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Central limit theorems for partial sums of bounded functionals of infinite-variance moving averages

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For $j = 1, \ldots, J$, let $K_j : \mathbb{R} \to \mathbb{R}$ be measurable bounded functions and $X_{j,n} = \int_{\mathbb{R}} a_j(n - c_j x) M(\mathrm{d}x)$, $n \geqslant 1$, be α-stable moving averages where $\alpha \in (0, 2)$, $c_j > 0$ for $j = 1, \ldots, J$, and $M(\mathrm{d}x)$ is an α-stable random measure on $\mathbb{R}$ with the Lebesgue control measure and skewness intensity $\beta \in [-1, 1]$. We provide conditions on the functions a_j and K_j, $j = 1, \ldots, J$, for the normalized partial sums vector $N_j^{-1/2} \sum_{n=1}^{N_j} (K_j(X_{j,n}) - \mathrm{E}K_j(X_{j,n}))$, $j = 1, \ldots, J$, to be asymptotically normal as $N_j \to \infty$. This extends a result established by Tailen Hsing in the context of causal moving averages with discrete-time stable innovations. We also consider the case of moving averages with innovations that are in the stable domain of attraction.

Keywords: central limit theorem; moving averages; stable distributions

1. Introduction

Our goal is to extend Hsing's (1999) result on the convergence of bounded functionals of infinite-variance moving averages. This extension is used in Pipiras *et al*. (2003) to establish asymptotic normality of some wavelet-based estimators in linear fractional stable motion.

We first recall Hsing's result and then describe our extension. Hsing (1999) considered moving average sequences

$$X_n = \sum_{j=1}^{\infty} a_j \epsilon_{n-j}, \qquad n \geqslant 1, \tag{1.1}$$

where $\{a_j\}_{j \geqslant 1}$ is a sequence of weights and $\{\epsilon_n\}_{n \in \mathbb{Z}}$ is a sequence of independent and identically distributed (i.i.d.) symmetric α-stable standard random variables with $\alpha \in (0, 2)$. Recall that a random variable ϵ is α-stable with $\alpha \in (0, 2)$ if its characteristic function has the form

$$\mathrm{E}\exp\{\mathrm{i}\theta\epsilon\} = \begin{cases} \exp\{-\sigma^{\alpha}|\theta|^{\alpha}(1 - \mathrm{i}\beta \mathrm{sign}(\theta)\tan(\alpha\pi/2)) + \mathrm{i}\mu\theta\} & \text{if } \alpha \neq 1, \\ \exp\{-\sigma|\theta|(1 + \mathrm{i}\beta(2/\pi)\mathrm{sign}(\theta)\ln|\theta|) + \mathrm{i}\mu\theta\} & \text{if } \alpha = 1 \end{cases}$$

where $\theta \in \mathbb{R}$, $\sigma > 0$ is a scale parameter, $\mu \in \mathbb{R}$ is a shift parameter and $\beta \in [-1, 1]$ is a skewness parameter. It is called symmetric (SαS, in short) if $\beta = 0$ and $\mu = 0$, and standard

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if $\sigma = 1$. Hsing (1999) obtained conditions on the weight sequence $\{a_j\}_{j\geq 1}$ for the normalized partial sums

$$\frac{S_N}{N^{1/2}} := \frac{1}{N^{1/2}} \sum_{n=1}^{N} (K(X_n) - \mathrm{E}K(X_n)), \tag{1.2}$$

where K is a bounded function, to converge in distribution to a Gaussian law, as $N \to \infty$. His result (Hsing 1999, Theorem 1) is stated below. The following notation is used:

$$X_{n,1,l} = \sum_{j=1}^{l} a_j \epsilon_{n-j}, \qquad l \geq 1,$$

and

$$S_{N,l} = \sum_{n=1}^{N} (K(X_{n,1,l}) - \mathrm{E}K(X_{n,1,l})), \qquad l \geq 1.$$

Theorem 1.1. *Let* $\alpha \in (0, 2)$ *and* $\{X_n\}_{n\geq 1}$ *be a* SαS *moving average sequence defined by* (1.1). *Suppose that* K *in* (1.2) *is a bounded function. Suppose also that*

$$\sum_{j=1}^{\infty} |a_j|^{\alpha/2} < \infty \tag{1.3}$$

and

$$\lim_{l\to\infty} \mathrm{E}(K(X_1) - K(X_{1,1,l}))^2 = 0. \tag{1.4}$$

Then

$$\lim_{l\to\infty} \limsup_{N\to\infty} N^{-1} \operatorname{var}(S_N - S_{N,l}) = 0$$

and

$$N^{-1/2} S_N \xrightarrow{d} \mathcal{N}(0, \sigma^2),$$

where

$$\sigma^2 = \lim_{N\to\infty} N^{-1} \operatorname{var}(S_N) = \lim_{l\to\infty} \lim_{N\to\infty} N^{-1} \operatorname{var}(S_{N,l}).$$

We extend this result in the following ways:

- We drop condition (1.4).
- We replace the discrete noise ϵ_n by a (more general) continuous-time one.
- We consider both symmetric and skewed noise.
- We consider two-sided moving averages, since in the stable case, these are not equivalent to one-sided ones.

- We develop a multivariate extension which will be used in Pipiras *et al*. (2003) to prove the asymptotic normality of wavelet-based estimators.
- Assuming that K is smooth, we show that the result extends to discrete-time moving averages with innovations that are in the domain of attraction of an α-stable distribution.

These extensions of Theorem 1.1 are formulated and proved in Sections 2 and 3 below.

2. Results

Consider α-stable moving average sequences $\{X_{j,n}\}_{n\geqslant 1}$, $j = 1, \ldots, J$, given by

$$X_{j,n} = \int_{\mathbb{R}} a_j(n - c_j x) M(\mathrm{d}x), \tag{2.1}$$

where $\alpha \in (0, 2)$, $c_j > 0$, $a_j \in L^\alpha(\mathbb{R}, \mathrm{d}x)$ and, in addition, $a_j \ln|a_j| \in L^1(\mathbb{R}, \mathrm{d}x)$ when $\alpha = 1$, and M is an α-stable random measure on $\mathbb{R}$ with the Lebesgue control measure $m(\mathrm{d}x) = \mathrm{d}x$ and the skewness intensity $\beta(x) \equiv \beta \in [-1, 1]$. Heuristically, $M(\mathrm{d}x)$, $x \in \mathbb{R}$, can be viewed as a sequence of independent α-stable random variables with the scale parameter $\mathrm{d}x$ and the skewness parameter β. The representation (2.1) means that the characteristic function of $\{X_{j,n}, j = 1, \ldots, J\}_{n\geqslant 1}$ can be expressed as

$$\mathrm{E}\exp\left\{\mathrm{i}\sum_{p=1}^{q}\theta_p X_{j_p,n_p}\right\} = \exp\left\{-\int_{\mathbb{R}}\left|f(\boldsymbol{\theta}, \mathbf{n}, \mathbf{j}, x)\right|^\alpha\left(1 - \mathrm{i}\beta\,\mathrm{sign}(f(\boldsymbol{\theta}, \mathbf{n}, \mathbf{j}, x))\tan\frac{\alpha\pi}{2}\right)\mathrm{d}x\right\}$$

if $\alpha \neq 1$, and

$$\mathrm{E}\exp\left\{\mathrm{i}\sum_{p=1}^{q}\theta_p X_{j_p,n_p}\right\}$$
$$= \exp\left\{-\int_{\mathbb{R}}\left|f(\boldsymbol{\theta}, \mathbf{n}, \mathbf{j}, x)\right|\left(1 + \mathrm{i}\beta\frac{2}{\pi}\,\mathrm{sign}(f(\boldsymbol{\theta}, \mathbf{n}, \mathbf{j}, x))\ln\left|f(\boldsymbol{\theta}, \mathbf{n}, \mathbf{j}, x)\right|\right)\mathrm{d}x\right\}$$

if $\alpha = 1$, where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_q) \in \mathbb{R}^q$, $\mathbf{n} = (n_1, \ldots, n_q) \in \mathbb{N}^q$, $\mathbf{j} = (j_1, \ldots, j_q) \in \{1, \ldots, J\}^q$ and

$$f(\boldsymbol{\theta}, \mathbf{n}, \mathbf{j}, x) = \sum_{p=1}^{q}\theta_p a_{j_p}(n_p - c_{j_p}x), \qquad x \in \mathbb{R}.$$

For more information on stable measures and stable processes, see Samorodnitsky and Taqqu (1994). In the wavelet setting considered in Pipiras *et al*. (2003), the indices c_j (usually taken equal to 2^{-j}) correspond to 'scale' and n to 'shift'.

Definition 2.1. *We will say that the moving average* $\{X_{j,n}\}_{n\geqslant 1}$ *is* causal *if* $a_j(x) = 0$ *for* $x < x_0$, *and* non-causal *if* $a_j(x) = 0$ *for* $x > x_0$, *where* $x_0 \in \mathbb{R}$. *When* $\{X_{j,n}\}_{n\geqslant 1}$ *is either causal or non-causal, we will say that it is* one-sided. *We will also call the moving average* $\{X_{j,n}\}_{n\geqslant 1}$ two-sided *if it is not one-sided.*

Now, fix n_j, $j = 1, \ldots, J$, and let N_j be positive integers such that

$$N_j \sim \frac{N}{n_j}, \tag{2.2}$$

as $N \to \infty$. For $j = 1, \ldots, J$, set

$$\frac{S_{j,N_j}}{N_j^{1/2}} = \frac{1}{N_j^{1/2}} \sum_{n=1}^{N_j} (K_j(X_{j,n}) - \mathrm{E}K_j(X_{j,n})), \tag{2.3}$$

where K_j is some measurable function. For $j = 1, \ldots, J$ and $n \geqslant 1$, define also the truncated integrals

$$X_{j,n,l_1,l_2} = \int_{c_j^{-1}(n-l_2)}^{c_j^{-1}(n-l_1+1)} a_j(n - c_j x) M(\mathrm{d}x), \qquad -\infty \leqslant l_1 \leqslant l_2 \leqslant \infty, \tag{2.4}$$

and $X_{j,n,l_1,l_2} = 0$ for $l_1 > l_2$. This ordering of the indices is motivated by the fact that, after a change of variables,

$$X_{j,n,l_1,l_2} = \int_{l_1-1}^{l_2} a_j(z) M\Big(\mathrm{d}(c_j^{-1}(n-z))\Big), \tag{2.5}$$

so the indices $l_1 - 1$ and l_2 refer to the range of the weights $a_j(z)$. It is best henceforth to view X_{j,n,l_1,l_2} as represented by (2.5); for example, one has

$$X_{j,n} = X_{j,n,-\infty,m-1} + X_{j,n,m,\infty}, \qquad \text{for all } m \in \mathbb{Z}. \tag{2.6}$$

Define also the corresponding partial sums

$$S_{j,N_j,l_1,l_2} = \sum_{n=1}^{N_j} (K_j(X_{j,n,l_1,l_2}) - \mathrm{E}K_j(X_{j,n,l_1,l_2})), \qquad -\infty \leqslant l_1 \leqslant l_2 \leqslant \infty.$$

The following result is our first extension of Hsing's Theorem 1.1. It is proved in Section 3 below.

