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Baum–Katz laws for certain weighted sums of independent and identically distributed random variables

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We consider weighted sums $\sum_k p_{nk} X_k$ of independent and identically distributed random variables (X_n) and compare the tail probabilities of these sums with the moment conditions on X_1, that is, we prove various results of Baum–Katz type. Some special examples of weights p_{nk} originating from summability are discussed.

Keywords: Baum–Katz laws; tail probabilities; weighted mean

1. Introduction

The tail behaviour of partial sums of independent and identically distributed (i.i.d.) random variables $X, X_1, X_2, \ldots$ depends on moment conditions on X. Whereas if the moment generating function exists in a neighbourhood of zero there are large-deviation principles giving precise information on the tail behaviour, at the other end of the scale, namely if only lower-order moments exist, we have less precise information on the tails of the distribution function of $S_n = \sum_{j=1}^n X_j$, which can be expressed in the so-called Baum–Katz laws. These theorems reflect exactly the moments available. Starting with papers by Erdös, Katz, and Baum and Katz, various results have evolved. We formulate one version; see, for example, Baum and Katz (1965). Throughout our paper X, X_k, $k \in \mathbb{N}$, denote i.i.d. random variables.

Theorem 1.

(a) *Let* $\gamma > 1$, $\max\{-\frac{1}{2}, \gamma^{-1} - 1\} < \beta \leqslant 0$. *Then the following statements are equivalent:*
  (i) $\mathrm{E}(|X|^\gamma) < \infty$, $\mathrm{E}(X) = \mu$.
  (ii) $\sum_{n=1}^\infty n^{\gamma(\beta+1)-2} P(|S_n - n\mu| > \varepsilon n^{\beta+1}) < \infty$, *for all* $\varepsilon > 0$.
  (iii) $\sum_{n=1}^\infty n^{\gamma(\beta+1)-2} P(\sup_{k \geqslant n} |S_k - k\mu| / k^{\beta+1} > \varepsilon) < \infty$, *for all* $\varepsilon > 0$.

(b) *Let* $\gamma > 0$, $\beta > \max\{0, \gamma^{-1} - 1\}$. *Then the following statements are equivalent:*
  (i) $\mathrm{E}(|X|^\gamma) < \infty$.
  (ii) $\sum_{n=1}^\infty n^{\gamma(\beta+1)-2} P(|S_n| > \varepsilon n^{\beta+1}) < \infty$, *for all* $\varepsilon > 0$.
  (iii) $\sum_{n=1}^\infty n^{\gamma(\beta+1)-2} P(\sup_{k \geqslant n} |S_k| / k^{\beta+1} > \varepsilon) < \infty$, *for all* $\varepsilon > 0$.

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Remark. Related statements are given in the literature for the case $\gamma = 1$ and other parameter constellations. Note that if $\beta < \gamma^{-1} - 1$ then the sums in (ii) and (iii) always converge.

The aim of the present paper is to give Baum–Katz type results for weighted sums T_n. There are a few scattered results of this type in the literature, for example in Wang *et al*. (1998), but there no equivalences are developed. The special case of Cesàro summability applied to random variables is discussed in Gut (1993). Liang and Su (1999) and Liang (2000) give versions of these results for random variables with a specific dependence structure. Doev (1989) treats some summability methods, such as Euler and Borel summability, but there are some problems with Doev's proofs. Our goal is twofold: on the one hand, we give some results on general classes of weights; on the other hand, we are interested in weights originating from summability. Here, in particular, the logarithmic method of summability, which plays an important role in the context of the almost sure central limit theorems, and methods of random walk type are discussed below, with the special cases of Euler and Borel. We also apply our results to Riesz methods. Exact moment conditions for complete convergence of the corresponding weighted sums are given as applications.

2. Main results

We provide various results depending on the structure of weights p_{nk}. First consider weights $p_k \geqslant 0$ with partial sums $P_n = \sum_{k=1}^{n} p_k$. Occasionally we will need the condition that there exist some $n_0 \in \mathbb{N}$ and constants $c_1, c_2 > 0$ such that

$$p_k \geqslant c_1 \max_{1 \leqslant \nu \leqslant n} p_\nu \text{ for at least } c_2 n \text{ indices } k \in \{1, \ldots, n\}, \text{ for all } n \geqslant n_0. \tag{2.1}$$

Theorem 2. *Let* $(p_k)_{k=1}^{\infty}$ *be a sequence of weights. For given* $\beta \in \mathbb{R}$, *define* $q_n = n^{\beta+1} \max_{1 \leqslant \nu \leqslant n} p_\nu$ *for* $n \in \mathbb{N}$. *Consider the following conditions:*

(i) $\mathrm{E}(|X|^\gamma) < \infty,\ \mathrm{E}(X) = 0$.
(i′) $\mathrm{E}(|X|^\gamma) < \infty$.
(ii) $\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2}\, P(|\sum_{k=1}^{n} p_k X_k| > \varepsilon q_n) < \infty$, *for all* $\varepsilon > 0$.
(iii) $\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2}\, P(|\sum_{k=1}^{n} p_{n+1-k} X_k| > \varepsilon q_n) < \infty$, *for all* $\varepsilon > 0$.
(iv) $\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2}\, P(\sup_{k \geqslant n} |\sum_{j=1}^{k} p_j X_j| / q_k > \varepsilon) < \infty$, *for all* $\varepsilon > 0$.

Then the following statements hold:

(a) *Let* $\gamma > 1$, $\max\{-\frac{1}{2}, \gamma^{-1} - 1\} < \beta \leqslant 0$. *Then* (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii). *Obviously* (iv) $\Rightarrow$ (ii). *Further,* (ii) $\Rightarrow$ (iv) *provided that* (i) *and* $\max_{1 \leqslant k \leqslant 2^j} p_k / \max_{1 \leqslant k \leqslant 2^{j+1}} p_k \geq \delta > 0$ *hold for all* $j \in \mathbb{N}$. *If in addition* (2.1) *holds, then* (i) $\Leftrightarrow$ (ii).
(b) *Let* $\gamma > 0$, $\beta > \max\{0, \gamma^{-1} - 1\}$. *Then the corresponding conclusions from* (a) *hold with* (i) *replaced by* (i′).

Remarks. We first remark that, for the conclusions concerning (i), (ii) and (iii), the weights p_k may obviously also depend on n.

Secondly, under condition (2.1) we have $q_n \asymp n^\beta P_n$, namely $1 \leqslant q_n/(n^\beta P_n) \leqslant 1/c_1c_2$, and then the result holds with q_n replaced by $n^\beta P_n$.

Thirdly, for the conclusion (ii) $\Rightarrow$ (i′) in (b) it is sufficient to have convergence of the sum in (ii) for one $\varepsilon_0 > 0$ only. If $\beta < 0$ the same holds true in (a) for (ii) $\Rightarrow$ (i).

