 2)$.
5. The process continues until the label $(r, r)$ is resembled from the $r^2$th (last) pool.

This will produce a BVRSS of size $r^2$. The procedure can be repeated $m$ times to obtain a sample of size $n = mr^2$.

Let $\{(X^z_{ijk}, Y^z_{ijk}),\ i = 1, \ldots, r,\ j = 1, \ldots, r,\ k = 1, \ldots, m, z = 1, 2, \ldots, r^2\}$, be $mr^4$ i.i.d. ordered pairs from a bivariate probability density function, say $f(x, y); (x, y) \in R^2$. Again, assume that the bivariate CDF $F$ is absolutely continuous with absolutely continuous marginal CDFs. Following the Al-Saleh and Zheng (2002) definition of BVRSS let $\{(X_{[i](j)k}, Y_{(i)[j]k}), i = 1, \ldots, r, j = 1, \ldots, r, k = 1, \ldots, m\}$ denote such a sample from $f(x, y)$. Note that the notation $(\cdot)$ indicates that the ranking is perfect and $[\cdot]$ indicates that the ranking is imperfect. Therefore, the pair $(X_{[i](j)}, Y_{(i)[j]})$ is the measurement of the matched-pair $(j, i)$ subjects with respect to $X$ and $Y$ characteristics, respectively, from the $k$th cycle, the $j$th perfect ranking order statistics with respect to $X$ and imperfect ranking with respect to $Y$ and the $i$th perfect ranking order statistic with respect to $Y$ and imperfect ranking with respect to $X$. Let $f_{X_{[i](j)},Y_{(i)[j]}}(x, y)$ be the joint PDF of $(X_{[i](j)k}, Y_{(i)[j]k})$, $k = 1, 2, \ldots, m$. Note that, as in Al-Saleh and Zheng (2002),

$$f_{X_{[i](j)},Y_{(i)[j]}}(x, y) = \frac{f_{Y_{(i)[j]}}(y) f_{X_{(j)}}(x) f_{Y|X}(y|x)}{f_{Y_{[j]}}(y)}, \tag{1}$$

where $f_{X_{(j)}}(x)$ is the density of the $j$th order statistic for a SRS sample of size $r$ from the marginal density of $f_X$ and given by $f_{X_{(j)}}(x) = \frac{r!}{(j-1)!(r-j)!} \times [F(x)]^{j-1}[1 - F(x)]^{r-j} f(x)$, $f_{Y_{[j]}}(y)$ is the density of the corresponding $Y$-value given by

$$f_{Y_{[j]}}(y) = \int_{-\infty}^{\infty} f_{X_{(j)}}(x) f_{Y|X}(y|x)\, dx,$$

and $f_{Y_{(i)[j]}}(y)$ is the density of the $i$th order statistic of an i.i.d. sample from $f_{Y_{[j]}}(y)$, that is,

$$f_{Y_{(i)[j]}}(y) = \frac{r!}{(i-1)!(r-i)!}\left[F_{Y_{[j]}}(y)\right]^{i-1}\left[1 - F_{Y_{[j]}}(y)\right]^{r-i} f_{Y_{[j]}}(y)$$

with $F_{Y_{[j]}}(x) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} f_{X_{(j)}}(x) f_{Y|X}(w|x)\, dx\, dw$.

Using the above results and that $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$ we can rewrite equation (1) as

$$\begin{aligned} &f_{X_{[i](j)},Y_{(i)[j]}}(x, y) \\ &\quad = c_1\left[F_{Y_{[j]}}(y)\right]^{i-1}\left[1 - F_{Y_{[j]}}(y)\right]^{r-i}[F(x)]^{j-1}[1 - F(x)]^{r-j} f(x, y), \end{aligned} \tag{2}$$

where $c_1 = \frac{r!r!}{(i-1)!(r-i)!(j-1)!(r-j)!}$.

Note that from Al-Saleh and Zheng (2002) we have the following results:

$$\frac{1}{r^2}\sum_{j=1}^{r}\sum_{i=1}^{r} f_{X_{[i](j)},Y_{(i)[j]}}(x, y) = f_{X,Y}(x, y),$$

$$f_X(x) = \frac{1}{r^2}\sum_{j=1}^{r}\sum_{i=1}^{r} f_{X_{[i](j)}}(x)$$

and

$$f_Y(y) = \frac{1}{r^2}\sum_{j=1}^{r}\sum_{i=1}^{r} f_{Y_{(i)[j]}}(y).$$

Samawi et al. (2006) used the idea of BVRSS to improve the efficiency of bivariate sign test for one-sample bivariate location model.

This paper introduces an optimal BVRSS designs (OBVRSS) for matched pairs sign test. Numerical comparisons between the performance of the OBVRSS matched pairs sign test and the performance of the BVSRS and BVRSS sign tests via Pitman's asymptotic efficiency and asymptotic power are investigated. The exact null distribution and the asymptotic null distribution and power of the OBVRSS sign test are derived. It will be shown that OBVRSS substantially improves the efficiency and the power of the matched pairs sign test. We also introduce a bootstrap method for finding the $P$-value of the matched pairs test for small sample sizes and demonstrate the procedure using real data from the Iowa 65+ Rural Health Study (RHS).

## 2 Alternative BVRSS designs for matched pairs sign test

An alternative bivariate ranked set sampling (ABVRSS) is a sampling protocol that quantifies the same order statistics in each pool using similar BVRSS protocol. Samawi at el. (2009) described ABVRSS as follows: Define $\mathcal{L}(A)$ to be the cardinality of a set $A$, then $\mathcal{L}(A) =$ the number of elements in a set $A$. Let $J_{\text{ABVRSS}} =$ {set of all possible alternative BVRSS designs} $= \{J_1, J_2, \ldots, J_{\mathcal{L}(J_{\text{ABVRSS}})}\}$, for example, when $r = 2$, then $J_{\text{ABVRSS}} = \{\{(1, 1)\}, \{(1, 2)\}, \{(2, 1)\}, \{(2, 2)\}, \{(1, 1), (1, 2)\}, \{(1, 1), (2, 1)\}, \{(1, 1), (2, 2)\}, \{(1, 2), (2, 1)\}, \{(1, 2), (2, 2)\}, \{(2, 1), (2, 2)\}, \{(1, 1), (1, 2), (2, 1)\}, \{(1, 1), (1, 2), (2, 2)\}, \{(1, 1), (2, 1), (2, 2)\}, \{(1, 2), (2, 1), (2, 2)\}, \{(1, 1), (1, 2), (2, 1), (2, 2)\}\}$.