Theorem 2.1. *Let* $\alpha \in (0, 2)$ *and* $\{X_{j,n}\}_{n\geqslant 1}$, $j = 1, \ldots, J$, *be* α*-stable moving averages defined by* (2.1). *Suppose that, for each* $j = 1, \ldots, J$, *the kernel* a_j *in* (2.1) *satisfies the condition*

$$\sum_{m=-\infty}^{\infty} \left(\int_{m-1}^{m} |a_j(x)|^\alpha \,\mathrm{d}x \right)^{1/2} < \infty. \tag{2.7}$$

Suppose also that, for each $j = 1, \ldots, J$, *the function* K_j *in* (2.3) *is bounded if* $\{X_{j,n}\}_{n\geqslant 1}$ *is one-sided, and is bounded and twice differentiable with bounded derivatives if* $\{X_{j,n}\}_{n\geqslant 1}$ *is two-sided. Then, for* $j = 1, \ldots, J$,

$$\lim_{(l_1,l_2)\to(-\infty,\infty)} \limsup_{N\to\infty} N_j^{-1} \operatorname{var}(S_{j,N_j} - S_{j,N_j,l_1,l_2}) = 0 \tag{2.8}$$

and

$$\left(N_j^{-1/2} S_{j,N_j}\right)_{j=1}^{J} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \boldsymbol{\sigma}). \tag{2.9}$$

The entries of the covariance matrix $\boldsymbol{\sigma} = (\sigma_{jk})_{j,k=1,\ldots,J}$ *can be obtained as*

$$\sigma_{jk} = \lim_{N\to\infty} \mathrm{E}\frac{S_{j,N_j}}{N_j^{1/2}}\frac{S_{k,N_k}}{N_k^{1/2}} \tag{2.10}$$

$$= \lim_{(l_1,l_2)\to(-\infty,\infty)} \lim_{N\to\infty} \mathrm{E}\frac{S_{j,N_j,l_1,l_2}}{N_j^{1/2}}\frac{S_{k,N_k,l_1,l_2}}{N_k^{1/2}} < \infty. \tag{2.11}$$

If $J = 1$, $c_1 = 1$, $\beta = 0$ and $a_1(x) = \sum_{k=1}^{\infty} a_k 1_{[k-1,k)}(x)$, $x \in \mathbb{R}$, $a_k \in \mathbb{R}$ in (2.1), then $X_{1,n} = \int_{\mathbb{R}} a(n-x)M(\mathrm{d}x) = \sum_{k=1}^{\infty} a_k \epsilon_{n-k}$ for some sequence $\{\epsilon_k\}$ of i.i.d. SαS random variables. Condition (2.7) becomes $\sum_{k=1}^{\infty} |a_k|^{\alpha/2} < \infty$, which is (1.3) in Hsing's Theorem 1.1 above. Observe, however, that Theorem 1.1 also requires condition (1.4). This condition is missing in Theorem 2.1 because, in fact, it *always* holds (and it can therefore be removed from Theorem 1.1). This is shown by the following results which are also used in the proof of Theorem 2.1.

Lemma 2.1. *For* $\alpha \in (0, 2)$*, let* X_n*,* $n \geqslant 0$*, be* α*-stable random variables such that* $X_n \to X_0$ *almost surely. Then, for any bounded measurable function K,*

$$\lim_{n\to\infty} \mathrm{E}(K(X_0) - K(X_n))^2 = 0. \tag{2.12}$$

Proof. Let f_n and ϕ_n, $n \geqslant 0$, be the density and characteristic functions of X_n, respectively. For any $\epsilon > 0$, there is a bounded and continuous function $\tilde{K}$ such that

$$\mathrm{E}(K(X_0) - \tilde{K}(X_0))^2 = \int_{\mathbb{R}} (K(x) - \tilde{K}(x))^2 f_0(x)\mathrm{d}x < \epsilon.$$

Indeed, $K(x)$ can be approximated uniformly by a continuous function on a compact interval, and this interval can be chosen large enough so that the measure of its complement is arbitrarily small. By using the Fourier inversion formula, we have

$$f_n(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}tx}\phi_n(t)\mathrm{d}t, \qquad x \in \mathbb{R},\ n = 0, 1, \ldots. \tag{2.13}$$

Since $X_n \to X_0$ a.s., we have $X_n \to X_0$ in distribution. Hence, $\phi_n(t) \to \phi_0(t)$, $t \in \mathbb{R}$, and $\gamma_n \to \gamma_0$, $\beta_n \to \beta_0$ and $\sigma_n \to \sigma_0$, where γ_n, β_n and σ_n, $n \geqslant 0$, are shift, skewness and scale parameters of X_n. Since $\sup_{n\geqslant 0}|\phi_n(t)| \leqslant \exp\{-C|t|^{\alpha}\}$, where C depends on γ_0, β_0 and σ_0 only, it follows from (2.13) that $f_n(x) \to f_0(x)$ for $x \in \mathbb{R}$. Then, by Scheffé's theorem (see, for example, Billingsley 1995),

$$\int_{\mathbb{R}} |f_0(x) - f_n(x)|\mathrm{d}x \to 0. \tag{2.14}$$

By using (2.14) and since K and $\tilde{K}$ are bounded functions, we have that, for large enough n,

$$\mathrm{E}(K(X_n) - \tilde{K}(X_n))^2 = \int_{\mathbb{R}} (K(x) - \tilde{K}(x))^2 f_n(x)\mathrm{d}x$$
$$\leqslant C\int_{\mathbb{R}} |f_0(x) - f_n(x)|\mathrm{d}x + \int_{\mathbb{R}} (K(x) - \tilde{K}(x))^2 f_0(x)\mathrm{d}x < \epsilon.$$

Then, by using the inequality $(a+b+c)^2 \leqslant 3a^2 + 3b^2 + 3c^2$ and the decomposition $K(X_0) - K(X_n) = K(X_0) - \tilde{K}(X_0) + (\tilde{K}(X_0) - \tilde{K}(X_n)) + \tilde{K}(X_n) - K(X_n)$, we obtain that, for large enough n,

$$\mathrm{E}(K(X_0) - K(X_n))^2 \leqslant 6\epsilon + 3\mathrm{E}(\tilde{K}(X_0) - \tilde{K}(X_n))^2.$$

Since $X_n \to X_0$ a.s. and the function $\tilde{K}$ is bounded and continuous, the dominated convergence theorem implies that $\mathrm{E}(\tilde{K}(X_0) - \tilde{K}(X_n))^2 \to 0$. The conclusion follows because $\epsilon > 0$ is arbitrarily small. □

Corollary 2.1. *Condition* (1.4) *in Theorem 1.1 always holds.*

Proof. This follows from Lemma 2.1 since, by Kolmogorov's three-series theorem, $X_{1,1,l} \to X_1$ a.s. as $l \to \infty$. □

In the following result, we extend Theorem 1.1 to two-sided moving averages with innovations that are in the domain of attraction of an α-stable distribution, $\alpha \in (0, 2)$, if K is bounded and is twice differentiable with bounded derivatives. We will assume that the innovations ϵ_j satisfy the assumption

$$L_1(z) := z^\alpha P(\epsilon_j \leqslant -z) \sim c_- L(z), \tag{2.15}$$

$$L_2(z) := z^\alpha P(\epsilon_j \geqslant z) \sim c_+ L(z), \tag{2.16}$$

as $z \to \infty$, where $c_-,\ c_+ \geqslant 0$, $c_- + c_+ > 0$ and L is a slowly varying function at infinity (for definition, see, for example, Bingham *et al*. 1987). The function L, for example, can behave like a constant or a logarithm for large z. When $1 < \alpha < 2$, we will suppose that $\mathrm{E}\epsilon_j = 0$. We will also use the notation

$$X_{n,l_1,l_2} = \sum_{j=l_1}^{l_2} a_j \epsilon_{n-j}, \qquad l_1 \leqslant l_2,$$

$$S_{N,l_1,l_2} = \sum_{n=1}^{N} (K(X_{n,l_1,l_2}) - \mathrm{E}K(X_{n,l_1,l_2})), \qquad l_1 \leqslant l_2.$$

Theorem 2.2. *Let*

$$X_n = \sum_{j=-\infty}^{\infty} a_j \epsilon_{n-j}, \qquad n \geqslant 1,$$

be a moving average with a sequence of i.i.d. innovations $\{\epsilon_j\}$ *satisfying assumptions above with* $\alpha \in (0, 2)$, *and let* S_N, $N \geqslant 1$, *be the partial sums defined by* (1.2). *Suppose that*

$$\sum_{j=-\infty}^{\infty} |a_j|^{\alpha/2}\left(L\left(\frac{1}{|a_j|}\right)\right)^{1/2} < \infty, \qquad \textit{when } \alpha \neq 1, \tag{2.17}$$

and

$$\sum_{j=-\infty}^{\infty} |a_j|^{1/2}\left(L\left(\frac{1}{|a_j|}\right)\right)^{1/2} + \sum_{j=-\infty}^{\infty} |a_j|^{1/2}\left|\gamma + H\left(\frac{1}{|a_j|}\right)\right|^{1/2} < \infty, \qquad \textit{when } \alpha = 1, \tag{2.18}$$

where H *and* γ *are defined in* (2.20) *and* (2.21) *below, and that* K *in* (1.2) *is a bounded function with its first two derivatives bounded. Then*

$$\lim_{(l_1,l_2)\to(-\infty,\infty)} \limsup_{N\to\infty} N^{-1}\operatorname{var}(S_N - S_{N,l_1,l_2}) = 0 \tag{2.19}$$

and

$$N^{-1/2}S_N \xrightarrow{d} \mathcal{N}(0, \sigma^2),$$

where

$$\sigma^2 = \lim_{N\to\infty} N^{-1}\operatorname{var}(S_N) = \lim_{(l_1,l_2)\to(-\infty,\infty)} \lim_{N\to\infty} N^{-1}\operatorname{var}(S_{N,l_1,l_2}).$$

