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Revisiting conditional limit theorems for the mortal simple branching process

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The various conditional limit theorems for the simple branching process are considered within a unified setting. In the subcritical case conditioning events have the form $\{\mathscr{H} \in n + \mathscr{S}\}$, where $\mathscr{H}$ is the time to extinction and $\mathscr{S}$ is a subset of the natural numbers. The resulting limit theorems contain all known forms, and collectively they are equivalent to the classical Yaglom form. In the critical case discrete limits exist provided $\mathscr{S}$ is a finite set. The principal results are extended to absorbing Markov chains. The Yaglom and Harris theorems for the critical case are generalized by considering the joint behaviour of generation sizes and total progeny conditioned by one-parameter families of events of the form $\{n < \mathscr{H} \leqslant \alpha n\}$ and $\{\mathscr{H} > \alpha n\}$, where $1 \leqslant \alpha \leqslant \infty$. A simple representation of the marginal limit laws of the population sizes relates the Yaglom and Harris limits. Analagous structure is elucidated for the marginal limit laws of the total progeny.

Keywords: branching process; conditional limit theorem; diffusion approximation; extinction time; total progeny

1. Introduction

We will adopt as a setting for our discussion the simple discrete-time branching process, although many of the general ideas and results will extend to other classes of branching process, and even to countable-state Markov chains and processes under appropriate conditions. So let $\{Z_n : n \in \mathbb{N}_0\}$ denote the successive generation sizes of the Bienaymé–Galton–Watson branching process having the offspring law $\{p_j : j \in \mathbb{N}_0\}$, where $p_j \neq 1$ for any j and $p_0 + p_1 < 1$. Denote the offspring probability generating function by $f(s) = \sum p_j s^j$. Let $m = f'(1-)$ be the mean per capita number of offspring. The number of ancestors is indicated using the notation $P_i(\cdot) = P(\cdot \mid Z_0 = i)$, and similarly for the expectation operator. A general initial law is always denoted by μ, with masses $\mu(i)$ where $\mu(0) = 0$, and $P_\mu(\cdot) = \sum_i \mu(i) P_i(\cdot)$, and similarly for the expectation operator.

Let $\mathscr{H}$ denote the hitting time of the zero state, i.e. the time to extinction of our branching process. The fundamental extinction theorem states that $P_i(\mathscr{H} < \infty) = q^i$, where q is the least non-negative solution of $f(s) = s$, *and* that $q = 1$ if and only if $m \leqslant 1$. Athreya and Ney (1972) is a good general reference for this and other basic facts about branching processes.

We will be interested in the behaviour of the conditional law of Z_n given that $\mathscr{H}$ takes

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values in certain subsets of $\{n+1, n+2, \ldots\}$. This condition entails that $\mathscr{H} < \infty$, i.e. our attention will be restricted to mortal family trees. Thus we can assume that $m \leqslant 1$ since the supercritical process conditioned on mortality is equivalent in terms of its finite-dimensional laws to the subcritical branching process whose offspring probability generating function is $f(sq)/q$. We make this assumption throughout this paper.

Scarcely less fundamental than the extinction theorem are several conditional limit theorems usually attributed to A.M. Yaglom. These theorems make assertions about the limit as $n \to \infty$ of the conditional law of Z_n given that $\mathscr{H} > n$ (written as $\mathscr{L}(Z_n|\mathscr{H} > n)$). When it exists, the limit is called the (singly) limiting conditional law. When $m = 1$ the generation sizes must be normed in order to give a non-trivial limit law. Yaglom obtained his results under certain moment assumptions which have since been removed by other investigators, and the results themselves have been supplemented by theorems allowing different conditioning events for $\mathscr{H}$. We give references below.

The essential contribution of this paper will be to unify and extend the scope of what is known about the behaviour of Z_n, and the total progeny $T_n = \sum_{j=0}^{n} Z_j$, when they are conditioned by events which imply that $\mathscr{H} > n$. In Section 2 we shall show that all known conditional limit theorems for $\{Z_n\}$ (no norming) can be embedded in just two assertions – Theorem 2.1 if $m < 1$ and Theorem 2.5 if $m = 1$. In Sections 3–5 we are concerned with the critical case $m = 1$ and $\upsilon = f''(1-)/2 < \infty$. In Section 3 we present new limit theorems for the random vectors (Z_n, T_n) when they are suitably normed and conditioned on certain events which imply that $\mathscr{H} > n$. These new results are contained in Corollary 3.1, Theorem 3.3 and Theorem 3.5. They are used in Sections 4 and 5 to give an extended and unified treatment of known conditional limiting results for T_n. Let us give some more detail.