Example. Let $p_k = k^\alpha$ for some $\alpha \geqslant 0$. Then

$$P_n \sim \frac{n^{\alpha+1}}{\alpha+1} = \frac{q_n\, n^{-\beta}}{\alpha+1} \qquad \text{as } n \to \infty.$$

Obviously condition (2.1) is satisfied.

The case of $p_k = k^\alpha$ with $-1 \leqslant \alpha < 0$ is not covered properly by Theorem 2. For this case we have the following theorem, where we restrict ourselves to the essentials. Naturally the results can be extended as in Theorem 2. Setting $\log^+ x = \max\{2, \log x\}$, we obtain:

Theorem 3.

(a) *Let* $-1 < \alpha < 0$ *and* $S_n = \sum_{k=1}^n k^\alpha X_k$. *Consider the moment condition*

$$\mathrm{E}(|X|^\gamma) < \infty. \tag{2.2}$$

(i) *If* $0 < \gamma < 1/|\alpha|$ *and* $\beta > \max\{0, \gamma^{-1} - 1\}$, *then* (2.2) *is equivalent to*

$$\sum_{n=1}^\infty n^{\gamma(\beta+1)-2}\, P(|S_n| > \varepsilon n^{\beta+\alpha+1}) < \infty, \qquad \textit{for all } \varepsilon > 0.$$

(ii) *If* $\gamma = 1/|\alpha|$ *and* $\beta > 0$, *then* (2.2) *is equivalent to*

$$\sum_{n=1}^\infty \frac{n^{\gamma(\alpha+\beta+1)-1}}{\log^+ n}\, P(|S_n| > \varepsilon n^{\beta+\alpha+1}) < \infty, \qquad \textit{for all } \varepsilon > 0.$$

(iii) *If* $\gamma > 1/|\alpha|$ *and* $\beta > 0$, *then* (2.2) *is equivalent to*

$$\sum_{n=1}^\infty n^{\gamma(\alpha+\beta+1)-1}\, P(|S_n| > \varepsilon n^{\beta+\alpha+1}) < \infty, \qquad \textit{for all } \varepsilon > 0.$$

If $\gamma > 1$ *and* $\max\{1/2, 1/\gamma - \alpha\} - 1 < \beta \leqslant 0$ *the respective statements hold true with condition* (2.2) *replaced by*

$$\mathrm{E}(|X|^\gamma) < \infty, \qquad \mathrm{E}(X) = 0. \tag{2.3}$$

(b) *Let* $\alpha = -1$, *that is,* $S_n = \sum_{k=1}^n X_k/k$. *Now the moment condition is*

$$\mathrm{E}(|X|^\gamma/(\log^+ X)^{2\gamma}) < \infty. \tag{2.4}$$

(i) *If* $0 < \gamma < 1$ *and* $\beta > \gamma^{-1} - 1$, *then* (2.4) *is equivalent to*

$$\sum_{n=1}^\infty \frac{n^{\gamma(\beta+1)-2}}{(\log^+ n)^\gamma}\, P(|S_n| > \varepsilon n^\beta \log n) < \infty, \qquad \textit{for all } \varepsilon > 0.$$

(ii) *If* $\gamma = 1$ *and* $\beta > 0$, *then* (2.4) *is equivalent to*

$$\sum_{n=1}^{\infty} \frac{n^{\beta-1}}{(\log^+ n)^2} P\left(|S_n| > \varepsilon n^{\beta} \log n\right) < \infty, \qquad \textit{for all } \varepsilon > 0.$$

(iii) *If* $\gamma > 1$ *and* $\beta > 0$, *then* (2.4) *is equivalent to*

$$\sum_{n=1}^{\infty} \frac{n^{\gamma\beta-1}}{(\log^+ n)^{\gamma}} P\big(|S_n| > \varepsilon n^{\beta} \log n\big) < \infty, \qquad \textit{for all } \varepsilon > 0.$$

Remarks. Part (a) with $\beta = 0$ is closely related to Theorem 2.2 in Gut (1993). Part (b) gives a Baum–Katz result for the logarithmic summability method applied to the sequence (X_k).

Note that Theorem 2.4 in Li *et al.* (1995), after using Corollary 1 (in Section 3 below) twice, essentially reduces to Theorems 2 and 3 (see also Theorem A in Liang 2000).

Next we deal with the general but typical situation where the weights follow a continuous pattern.

Theorem 4. *Let* X, X_k, $k \in \mathbb{Z}$, *be i.i.d. random variables and* $\phi\colon \mathbb{R} \to \mathbb{R}$ *be continuous, decreasing and integrable on* $[0, \infty)$. *Further, assume that* $\phi(x) = \phi(-x)$ *for all* $x \geq 0$. *Define*

$$T_n = \sum_{k=-\infty}^{\infty} \phi\left(\frac{k}{n^{\alpha}}\right) \frac{1}{n^{\alpha}} X_k$$

with some $\alpha \in (0, 1]$ *(where we implicitly assume almost sure convergence of the sums).*

(a) *Assume* $\gamma \geq 1$ *and* $\beta > 0$. *Then the following are equivalent:*

(i) $\mathrm{E}(|X|^{\gamma}) < \infty$.

(ii) $\sum_{n=1}^{\infty} n^{\gamma(\alpha+\beta)-1-\alpha} P\big(|T_n| > n^{\beta}\varepsilon\big) < \infty$, *for all* $\varepsilon > 0$.

If $\gamma > 1$ *and* $\max\{-\alpha/2, \alpha(1/\gamma - 1)\} < \beta \leqslant 0$, *then* (ii) *is equivalent to* $\mathrm{E}(|X|^{\gamma}) < \infty$ *and* $\mathrm{E}(X) = 0$.

(b) *Assume* $0 < \gamma < 1$ *and* $\beta > \alpha(1/\gamma - 1)$ *and that* $\int_0^{\infty} (\phi(t))^{\gamma}\, \mathrm{d}t < \infty$. *Then* (i) *and* (ii) *are equivalent.*

Remark. The same result holds for

$$T_n' := \sum_{k=-m_n}^{m_n} \phi\left(\frac{k}{n^{\alpha}}\right) \frac{1}{n^{\alpha}} X_k$$

for $m_n \in \mathbb{N}$ satisfying $m_n / n^{\alpha} \to \infty$.

3. Proofs

We begin with a lemma which can be found in a paper by Sztencel (1981). For the sake of completeness we repeat its proof.