The function H and the constant γ in (2.18) of Theorem 2.2 are defined as

$$H(z) = \int_0^z \frac{xL_1(x)}{1+x^2}\,\mathrm{d}x - \int_0^z \frac{xL_2(x)}{1+x^2}\,\mathrm{d}x \tag{2.20}$$

and

$$\gamma = \int_{\mathbb{R}} \left(\frac{x}{1+x^2} + \operatorname{sign}(x)\int_0^{|x|} \frac{2u^2}{(1+u^2)^2}\,\mathrm{d}u\right) G(\mathrm{d}x), \tag{2.21}$$

where G denotes the distribution function of ϵ_j. They appear in the representation of a characteristic function of a random variable in the domain of attraction of a 1-stable random variable (see Aaronson and Denker 1998). Observe also that $H \equiv 0$ and $\gamma = 0$ when ϵ_j (or its distribution function G) is symmetric.

Remark 2.1. Conditions (2.17) and (2.18) are equivalent to

$$\sum_{j=-\infty}^{\infty} |a_j|^{\alpha/2}\left(\tilde{L}(1/|a_j|)\right)^{1/2} < \infty,$$

where

$$\tilde{L}(z) = \begin{cases} L(z) & \text{if } \alpha \neq 1, \\ L(z) + |H(z) + \gamma| & \text{if } \alpha = 1. \end{cases} \tag{2.22}$$

The function $\tilde{L}$ is slowly varying at infinity because the functions H (see Lemma 3 in Aaronson and Denker 1998) and L are slowly varying at infinity.

Remark 2.2. When $1 < \alpha < 2$, suppose $\mu = \mathrm{E}\epsilon_j \neq 0$ and $\sum_j |a_j| < \infty$ so that the process X_n is well defined. Theorem 2.2 continues to hold since one can replace $K(x)$ by $K_1(x) = K(x - \mu \sum_j a_j)$.

3. Proofs

The following three elementary lemmas play an important role. The first lemma is implicit in Hsing (1999) and the second and third lemmas amplify Lemma 3 in Hsing (1999). We will use the notation $f^{(j)} = \mathrm{d}^j f / \mathrm{d}x^j$ for the jth derivative of a function f.

Lemma 3.1. *Let X, Y be two random variables such that* $\mathrm{E}X^2 < \infty$ *and* $\mathrm{E}Y^2 < \infty$. *Also let* $\{\mathcal{F}_n\}_{n \in \mathbb{Z}}$ *be a monotone sequence of* σ*-algebras, that is, for* $n_1 \leqslant n_2$, *either* $\mathcal{F}_{n_1} \subset \mathcal{F}_{n_2}$ *or* $\mathcal{F}_{n_1} \supset \mathcal{F}_{n_2}$. *Then, for all* $n, m \in \mathbb{Z}$ *such that* $n \neq m$, *we have*

$$\mathrm{E}\{(\mathrm{E}(X|\mathcal{F}_n) - \mathrm{E}(X|\mathcal{F}_{n-1}))(\mathrm{E}(Y|\mathcal{F}_m) - \mathrm{E}(Y|\mathcal{F}_{m-1}))\} = 0. \tag{3.1}$$

Proof. Consider the case when $\mathcal{F}_{n_1} \subset \mathcal{F}_{n_2}$ for $n_1 \leqslant n_2$ and take, for example, $n < m$. Denote the left-hand side of (3.1) by I. Since the random variable $\mathrm{E}(X|\mathcal{F}_n) - \mathrm{E}(X|\mathcal{F}_{n-1})$ is $\mathcal{F}_n$-measurable and $n < m$, it is also measurable with respect to $\mathcal{F}_{m-1}$ and $\mathcal{F}_m$. Then, by the definition of conditional expectation,

$$\begin{aligned} I &= \mathrm{E}\{(\mathrm{E}(X|\mathcal{F}_n) - \mathrm{E}(X|\mathcal{F}_{n-1}))\mathrm{E}(Y|\mathcal{F}_m)\} - \mathrm{E}\{(\mathrm{E}(X|\mathcal{F}_n) - \mathrm{E}(X|\mathcal{F}_{n-1}))\mathrm{E}(Y|\mathcal{F}_{m-1})\} \\ &= \mathrm{E}\{(\mathrm{E}(X|\mathcal{F}_n) - \mathrm{E}(X|\mathcal{F}_{n-1}))Y\} - \mathrm{E}\{(\mathrm{E}(X|\mathcal{F}_n) - \mathrm{E}(X|\mathcal{F}_{n-1}))Y\} = 0. \end{aligned}$$

□

Lemma 3.2. *Let* $g(x) = \mathrm{E}G(x + X)$, *where* $x \in \mathbb{R}$ *and X is an* α*-stable standard random variable with* $\alpha \in (0, 2)$, *scale parameter* $\sigma > 0$, *skewness parameter* $\beta \in [-1, 1]$ *and shift parameter* $\mu = 0$. *If G is a bounded function, then g is infinitely differentiable and, for all* $x \in \mathbb{R}$ *and* $j = 0, 1, \ldots,$

$$|g^{(j)}(x)| \leqslant \begin{cases} C_j \sigma^{-j} & \text{if } \alpha \neq 1, \\ C_j \sigma^{-j}(1 + |\ln \sigma| + |\ln \sigma|^2) & \text{if } \alpha = 1, \end{cases} \tag{3.2}$$

where the constant C_j *does not depend on* σ *and* β.

Proof. Relation (3.2) holds for $j = 0$ since G is bounded. Now suppose $j \geqslant 1$. We have $X = \sigma Z$, where Z is an α-stable random variable with scale parameter 1, skewness parameter β and shift parameter $\mu = 0$ when $\alpha \neq 1$, and $\mu = \beta(2/\pi)\ln \sigma$ when $\alpha = 1$. Let $f(z)$ and $\phi(z)$ be the density and characteristic functions of Z, respectively. By using the

Fourier inversion formula and the integration by parts formula twice, we can express $f^{(j)}$ as in Hsing (1999):

$$f^{(j)}(z) = \frac{(-\mathrm{i})^j}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}tz}\phi_j(t)\mathrm{d}t, \tag{3.3a}$$

$$f^{(j)}(z) = -\frac{(-\mathrm{i})^j}{2\pi z^2}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}tz}\phi_j^{(2)}(t)\mathrm{d}t, \tag{3.3b}$$

where $\phi_j(t) = t^j\phi(t),\ t \in \mathbb{R}$.

When $\alpha = 1$, relation (3.3a) implies that $|f^{(j)}(z)| \leqslant \tilde{C}_j$ for all z, where $\tilde{C}_j$ does not depend on σ and β. By computing $\phi_j^{(2)}$ and using relation (3.3b), we can conclude that $|f^{(j)}(z)| \leqslant \tilde{C}_j(1 + |\ln\sigma| + |\ln\sigma|^2)z^{-2}$. Hence, when $\alpha = 1$,

$$|f^{(j)}(z)| \leqslant \frac{C_j(1 + |\ln\sigma| + |\ln\sigma|^2)}{1+z^2}, \qquad z \in \mathbb{R}. \tag{3.4}$$

Similarly, in the case $\alpha \neq 1$, one obtains

$$|f^{(j)}(z)| \leqslant \frac{C_j}{1+z^2}, \qquad z \in \mathbb{R}, \tag{3.5}$$

where the constant C_j does not depend on β.

Since, by (3.4) and (3.5), $f^{(j)} \in L^1(\mathbb{R})$, we conclude as in Lemma 3 of Hsing (1999) that

$$\begin{aligned} g^{(j)}(x) &= \frac{\mathrm{d}^j}{\mathrm{d}x^j}\int_{\mathbb{R}} G(x+\sigma z)f(z)\mathrm{d}z \\ &= \frac{\mathrm{d}^j}{\mathrm{d}x^j}\int_{\mathbb{R}} G(y)f(\sigma^{-1}(y-x))\mathrm{d}y \\ &= \int_{\mathbb{R}} G(y)\frac{\mathrm{d}^j}{\mathrm{d}x^j}\big(f(\sigma^{-1}(y-x))\big)\mathrm{d}y \\ &= (-1)^j\sigma^{-j}\int_{\mathbb{R}} G(x+\sigma z)f^{(j)}(z)\mathrm{d}z, \end{aligned} \tag{3.6}$$

where in the last step we used the fact that

$$\frac{\mathrm{d}^j}{\mathrm{d}x^j}(f(\sigma^{-1}(y-x))) = (-1)^j\sigma^{-j}f^{(j)}(\sigma^{-1}(y-x))$$

and a change of variables. Inequality (3.2) follows from (3.6) by using (3.4) when $\alpha = 1$ and (3.5) when $\alpha \neq 1$, since G is bounded. □

Lemma 3.3. *Let* $h(x) = \mathrm{E}H(x+X)$, *where* $x \in \mathbb{R}$ *and* X *is a random variable. If* H *is bounded and differentiable up to order* r *with its derivatives bounded, then for all* $x \in \mathbb{R}$, *all random variables* X *and* $j = 0, 1, 2, \ldots, r$,

$$|h^{(j)}(x)| \leqslant C_j, \tag{3.7}$$

where $C_j = \sup_{x\in\mathbb{R}} |H^{(j)}(x)|$.