Then for $r = 2$, $\mathcal{L}(J_{\text{ABVRSS}}) = \sum_{i=1}^{4}\binom{4}{i} = 15$. In general, for a set of size $r$, $\mathcal{L}(J_{\text{ABVRSS}}) = \sum_{i=1}^{r^2}\binom{r^2}{i} = 2^{r^2} - 1$.

Now, for an integer $s$, $s \in \{1, 2, \ldots, 2^{r^2} - 1\}$ let $J_s \in J_{\text{ABVRSS}}$ be the set of judgment ranks of ordered pairs labels for the observations to be quantified. Our sampling protocol involves selecting $m\mathcal{L}(J_s)r^2$ units from an infinite population. These units are partitioned into $m\mathcal{L}(J_s)$ pools each having $r^2$ units.

From each pool, by using the same procedure discussed for a BVRSS protocol by Al-Saleh and Zheng (2002), we quantify only one of the ordered pair's labels in $J_s$; therefore, they are mutually independent. To insure a balance design we need to have equal number of measurement from each label in the symmetric design.

Let $J_s = \{(c_1, d_1), \ldots, (c_{\mathcal{L}(J_s)}, d_{\mathcal{L}(J_s)})\}$, then

$$\left\{\left(X_{[c_1](d_1)k}, Y_{(c_1)[d_1]k}\right), \ldots, \left(X_{[c_{\mathcal{L}(J_s)}](d_{\mathcal{L}(J_s)})k}, Y_{(c_{\mathcal{L}(J_s)})[d\mathcal{L}(J_s)]k}\right)\right\},$$

where $k = 1, \ldots, m$, will be ABVRSS from with $F(x, y)$ with $n = m\mathcal{L}(J_s)$. The ABVRSS sign test statistic can be defined as $T_{\text{ABVRSS}} =$ the number of "+" pairs, or $T_{\text{ABVRSS}} = \#(X_{[c_u](d_u)k} < Y_{(c_u)[d_u]k})$ for all $u$ and $k$, that is,

$$T_{\text{ABVRSS}} = \sum_{u=1}^{\mathcal{L}(J_s)} \sum_{k=1}^{m} I\left(X_{[c_u](d_u)k} < Y_{(c_u)[d_u]k}\right) = \sum_{u=1}^{\mathcal{L}(J_s)} T_u, \tag{3}$$

where

$$T_u = \sum_{k=1}^{m} I\left(X_{[c_u](d_u)k} < Y_{(c_u)[d_u]k}\right).$$

Clearly, $T_u,\ u = 1, 2, \ldots, \mathcal{L}(J_s)$, are stochastically independent and each has a binomial distribution with parameters $m$ and $p_u = P(X_{[c_u](d_u)} < Y_{(c_u)[d_u]})$. The mean and the variance of are, respectively,

$$E(T_{\text{ABVRSS}}) = E\left[\sum_{u=1}^{\mathcal{L}(J_s)} \sum_{k=1}^{m} I\left(X_{[c_u](d_u)k} < Y_{(c_u)[d_u]k}\right)\right] = m \sum_{u=1}^{\mathcal{L}(J_s)} P_u \tag{4}$$

and

$$\begin{aligned} V(T_{\text{ABVRSS}}) &= V\left[\sum_{u=1}^{\mathcal{L}(J_s)} \sum_{k=1}^{m} I\left(X_{[c_u](d_u)k} < Y_{(c_u)[d_u]k}\right)\right] \\ &= m \sum_{u=1}^{\mathcal{L}(J_s)} P_u(1 - P_u). \end{aligned} \tag{5}$$

Also, the exact distribution of $T_{\text{ABVRSS}}$ is given by

$$P(T_{\text{ABVRSS}} = t) = \sum_{l_{xy}} \prod_{u=1}^{\mathcal{L}(J_s)} \binom{m}{v_u} P_u^{v_u} (1 - P_u)^{m - v_u} \tag{6}$$

for $t = 0, 1, 2, \ldots, \mathcal{L}(J_s)$, where $l_{xy} = \{v_u : u = 1, 2, \ldots, \mathcal{L}(J_s)\}$.

According to our setting, ordinary BVRSS is a special case when $J_s = J_{\mathcal{L}(J_{\text{ABVRSS}})}$.

Unfortunately, most of the exact distributions in (6) and the results in (4) and (5) from the ABVRSS designs, $J_s$, $s = 1, 2, \ldots, \mathcal{L}(J_{\text{ABVRSS}})$, depend on the given underlying bivariate distribution function even under the null hypothesis. Thus, finding the exact critical value and the $P$-value of the test requires knowledge of the underlying distribution function. However, under the null hypothesis, for some of the alternative designs, $T_{\text{ABVRSS}}$, has binomial distribution. Similar to univariate case described by Öztürk (2001), ABVRSS design is called symmetric if $c_u + c_{r-u+1} = r + 1$ and $d_u + d_{r-u+1} = r + 1$ or $c_u = d_u = \frac{r+1}{2}$ (when $r$ is odd). The first result we need to show is that the symmetric design preserves the property of diagonally symmetric of the underlying distribution if needed. We only state some of the symmetric design. The other symmetric designs can be showed in similar manner.

**Theorem 1.** *Assume that* $f(x, y) = f(-x, -y)$ *is diagonally symmetric about* 0. *If ABVRSS design is symmetric then the following are true*:

(i) *When r is odd*,

$$f_{X_{[(r+1)/2]((r+1)/2)},Y_{((r+1)/2)[(r+1)/2]}}(x, y) = f_{X_{[(r+1)/2]((r+1)/2)},Y_{((r+1)/2)[(r+1)/2]}}(-x, -y);$$

*diagonally symmetric*.