Lemma 1. *Suppose we have symmetric, independent random variables $X_1, X_2, \dots, X_n$ and weights $0 \leqslant a_k \leqslant 1$, $1 \leqslant k \leqslant n$. Then, for all $\varepsilon > 0$,*

$$\frac{1}{2} P\left(\min_{1\leqslant k\leqslant n}\{a_k\}\left|\sum_{k=1}^{n} X_k\right| > \varepsilon\right) \leqslant P\left(\left|\sum_{k=1}^{n} a_k X_k\right| > \varepsilon\right) \leqslant 2P\left(\left|\sum_{k=1}^{n} X_k\right| > \varepsilon\right).$$

Proof. For the second inequality assume, without loss of generality, that $0 \leqslant a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n \leqslant 1$ and put $a_0 = 0$. Then we have, with $S_j = \sum_{k=1}^{j} X_k$, $b_j = a_j - a_{j-1}$, $1 \leqslant j \leqslant n$, that

$$\sum_{k=1}^{n} a_k X_k = \sum_{k=1}^{n} b_k S_k \quad \text{and} \quad \sum_{k=1}^{n} b_k = a_n \leqslant 1.$$

Now use Lévy's inequality, and note that $S_j \leqslant \varepsilon$, $1 \leqslant j \leqslant n$ implies $\sum_{j=1}^{n} b_j S_j \leqslant \varepsilon$, hence

$$P\left(\sum_{k=1}^{n} a_k X_k > \varepsilon\right) = P\left(\sum_{j=1}^{n} b_j S_j > \varepsilon\right) \leqslant P\left(\max_{1\leqslant j\leqslant n} S_j > \varepsilon\right) \leqslant 2\,P(S_n > \varepsilon).$$

To verify the first inequality put $\tilde{X}_k = a_k X_k$ and use the second inequality to obtain

$$P\left(\sum_{j=1}^{n} \frac{\min_{1\leqslant j\leqslant n}\{a_j\}}{a_j}\,\tilde{X}_j > \varepsilon\right) \leqslant 2P\left(\sum_{j=1}^{n} \tilde{X}_j > \varepsilon\right),$$

which contains the desired inequality. Note that the first inequality is trivial if $a_j = 0$ for some j. □

Corollary 1. *Suppose we have given symmetric, independent random variables $X_1, X_2, \dots, X_n$ and weights $0 \leqslant \tilde{a}_j \leqslant a_j$, $1 \leqslant j \leqslant n$, $n \in \mathbb{N}$. Then*

$$P\left(\sum_{k=1}^{n} a_k X_k > \varepsilon\right) \geqslant \frac{1}{2} P\left(\sum_{k=1}^{n} \tilde{a}_k X_k > \varepsilon\right).$$

Proof. Put $Y_k = a_k X_k$, so $\tilde{a}_k / a_k\, Y_k = \tilde{a}_k X_k$ and the second inequality in Lemma 1 yields the corollary. □

Remark. If the sums in Corollary 1 converge as $n \to \infty$, the inequality holds with $n = \infty$ as well.

Next we state some auxiliary results that are needed to find appropriate moment conditions.

Lemma 2. *Let $X, X_1, X_2, \dots$ be i.i.d. and symmetric, $(m_n)_{n=1}^{\infty}$ a sequence of positive integers, p_{nk}, $1 \leqslant k \leqslant m_n$, $n \in \mathbb{N}$, positive weights, and $(\lambda_n)_{n=1}^{\infty}$ a positive sequence. Then $P(|\sum_{k=1}^{m_n} p_{nk} X_k| > \lambda_n) \to 0$, $n \to \infty$, implies*

$$m_n P\left(|X| > 2\lambda_n \max_{1 \leqslant k \leqslant m_n} \frac{1}{p_{nk}}\right) \leqslant \sum_{k=1}^{m_n} P(|p_{nk} X_k| > 2\lambda_n) \to 0, \qquad n \to \infty.$$

Proof. By Lévy's inequality we have

$$P\left(\max_{1 \leqslant j \leqslant m_n} \left|\sum_{k=1}^{j} p_{nk} X_k\right| > \lambda_n\right) \leqslant 2P\left(\left|\sum_{k=1}^{m_n} p_{nk} X_k\right| > \lambda_n\right) \to 0, \qquad n \to \infty.$$

Hence, as $n \to \infty$,

$$\prod_{j=1}^{m_n} P(|p_{nj} X_j| \leqslant 2\lambda_n) = P\Big(\max_{1 \leqslant j \leqslant m_n} |p_{nj} X_j| \leqslant 2\lambda_n\Big) \geqslant P\left(\max_{1 \leqslant j \leqslant m_n} \left|\sum_{k=1}^{j} p_{nk} X_k\right| \leqslant \lambda_n\right) \to 1.$$

Using the fact that $P(|p_{nj} X_j| \leqslant 2\lambda_n) \to 1$, $n \to \infty$ uniformly in j and that $\log(1+x) \sim x$ as $x \to 0$, we find that

$$m_n P\left(|X| > 2\lambda_n \max_{1 \leqslant k \leqslant m_n} \frac{1}{p_{nk}}\right) \leqslant \sum_{j=1}^{m_n} P(|p_{nj} X_j| > 2\lambda_n) \to 0, \qquad n \to \infty.$$

□

Remark. Under these assumptions we have, uniformly in $1 \leqslant l \leqslant m_n$,

$$P\left(\left|\sum_{\substack{k=1 \\ k \neq l}}^{m_n} p_{nk} X_k\right| \leqslant 3\lambda_n\right) \to 1, \qquad n \to \infty,$$

since

$$\begin{aligned} P\left(\left|\sum_{\substack{k=1 \\ k \neq l}}^{m_n} p_{nk} X_k\right| > 3\lambda_n\right) &\leqslant P\left(\left|\sum_{k=1}^{m_n} p_{nk} X_k\right| > \lambda_n\right) + P(|p_{nl} X_l| > 2\lambda_n) \\ &\leqslant P\left(\left|\sum_{k=1}^{m_n} p_{nk} X_k\right| > \lambda_n\right) + \sum_{k=1}^{m_n} P(|p_{nk} X_k| > 2\lambda_n) \\ &\to 0, \qquad n \to \infty, \end{aligned}$$

where the last expression does not depend on l.

Lemma 3. *Under the assumptions of Lemma 2 there exists some* $n_0 \in \mathbb{N}$ *such that*

$$m_n P\left(|X| > 4\lambda_n \max_{1 \leqslant l \leqslant m_n} \frac{1}{p_{nl}}\right) \leqslant \sum_{k=1}^{m_n} P(|p_{nk} X_k| > 4\,\lambda_n) \leqslant 2 \cdot P\left(\left|\sum_{k=1}^{m_n} p_{nk} X_k\right| > \lambda_n\right)$$

for all $n \geq n_0$.

Remark. This implies a version of Theorem 2.5 in Gut (1992) in the case of symmetric random variables with explicit inequalities.