Proof. Let X be any random variable and F_X denote its distribution function. If H is a bounded and twice differentiable function with its first r derivatives bounded, then one can show by using relation (16) in Lemma 3 of Hsing (1999) that, for $j = 0, 1, \ldots, r$,

$$h^{(j)}(x) = \int_{\mathbb{R}} H^{(j)}(x+z)\mathrm{d}F_X(z).$$

Inequality (3.7) follows, since the functions $H^{(j)}$ are bounded. □

Proof of Theorem 2.1. We will consider the case $J = 2$ only. To show (2.9) with (2.10) and (2.11), it is then enough to verify that, for all fixed $b_1,\ b_2 \in \mathbb{R}$, the random variables $b_1 N_1^{-1/2} S_{1,N_1} + b_2 N_2^{-1/2} S_{2,N_2}$ converge in distribution to a Gaussian law as $N \to \infty$, whose variance can be expressed as

$$\lim_{N\to\infty} \mathrm{E}\left(b_1 \frac{S_{1,N_1}}{N_1^{1/2}} + b_2 \frac{S_{2,N_2}}{N_2^{1/2}}\right)^2 = \lim_{(l_1,l_2)\to(-\infty,\infty)} \lim_{N\to\infty} \mathrm{E}\left(b_1 \frac{S_{1,N_1,l_1,l_2}}{N_1^{1/2}} + b_2 \frac{S_{2,N_2,l_1,l_2}}{N_2^{1/2}}\right)^2.$$

We will do this as in the proof of Theorem 1 in Hsing (1999), by arguing first that, for all $-\infty \leqslant l_1 \leqslant l_2 \leqslant \infty$,

$$b_1 \frac{S_{1,N_1,l_1,l_2}}{N_1^{1/2}} + b_2 \frac{S_{2,N_2,l_1,l_2}}{N_2^{1/2}} \xrightarrow{d} \mathcal{N}(0, \sigma^2_{l_1,l_2}) \tag{3.8}$$

with

$$\sigma^2_{l_1,l_2} = \lim_{N\to\infty} \mathrm{E}\left(b_1 \frac{S_{1,N_1,l_1,l_2}}{N_1^{1/2}} + b_2 \frac{S_{2,N_2,l_1,l_2}}{N_2^{1/2}}\right)^2, \tag{3.9}$$

and then verifying that, for $j = 1, 2$, the limit relation (2.8) holds.

To prove (3.8) with (3.9), suppose that $n_1 \leqslant n_2$ in (2.2) and assume for simplicity that $N_j = N/n_j$ and $\mathrm{E}K_j(X_{j,1,l_1,l_2}) = 0$ for $j = 1, 2$. Since $N_2 \leqslant N_1$, the sequence in (3.8) can be written as

$$N_2^{-1/2} \sum_{n=1}^{N_2} \left(\frac{b_1 n_1^{1/2}}{n_2^{1/2}} K_1(X_{1,n,l_1,l_2}) + b_2 K_2(X_{2,n,l_1,l_2}) \right) + N_1^{-1/2} \sum_{n=N_2+1}^{N_1} b_1 K_1(X_{1,n,l_1,l_2}). \tag{3.10}$$

Now consider the random variables X_{j_1,n,l_1,l_2} and X_{j_2,n',l_1,l_2} in (2.4), where $j_1, j_2 = 1, \ldots, J$. These are independent when $n - n' > n_0$ for some fixed large enough n_0 because the corresponding kernel functions $a_{j_1}(n - c_{j_1}x)$ for $c_{j_1}^{-1}(n - l_2) \leqslant x < c_{j_1}^{-1}(n - l_1 + 1)$ and $a_{j_2}(n' - c_{j_2}x)$ for $c_{j_2}^{-1}(n' - l_2) \leqslant x < c_{j_2}^{-1}(n' - l_1 + 1)$ have disjoint supports (see Samorodnitsky and Taqqu 1994, Theorem 3.5.3). It follows, using the so-called 'm-dependent central limit theorem', that each of the two terms in (3.10) converges in distribution to a Gaussian law with the corresponding limiting variances. By summing the second term from

N_2' to N_1 where the difference $N_2' - N_2$ is large enough but of fixed length, we observe that these two terms are also asymptotically independent. This then implies the convergence (3.8) with (3.9).

We show (2.8) with, for example, $j = 1$. The proof below involves the random sequence X_{1,n,m_1,m_2}, $n \geq 1$, $-\infty \leq m_1 \leq m_2 \leq \infty$. Since

$$\{X_{1,n,m_1,m_2}\}_{n,m_1,m_2} \stackrel{d}{=} \left\{ c_1^{-1/\alpha} \int_{n-m_2}^{n-m_1+1} a_1(n-x)M(\mathrm{d}x) \right\}_{n,m_1,m_2}, \tag{3.11}$$

after a change of variables, we can suppose without loss of generality that $c_1 = 1$ in (2.4). We will also assume for simplicity that $n_1 = 1$, $N_1 = N$ and use the notation $a_1 = a$, $K_1 = K$, $S_{1,N} = S_N$, $X_{1,n} = X_n$,

$$X_{1,n,m_1,m_2} = X_{n,m_1,m_2} = \int_{n-m_2}^{n-m_1+1} a(n-x)M(\mathrm{d}x)$$

and

$$S_{1,N,l_1,l_2} = S_{N,l_1,l_2} = \sum_{n=1}^{N} (K(X_{n,l_1,l_2}) - \mathrm{E}K(X_{n,l_1,l_2})), \qquad -\infty \leq l_1 \leq l_2 \leq \infty.$$

Recall that our goal is to establish

$$\lim_{(l_1,l_2)\to(-\infty,\infty)} \limsup_{N\to\infty} N^{-1} \operatorname{var}(S_N - S_{N,l_1,l_2}) = 0. \tag{3.12}$$

We will prove (3.12) for causal moving averages, non-causal moving averages and two-sided moving averages separately.

Causal moving averages. We assume for simplicity that $x_0 = 0$ in Definition 2.1 of a causal moving average X_n, that is, $a(x) = 0$ for $x < 0$. We may also suppose without loss of generality that $\int_0^1 |a(x)|^\alpha \, \mathrm{d}x \neq 0$. We need to establish (3.12), which now becomes

$$\lim_{l\to\infty} \limsup_{N\to\infty} N^{-1} \operatorname{var}(S_{N,1,\infty} - S_{N,1,l}) = 0. \tag{3.13}$$

The proof uses ideas due to Hsing (1999). Let $\mathcal{F}_{k-1}$, $k \in \mathbb{Z}$, be σ-algebras generated by the 'causal' random variables $\int_{-\infty}^{k} f(x)M(\mathrm{d}x)$, $f \in L^\alpha(\mathbb{R}, \mathrm{d}x)$. Then, for instance, $X_{n,1,l} = \int_{n-l}^{n} a(n-x)M(\mathrm{d}x)$, $l \geq 1$, is measurable with respect $\mathcal{F}_{n-1}$. Using the relations

$$\mathrm{E}(K(X_{n,1,\infty})|\mathcal{F}_{n-1}) = K(X_{n,1,\infty}),$$

$$\mathrm{E}(K(X_{n,1,\infty})|\mathcal{F}_{-\infty}) = \mathrm{E}K(X_{n,1,\infty})$$

and, for any $l \geq 1$,

$$\mathrm{E}(K(X_{n,1,l})|\mathcal{F}_{n-1}) = K(X_{n,1,l}),$$

$$\mathrm{E}(K(X_{n,1,l})|\mathcal{F}_{n-(l+1)}) = \mathrm{E}K(X_{n,1,l}),$$

we can express

$$S_{N,1,\infty} = \sum_{n=1}^{N} (K(X_{n,1,\infty}) - \mathrm{E}K(X_{n,1,\infty}))$$

and

$$S_{N,1,l} = \sum_{n=1}^{N} (K(X_{n,1,l}) - \mathrm{E}K(X_{n,1,l}))$$

as telescoping sums,

$$S_{N,1,\infty} = \sum_{n=1}^{N} \sum_{m=1}^{\infty} \big(\mathrm{E}(K(X_{n,1,\infty})|\mathcal{F}_{n-m}) - \mathrm{E}(K(X_{n,1,\infty})|\mathcal{F}_{n-(m+1)})\big),$$

$$S_{N,1,l} = \sum_{n=1}^{N} \sum_{m=1}^{l} \big(\mathrm{E}(K(X_{n,1,l})|\mathcal{F}_{n-m}) - \mathrm{E}(K(X_{n,1,l})|\mathcal{F}_{n-(m+1)})\big).$$

We obtain

$$S_{N,1,\infty} - S_{N,1,l} = \sum_{n=1}^{N} \sum_{m=1}^{\infty} U_{n,m,l},$$

where

$$U_{n,m,l} = \big(\mathrm{E}(K(X_{n,1,\infty})|\mathcal{F}_{n-m}) - \mathrm{E}(K(X_{n,1,\infty})|\mathcal{F}_{n-(m+1)})\big) - \big(\mathrm{E}(K(X_{n,1,l})|\mathcal{F}_{n-m}) - \mathrm{E}(K(X_{n,1,l})|\mathcal{F}_{n-(m+1)})\big)1_{\{1\leq m\leq l\}}. \tag{3.14}$$

By Lemma 3.1, the random variables $U_{n,m,l}$ satisfy

$$\mathrm{cov}(U_{n,m,l}, U_{n',m',l}) = 0 \qquad \text{unless } n - m = n' - m'.$$