(ii) *When r is odd or even*,

$$f_{X_{[c_u](d_u)},Y_{(c_u)[d_u]}}(x, y) = f_{X_{[r-c_u+1](r-d_u+1)},Y_{(r-c_u+1)[r-d_u+1]}}(-x, -y).$$

**Proof.** 1. Using equation (2) and let $c = \frac{r!}{((r-1)/2)!((r-1)/2)!}$,

$$\begin{aligned} &f_{X_{[(r+1)/2]((r+1)/2)},Y_{((r+1)/2)[(r+1)/2]}}(-x, -y) \\ &\quad = c_1\big[F_{Y_{[(r+1)/2]}}(-y)\big]^{(r-1)/2}\big[1 - F_{Y_{[(r+1)/2]}}(-y)\big]^{(r-1)/2}[F(-x)]^{(r-1)/2} \\ &\qquad \times [1 - F(-x)]^{(r-1)/2} f(-x, -y) \end{aligned}$$

now, $F_{Y_{[(r+1)/2]}}(-y) = 1 - F_{Y_{[(r+1)/2]}}(y)$, because

$$\begin{aligned} f_{Y_{[(r+1)/2]}}(-y) &= \int_{-\infty}^{\infty} c[F(-x)]^{(r-1)/2}[1 - F(-x)]^{(r-1)/2} f(-x, -y)\,dx \\ &= \int_{-\infty}^{\infty} c[1 - F(x)]^{(r-1)/2}[F(x)]^{(r-1)/2} f(x, y)\,dx = f_{Y_{[(r+1)/2]}}(y) \end{aligned}$$

is symmetric about 0. Therefore, by the diagonally symmetric assumption of $f(x, y)$,

$$f_{X_{[(r+1)/2]((r+1)/2)},Y_{((r+1)/2)[(r+1)/2]}}(-x, -y) = f_{X_{[(r+1)/2]((r+1)/2)},Y_{((r+1)/2)[(r+1)/2]}}(x, y).$$

2. Again by (2)

$$\begin{aligned} &f_{X_{[c_u](d_u)},Y_{(c_u)[d_u]}}(-x, -y) \\ &\quad = c_1\big[F_{Y_{[d_u]}}(-y)\big]^{c_u-1}\big[1 - F_{Y_{[d_u]}}(-y)\big]^{r-c_u}[F(-x)]^{d_u}[1 - F(-x)]^{d_u} \\ &\qquad \times f(-x, -y) \end{aligned}$$

by using similar argument as in Proof 1, we have

$$f_{X_{[c_u](d_u)},Y_{(c_u)[d_u]}}(-x,-y) = c_1\big[1-F_{Y_{[r-d_u+1]}}(y)\big]^{c_u-1}\big[F_{Y_{[r-d_u+1]}}(y)\big]^{r-c_u} \times [1-F(x)]^{d_u}[F(x)]^{d_u} f(x,y) = f_{X_{[r-c_u+1](r-d_u+1)},Y_{(r-c_u+1)[r-d_u+1]}}(x,y)$$

because also,

$$f_{Y_{[d_u]}}(-y) = \int_{-\infty}^{\infty} c[F(-x)]^{d_u-1}[1-F(-x)]^{r-d_u} f(-x,-y)\,dx = \int_{-\infty}^{\infty} c[1-F(x)]^{d_u-1}[F(x)]^{r-d_u} f(x,y)\,dx = f_{Y_{[r-d_u+1]}}(y). \qquad \square$$

**Theorem 2.** *Assuming no tied pairs* $(X_{[c_u](d_u)k} = Y_{(c_u)[d_u]k})$. *Under the null hypothesis* $H_0: P(+) = P(-) = \frac{1}{2}$ *and the assumption of symmetric ABVRSS design, for fixed* $r$ *we have the following*:

1. $E(T_{\text{ABVRSS}}) = \frac{n}{2} = \frac{m\mathcal{L}(J_s)}{2}$.
2. $V_o = V(T_{\text{ABVRSS}}) = \frac{m\mathcal{L}(J_s)}{2}[1 - \frac{2}{\mathcal{L}(J_s)}\sum_{u=1}^{\mathcal{L}(J_s)} P_u^2] = \frac{m\mathcal{L}(J_s)}{4}[2 - \frac{4}{\mathcal{L}(J_s)} \times \sum_{u=1}^{\mathcal{L}(J_s)} P_u^2]$.
   *A special case is* $V_o = \frac{m\mathcal{L}(J_s)}{4}$ $\{$*if* $J_s = (\frac{r+1}{2}, \frac{r+1}{2})$, *when* $r$ *is odd*$\}$.
3. *For large* $m$, $T_{\text{ABVRSS}}$ *has approximately* $N(\frac{n}{2}, V_o)$ *where* $n = m\mathcal{L}(J_s)$.

**Proof.** Proving 1 and 2 is just simple algebra.

Proof 3. Discard all tied pairs and let $n$ equal the number of pairs that are not ties. From parts 1 and 2 in the theorem, and noting that $[2 - \frac{4}{\mathcal{L}(J_s)}\sum_{u=1}^{\mathcal{L}(J_s)} P_u^2]$ is finite fixed number. Therefore, using similar argument as in Hettmansperger (1995) the proof follows.

Clearly, when $r$ is odd, $T_{\text{ABVRSS}}$ has a binomial distribution with $n = m\mathcal{L}(J_s) =$ number of trials, and probability of success $p = 0.5$. Again, similar to Conover (1980), discard all tied pairs and let $n$ equal the number of the remaining pairs. Depending on whether the alternative hypothesis is one-tailed or two-tailed, and if $n \le 20$, then use the binomial distribution with the values $n$ and $p = 0.5$ for finding the critical region of approximately test size $\alpha$. For $n$ larger than 20 use normal approximation in Theorem 2 part 3. For $r$ even, a consistent estimator for $V_o$ is given by $\hat{V}_o = \frac{m\mathcal{L}(J_s)}{4}[2 - \frac{4}{\mathcal{L}(J_s)}\sum_{u=1}^{\mathcal{L}(J_s)} \hat{P}_u^2]$, where $\hat{P}_u = \frac{1}{m}\sum_{k=1}^{m} I(X_{[c_u](d_u)k} < Y_{(c_u)[d_u]k})$.