Proof. The left inequality follows from Lemma 2. To prove the right inequality, note that with

$$A_k = \{|p_{nk}X_k| > 4\lambda_n\}, \qquad B_k = \left\{\left|\sum_{\substack{j=1\\ j\neq k}}^{m_n} p_{nj}X_j\right| \leqslant 3\lambda_n\right\},$$

we have

$$\begin{aligned} P\left(\left|\sum_{k=1}^{m_n} p_{nk}X_k\right| > \lambda_n\right) &\geqslant P\left(\bigcup_{k=1}^{m_n}(A_k \cap B_k)\right) \\ &\geqslant \sum_{k=1}^{m_n}\left(p(A_k \cap B_k) - P\left(A_k \cap \bigcup_{\nu=1}^{k-1}(A_\nu \cap B_\nu)\right)\right) \\ &\geqslant \sum_{k=1}^{m_n}\left(P(A_k \cap B_k) - \sum_{\nu=1}^{k-1} P(A_k \cap A_\nu)\right) \\ &\geqslant \sum_{k=1}^{m_n} P(A_k)\left(P(B_k) - \sum_{\nu=1}^{k-1} P(A_\nu)\right) \\ &\geqslant \frac{1}{2}\sum_{k=1}^{m_n} P(A_k) \end{aligned}$$

for n large enough, by Lemma 2 and the remark following its proof. □

The following lemma will be useful in showing the necessity of the moment conditions. Its proof consists of a well-known argument but is stated for the sake of completeness.

Lemma 4. *Suppose we have some* $\zeta > -1$ *and a strictly increasing function* ψ: $(0, \infty) \to (0, \infty)$. *Then*

$$\sum_{n=1}^{\infty} n^{\zeta} P(|X| > \psi(n)) < \infty \text{ if and only if } \mathrm{E}\left(\left(\psi^{-1}(|X|)\right)^{\zeta+1}\right) < \infty.$$

Proof. It is well known that, for $\zeta > -1$,

$$\mathrm{E}|Y|^{\xi+1} \Leftrightarrow \sum_{n=1}^{\infty} n^{\xi} P(|Y| > n) < \infty.$$

Taking into account that $P(|X| > \psi(n)) = P\big(\psi^{-1}(|X|) > n\big)$ this proves the lemma when applied to $Y = \psi^{-1}(|X|)$. □

Proof of Theorem 2. (ii) $\Leftrightarrow$ (iii) is obvious since the sums involved have the same distribution function. In the case where X is symmetric the inclusion (i) $\Rightarrow$ (ii) follows from Theorem 1 and the observation

$$P\left(\left|\sum_{k=1}^{n} p_k X_k\right| > \varepsilon q_n\right) = P\left(\left|\sum_{k=1}^{n} \frac{p_k}{\max_{1\leqslant\nu\leqslant n} p_\nu} X_k\right| > \varepsilon n^{\beta+1}\right) \leqslant 2P\left(\left|\sum_{k=1}^{n} X_k\right| > \varepsilon n^{\beta+1}\right)$$

by an application of Lemma 1.

In this proof we discuss in detail how the general case works. In later proofs the same arguments can be applied and will be omitted. Note that

$$\frac{1}{q_n}\sum_{k=1}^{n} p_k X_k \xrightarrow{p} 0, \qquad n \to \infty. \tag{3.1}$$

Using the Marcinkiewicz–Zygmund (cf. Chow and Teicher 1978) and c_r inequalities (cf. Loève 1977), we can prove (3.1) for $0 < \gamma < 2$ by observing that

$$\begin{aligned} P\left(\left|\frac{1}{q_n}\sum_{k=1}^{n} p_k X_k\right| > \varepsilon\right) &\leqslant \frac{1}{(\varepsilon q_n)^{\gamma}} \mathrm{E}\left(\left|\sum_{k=1}^{n} p_k X_k\right|^{\gamma}\right) \\ &\leqslant \frac{1}{(\varepsilon q_n)^{\gamma}} \mathrm{E}\left(\left(\sum_{k=1}^{n} p_k |X_k|\right)^{\gamma}\right) \\ &\leqslant \frac{1}{(\varepsilon q_n)^{\gamma}} \sum_{k=1}^{n} p_k^{\gamma} \mathrm{E}(|X|^{\gamma}) \\ &\leqslant n^{1-(\beta+1)\gamma} \mathrm{E}(|X|^{\gamma}) \to 0 \quad (n \to \infty); \end{aligned}$$

If $\gamma \geqslant 2$ the Markov inequality yields

$$P\left(\left|\frac{1}{q_n}\sum_{k=1}^{n} p_k X_k\right| > \varepsilon\right) \leqslant \frac{1}{\varepsilon^2 n^{1+2\beta}} \mathrm{E}(X^2) \to 0, \quad n \to \infty.$$

Thus with $m(Y)$ denoting a median of the random variable Y, for any $\varepsilon > 0$ we have

$$m\left(\frac{1}{q_n}\sum_{k=1}^{n} p_k X_k\right) \leqslant \frac{\varepsilon}{2}$$

for all large enough n. For these n it follows that

$$P\left(\left|\sum_{k=1}^{n} p_k X_k\right| > \varepsilon q_n\right) \leqslant P\left(\left|\frac{1}{q_n}\sum_{k=1}^{n} p_k X_k - m\left(\frac{1}{q_n}\sum_{k=1}^{n} p_k X_k\right)\right| > \frac{\varepsilon}{2}\right)$$

$$\leqslant 2P\left(\left|\frac{1}{q_n}\sum_{k=1}^{n} p_k X_k^s\right| > \frac{\varepsilon}{2}\right),$$

where the last step is due to the symmetrization inequality (e.g. Lemma 6.16 in Petrov 1995); note that $(\sum p_k X_k)^s \stackrel{d}{=} \sum p_k X_k^s$.

For the proof of (ii) $\Rightarrow$ (i) note that a successive application of the symmetrization inequality, Corollary 1 (twice) and Lemma 3 yields

$$\infty > \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2}\, P\left(\left|\sum_{k=1}^{n} p_k X_k^s\right| > \varepsilon q_n\right)$$

$$\geqslant \frac{1}{2}\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2}\, P\left(\left|\sum_{k:\, p_k \geqslant c_1 \max\limits_{1\leqslant\nu\leqslant n} p_\nu} \frac{p_k}{\max\limits_{1\leqslant\nu\leqslant n} p_\nu} X_k^s\right| > \varepsilon n^{\beta+1}\right)$$

$$\geqslant \frac{1}{4}\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2}\, P\left(\left|\sum_{k:\, p_k \geqslant c_1 \max\limits_{1\leqslant\nu\leqslant n} p_\nu} X_k^s\right| > \varepsilon n^{\beta+1}/c_1\right)$$

$$\geqslant \frac{c_2}{4c}\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-1}\, P\big(|X^s| > 4\varepsilon n^{\beta+1}/c_1\big).$$

For the application of Lemma 3 note that $\gamma(\beta+1) - 2 > -1$ and that the argument from Baum and Katz (1965, p. 110), can be applied to show that

$$P\left(\left|\sum_{k:\, p_k \geqslant c_1 \max_{1\leqslant\nu\leqslant n} p_\nu} X_k^s\right| > \varepsilon n^{\beta+1}/c_1\right) \to 0, \qquad n \to \infty.$$

Finally by Lemma 4 the inequality above yields $\mathrm{E}|X^s|^\gamma < \infty$ and thus $\mathrm{E}|X|^\gamma < \infty$.