By writing

$$\mathrm{E}(S_{N,1,\infty} - S_{N,1,l})^2 \leq 3\mathrm{E}\left(\sum_{n=1}^{N} U_{n,1,l}\right)^2 + 3\mathrm{E}\left(\sum_{n=1}^{N}\sum_{m=2}^{l} U_{n,m,l}\right)^2 + 3\mathrm{E}\left(\sum_{n=1}^{N}\sum_{m=l+1}^{\infty} U_{n,m,l}\right)^2,$$

and then using the decorrelation of $\{U_{n,m,l}\}$ and the Cauchy–Schwarz inequality, we obtain

$$\mathrm{E}(S_{N,1,\infty} - S_{N,1,l})^2 \leq 3Q_{N,1,l} + 3Q_{N,2,l} + 3Q_{N,3,l},$$

where

$$Q_{N,1,l} = \sum_{n=1}^{N} \mathrm{E}U_{n,1,l}^2,$$

$$Q_{N,2,l} = \sum_{n=1}^{N} \sum_{m=2}^{l} \sum_{m'=2}^{l} (\mathrm{E}U_{n,m,l}^2)^{1/2} (\mathrm{E}U_{n',m',l}^2)^{1/2},$$

$$Q_{N,3,l} = \sum_{n=1}^{N} \sum_{m=l+1}^{\infty} \sum_{m'=l+1}^{\infty} (\mathrm{E}U_{n,m,l}^2)^{1/2} (\mathrm{E}U_{n',m',l}^2)^{1/2},$$

with $n' = n - m + m'$. One then needs to show that

$$\lim_{l\to\infty} \limsup_{N\to\infty} N^{-1} Q_{N,i,l} = 0, \qquad \text{for } i = 1, 2, 3. \tag{3.15}$$

We first prove (3.15) for $i = 1$. We have

$$\mathrm{E}(\mathrm{E}(X|\mathcal{G}) - \mathrm{E}(Y|\mathcal{F}))^2 \leqslant 2\mathrm{E}(\mathrm{E}(X|\mathcal{G}))^2 + 2\mathrm{E}(\mathrm{E}(Y|\mathcal{F}))^2 \leqslant 2\mathrm{E}(\mathrm{E}(X^2|\mathcal{G})) + 2\mathrm{E}(\mathrm{E}(Y^2|\mathcal{F}))$$
$$= 2(\mathrm{E}X^2 + \mathrm{E}Y^2).$$

Applying this inequality to (3.14) gives

$$N^{-1} Q_{N,1,l} \leqslant 4N^{-1} \sum_{n=1}^{N} \mathrm{E}(K(X_{n,1,\infty}) - K(X_{n,1,l}))^2 = 4\mathrm{E}(K(X_{1,1,\infty}) - K(X_{1,1,l}))^2.$$

Since, by Kolmogorov's three-series theorem, $X_{1,1,l} \to X_{1,1,\infty}$ a.s. as $l \to \infty$, it follows from Lemma 2.1 that $\mathrm{E}(K(X_{1,1,\infty}) - K(X_{1,1,l}))^2 \to 0$ and hence that (3.15) holds with $i = 1$.

We now prove (3.15) for $i = 2, 3$. Since the function a satisfies (2.7), it is enough to show that

$$\mathrm{E}U_{n,m,l}^2 \leqslant C\left(\int_{m-1}^{m} |a(x)|^\alpha \, \mathrm{d}x\right)\left(\int_{l}^{\infty} |a(x)|^\alpha \, \mathrm{d}x\right), \qquad \text{for } 2 \leqslant m \leqslant l, \tag{3.16}$$

$$\mathrm{E}U_{n,m,l}^2 \leqslant C\left(\int_{m-1}^{m} |a(x)|^\alpha \, \mathrm{d}x\right), \qquad \text{for } m \geqslant l + 1. \tag{3.17}$$

We will first establish (3.16). As in (2.6), we have $X_{n,1,\infty} = X_{n,1,m-1} + X_{n,m,\infty}$ and $X_{n,1,l} = X_{n,1,m-1} + X_{n,m,l}$. Observe that in the decomposition of $X_{n,1,\infty}$, for example, the terms $X_{n,1,m-1}$ and $X_{n,m,\infty}$ are respectively independent and measurable with respect to the σ-algebra $\mathcal{F}_{n-m}$. Hence setting

$$k_m(x) = \mathrm{E}K(x + X_{n,1,m-1}), \tag{3.18}$$

we obtain $\mathrm{E}(K(X_{n,1,\infty})|\mathcal{F}_{n-m}) = k_m(X_{n,m,\infty})$ and $\mathrm{E}(K(X_{n,1,l})|\mathcal{F}_{n-m}) = k_m(X_{n,m,l})$. In view of (3.14), we can then write

$$U_{n,m,l} = k_m(X_{n,m,\infty}) - k_{m+1}(X_{n,m+1,\infty}) - (k_m(X_{n,m,l}) - k_{m+1}(X_{n,m+1,l})). \tag{3.19}$$

Now denote the distribution functions corresponding to X_{n,m_1,m_2} and $X_{n,m,m}$ by F_{m_1,m_2} and F_m, respectively. Since

$$X_{n,m,\infty} = X_{n,m,m} + X_{n,m+1,l} + X_{n,l+1,\infty},$$
$$X_{n,m+1,\infty} = X_{n,m+1,l} + X_{n,l+1,\infty},$$
$$X_{n,m,l} = X_{n,m,m} + X_{n,m+1,l},$$

we obtain from (3.19) that

$$\begin{aligned}\mathrm{E}U^2_{n,m,l} = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} &\{(k_m(u+v+w) - k_{m+1}(v+w)) \\ &- (k_m(u+v) - k_{m+1}(v))\}^2 \,\mathrm{d}F_m(u)\mathrm{d}F_{m+1,l}(v)\mathrm{d}F_{l+1,\infty}(w).\end{aligned}$$

By using the relation $k_{m+1}(z) = \mathrm{E}K(z + X_{n,1,m}) = \mathrm{E}K(z + X_{n,1,m-1} + X_{n,m,m}) = \int k_m(z + x)\mathrm{d}F_m(x)$, we obtain further that

$$\begin{aligned}\mathrm{E}U^2_{n,m,l} &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \left\{\int_{\mathbb{R}} D(u, v, w, x)\mathrm{d}F_m(x)\right\}^2 \mathrm{d}F_m(u)\mathrm{d}F_{m+1,l}(v)\mathrm{d}F_{l+1,\infty}(w) \\ &\leqslant \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} D(u, v, w, x)^2 \,\mathrm{d}F_m(u)\mathrm{d}F_{m+1,l}(v)\mathrm{d}F_{l+1,\infty}(w)\mathrm{d}F_m(x), \qquad (3.20)\end{aligned}$$

where

$$D(u, v, w, x) = k_m(u+v+w) - k_m(v+w+x) - (k_m(u+v) - k_m(v+x)).$$

Observe now that $\lambda_{1,m-1} = (\int_0^{m-1} |a(x)|^\alpha \,\mathrm{d}x)^{1/\alpha}$ is the scale parameter of the α-stable random variables $X_{n,1,m-1}$. Since $\lambda_{1,m-1}$, $m \geqslant 2$, is uniformly bounded away from 0 and from infinity, relation (3.2) of Lemma 3.2 implies that the functions $k_m(x)$ have their first two derivatives bounded uniformly for $m \geqslant 2$ and $x \in \mathbb{R}$. Then, by the mean value theorem,

$$D(u, v, w, x)^2 \leqslant C \min\{1, w^2, (u-x)^2, (u-x)^2 w^2\}. \qquad (3.21)$$

To show, for example, that $D(u, v, w, x)^2 \leqslant Cw^2$, write $D(u, v, w, x) = g_2(w) - g_2(0)$ with $g_2(w) = k_m(u+v+w) - k_m(v+w+x)$ and apply the mean value theorem using the uniform boundedness of the derivatives $k^{(1)}_m(x)$. To obtain $D(u, v, w, x)^2 \leqslant C(u-x)^2$, write $D(u, v, w, x) = g_1(u) - g_1(x)$ with $g_1(z) = k_m(z+v+w) - k_m(z+v)$ and apply the mean value theorem, and to obtain $D(u, v, w, x)^2 \leqslant C(u-x)^2 w^2$, use the previous mean value relationship $D(u, v, w, x) = g^{(1)}_1(z^*)(u-x)$ and again apply the mean value theorem to $g^{(1)}_1(z^*) = k^{(1)}_m(z^* + v + w) - k^{(1)}_m(z^* + v)$. Then, by using (3.20) and (3.21), we obtain that

$$\begin{aligned}\mathrm{E}U_{n,m,l}^2 \leqslant\; & C\left(\iint_{|u-x|\leqslant 1}(u-x)^2\,\mathrm{d}F_m(u)\mathrm{d}F_m(x)\right)\left(\int_{|w|\leqslant 1} w^2\,\mathrm{d}F_{l+1,\infty}(w)\right)\\ &+ C\left(\iint_{|u-x|>1}\mathrm{d}F_m(u)\mathrm{d}F_m(x)\right)\left(\int_{|w|\leqslant 1} w^2\,\mathrm{d}F_{l+1,\infty}(w)\right)\\ &+ C\left(\iint_{|u-x|\leqslant 1}(u-x)^2\,\mathrm{d}F_m(u)\mathrm{d}F_m(x)\right)\left(\int_{|w|>1}\mathrm{d}F_{l+1,\infty}(w)\right)\\ &+ C\left(\iint_{|u-x|>1}\mathrm{d}F_m(u)\mathrm{d}F_m(x)\right)\left(\int_{|w|>1}\mathrm{d}F_{l+1,\infty}(w)\right). \end{aligned} \tag{3.22}$$

We will now use the fact that, for an α-stable standard random variable Z, $P(\lambda|Z|>1)\leqslant C\lambda^\alpha$ and $\mathrm{E}((\lambda Z)^2 1_{\{\lambda|Z|\leqslant 1\}})\leqslant C\lambda^\alpha$ for any $\lambda>0$, where the constant C does not depend on the skewness parameter of Z. These bounds follow from the asymptotic behaviour of the tails $P(|Z|>z)\sim cz^{-\alpha}$ as $z\to\infty$ (see, for example, Samorodnitsky and Taqqu 1994, p. 16) and the relation $\mathrm{E}X^2 1_{\{|X|\leqslant x_0\}}\leqslant\int_0^{x_0}2xP(|X|>x)\mathrm{d}x$ which yields $\mathrm{E}((\lambda Z)^2 1_{\{\lambda|Z|\leqslant 1\}})=\lambda^2\mathrm{E}Z^2 1_{\{|Z|\leqslant 1/\lambda\}}\leqslant\lambda^2 C\lambda^{\alpha-2}=C\lambda^\alpha$. Since $\lambda_m Z_1$ has the distribution F_m with $\lambda_m^\alpha=\int_{m-1}^m|a(x)|^\alpha\,\mathrm{d}x$ and $\lambda_{l+1,\infty}Z_2$ has the distribution $F_{l+1,\infty}$ with $\lambda_{l+1,\infty}^\alpha=\int_l^\infty|a(x)|^\alpha\,\mathrm{d}x$, where Z_1 and Z_2 are α-stable standard random variables (with perhaps different skewness parameters), we conclude that each of the four terms in (3.22) is bounded by $C\lambda_m^\alpha\lambda_{l+1,\infty}^\alpha$ and hence that the bound (3.16) holds. One may prove the bound (3.17) in a similar (in fact, simpler) manner.