Depending on whether the alternative hypothesis it is one-tailed or two-tailed and if $n \ge 20$, the asymptotic test procedure is to reject the null hypothesis in $H_0: P(+) = P(-) = \frac{1}{2}$ favor of the alternative $\{$e.g., $H_{\text{a}}: P(+) > P(-)\}$ if $Z_0 = \frac{T_{\text{ABVRSS}} - n/2}{\sqrt{\hat{V}_o}} > Z_\alpha$, where $Z_\alpha$ is the $100(1-\alpha)\%$ quantile of the standard normal distribution. $\square$

## 3 The asymptotic relative efficiency

The performance of the matched pairs sign test using ABVRSS will be compared with the matched pairs sign test using BVSRS based on the criterion of Pitman's asymptotic relative efficiency (ARE). The Pitman's regularity conditions are satisfied for both $T_{\mathrm{ABVRSS}}$ and $T_{\mathrm{BVSRS}}$ because all moments of the tests are in terms of probabilities, and hence are bounded above by 1. Let $e(T)$ denotes the asymptotic efficiency of a test statistics $T$. Then Pitman's efficiency of a test statistics $T$ is given by

$$e(T) = \operatorname*{Lim}_{n\to\infty} \left.\frac{\frac{dE(T)}{d\theta}}{\sqrt{nV(TY)}}\right|_{H_0},$$

where $\theta$ could be the central parameter or the shifted parameter such that $P(X < Y + \theta) = 0.5$ under the null hypothesis.

Therefore, the Pitman's ARE of $T_{\mathrm{ABVRSS}}$ versus $T_{\mathrm{BVSRS}}$ is

$$ARE(T_{\mathrm{ABVRSS}}, T_{\mathrm{BVSRS}}) = \frac{e^2(T_{\mathrm{ABVRSS}})}{e^2(T_{\mathrm{BVSRS}})}. \tag{7}$$

Using the above definition and letting $F_D(\cdot)$ denote the CDF of the random variable $D$, where $D = X - Y$ and $F_D(0) = P(X < Y) = P(X - Y < 0) = P(D < 0)$, then the efficiencies of $T_{\mathrm{ABVRSS}}$ and $T_{\mathrm{BVSRS}}$ are obtained as

$$e(T_{\mathrm{ABVRSS}}) = \left.\frac{2\sum_{u=1}^{\mathcal{L}(J_s)} f_{D_u(0)}/\mathcal{L}(J_s)}{\sqrt{[2-(4/\mathcal{L}(J_s))\sum_{u=1}^{\mathcal{L}(J_s)} P_u^2]}}\right|_{H_0} \tag{8}$$

and

$$e(T_{\mathrm{BVSRS}}) = 2f_D(0), \tag{9}$$

respectively. Therefore, by (7), (8) and (9)

$$ARE(T_{\mathrm{ABVRSS}}, T_{\mathrm{BVSRS}}) = \left.\frac{(\sum_{u=1}^{\mathcal{L}(J_s)} f_{D_u(0)}/\mathcal{L}(J_s))^2}{f_D^2(0)[2-(4/\mathcal{L}(J_s))\sum_{u=1}^{\mathcal{L}(J_s)} P_u^2]}\right|_{H_0},$$

where

$$F_{D_u}(0) = P\big(X_{[c_u](d_u)} < Y_{(c_u)[d_u]}\big) = P\big(X_{[c_u](d_u)} - Y_{(c_u)[d_u]} < 0\big) = P(D_u < 0).$$

### 3.1 Numerical comparisons

Two types of underlying bivariate distributions are used to investigate the performance of the matched-pair sign test by using different symmetric BVRSS designs with respect to Pitman's asymptotic relative efficiency.

1. Assume that the bivariate random variable $(X, Y)$ has a bivariate normal distribution. We computed $ARE(T_{\text{ABVRSS}}, T_{\text{BVSRS}})$ for $\{r = 2, 3$, and 4, and correlation coefficient ($\rho = \pm 0.5$ and $\rho = \pm 0.9$)$\}$. Note that bivariate normal distribution has the property of diagonal symmetry.
2. However, to depart from diagonal symmetry assumptions and symmetrical marginal distributions, Gumbel's bivariate exponential distribution is used. Note that we only need to assume that the marginal's have the same median under the null hypothesis (see Johnson (1987)).

The density function is given by

$$f(x, y; \theta) = [(1 + \theta x)(1 + \theta y)] \exp[-x - y - \theta xy], \qquad x, y > 0;\ 0 \le \theta \le 1.$$

The marginal distributions are stander exponential distribution, and $\theta$ equals zero then the components are independent. However, when $\theta = 1$, the correlation between $X$ and $Y$ is $-0.43$, which indicates a relatively weak negative dependence. Two values of $\theta = 0.1, 1$ are considered.

Table 1 shows Pitman's asymptotic relative efficiency $ARE(T_{\text{ABVRSS}}, T_{\text{BVSRS}})$, for only the most efficient ABVRSS symmetrical designs, for both underlying bivariate distributions.

From Table 1 an optimal bivariate ranked set sample designs for matched pairs sign test are those with quantifying order statistics with labels $\{(\frac{r+1}{2}, \frac{r+1}{2})\}$ when the set size $r$ is odd and $\{(\frac{r}{2}, \frac{r}{2}), (\frac{r}{2}+1, \frac{r}{2}+1)\}$ when the set size $r$ is even. The matched pairs sign tests using this optimal designs will be denoted by $T_{\text{OBVRSS}}$. Clearly, via Pitman's asymptotic relative efficiency, the performance of $T_{\text{OBVRSS}}$ is superior to $T_{\text{BVSRS}}$, the ordinary $T_{\text{BVRSS}}$ and all alternative symmetrical designs ($T_{\text{ABVRSS}}$). However, in order to show other designs that not symmetrical and slightly more efficient than the proposed designs, Table 1 shows that $\{(2, 1), (2, 3)\}$ is slightly more efficient than the optimal design for low negative correlation in

**Table 1** *Pitman's asymptotic relative efficiency* $ARE(T_{\text{ABVRSS}}, T_{\text{BVSRS}})$

| | | Bivariate Normal | | Bivariate Gumble's | |
|---|---|---|---|---|---|
| ABVRSS designs | $r$ | $\rho = \pm 0.5$ | $\rho = \pm 0.9$ | $\theta = 0.1$ | $\theta = 1$ |
| Ordinary BVRSS | 2 | 1.15 (**1.43**) | 1.04 (**1.59**) | 1.31 | 1.45 |
| {(1, 2), (2, 1)} | 2 | 1.08 (**1.12**) | 1.02 (**1.06**) | 0.50 | 1.17 |
| {(1, 1), (2, 2)} | 2 | 1.23 (**1.80**) | 1.05 (**2.14**) | 1.83 | 2.75 |
| Ordinary BVRSS | 3 | 1.29 (**1.77**) | 1.09 (**2.38**) | 1.55 | 1.81 |
| {(2, 1), (2, 3)} | 3 | 1.42 (**1.67**) | 1.10 (**1.57**) | 2.34 | 2.70 |
| {(2, 2)} | 3 | 1.67 (**3.02**) | 1.16 (**4.28**) | 2.32 | 3.28 |
| Ordinary BVRSS | 4 | 1.43 (**2.14**) | 1.12 (**2.56**) | 1.76 | 2.11 |
| {(2, 2), (3, 3)} | 4 | 1.96 (**3.94**) | 1.24 (**5.97**) | 2.41 | 4.12 |