Now in the case $\gamma > 1$ it follows immediately from Theorem 1 of Pruitt (1966) that

$$\frac{1}{P_n}\sum_{k=1}^{n} p_k X_k \xrightarrow{p} \mathrm{E}(X) = \mu, \quad n \to \infty.$$

Now assume $\mu \neq 0$. Since we know that, by (2.1),

$$c_1 c_2 \leqslant \frac{P_n}{n \max_{1\leqslant k\leqslant n} p_k} \leqslant 1$$

it follows that

$$P\left(\left|\frac{1}{q_n}\sum_{k=1}^{n} p_k X_k\right| \geqslant \frac{c_1 c_2|\mu|}{2}\right) \to 1, \qquad n \to \infty,$$

and the series in (ii) diverges for $\varepsilon < c_1 c_2|\mu|/2$. This contradicts (ii) and hence $\mathrm{E}(X) = 0$.

It remains to show that (ii) $\Rightarrow$ (iv) in the case of symmetric random variables. Note that the converse direction is obvious. Following the method of Baum and Katz (1965), we use Lévy's inequality after some manipulations and find

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2}\, P\left(\sup_{k\geqslant n}\left|\sum_{j=1}^{k} p_j X_j\right|/q_k > \varepsilon\right)\\
&\quad\leqslant \sum_{i=1}^{\infty} 2^i \cdot 2^{(i+1)(\gamma(\beta+1)-2)}\, P\left(\sup_{k\geqslant 2^i}\left|\sum_{\nu=1}^{k} p_\nu X_\nu\right|/q_k > \varepsilon\right)\\
&\quad\leqslant \sum_{i=1}^{\infty} 2^{(i+1)(\gamma(\beta+1)-1)} \sum_{j=i}^{\infty} P\left(\sup_{2^j\leqslant k<2^{j+1}}\left|\sum_{\nu=1}^{k} p_\nu X_\nu\right| > \varepsilon q_{2^j}\right)\\
&\quad\leqslant c\sum_{j=1}^{\infty} 2^{(j+1)(\gamma(\beta+1)-1)}\, P\left(\sup_{2^j\leqslant k<2^{j+1}}\left|\sum_{\nu=1}^{k} p_\nu X_\nu\right| > \varepsilon q_{2^j}\right)\\
&\quad\leqslant c\sum_{j=1}^{\infty} 2^{(j+1)(\gamma(\beta+1)-1)}\, 2P\left(\left|\sum_{\nu=1}^{2^{j+1}} p_\nu X_\nu\right| > \varepsilon q_{2^{j+1}}\frac{q_{2^j}}{q_{2^{j+1}}}\right)\\
&\quad\leqslant 2c\sum_{j=1}^{\infty} 2^{(j+1)(\gamma(\beta+1)-1)}\, P\left(\left|\sum_{\nu=1}^{2^{j+1}} p_\nu X_\nu\right| > \varepsilon\delta q_{2^{j+1}}\right)\\
&\quad\leqslant \tilde{c}\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2}\, P\left(\left|\sum_{\nu=1}^{n} p_\nu X_\nu\right| > \varepsilon\delta q_n\right) < \infty,
\end{aligned}$$

where we have used that, for $2^j \leqslant n < 2^{j+1}$,

$$\begin{aligned}
P\left(\sum_{k=1}^{n} p_k X_k > \lambda\right) &\geqslant P\left(\sum_{k=1}^{2^j} p_k X_k > \lambda,\ \sum_{k=2^j+1}^{n} p_k X_k \geqslant 0\right)\\
&\geqslant \frac{1}{2} P\left(\sum_{k=1}^{2^j} p_k X_k > \lambda\right)
\end{aligned}$$

and that

$$\sum_{n=2^j}^{2^{j+1}-1} n^{\gamma(\beta+1)-2} \geqslant c^* 2^{(j+1)(\gamma(\beta+1)-1)}.$$

□

Proof of Theorem 3. We shall only show part (b). The proof of part (a) follows along the same lines (see also Gut 1993, Theorem 2.2). For the sufficiency part we choose, without loss of generality, $\varepsilon = 3^j$ with some fixed value of j to be specified later. Then we use the Hoffmann-Jørgensen inequality (see Hoffmann-Jørgensen 1974) in its iterated form (cf. Jain 1975), yielding that, for any $j \in \mathbb{N}$, there exist constants $c_{1j}, c_{2j} > 0$ such that

$$\begin{aligned} P(|S_n| > 3^j n^\beta \log n) &\leqslant c_{1j} P\left(\max_{1\leqslant k\leqslant n}\left|\frac{1}{k}X_k\right| > n^\beta \log n\right) + c_{2j}(P(|S_n| > n^\beta \log n))^{2^j} \\ &\leqslant c_{1j}\sum_{k=1}^{n} P(|X| > k\, n^\beta \log n) + c_{2j}(P(|S_n| > n^\beta \log n))^{2^j} \\ &= I + II \quad \text{(say)}. \end{aligned}$$

By the Markov inequality we find that

$$P(|S_n| > n^\beta \log n) = O(1)\begin{cases} n^{1-\gamma\,(1+\beta)}, & 0<\gamma<1, \\ n^{-\beta\,\gamma}, & \gamma = 1 \\ n^{-\beta\,\gamma'}, & \text{with } 1<\gamma'<\gamma,\ \gamma'\leqslant 2, \end{cases} \tag{3.2}$$

where we have used the c_r inequality (cf. Loève 1977) and, in the case $\gamma > 1$, also the Marcinkiewicz–Zygmund inequality (cf. Chow and Teicher 1978). So, under our assumptions the term II is sufficiently small after choosing j large enough. The main term is the first one. For example, in case $0 < \gamma < 1$, we have

$$\sum_{n=1}^{\infty}\frac{n^{\gamma(\beta+1)-2}}{(\log^+ n)^\gamma}\sum_{k=1}^{n} P(|X| > kn^\beta \log n) = \sum_{n=1}^{\infty}\frac{n^{\gamma(\beta+1)-2}}{(\log^+ n)^\gamma}\sum_{k=1}^{n}\sum_{\nu > kn^\beta \log n} b_\nu,$$

where $b_\nu = P(\nu - 1 \leqslant |X| < \nu)$. Now an asymptotic evaluation of this sum shows that it converges under the given moment condition. Similar arguments give the other cases.