Non-causal moving averages. Assuming for simplicity that $x_0=0$ in Definition 2.1 for a non-causal moving average X_n, we have $X_n=\int_n^\infty a(n-x)M(\mathrm{d}x)$. We need to prove that

$$\lim_{l\to-\infty}\limsup_{N\to\infty}N^{-1}\operatorname{var}(S_{N,-\infty,0}-S_{N,l,0})=0, \tag{3.23}$$

where

$$S_{N,-\infty,0}=\sum_{n=1}^N(K(X_{n,-\infty,0})-\mathrm{E}K(X_{n,-\infty,0}))$$

$$S_{N,l,0}=\sum_{n=1}^N(K(X_{n,l,0})-\mathrm{E}K(X_{n,l,0})).$$

We will show that the proof of (3.23) can be reduced to that of (3.13). While $(X_{n,-\infty,0},X_{n,l,0},\,l\leqslant 0)_{n=1,\ldots,N}$ does not have the same distribution as $(X_{N-n+1,-\infty,0},X_{N-n+1,l,0},\,l\leqslant 0)_{n=1,\ldots,N}$, one has, for fixed N,

$$S_{N,-\infty,0} = \sum_{n=1}^{N}(K(X_{N-n+1,-\infty,0}) - \mathrm{E}K(X_{N-n+1,-\infty,0})) \tag{3.24}$$

and

$$S_{N,l,0} = \sum_{n=1}^{N}(K(X_{N-n+1,l,0}) - \mathrm{E}K(X_{N-n+1,l,0})). \tag{3.25}$$

Now, for fixed N, by making the change of variables $N+1-x=y$ below,

$$\begin{aligned}
&(X_{N-n+1,-\infty,0},\ X_{N-n+1,l,0},\ l \leqslant 0)_{n=1,\dots,N}\\
&= \left(\int_{N-n+1}^{\infty} a(N-n+1-x)M(\mathrm{d}x),\ \int_{N-n+1}^{N-n+2-l} a(N-n+1-x)M(\mathrm{d}x),\ l \leqslant 0\right)_{n=1,\dots,N}\\
&= \left(-\int_{-\infty}^{n} a(y-n)M(\mathrm{d}(N+1-y)),\ -\int_{n+l-1}^{n} a(y-n)M(\mathrm{d}(N+1-y)),\ l \leqslant 0\right)_{n=1,\dots,N}\\
&\overset{d}{=} \left(\int_{-\infty}^{n} \tilde{a}(n-y)\tilde{M}(\mathrm{d}y),\ \int_{n+l-1}^{n} \tilde{a}(n-y)\tilde{M}(dy),\ l \leqslant 0\right)_{n=1,\dots,N}\\
&= \left(\tilde{X}_{n,1,\infty},\ \tilde{X}_{n,1,-l+1},\ l \leqslant 0\right)_{n=1,\dots,N},
\end{aligned}$$

where $\tilde{M}$ is an α-stable random measure with the Lebesgue control measure and the skewness intensity β, $\tilde{a}(z)=a(-z)$ for $z\in\mathbb{R}$ and $\tilde{X}_{n,m_1,m_2}=\int_{n-m_2}^{n-m_1+1}\tilde{a}(n-y)\tilde{M}(\mathrm{d}y)$, $-\infty \leqslant m_1 \leqslant m_2 \leqslant \infty$. Then, by using (3.24) and (3.25), we obtain that, for fixed N, $(S_{N,-\infty,0}, S_{N,l,0}) =_d (\tilde{S}_{N,1,\infty}, \tilde{S}_{N,1,-l+1})$ with $\tilde{S}_{N,m_1,m_2}=\sum_{n=1}^{N}(K(\tilde{X}_{n,m_1,m_2})-\mathrm{E}K(\tilde{X}_{n,m_1,m_2}))$ and hence

$$N^{-1}\operatorname{var}(S_{N,-\infty,0}-S_{N,l,0}) = N^{-1}\operatorname{var}(\tilde{S}_{N,1,\infty}-\tilde{S}_{N,1,-l+1}).$$

The convergence (3.23) then follows from (3.13) since $\tilde{a}$ satisfies condition (2.7) and $\tilde{X}$ is causal.

Two-sided moving averages. We need to prove (3.12) for $X_n=\int_{-\infty}^{\infty}a(n-x)M(\mathrm{d}x)$ and for a function K which is twice differentiable with bounded derivatives. As in the case of causal moving averages, we can write

$$S_N = \sum_{n=1}^{N}\sum_{m=-\infty}^{\infty}\left(\mathrm{E}(K(X_n)|\mathcal{F}_{n-m}) - \mathrm{E}(K(X_n)|\mathcal{F}_{n-(m+1)})\right),$$

$$S_{N,l_1,l_2} = \sum_{n=1}^{N}\sum_{m=l_1}^{l_2}\left(\mathrm{E}(K(X_{n,l_1,l_2})|\mathcal{F}_{n-m}) - \mathrm{E}(K(X_{n,l_1,l_2})|\mathcal{F}_{n-(m+1)})\right)$$

and hence $S_N - S_{N,l_1,l_2} = \sum_{n=1}^{N}\sum_{m=-\infty}^{\infty}U_{n,m,l_1,l_2}$, where the random variables

$$U_{n,m,l_1,l_2} = \big(\mathrm{E}(K(X_n)|\mathcal{F}_{n-m}) - \mathrm{E}(K(X_n)|\mathcal{F}_{n-(m+1)})\big) - \big(\mathrm{E}(K(X_{n,l_1,l_2})|\mathcal{F}_{n-m}) - \mathrm{E}(K(X_{n,l_1,l_2})|\mathcal{F}_{n-(m+1)})\big)1_{\{l_1 \leqslant m \leqslant l_2\}} \tag{3.26}$$

satisfy $\mathrm{cov}(U_{n,m,l_1,l_2}, U_{n',m',l_1,l_2}) = 0$ unless $n - m = n' - m'$. Then $\mathrm{E}(S_N - S_{N,l_1,l_2})^2 \leqslant 3Q_{N,1,l_1,l_2} + 3Q_{N,2,l_1,l_2}$, where

$$Q_{N,1,l_1,l_2} = \sum_{n=1}^{N}\sum_{m=l_1}^{l_2}\sum_{m'=l_1}^{l_2}(\mathrm{E}U^2_{n,m,l_1,l_2})^{1/2}(\mathrm{E}U^2_{n',m',l_1,l_2})^{1/2},$$

$$Q_{N,2,l_1,l_2} = \sum_{n=1}^{N}\left(\sum_{m=-\infty}^{l_1-1}\sum_{m'=-\infty}^{l_1-1} + \sum_{m=l_2+1}^{\infty}\sum_{m'=l_2+1}^{\infty}\right)(\mathrm{E}U^2_{n,m,l_1,l_2})^{1/2}(\mathrm{E}U^2_{n',m',l_1,l_2})^{1/2}$$

with $n' = n - m + m'$, and one needs to show that

$$\lim_{(l_1,l_2)\to(-\infty,\infty)}\limsup_{N\to\infty} N^{-1}Q_{N,i,l_1,l_2} = 0, \quad \text{for } i = 1, 2. \tag{3.27}$$

The convergence (3.27) will follow from the bounds

$$\mathrm{E}U^2_{n,m,l_1,l_2} \leqslant C\left(\int_{m-1}^{m}|a(x)|^\alpha\,\mathrm{d}x\right)\left(\int_{-\infty}^{l_1-1}|a(x)|^\alpha\,\mathrm{d}x + \int_{l_2}^{\infty}|a(x)|^\alpha\,\mathrm{d}x\right), \qquad \text{for } l_1 \leqslant m \leqslant l_2, \tag{3.28}$$

$$\mathrm{E}U^2_{n,m,l_1,l_2} \leqslant C\left(\int_{m-1}^{m}|a(x)|^\alpha\,\mathrm{d}x\right), \qquad \text{for } m \leqslant l_1 - 1 \text{ or } m \geqslant l_2 + 1. \tag{3.29}$$

We will first prove (3.28). Setting

$$k_{m,l_1}(z) = \mathrm{E}K(z + X_{n,l_1,m-1}), \tag{3.30}$$

we can write $\mathrm{E}(K(X_{n,l_1,l_2})|\mathcal{F}_{n-m}) = k_{m,l_1}(X_{n,m,l_2})$ and

$$\begin{aligned}\mathrm{E}(K(X_n)|\mathcal{F}_{n-m}) &= \mathrm{E}K(z + X_{n,-\infty,m-1})\big|_{z=X_{n,m,\infty}}\\ &= \mathrm{E}K(z + X_{n,-\infty,l_1-1} + X_{n,l_1,m-1})\big|_{z=X_{n,m,\infty}}\\ &= \int_{\mathbb{R}} k_{m,l_1}(z+y)\mathrm{d}F_{-\infty,l_1-1}(y)\,\big|_{z=X_{n,m,\infty}}\end{aligned}$$

and hence, in view of (3.26),

$$\begin{aligned}U_{n,m,l_1,l_2} =& \int_{\mathbb{R}} k_{m,l_1}(z+y)\mathrm{d}F_{-\infty,l_1-1}(y)\,\big|_{z=X_{n,m,\infty}} - \int_{\mathbb{R}} k_{m+1,l_1}(z+y)\mathrm{d}F_{-\infty,l_1-1}(y)\,\big|_{z=X_{n,m+1,\infty}}\\ &- (k_{m,l_1}(X_{n,m,l_2}) - k_{m+1,l_1}(X_{n,m+1,l_2})).\end{aligned}$$