The results for negative correlation coefficients are in bold and parenthesis.

case of bivariate Gumbel's distribution and $r = 3$. However, this design is not symmetrical, therefore, the expected value of the sign test depends on the underlying distribution (not free of parameters) even under the null hypothesis. Moreover, note that all proposed alternative designs including ordinary BVRSS performed better for negative correlation in bivariate normal case. The asymptotic relative efficiency $ARE(T_{\text{ABVRSS}}, T_{\text{BVSRS}})$ increases as the set size $r$ increases in all cases. Also, it is clear that the $ARE(T_{\text{ABVRSS}}, T_{\text{BVSRS}})$ increases as the negative $\rho$ decreases away from zero. When the correlation coefficient $\rho$ is positive, $ARE(T_{\text{ABVRSS}}, T_{\text{BVSRS}})$, decreases as $\rho$ increases in case of bivariate normal distribution. However, ABVRSS is still more efficient than BVSRS. For Gumbel's bivariate, the efficiency increases as $r$ increases. Also, the efficiency increases as $\theta$ increases away from zero.

## 4 Matched pairs sign tests using optimal designs

In this section we introduce the optimal bivariate ranked set sampling protocols (OBVRSS) for the matched pairs sign test. Some theoretical results of the test using those (OBVRSS) designs are derived. Also, we investigate the power of the test for those designs.

*Case* 1. Set size $r$ is odd.

Let $\{(X_{[(r+1)/2]((r+1)/2)k}, Y_{((r+1)/2)[(r+1)/2]k}), k = 1, 2, \ldots, n\}$ be $\text{OBVRSS}_\text{o}$ from $F(x, y)$. Then the sign test statistic based on $\text{OBVRSS}_\text{o}$ is

$$T_{\text{OBVRSS}_\text{o}} = \sum_{k=1}^{n} I\big(X_{[(r+1)/2]((r+1)/2)k} < Y_{((r+1)/2)[(r+1)/2]k}\big). \tag{10}$$

By Theorem 2 and under the null hypothesis, $T_{\text{OBVRSS}_\text{o}}$ has a binomial distribution with parameters $n$ and $P = \frac{1}{2}$. Also, under the alternative hypothesis, $H_a : P(+) > P(-)$, $T_{\text{OBVRSS}_\text{o}}$ has a binomial distribution with parameters $n$ and $p_0 = P(X_{[(r+1)/2]((r+1)/2)} < Y_{((r+1)/2)[(r+1)/2]})$.

Moreover, for $n > 20$, by Theorem 2, the asymptotic power of testing the hypothesis $H_0 : P(+) = P(-)$ versus the alternative {without loss of generality consider $H_a : P(+) > P(-)$} for $T_{\text{OBVRSS}_\text{o}}$ and $T_{\text{BVSRS}}$ are defined by

$$\beta_{\text{OBVRSS}_\text{o}} = 1 - \Phi\left[\left(z_\alpha\sqrt{\frac{n}{4}} + \frac{n}{2} - np_0\right)\Big/\sqrt{V_{a0}}\right],$$

where $V_{a0} = np_0(1 - p_0)$ is the variance of $T_{\text{OBVRSS}_\text{o}}$ under the alternative hypotheses and

$$\beta_{\text{BVSRS}} = 1 - \Phi\left[\left(z_\alpha\sqrt{\frac{n}{4}} + \frac{n}{2} - nP(X < Y)\right)\Big/\sqrt{nP(X < Y)\big(1 - P(X < Y)\big)}\right].$$

Therefore, under the null hypothesis $\beta_{\text{OBVRSS}_\text{o}} = 1 - \Phi[z_\alpha] = \alpha$ and $\beta_{\text{BVSRS}} = 1 - \Phi[z_\alpha] = \alpha$.

*Case* 2. Set size $r$ is even.

Let $\{(X_{[r/2](r/2)k}, Y_{(r/2)[r/2]k}), (X_{[r/2+1](r/2+1)k}, Y_{(r/2+1)[r/2+1]k}), k = 1, 2, \dots, m\}$ be OBVRSS$_E$ from $F(x, y)$. Then the sign test statistic based on OBVRSS$_E$ is

$$T_{\mathrm{OBVRSS_E}} = \sum_{k=1}^{m} \big[ I\big(X_{[r/2](r/2)k} < Y_{(r/2)[r/2]k}\big) + I\big(X_{[r/2+1](r/2+1)k} < Y_{(r/2+1)[r/2+1]k}\big)\big]. \tag{11}$$

Also, the exact distribution of $T_{\mathrm{OBVRSS_E}}$ is given by

$$P(T_{\mathrm{OBVRSS_E}} = t) = \begin{cases} \sum_{j=0}^{t} \binom{m}{j} p_{10}^{j}(1-p_{10})^{m-j} \binom{m}{t-j} p_{20}^{t-j}(1-p_{20})^{m-1+j}, \\ \quad \text{if } t \le m, \\ \sum_{j=t-m}^{m} \binom{m}{j} p_{10}^{j}(1-p_{10})^{m-j} \binom{m}{t-j} p_{20}^{t-j}(1-p_{20})^{m-1+j}, \\ \quad \text{if } m < t \le n, \end{cases} \tag{12}$$

where

$$p_{10} = P\big(X_{[r/2](r/2)} < Y_{(r/2)[r/2]}\big)$$

and

$$p_{20} = P\big(X_{[r/2+1](r/2+1)} < Y_{(r/2+1)[r/2+1]}\big).$$

Unfortunately, the exact null distribution as well as the alternative distribution depends on the underlying bivariate distribution. Therefore, for large $m$ use the asymptotic $z$-test similar to that introduced in Section 2. For small $m < 20$, a bootstrap algorithm will be introduced.