For the necessity part we use Lemma 3, that is, we have, for $0 < \gamma < 1$,

$$\sum_{n=1}^{\infty}\frac{n^{\gamma(\beta+1)-2}}{(\log^+ n)^\gamma}\sum_{k=1}^{n} P\big(|X| > kn^\beta \log n\big) < \infty,$$

and note that replacing k by n yields the moment condition. In the corresponding calculations, for $\gamma > 1$ use just the first summand in the inner sum, whereas for $\gamma = 1$ the whole sum has to be used. □

Proof of Theorem 4. We may assume X to be symmetric, with the general case following by the same argument as used in the proof of Theorem 2. First we show (i) $\Rightarrow$ (ii). Using the

Marcinkiewicz–Zygmund (cf. Chow and Teicher 1978) and c_r inequalities (cf. Loève 1977), we find

$$\begin{aligned} \mathrm{E}(|T_n|^\gamma) &\leqslant \sum_{k=-\infty}^{\infty} \left(\phi\left(\frac{k}{n^\alpha}\right)\frac{1}{n^\alpha}\right)^\gamma \mathrm{E}(|X|^\gamma) \\ &\leqslant \frac{c}{n^{\alpha(\gamma-1)}} \int_{-\infty}^{\infty} (\phi(t))^\gamma \,\mathrm{d}t < \infty, \end{aligned} \tag{3.3}$$

for $0 < \gamma \leqslant 2$.

Further, we find by the Hoffmann–Jørgensen inequality (Hoffmann–Jørgensen 1974) in its iterated form (Jain 1975) that, for any $j \in \mathbb{N}$, there exist constants c_{1j}, $c_{2j} > 0$ such that

$$\begin{aligned} P\big(|T_n| > 3^j n^\beta\big) &\leqslant c_{1j} P\left(\max_{k\in\mathbb{Z}} \left|\phi\left(\frac{k}{n^\alpha}\right)\frac{1}{n^\alpha} X_k\right| > n^\beta\right) + c_{2j}\big(P\big(|T_n| > n^\beta\big)\big)^{2^j} \\ &\leqslant c_{1j} \sum_{j\in\mathbb{Z}} P\left(|X| > \frac{n^{\alpha+\beta}}{\phi(j/n^\alpha)}\right) + c_{2j}\big(P\big(|T_n| > n^\beta\big)\big)^{2^j}. \end{aligned}$$

Thus we obtain (we choose, again without loss of generality, $\varepsilon = 3^j$ with some fixed value of j to be specified later on)

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{\gamma(\alpha+\beta)-1-\alpha}\, P\big(T_n > 3^j n^\beta\big) \\ &\quad\leqslant c_{1j} \sum_{n=1}^{\infty} n^{\gamma(\alpha+\beta)-1-\alpha} \sum_{j\in\mathbb{Z}} P\left(|X| > \frac{n^{\alpha+\beta}}{\phi(j/n^\alpha)}\right) + c_{2j} \sum_{n=1}^{\infty} n^{\gamma(\alpha+\beta)-1-\alpha}\big(P\big(|T_n| > n^\beta\big)\big)^{2^j} \\ &\quad= I + II. \end{aligned}$$

By (3.3) we find, for $\gamma \in (0, 2)$,

$$II \leqslant c_{2j} \sum_{n=1}^{\infty} n^{\gamma(\alpha+\beta)-1-\alpha} c_\gamma n^{-2^j((\alpha+\beta)\gamma-\alpha)} < \infty,$$

if we choose j large enough since $(\alpha+\beta)\gamma - \alpha > 0$ under our assumptions in (a) and (b). For $\gamma \geqslant 2$ we obtain, again for large enough j,

$$II \leqslant c_{2j} c_2 \sum_{n=1}^{\infty} n^{\gamma(\alpha+\beta)-1-\alpha} n^{-2^j(\alpha+2\beta)} < \infty.$$

For the first term we set $b_\nu = P(\nu - 1 \leqslant |X| < \nu)$ and obtain with a constant c, possibly varying from line to line,

$$
\begin{aligned}
\mathrm{I} &\leqslant c\sum_{n=1}^{\infty} n^{\gamma(\alpha+\beta)-1-\alpha}\sum_{k\in\mathbb{Z}} P\left(\left|\phi\left(\frac{k}{n^{\alpha}}\right)\frac{1}{n^{\alpha}}X_k\right| > n^{\beta}\right)\\
&\leqslant c\sum_{n=1}^{\infty} n^{\gamma(\alpha+\beta)-1-\alpha}\sum_{k\in\mathbb{Z}}\ \sum_{\nu\geqslant\max\{n^{\alpha+\beta}/\phi(k/n^{\alpha}),1\}} b_\nu\\
&\leqslant c\sum_{n=1}^{\infty} n^{\gamma(\alpha+\beta)-1-\alpha}\sum_{\nu=1}^{\infty} b_\nu \sum_{\phi(k/n^{\alpha})\geqslant n^{\alpha+\beta}/\nu} 1.
\end{aligned}
$$

Note that the inner sum vanishes if $n^{\alpha+\beta}/\nu > \phi(0)$, that is, $n > (\nu\phi(0))^{1/(\alpha+\beta)}$. Thus

$$
\begin{aligned}
\mathrm{I} &\leqslant c\sum_{\nu=1}^{\infty} b_\nu \sum_{1\leqslant n\leqslant(\nu\phi(0))^{1/(\alpha+\beta)}} n^{\gamma(\alpha+\beta)-1-\alpha}\sum_{|k|\leqslant\phi^{-1}(n^{\alpha+\beta}/\nu)n^{\alpha}} 1\\
&\leqslant c\sum_{\nu=1}^{\infty} b_\nu \sum_{1\leqslant n\leqslant(\nu\phi(0))^{1/(\alpha+\beta)}} n^{\gamma(\alpha+\beta)-1}\phi^{-1}\left(\frac{n^{\alpha+\beta}}{\nu}\right).
\end{aligned}
$$

Now assume $\gamma \geqslant 1$ and observe that since by monotonicity and $\phi \in L^1[0,\infty)$ we have $\lim_{y\to\infty} y\phi(y) = 0$, implying $\phi^{-1}(\nu^{-1}) = o(\nu)$ as $\nu\to\infty$.