Then, by writing, as in the causal case,

$$X_{n,m,\infty} = X_{n,m,m} + X_{n,m+1,l_2} + X_{n,l_2+1,\infty},$$

$$X_{n,m+1,\infty} = X_{n,m+1,l_2} + X_{n,l_2+1,\infty},$$

$$X_{n,m,l_2} = X_{n,m,m} + X_{n,m+1,l_2},$$

and by using the relation $k_{m+1,l_1}(z) = \int k_{m,l_1}(z+x)\mathrm{d}F_m(x)$ and the Cauchy–Schwarz inequality, we obtain that

$$\begin{aligned}\mathrm{E}U^2_{n,m,l_1,l_2} &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\left\{\int_{\mathbb{R}}\int_{\mathbb{R}} D(u,\, v,\, w,\, x,\, y)\mathrm{d}F_{-\infty,l_1-1}(y)\mathrm{d}F_m(x)\right\}^2 \\ &\quad\times \mathrm{d}F_m(u)\mathrm{d}F_{m+1,l_2}(v)\mathrm{d}F_{l_2+1,\infty}(w) \\ &\leqslant \int_{\mathbb{R}}\ldots\int_{\mathbb{R}} D(u,\, v,\, w,\, x,\, y)^2\,\mathrm{d}F_m(u)\mathrm{d}F_{m+1,l_2}(v)\mathrm{d}F_{l_2+1,\infty}(w)\mathrm{d}F_m(x)\mathrm{d}F_{-\infty,l_1-1}(y),\end{aligned} \tag{3.31}$$

where

$$\begin{aligned}D(u,\, v,\, w,\, x,\, y) &= k_{m,l_1}(u+v+w+y) - k_{m,l_1}(v+w+x+y) \\ &\quad - (k_{m,l_1}(u+v) - k_{m,l_1}(v+x)).\end{aligned}$$

By assumption on K and by using (3.7) in Lemma 3.3, we obtain that the functions $k_{m,l_1}(x)$ have their first two derivatives bounded uniformly for $m,\, l_1 \in \mathbb{Z}$ and $x \in \mathbb{R}$. Then, as in (3.21) with w replaced by $w+y$, we obtain that

$$D(u,\, v,\, w,\, x,\, y)^2 \leqslant C\min\{1,\, (w+y)^2,\, (u-x)^2,\, (u-x)^2(w+y)^2\}$$

and hence, by (3.31),

$$\begin{aligned}\mathrm{E}U^2_{n,m,l_1,l_2} &\leqslant C\left(\iint_{|u-x|\leqslant 1}(u-x)^2\,\mathrm{d}F_m(u)\mathrm{d}F_m(x)\right) \\ &\quad\times\left(\iint_{|w+y|\leqslant 1}(w+y)^2\,\mathrm{d}F_{l_2+1,\infty}(w)\mathrm{d}F_{-\infty,l_1-1}(y)\right) \\ &\quad + C\left(\iint_{|u-x|>1}\mathrm{d}F_m(u)\mathrm{d}F_m(x)\right)\left(\iint_{|w+y|\leqslant 1}(w+y)^2\,\mathrm{d}F_{l_2+1,\infty}(w)\mathrm{d}F_{-\infty,l_1-1}(y)\right) \\ &\quad + C\left(\iint_{|u-x|\leqslant 1}(u-x)^2\,\mathrm{d}F_m(u)\mathrm{d}F_m(x)\right)\left(\iint_{|w+y|>1}\mathrm{d}F_{l_2+1,\infty}(w)\mathrm{d}F_{-\infty,l_1-1}(y)\right) \\ &\quad + C\left(\iint_{|u-x|>1}\mathrm{d}F_m(u)\mathrm{d}F_m(x)\right)\left(\iint_{|w+y|>1}\mathrm{d}F_{l_2+1,\infty}(w)\mathrm{d}F_{-\infty,l_1-1}(y)\right).\end{aligned} \tag{3.32}$$

Since $\lambda_{-\infty,l_1-1}Z_1$ has the distribution $F_{-\infty,l_1-1}$ with $\lambda^\alpha_{-\infty,l_1-1} = \int_{-\infty}^{l_1-1}|a(x)|^\alpha\,\mathrm{d}x$, $\lambda_{l_2+1,\infty}Z_2$ has the distribution $F_{l_2+1,\infty}$ with $\lambda^\alpha_{l_2+1,\infty} = \int_{l_2}^{\infty}|a(x)|^\alpha\,\mathrm{d}x$ and $\lambda_m Z_3$ has the distribution F_m with $\lambda^\alpha_m = \int_{m-1}^{m}|a(x)|^\alpha\,\mathrm{d}x$, where Z_1, Z_2 and Z_3 are α-stable standard random variables (with perhaps different skewness parameters), we conclude as in the case of (3.22) that each of the four terms in (3.32) is bounded by $C\lambda^\alpha_m(\lambda^\alpha_{-\infty,l_1-1}+\lambda^\alpha_{l_2+1,\infty})$ and hence that the bound (3.28) holds. One can prove the bound (3.29) in a similar way. □

Remark 3.1. The proof for two-sided moving averages does not apply to partial sums with any bounded function K because the scale parameters of α-stable random variables

$$X_{n,l_1,m-1} = \int_{n-m+1}^{n-l_1+1} a(n-x)M(\mathrm{d}x),$$

namely, $\lambda_{l_1,m-1} = (\int_{l_1-1}^{m-1}|a(x)|^\alpha\,\mathrm{d}x)^{1/\alpha}$, are not bounded away from zero uniformly in m and l_1, since $\lambda_{l_1,m-1}\to 0$ as $l_1\to-\infty$ and $m\to-\infty$. Hence, the first and second derivatives of the functions $k_{m,l_1}(x)$ in (3.30) are not necessarily bounded uniformly in m and l_1 (see Lemma 3.2, which involves a scale parameter $\sigma>0$). Whether Theorem 2.1 is valid for two-sided moving averages with any bounded functions K_j is still an open question.

Proof of Theorem 2.2. Observe first that the moving average X_n is well defined since, by assumptions (2.17) and (2.18), $\sum_j|a_j|^\delta<\infty$ for some $0<\delta<\min\{1,\alpha\}$ and hence $\mathrm{E}|X_n|^\delta\leqslant\sum_j|c_j|^\delta\mathrm{E}|\epsilon_{n-j}|^\delta = \mathrm{E}|\epsilon_1|^\delta\sum_j|c_j|^\delta<\infty$. As in the proof of Theorem 2.1, it is enough to show the convergence (2.19). Let $\mathcal{F}_k$, $k\in\mathbb{Z}$, be σ-algebras generated by innovations ϵ_i, $i\leqslant k$. (This notation corresponds to that used in the continuous case with $\epsilon_i=\int_i^{i+1}M(\mathrm{d}x)$.) We set $X_{n,m_1,m_2}=\sum_{m_1\leqslant j\leqslant m_2}a_j\epsilon_{n-j}$. Then, by proceeding as in the proof of Theorem 2.1 for two-sided moving averages, it is enough to prove (3.27), which will follow here (compare with (3.28) and (3.29)) from the bounds

$$\mathrm{E}U^2_{n,m,l_1,l_2}\leqslant C|a_m|^\alpha\tilde{L}\left(\frac{1}{|a_m|}\right)\left(\sum_{j=-\infty}^{l_1-1}+\sum_{j=l_2+1}^{\infty}\right)|a_j|^\alpha\tilde{L}\left(\frac{1}{|a_j|}\right),\qquad \text{for } l_1\leqslant m\leqslant l_2, \tag{3.33}$$

$$\mathrm{E}U^2_{n,m,l_1,l_2}\leqslant C|a_m|^\alpha\tilde{L}\left(\frac{1}{|a_m|}\right),\qquad \text{for } m\leqslant l_1-1 \text{ or } m\geqslant l_2+1, \tag{3.34}$$

where $\tilde{L}$ is defined in (2.22).

To show (3.33), observe that we can bound $\mathrm{E}U^2_{n,m,l_1,l_2}$ as in (3.32) and hence it is enough to prove that there is a constant C such that, for every $-\infty\leqslant m_1\leqslant m_2\leqslant\infty$,

$$P\left(\left|\sum_{j=m_1}^{m_2}b_j\epsilon_j\right|>1\right)\leqslant C\sum_{j=m_1}^{m_2}|b_j|^\alpha\tilde{L}\left(\frac{1}{|b_j|}\right), \tag{3.35}$$

$$\mathrm{E}\left(\left|\sum_{j=m_1}^{m_2}b_j\epsilon_j\right|^2 1_{\{|\sum_{j=m_1}^{m_2}b_j\epsilon_j|<1\}}\right)\leqslant C\sum_{j=m_1}^{m_2}|b_j|^\alpha\tilde{L}\left(\frac{1}{|b_j|}\right), \tag{3.36}$$

where $\{b_j\}$ satisfies $\sum_j|b_j|^\alpha\tilde{L}(1/|b_j|)<\infty$. For example,

$$\iint_{|w+y|>1} \mathrm{d}F_{l_2+1,\infty}(w)\mathrm{d}F_{-\infty,l_1-1}(y) = P\left(\left|\sum_{j=-\infty}^{l_1-1} a_j\epsilon'_{n-j} + \sum_{j=l_2+1}^{\infty} a_j\epsilon''_{n-j}\right| > 1\right)$$

$$= P\left(\left|\sum_{j=-\infty}^{\infty} b_j\epsilon_j\right| > 1\right),$$

where ϵ', ϵ'' are independent copies of the sequence ϵ, and $b_j = a_j$ for $-\infty < j \leqslant l_1 - 1$ and $l_2 + 1 \leqslant j < \infty$ and 0 otherwise.