Moreover, by Theorem 2, the asymptotic power of testing the hypothesis $H_0 : P(+) = P(-)$ versus the alternative {without loss of generality consider $H_a : P(+) > P(-)$} for $T_{\mathrm{OBVRSS_E}}$ is defined by

$$\beta_{\mathrm{OBVRSS_E}} = 1 - \Phi\left[\left(z_\alpha \sqrt{V_o} + \frac{n}{2} - m(p_1 + p_2)\right) \Big/ \sqrt{V_a}\right],$$

where $V_a = m[p_1(1 - p_1) + p_2(1 - p_2)]$ is the variance of $T_{\mathrm{OBVRSS_E}}$ under the alternative hypotheses $p_1 = P(X_{[r/2](r/2)} < Y_{(r/2)[r/2]})$ and $p_2 = P(X_{[r/2+1](r/2+1)} < Y_{(r/2+1)[r/2+1]})$. Also, $V_0$ (as in Theorem 2).

Table 2 shows the asymptotic power for $\{(r = 2, m = 10), (r = 2, m = 12), (r = 3, m = 27)$ and $(r = 4, m = 18)\}$, shifted parameter of center of the two marginal distributions ($\theta = 0, 0.1, 0.5,$ and 1), level of significance $\{\alpha = 0.5\}$ and correlation coefficient ($\rho = \pm 0.5, \rho = \pm 0.9$) for the case of bivariate normal distribution.

**Table 2** *Asymptotic power for $T_{\text{OBVRSS}}$ and $T_{\text{BVSRS}}$, when $\alpha = 0.5$*

| | | $\rho = \pm 0.5$ | | $\rho = \pm 0.9$ | |
|---|---|---|---|---|---|
| $n = mr^2$ | $\theta$ | OBVRSS | BVSRS | OBVRSS | BVSRS |
| $n = 24$ ($r = 2$, $m = 12$) | 0 | 0.0500 (**0.0500**) | 0.0500 (**0.0500**) | 0.0500 (**0.0500**) | 0.0500 (**0.0500**) |
| | 0.1 | 0.3083 (**0.2904**) | 0.3033 (**0.2807**) | 0.3741 (**0.2867**) | 0.3708 (**0.2768**) |
| | 0.5 | 0.5822 (**0.4677**) | 0.5431 (**0.4101**) | 0.9688 (**0.4605**) | 0.9596 (**0.3901**) |
| | 1.0 | 0.9564 (**0.7414**) | 0.9023 (**0.5977**) | 1.0000 (**0.7262**) | 1.0000 (**0.5528**) |
| $n = 24$ ($r = 2$, $m = 12$) | 0 | 0.0500 (**0.0500**) | 0.0500 (**0.0500**) | 0.0500 (**0.0500**) | 0.0500 (**0.0500**) |
| | 0.1 | 0.3323 (**0.3170**) | 0.3170 (**0.2938**) | 0.3964 (**0.3197**) | 0.3859 (**0.2898**) |
| | 0.5 | 0.6668 (**0.5628**) | 0.5609 (**0.4259**) | 0.9879 (**0.5828**) | 0.9664 (**0.4056**) |
| | 1.0 | 0.9968 (**0.9141**) | 0.9141 (**0.6160**) | 1.0000 (**0.9456**) | 1.0000 (**0.5707**) |

The results for negative correlation coefficients are in bold and are in parenthesis.

Table 2 gives evidence towards $T_{\text{BVSRS}}$ being unbiased and consistent in this case, although such evidence is not a conclusive proof. The power of $T_{\text{BVSRS}}$ increases as the sample size $n$ increases and the shift parameter on the variable $Y$ increases away from 0.

Also, Table 2 shows that $T_{\text{OBVRSS}}$ is more powerful than $T_{\text{BVSRS}}$ for all studied sample sizes, shifted parameter and $\rho$ values. There is a draw back of power when $\rho$ is negative, $T_{\text{OBVRSS}}$ gained more efficiency when $\rho$ is negative than when $\rho$ is positive (see also Table 1). Also, the superiority of $T_{\text{OBVRSS}}$ over $T_{\text{BVSRS}}$ increases as the set size $r$ increases. There is evidence towards $T_{\text{OBVRSS}}$ being unbiased and consistent in this case; such evidence is again not a conclusive proof. From Theorem 2 $T_{\text{OBVRSS}}$ has a similar asymptotic distribution as $T_{\text{BVSRS}}$ but with smaller asymptotic variance. Therefore, it is safe to say that $T_{\text{OBVRSS}}$ has similar asymptotic properties as for testing $H_0 : P(+) = P(-)$, that is, $T_{\text{OBVRSS}}$ is also unbiased and a consistent test procedure. However, $T_{\text{OBVRSS}}$ is more efficient and more powerful than $T_{\text{BVSRS}}$ and $T_{\text{BVRSS}}$.

**Remark.** This paper provides both cases when $r$ is odd and when $r$ is even. However, in practice, since the case when $r$ is odd, OBVRSS is similar to BVSRS without any extra complications in computation, and using OBVRSS for matched pairs sign test is more efficient and more powerful than using BVSRS, it is recommended to choose $r$ to be odd.

### 4.1 Bootstrap algorithm for estimating the $P$-value of the test

The null distribution of our nonparametric tests, $T_{\text{OBVRSS}}$, in Section 4 depend on the underlying bivariate distribution function, especially when $r$ is even. Thus, the exact $P$-value calculation for sample size $n < 20$ is not feasible without knowing the underlying distribution. In this section we introduce a simple bootstrap method for calculating the $P$-value of the sign test for any given bivariate data. For general description of the bootstrap method of estimation see Efron and Tibshirani (1993).

Suppose that a bivariate random sample of size $n < 20$ is drawn from a population using the BVSRS sampling method. This implies that $\{(X_i, Y_i), i = 1, 2, \ldots, n\}$ is a random sample. The bootstrap algorithm for approximating the bootstrap $P$-value of the test for testing the hypothesis $H_0 : P(+) = P(-)$ versus the alternative {e.g. $H_a : P(+) > P(-)$} is:

(1) Calculate the sample test statistic (say $T = \sum_{i=1}^{n} I(X_i < Y_i)$) from the original sample.
(2) Estimate $\theta$ from the data, say $\hat{\theta}$. Shift $Y_i$, to $Y_i - \hat{\theta}$, $i = 1, 2, \ldots, n$, where $\theta = median(Y - X)$.
(3) Define $\hat{F}(x, y)$ by placing a mass probability $p_i = \frac{1}{n}$ on $(X_i, Y_i)$, $i = 1, 2, \ldots, n$.
(4) Generate a resample $(X_i^*, Y_i^*)$, $i = 1, 2, \ldots, n$, from $\hat{F}(x, y)$.
(5) Find $T_b^* = \sum_{i=1}^{n} I(X_i^* < Y_i^*)$.
(6) Repeat steps 3 and 4 $B$ times.