Substituting $w = t^{\alpha+\beta}/\nu$ and later $v = \phi^{-1}(w)$, it is easy to see that

$$
\begin{aligned}
\sum_{1\leqslant n\leqslant(\nu\phi(0))^{1/(\alpha+\beta)}} n^{\gamma(\alpha+\beta)-1}\phi^{-1}\left(\frac{n^{\alpha+\beta}}{\nu}\right) &\leqslant \phi^{-1}\left(\frac{1}{\nu}\right) + c\int_1^{(\nu\phi(0))^{1/(\alpha+\beta)}} t^{\gamma(\alpha+\beta)-1}\phi^{-1}\left(\frac{t^{\alpha+\beta}}{\nu}\right)\mathrm{d}t\\
&= o(\nu) + c\int_{1/\nu}^{\phi(0)} (\nu w)^{\gamma-1/(\alpha+\beta)}\phi^{-1}(w)(\nu\, w)^{1/(\alpha+\beta)}\,\frac{\mathrm{d}w}{w}\\
&= o(\nu) + c\nu^{\gamma}\int_{1/\nu}^{\phi(0)} w^{\gamma-1}\phi^{-1}(w)\,\mathrm{d}w\\
&= o(\nu) + c\nu^{\gamma}\int_0^{\phi^{-1}(1/\nu)} w(\phi(w))^{\gamma-1}\,\mathrm{d}\phi(w)\\
&\leqslant c\nu^{\gamma},
\end{aligned}
$$

since the integral is bounded due to the continuity of ϕ and the fact that $\int_0^\infty \phi^{\gamma}(v)\,\mathrm{d}v < \infty$. Hence $I < \infty$. Similar arguments apply for the case $\gamma \in (0,1)$.

For the proof of (ii) $\Rightarrow$ (i) we proceed as in Theorem 1. Without loss of generality, we assume $\phi(1) > 0$ and apply the Corollary 1 twice and Lemma 3 to obtain, with some constant $c > 0$,

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{\gamma(\alpha+\beta)-1-\alpha}\, P\big(|T_n| > \varepsilon n^{\beta}\big) = 2\sum_{n=1}^{\infty} n^{\gamma(\alpha+\beta)-1-\alpha}\, P\big(T_n > \varepsilon n^{\beta}\big)\\
&\geqslant \sum_{n=1}^{\infty} n^{\gamma(\alpha+\beta)-1-\alpha}\, P\left(\sum_{|k|\leqslant n^{\alpha}} \phi\left(\frac{k}{n^{\alpha}}\right)\frac{1}{n^{\alpha}} X_k > \varepsilon n^{\beta}\right)\\
&\geqslant \frac{1}{2}\sum_{n=1}^{\infty} n^{\gamma(\alpha+\beta)-1-\alpha}\, P\left(\sum_{|k|\leqslant n^{\alpha}} X_k > \frac{\varepsilon n^{\alpha+\beta}}{\phi(1)}\right)\\
&\geqslant c\sum_{n=1}^{\infty} n^{\gamma(\alpha+\beta)-1}\, P\left(|X| > 4\frac{\varepsilon n^{\alpha+\beta}}{\phi(1)}\right),
\end{aligned}$$

and by Lemma 4 we have $\mathrm{E}|X|^{\gamma} < \infty$. □

4. Application to weights originating from summability

Given a matrix $P = (p_{nk})_{n,k=0}^{\infty}$, we say that a real sequence (s_n) converges to a limit s with respect to the summability method P, $s_n \to s(P)$, if $\sum_k p_{nk}\, s_k \to s$, $n \to \infty$, (see Zeller and Beekmann 1970). The Cesàro methods (C_α) with $p_{nk} = \binom{n-k+\alpha-1}{n-k}/\binom{n+\alpha}{n}$ for $0 \leqslant k \leqslant n$ and $\alpha > 0$ are well known, in particular the arithmetic mean (C_1).

4.1. Euler, Borel and random walk methods of summability

In the case of the Euler method we have the weights

$$p_{nk} = \binom{n}{k} p^k (1-p)^{n-k}, \qquad 0 \leqslant k \leqslant n,$$

for some $p \in (0, 1)$ whereas in the case of the Borel method we have

$$p_{nk} = \mathrm{e}^{-n}\frac{n^k}{k!}, \qquad k \in \mathbb{N}_0.$$

More generally, one can consider i.i.d. integer-valued random variables Y_i with partial sums W_n, and define weights $p_{nk} = P(W_n = k)$ to obtain so-called random walk methods of summability; see, for example, Bingham and Maejima (1985). Under appropriate conditions on Y_i it is well known (see Petrov 1975) that

$$p_{nk} \asymp 1/\sqrt{n} \qquad \text{for } |k - nE(Y_1)| \leqslant M\sqrt{n} \tag{4.1}$$

for an arbitrary constant $M > 0$. Furthermore, under stronger conditions, we obtain an asymptotic expansion of the weights which can also be found in Petrov (1975). For example, in the case of the Euler method we have, say,

$$p_{nk} = \frac{1}{\sqrt{2\pi np(1-p)}} \exp\left\{-\frac{(k-np)^2}{2np(1-p)}\right\}\left(1 + \frac{c_3\, H_3\left(\frac{k-np}{\sqrt{np(1-p)}}\right)}{\sqrt{n}}\right)$$

$$+ \frac{1}{1+\left|\frac{k-np}{\sqrt{np(1-p)}}\right|^3} \cdot o\left(\frac{1}{n}\right) =: \sum_{\nu=1}^{3} p_{nk}^{(\nu)}$$

uniformly in k with some constant c_3 and the third Hermite polynomial H_3. A similar Edgeworth expansion holds for Borel weights (and also for weights deriving from suitable random walk methods). This leads to the following result.

Corollary 2. *With the notation introduced above, we have for the Euler or the Borel method with associated weighted sums T_n,* for $\gamma > 1$,

$$\mathrm{E}(|X|^\gamma) < \infty, \qquad \mathrm{E}(X) = 0 \Leftrightarrow \sum_{n=1}^{\infty} n^{(\gamma-3)/2} P(|T_n| > \varepsilon) < \infty, \quad \text{for all } \varepsilon > 0.$$

Proof. Decomposing $T_n = \sum p_{nk} X_k$ into the sums corresponding to the weights $p_{nk}^{(\nu)}$ and applying Theorem 4 to each of the summands shows the sufficiency of the moment condition. For example, for $\nu = 1$ we have $\alpha = \frac{1}{2}$ and $\phi(x) = \mathrm{e}^{-x^2/(2p(1-p))}/\sqrt{2\pi p(1-p)}$, and similarly for $\nu = 2$. For the remainder term ($\nu = 3$) we proceed as follows: Define

$$p_{nk}^* := n^{-1/2}\left(1 + \left|\frac{k-np}{\sqrt{np(1-p)}}\right|^3\right)^{-1}.$$

Then use Corollary 1 to conclude

$$P\left(\left|\sum_k p_{nk}^{(3)} X_k\right| > \varepsilon\right) \leqslant P\left(\sum_k |p_{nk}^{(3)}|\, |X_k| > \varepsilon\right) \leqslant P\left(\sum_k p_{nk}^*\, |X_k| > \varepsilon n^{1/2}\right)$$

$$\leqslant P\left(\left|\sum_k p_{nk}^* (|X_k| - E(|X_k|))\right| > \varepsilon n^{1/2}/2\right)$$

$$+ P\left(\left|\sum_k p_{nk}^*\, E(|X_k|)\right| > \varepsilon n^{1/2}/2\right)$$

$$\leqslant P\left(\left|\sum_k p_{nk}^* (|X_k| - E(|X_k|))\right| > \varepsilon n^{1/2}/2\right) + 0,$$

for n large enough. Then apply Theorem 4 again. Note that the shift in the argument of the function $\phi(\cdot)$ does not matter for i.i.d. random variables.