Consider inequality (3.35) first. By using

$$P(|Z| \geqslant x) \leqslant \frac{x}{2}\int_{-2/x}^{2/x} (1 - \mathrm{E}e^{\mathrm{i}\theta Z})\mathrm{d}\theta, \qquad x > 0,$$

(Billingsley 1995, p. 350), we obtain that

$$P\left(\left|\sum_{j=m_1}^{m_2} b_j\epsilon_j\right| > 1\right) \leqslant C\int_{-2}^{2}\left|1 - \mathrm{E}e^{\mathrm{i}\theta\sum_{j=m_1}^{m_2} b_j\epsilon_j}\right|\mathrm{d}\theta$$

$$= C\int_{-2}^{2}\left|1 - \prod_{j=m_1}^{m_2} \mathrm{E}e^{\mathrm{i}\theta b_j\epsilon_1}\right|\mathrm{d}\theta \leqslant C\sum_{j=m_1}^{m_2}\int_{-2}^{2}\left|1 - \mathrm{E}e^{\mathrm{i}\theta b_j\epsilon_1}\right|\mathrm{d}\theta, \qquad (3.37)$$

by applying the inequality $|1 - \prod_{j=m_1}^{m_2} c_j| \leqslant \sum_{j=m_1}^{m_2}|1 - c_j|$ valid for $|c_j| \leqslant 1$.

Consider first the case $\alpha \neq 1$. By using (2.15), (2.16) and Ibragimov and Linnik (1971, Theorem 2.6.5), there exist $c > 0$, $\beta \in [-1, 1]$ and a function $h(u) = h_1(u) + \mathrm{i}h_2(u)$ with $h(u) \to 1$, as $u \to 0$, such that

$$\mathrm{E}\exp\{\mathrm{i}u\epsilon_1\} = \exp\left\{-c|u|^\alpha \widetilde{L}\left(\frac{1}{|u|}\right)\left(1 - \mathrm{i}\beta\,\mathrm{sign}(u)\tan\frac{\alpha\pi}{2}\right)h(u)\right\}$$

$$= \exp\left\{-c|u|^\alpha \widetilde{L}\left(\frac{1}{|u|}\right)\left(h_1(u) + \beta\,\mathrm{sign}(u)\tan\frac{\alpha\pi}{2}\,h_2(u)\right)\right.$$

$$\left. + \mathrm{i}c|u|^\alpha \tilde{L}\left(\frac{1}{|u|}\right)\left(\beta\,\mathrm{sign}(u)\tan\frac{\alpha\pi}{2}\,h_1(u) - h_2(u)\right)\right\},$$

where $\tilde{L}(z) = L(z)$. Then, by using the fact that $h_1(u) + \beta\,\mathrm{sign}(u)\tan(\alpha\pi/2)h_2(u) \to 1$, as $u \to 0$, and the inequalities $|1 - c_1c_2| \leqslant |1 - c_1| + |1 - c_2|$ for $|c_1|, |c_2| \leqslant 1$, $|1 - e^{-x}| \leqslant x$ for $x > 0$, and $|1 - e^{\mathrm{i}x}| \leqslant |x|$ for $x \in \mathbb{R}$, we conclude that

$$\left|1 - \mathrm{E}e^{\mathrm{i}u\epsilon_1}\right| \leqslant C|u|^\alpha \tilde{L}\left(\frac{1}{|u|}\right), \qquad (3.38)$$

for small enough u. We may suppose without loss of generality that (3.38) holds for all u. Indeed, since the function $\tilde{L}(z) = L(z)$ need not be defined around $z = 0$ (see (2.15) and (2.16)), we can define it as

$$z^{-\alpha}\tilde{L}(z) \sim 1, \tag{3.39}$$

as $z \to 0$. With this non-restricting assumption in mind, by observing that the the left-hand side of (3.38) is bounded by 2 and by increasing C, we obtain that the bound (3.38) holds for all u.

For $\alpha = 1$, by using Aaronson and Denker (1998, Theorem 2), we can write

$$\begin{aligned}\mathrm{E}\exp\{\mathrm{i}u\epsilon_1\} &= \exp\left\{-c|u|L\left(\frac{1}{|u|}\right)\left(1-\mathrm{i}\frac{2\beta}{c}C\operatorname{sign}(u)\right)h(u)+\mathrm{i}u\left(\gamma+H\left(\frac{1}{|u|}\right)\right)\right\}\\ &= \exp\left\{-c|u|L\left(\frac{1}{|u|}\right)\left(h_1(u)+\frac{2\beta}{c}C\operatorname{sign}(u)h_2(u)\right)\right.\\ &\quad\left.+\mathrm{i}c|u|L\left(\frac{1}{|u|}\right)\left(\frac{2\beta}{c}C\operatorname{sign}(u)h_1(u)-h_2(u)\right)+\mathrm{i}u\left(\gamma+H\left(\frac{1}{|u|}\right)\right)\right\},\end{aligned}$$

where c, C, β are constants, $h(u) = h_1(u) + \mathrm{i}h_2(u) \to 1$ as $u \to 0$, and the function H and the constant γ are defined by (2.20) and (2.21). Then, by arguing as in the case $\alpha \neq 1$, we can conclude that (3.38) holds for small enough u with $\tilde{L}(z) = L(z) + |\gamma + H(z)|$. Recall from Remark 2.1 that $\tilde{L}$ is a slowly varying function at infinity. By choosing $L(z)$ such that $z^{-1}L(z) \sim 1 - |\gamma + H(z)|z^{-1}$ as $z \to 0$, we can suppose without loss of generality that (3.39) holds and hence that (3.38) holds for all u.

By using (3.38), we obtain from (3.37) that

$$P\left(\left|\sum_{j=m_1}^{m_2} b_j\epsilon_j\right| > 1\right) \leqslant C\sum_{j=m_1}^{m_2}|b_j|^\alpha\int_{-2}^{2}|\theta|^\alpha\tilde{L}\left(\frac{1}{|b_j\theta|}\right)\mathrm{d}\theta.$$

The bound (3.35) follows since, by Potter's bounds (see, for example, Bingham *et al*. 1987), $\tilde{L}(1/|\theta b_j|)(\tilde{L}(1/|b_j|))^{-1}$ is bounded by $C|\theta|^{-\epsilon}$ with any $\epsilon > 0$, uniformly for j and $|\theta| \leqslant 2$.

To prove inequality (3.36), observe that $\mathrm{E}X^2 1_{\{0<|X|<1\}} \leqslant \int_0^1 2xP(|X| > x)\mathrm{d}x$ and hence that

$$\mathrm{E}\left(\left|\sum_{j=m_1}^{m_2} b_j\epsilon_j\right|^2 1_{\{|\sum_{j=m_1}^{m_2} b_j\epsilon_j|<1\}}\right) \leqslant \int_0^1 2xP\left(\left|\sum_{j=m_1}^{m_2} b_j\epsilon_j\right| > x\right)\mathrm{d}x. \tag{3.40}$$

By arguing as above, we can conclude that

$$\begin{aligned}P\left(\left|\sum_{j=m_1}^{m_2} b_j\epsilon_j\right| > x\right) &\leqslant Cx\sum_{j=m_1}^{m_2}|b_j|^\alpha\int_{-2/x}^{2/x}|\theta|^\alpha\tilde{L}\left(\frac{1}{|b_j\theta|}\right)\mathrm{d}\theta\\ &\leqslant Cx\sum_{j=m_1}^{m_2}|b_j|^{-1}\int_{-2|b_j|/x}^{2|b_j|/x}|\nu|^\alpha\tilde{L}\left(\frac{1}{|\nu|}\right)\mathrm{d}\nu.\end{aligned}$$

Then, by using Karamata's theorem for the case $2|b_j| < x$ and by making the change of variables $z = 1/\nu$, we obtain that

$$P\left(\left|\sum_{j=m_1}^{m_2} b_j\epsilon_j\right| > x\right) \leqslant Cx\sum_{j=m_1}^{m_2}|b_j|^{-1}\left(\frac{|b_j|}{x}\right)^{\alpha+1}\tilde{L}\left(\frac{x}{|b_j|}\right)1_{\{2|b_j|<x\}}$$

$$+ Cx\sum_{j=m_1}^{m_2}|b_j|^{-1}\int_{x/(2|b_j|)}^{\infty} z^{-2}z^{-\alpha}\tilde{L}(z)\mathrm{d}z1_{\{2|b_j|\geqslant x\}}$$

$$\leqslant C'\sum_{j=m_1}^{m_2}|b_j|^{\alpha}x^{-\alpha}\tilde{L}\left(\frac{x}{|b_j|}\right)1_{\{2|b_j|<x\}} + C'\sum_{j=m_1}^{m_2}1_{\{2|b_j|\geqslant x\}}. \tag{3.41}$$

The last term in (3.41) follows from (3.39), yielding the bound

$$\int_{x/(2|b_j|)}^{\infty} z^{-2}z^{-\alpha}\tilde{L}(z)\mathrm{d}z \leqslant C_1\int_{x/(2|b_j|)}^{\infty} z^{-2}\,\mathrm{d}z \leqslant C_2\frac{|b_j|}{x}.$$

By substituting the bound (3.41) into (3.40) and by using Karamata's theorem with a change of variables as above, we obtain that

$$\mathrm{E}\left(\left|\sum_{j=m_1}^{m_2} b_j\epsilon_j\right|^2 1_{\left\{|\Sigma_{j=m_1}^{m_2} b_j\epsilon_j|<1\right\}}\right) \leqslant C\sum_{j=m_1}^{m_2}\left\{\int_0^{2|b_j|} x\,\mathrm{d}x + |b_j|^{\alpha}\int_{2|b_j|}^{1} x^{1-\alpha}\tilde{L}\left(\frac{x}{|b_j|}\right)\mathrm{d}x\right\}$$

$$\leqslant C'\sum_{j=m_1}^{m_2}\left\{|b_j|^2 + |b_j|^2\int_2^{1/|b_j|} y^{1-\alpha}\tilde{L}(y)dy\right\}$$

$$\leqslant C''\sum_{j=m_1}^{m_2}\left\{|b_j|^2 + |b_j|^{\alpha}\tilde{L}\left(\frac{1}{|b_j|}\right)\right\}$$

$$\leqslant C\sum_{j=m_1}^{m_2}|b_j|^{\alpha}\tilde{L}\left(\frac{1}{|b_j|}\right).$$

Finally, the bound (3.34) can be proved in a similar way by using (3.35) and (3.36). □

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