Then the bootstrap $P$-value, $P^* = P(T^* \geq T | \hat{F}(x, y))$, can be approximated by $\hat{P}^* = \frac{1}{B} \sum_{b=1}^{B} I(T_b^* \geq T)$. However, since OBVRSS$_E$ has two different ordered pair labels, slight modification of the above algorithm is needed as follows:

1. Divide the sample into 2 mutually exclusive strata each containing $m$ i.i.d. (only one of the labels) ordered pair labels.
2. Independently from each stratum generate a resample with replacement of size $m$ by placing a mass probability $(\frac{1}{m})$ on each original observation in that stratum.
3. Combine both resamples and does similar steps as in (5) and (6) above.

## 5 Illustration using real data from the Iowa Rural Health Study (RHS)

The Iowa Rural Heath Study (RHS) is a prospective longitudinal cohort study of 8 years from 1981 to 1989 of 3673 individuals (1420 men and 2253 women) aged 65 or older living in Washington and Iowa counties of the state of Iowa. This study is one of four supported by the National Institute on Aging and collectively referred to as Established Populations for Epidemiological Studies of the Elderly (EPESE); see Rubenstein and Lemke (1993) and Brock et al. (1986).

The life histories of 2717 noninstitutional individuals who could walk across a small room without any help were obtained from RHS and divided into two cohorts, one containing 1134 who exercised daily by walking and the other containing 1583 who did not exercise daily by walking. The purpose of this illustration is to test the hypothesis that those elderly people (age 65+) who exercise by outdoor daily walking tend to be younger than those who do not exercise by outdoor daily walking in the Iowa 65+ RHS. Hence, we are interested in testing $H_0: P(+) = P(-)$ versus the alternative $H_a: P(+) > P(-)$, where "+" if $X_i < Y_i$, "−" if $X_i > Y_i$. Although we rounded ages to the nearest integer, age is considered continuous variable. Therefore, we assume no ties.

We created matched pairs of the cohort of daily walking with the cohort of non-daily walking based on their gender and some health conditions. Thus the total number of pairs included in our target population was 1134. Let the random variable $X$ represent the age at baseline of the elderly who exercised by outdoor daily walking and $Y$ represent the age at baseline of the elderly who did not exercise by outdoor daily walking. Due to the availability of the age at baseline for all matched pairs in this illustration, ranking was done on both variables $X$ and $Y$ using the actual ages. However, in real life situation, selecting BVRSS should be done as follows: for example, when $r = 3$, nine prematched pairs of the elderly should be randomly selected. From the first three pairs select the pair with the second youngest age with respect to $X$ (age at baseline of the elderly who exercised by outdoor daily walking). From the second three pairs, again the pair with the second youngest age with respect to $X$ and so on. Quantify the values of $X$ and $Y$ for the pair with the second youngest age with respect to $Y$ (the age at baseline of the elderly who did not exercise by outdoor daily walking). This resemble the label (2, 2). Repeat this 16 times. Two samples, OBVRSS and BVSRS, of size $n = 16$, were drawn from the population of matched pairs (Table 3).

The observed test statistics from the observed samples are $T_{\text{OBVRSS}} = 13$ and $T_{\text{BVSRS}} = 13$. The exact $P$-values of $T_{\text{OBVRSS}}$ and $T_{\text{BVSRS}}$ and can be obtained by using the binomial distribution for $n = 16$ and $p = 0.5$. The $P$-value for both tests is found to be 0.0105. For illustration purposes, we use the bootstrap method with 5000 bootstrap replications to obtain the $P$-value for both $T_{\text{OBVRSS}}$ and $T_{\text{BVSRS}}$. The results of our simulations are as follows: the approximate bootstrap $P$-value of $T_{\text{OBVRSS}}$ is 0.0192 with bootstrap MSE (based on 1000 iterations) equal 0.000019. However, the approximate bootstrap $P$-value of $T_{\text{BVSRS}}$ is 0.0281 with bootstrap MSE (based on 1000 iterations) equal 0.000052. Thus the $P$-value of the RSS is less than the $P$-value of the SRS and therefore one is more likely to reject $H_0$ with RSS and that may be due the fact that $T_{\text{OBVRSS}}$ is more powerful than $T_{\text{BVSRS}}$. It seems that bootstrap method for estimating the $P$-value, tends to over estimating the $P$-value of the test.

In conclusion, whenever OBVRSS, especially when $r$ is odd, can be obtained, it is recommended to be used instead of BVSRS for the bivariate matched pairs sign test.

**Table 3** *The drawn samples of size 16*

| | OBVRSS sample of ($r = 3$, and $m = 16$) | | | | BVSRS of size $n = 16$ | | | |
|---|---|---|---|---|---|---|---|---|
| | Daily walking | | Non-daily walking | | Daily walking | | Non-daily walking | |
| No. | Age $X$ | Gender | Age $Y$ | Gender | Age $X$ | Gender | Age $Y$ | Gender |
| 1 | 73 | Male | 72 | Male | 67 | Male | 72 | Male |
| 2 | 72 | Male | 74 | Male | 72 | Male | 74 | Male |
| 3 | 74 | Male | 77 | Male | 74 | Male | 77 | Male |
| 4 | 68 | Male | 74 | Male | 76 | Male | 83 | Male |
| 5 | 66 | Male | 67 | Male | 66 | Male | 67 | Male |
| 6 | 67 | Male | 71 | Male | 79 | Male | 66 | Male |
| 7 | 75 | Male | 74 | Male | 66 | Female | 68 | Female |
| 8 | 77 | Female | 79 | Female | 83 | Female | 78 | Female |
| 9 | 77 | Female | 69 | Female | 79 | Female | 78 | Female |
| 10 | 75 | Female | 79 | Female | 72 | Female | 74 | Female |
| 11 | 70 | Female | 72 | Female | 66 | Female | 73 | Female |
| 12 | 66 | Female | 72 | Female | 76 | Female | 81 | Female |
| 13 | 67 | Female | 69 | Female | 68 | Female | 75 | Female |
| 14 | 72 | Female | 80 | Female | 67 | Female | 76 | Female |
| 15 | 71 | Female | 78 | Female | 79 | Female | 86 | Female |
| 16 | 67 | Female | 73 | Female | 67 | Female | 73 | Female |

*The parenthesis is the observation label according to the OBVRSS procedure.