Necessity follows with the help of (4.1), Corollary 1 (set weights to zero for

$|k - n\mu| > \sqrt{n}$), then Lemma 3 with $m_n = c\sqrt{n}$ with some $c > 0$, and finally Lemma 4 with $\psi(x) = c_\varepsilon x^{1/2}$. □

Remarks. We first observe that a similar theorem can be proved for $\gamma \in (0, 1]$ using a higher-order Edgeworth expansion, and also for other random walk methods.

Secondly, it is well known that $X_n \xrightarrow{\text{a.s.}} \mu(C_1)$ if and only if $E(|X|) < \infty$ and $\mathrm{E}(X) = \mu$, but $X_n \xrightarrow{\text{a.s.}} \mu(E_p)$ if and only if $\mathrm{E}(X^2) < \infty$ and $\mathrm{E}(X) = \mu$ (for $0 < p < 1$). For complete convergence we have in the case of the C_1 mean that the existence of the second moment is necessary and sufficient (see Gut 1993, Theorem 2.1) and for the Euler or Borel means that the third moment exists. This follows from Corollary 2.

4.2. Riesz and related methods

Let $p > 1$ and define $\lambda_n = n^{1/p}\exp(n^{1-1/p})$ and $\phi(n) = n^{1/p}$, $n \in \mathbb{N}$. Then the following summability methods are closely related to the $C_{1/p}$ method: the Riesz method, R_p, where we say that $s_n \to s(R_p)$ if, with $a_k = s_k - s_{k-1}$, $k \in \mathbb{N}(s_0 := 0)$,

$$\frac{1}{\lambda_n}\sum_{k=1}^{n}(\lambda_n - \lambda_k)a_k \to s, \qquad n \to \infty;$$

and the moving average method, M_p, where we say that $s_n \to s(M_p)$ if

$$\frac{1}{u\phi_n}\sum_{n<k\leqslant n+u\phi(n)} s_k \to s, \qquad n \to \infty, \text{ for all } u > 0.$$

It is well known that the R_p and M_p methods are equivalent (Bingham and Goldie 1988) and are both weaker than the Cesàro method $C_{1/p}$ (Jurkat *et al*. 1975). However, applied to i.i.d. random variables for fixed p, all these methods are equivalent – for example,

$$X_k \xrightarrow{\text{a.s.}} \mu(R_p, M_p, C_{1/p}) \Leftrightarrow \mathrm{E}(|X|^p), \mathrm{E}(X) = \mu$$

holds (Bingham and Goldie 1988).

What can be said about Baum–Katz type results? Note that the following statements are equivalent:

$$\sum_{n=1}^{\infty} n^\rho P\left(\left|\frac{1}{u\phi(n)}\sum_{n<k\leqslant n+u\phi(n)} X_k\right| > \varepsilon n^\beta\right) < \infty, \qquad \text{for all } u,\ \varepsilon > 0;$$

$$\sum_{\ell=1}^{\infty} \ell^{p(\rho+1)-1} P\left(\left|\frac{1}{\ell}\sum_{k=1}^{\ell} X_k\right| > \varepsilon \ell^{p\beta}\right) < \infty, \qquad \text{for all } \varepsilon > 0.$$

Hence the M_ϕ case can be embedded in the usual arithmetic mean case. If $\rho = \gamma(\beta + 1/p) - 1 - 1/p$, we can use Theorem 1 to deal with the second statement. For the Riesz method we have:

Theorem 5. *Let be given* $p > 1$, $1/(2p) < \zeta \leqslant 1/p$ *and* $\gamma > 1/(p\zeta)(\geqslant 1)$ *and define* $X_0 = 0$. *Then the following statements are equivalent:*

(i) $\mathrm{E}(|X|^\gamma) < \infty$, $\mathrm{E}(X) = 0$,

(ii) $$\sum_{n=1}^{\infty} n^{\zeta\gamma-1-1/p} P\left(\left|\frac{1}{\exp(n^{1-1/p})}\sum_{k=1}^{n}(\lambda_n - \lambda_k)(X_k - X_{k-1})\right| > n^{\zeta}\right) < \infty,$$

(iii) $$\sum_{n=1}^{\infty} n^{\zeta\gamma-1-1/p} P\left(\left|\sum_{k=1}^{n}\exp(k^{1-1/p})X_k\right| > \exp(n^{1-1/p})n^{\zeta}\right) < \infty.$$

Note that (iii) cannot be treated with Theorem 1 since condition (2.1) is not satisfied. A corresponding result holds for $\zeta > 1/p$, but then condition (i) should read $\mathrm{E}(|X|^\gamma) < \infty$ only.

Proof. Comparing (i) with (iii), one uses the same arguments as in the proofs of Theorem 2 and 3. That (ii) and (iii) are equivalent follows with Abel's partial summation and Corollary 1. □

Corollary 3. *Let* $p > 1$. *A sequence* (X_n) *of i.i.d. random variables converges completely in the sense of:*

(i) *the* R_p *or the* M_p *method if and only if* $\mathrm{E}(|X_1|^{p+1}) < \infty$*;*

(ii) *the Cesàro method* $C_{1/p}$ if and only if $\mathrm{E}(|X_1|^{\max\{2,p\}}) < \infty$ *for* $p \neq 2$ *and* $\mathrm{E}(|X_1|^2 \log^+|X_1|) < \infty$ *for* $p = 2$*, respectively.*

Proof. Use Theorem 1.2 in Gut (1993) for part (ii). For (i) use Theorem 5 above with $\zeta = 1/p$ for the Riesz means and the equivalence above Theorem 5 with $\rho = \gamma(\beta + 1/p) - 1/p - 1$ together with Theorem 2 for the moving average case. □

Hence we may not distinguish between the Riesz and the moving average method, but both can be distinguished from the Cesàro method with the help of a Baum–Katz theorem.

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