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## References

Al-Saleh, M. F. and Zheng, G. (2002). Estimation of multiple characteristics using ranked set sampling. *Australian and New Zealand Journal of Statistics* **44** 221–232. MR1965150

Barabesi, L. (1998). The computation of the distribution of the sign test for ranked set sampling. *Communication in Statistics Simulation and Computation* **27** 833–842.

Bohn, L. L. and Wolfe, D. A. (1992). Nonparametric two-sample procedures for ranked-set samples data. *Journal of the American Statistical Association* **87** 522–561.

Bohn, L. L. and Wolfe, D. A. (1994). The effect of imperfect judgment on ranked-set samples analog of the Mann–Whitney–Wilcoxon statistics. *Journal of the American Statistical Association* **89** 168–176. MR1266294

Brock, D. B., Wineland, T., Freeman, D. H., Lemke, J. H. and Scherr, P. A. (1986). Demographic characteristics. In *Established Population for Epidemiologic Studies of the Elderly. Resource Data Book* (J. Cornoni-Huntley, D. B. Brock, A. M. Ostfeld, J. O. Taylor and R. B. Wallace, eds.). *National Institute on Aging, NIH Publication No. 86-2443*. U.S. Government Printing Office, Washington, DC.

Chen, Z. (2000). On ranked-set sample quantiles and their application. *Journal of Statistical Planning Inference* **83** 125–135. MR1741448

Chen, Z. (2001). Optimal ranked-set sampling scheme for inference on population quantiles. *Statistica Sinica* **11** 23–37. MR1819998

Conover, W. J. (1980). *Practical Nonparametric Statistics*, 2nd ed. Wiley, New York.

Dell, T. R. and Clutter, J. L. (1972). Ranked set sampling theory with order statistics background. *Biometrics* **28** 545–555.

Efron, B. and Tibshirani, R. (1993). *An Introduction to the Bootstrap*. Chapman and Hall, New York. MR1270903

Halls, L. K. and Dell, T. R. (1966). Trial of ranked set sampling for forage yields. *Forest Science* **12** 22–26.

Hennekens, C. H. and Buring, J. E. (1987). *Epidemiology in Medicine*. Little, Brown and Company, Boston/Toronto.

Hettmansperger, T. P. (1984). *Statistical Inference Based on Ranks*. Wiley, New York.

Hettmansperger, T. P. (1995). The ranked-set sample sign test. *Journal of Nonparametric Statistics* **4** 263–270. MR1366773

Johnson, M. E. (1987). *Multivariate Statistical Simulation*. Wiley, New York.

Kaur, A., Patil, G. P., Sinha, A. K. and Taillie, C. (1995). Ranked set sampling: An annotated bibliography. *Environmental and Ecological Statistics* **2** 25–54.

Koti, K. M. and Babu, G. J. (1996). Sign test for ranked-set sampling. *Communications in Statistics. Theory and Methods* **25** 1617–1630. MR1411102

Kvam, P. H. and Samaniego, F. J. (1994). Nonparametric maximum likelihood estimation based on ranked set samples. *Journal of the American Statistical Association* **89** 526–537. MR1294079

McIntyre, G. A. (1952). A method for unbiased selective sampling, using ranked sets. *Australian Journal of Agriculture Research* **3** 385–390.

Öztürk, Ö. (1999). One and two sample sign tests for ranked set samples with selective designs. *Communications in Statistics. Theory and Methods* **28** 1231–1245. MR1704361

Öztürk, Ö. (2001). A nonparametric test of symmetry versus asymmetry for ranked set samples. *Communications in Statistics. Theory and Methods* **30** 2117–2133. MR1862950

Öztürk, Ö. and Wolfe, D. A. (2000). Alternative ranked set sampling protocols for the sign test. *Statistics & Probability Letters* **47** 15–23. MR1745675

Patil, G. P., Sinha, A. K. and Tillie, C. (1999). Ranked set sampling: A bibliography. *Environmental Ecological Statistics* **6** 91–98.

Rubenstein, L. M. and Lemke, J. H. (1993). The construction of self-reported medical condition histories: The Iowa 65+ rural health study. Technical Report No. 93-2. University of Iowa, Dept. Preventive Med. and E. H.

Samawi, H. M. (2001). On quantiles estimation using ranked samples with some applications. *Journal of the Korean Statistical Society* **30** 667–678. MR1898619

Samawi, H. M. and Abu-Dayyeh, W. (2003). More powerful sign test using median ranked set sample: Finite sample power comparison. *Journal of Statistical Computation and Simulation* **73** 697–708. MR2009432

Samawi, H. M., Al-haj Ebrahim, M., Al-Zubaidin and Abu-Dayyeh, W. (2009). Optimal sign test for one sample biavrіate location model using an alternative bivariate ranked set sample. *Journal of Applied Statistics*. To appear.

Samawi, H. M. and Al-Saleh, M. F. (2004). On biavriate ranked set sampling for distribution and quantile estimation and quantile interval estimation using ratio estimator. *Communications in Statistics Theory and Methods* **33** 1801–1819. MR2065175

Samawi, H. M., Al-Saleh, M. F. and Al-Saidy, O. (2006). Bivariate sign test for on-sample bivariate location model using ranked set sample. *Communications in Statistics. Theory and Methods* **35** 1071–1083. MR2256264

Samawi, H. M., Al-Saleh, M. F. and Al-Saidy, O. (2008). The matched pairs sign test using bivariate ranked set sampling. *African Journal of Environmental Science and Technology* **2** 1–9.

Stokes, S. L. and Sager, T. (1988). Characterization of a ranked set sample with application to estimating distribution functions. *Journal of the American Statistical Association* **83** 374–381. MR0971362

Takahasi, K. and Wakimoto, K. (1968). On unbiased estimates of the population mean based on the stratified sampling by means of ordering. *Annals of the Institute of Statistical Mathematics* **20** 1–31. MR0228143

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