In Section 2 we will allow conditioning events of the form $\mathscr{A}_n = \{\mathscr{H} \in n + \mathscr{S}\}$, where $\mathscr{S} \subset \mathbb{N}$ is chosen to give discrete limit laws. This builds on some earlier work of Pakes (1998) by allowing initial laws other than point masses, and tailoring the proofs to emphasize the fact that our generalizations are equivalent to certain of the known results. When $m < 1$ we allow $\mathscr{S}$ to be quite arbitrary – see Theorem 2.1. We also establish some links with the doubly limiting conditional law in Theorems 2.2–2.4. This is the weak limit law of $(Z_n|\mathscr{H} > n + \nu)$ as $\nu \to \infty$ and then as $n \to \infty$. These results provide some compelling probabilistic expressions of the notion of 'quasi-stationarity'. In Theorem 2.5 we show when $m = 1$ that a non-trivial limiting conditional law exists if and only if $\mathscr{S}$ has finite cardinality.

In Sections 3–5 we are concerned with the critical case, and $\mathscr{S}$ is an infinite set of a fairly simple kind. The Yaglom theorem asserts convergence of $\mathscr{L}(Z_n/\upsilon n|\mathscr{H} > n)$ to the standard exponential law. Pakes (1998) observes that known results on diffusion approximations imply the following extension. For $1 \leqslant \alpha < \infty$, $\mathscr{L}(Z_n/\upsilon n|\mathscr{H} > \alpha n)$ has a limit law which can be represented as

$$\lambda = \varepsilon_1 + (1 - \alpha^{-1})\varepsilon_2, \tag{1.1}$$

where ε_1 and ε_2 are independent random variables having the standard exponential law. See Lamperti and Ney (1968) for the usual form of this result, and Athreya and Ney (1972, p. 61). Khalili (1981) gives a multivariate extension of the Lamperti–Ney theorem. The case $\alpha = 1$ is the critical Yaglom theorem and the case $\alpha \to \infty$ can be interpreted to be the limit law of $\mathscr{L}(Z_n/\upsilon n|\mathscr{H} = \infty)$, where the conditioning event is taken to mean

$$\mathscr{L}(Z_n|\mathscr{H}=\infty)=\lim_{\nu\to\infty}\mathscr{L}(Z_n|\mathscr{H}>n+\nu). \tag{1.2}$$

The resulting iterated limit, $\alpha=\infty$ in (1.1), is due to Harris (1951).

A principal aim of this paper is to investigate analogous structure for the total progeny T_n in the critical case. In particular, we seek to link together the limit theorems for $\mathscr{L}(T_n/\upsilon n^2|\mathscr{H}>n)$ and $\mathscr{L}(T_n/\upsilon n^2|\mathscr{H}=\infty)$, first found by Pakes (1971a). This requires the determination of the limit of the joint conditional law $\mathscr{L}(Z_n/\upsilon n,\, T_n/\upsilon n^2|\mathscr{H}>\alpha n)$ where $\alpha\geqslant 1$. Our principal results in this direction are Theorems 3.1 and 3.2. Their proof is based on Durrett's (1978) conditional functional limit theorem, but we will use a different time parametrization – see (3.11). Theorem 3.2 is strengthened in Corollary 3.1 to allow conditioning events of the form $\{\mathscr{H}>\alpha n+o(n)\}$. Next, in Theorem 3.3 we use Theorem 3.1 to obtain the limit law of $(Z_n/\upsilon n,\, T_n/\upsilon n^2|\mathscr{H}>n+\nu)$ in which $\nu/n\to\infty$. The proof introduces a branching process augmented by independent immigrations whose numbers have the probability generating function $f'(s)$. Only a little more effort is needed to obtain a limit theorem for the joint law of the size and total progeny in the general case, and we include this as Theorem 3.4. Theorem 3.2 conditions on extinction *after* αn. Theorem 3.5 complements this result by conditioning on extinction occurring at or *before* αn.

Section 4 is devoted to elucidating the nature of the marginal limit laws for T_n in Theorems 3.2 and 3.5. The cases $\alpha=1$ and $\alpha=\infty$ for the conditioning $\{\mathscr{H}>\alpha n\}$ were those treated by Pakes (1971a). Their representations in terms of series of independent exponential random variables are discussed by Pakes (1997). First, we relate the limit laws of Theorems 3.2 to the laws of certain functionals of Brownian motion. We then seek an analogue of the limit law (1.1) of the generation sizes. The result, (4.8) together with Theorem 4.2, is more complicated. But it still retains the fundamental structure of (1.1), i.e. as the sum of the limit for $\alpha=1$ plus an independent component which increases stochastically with α. A closely related representation is found for the conditioning events $\{n<\mathscr{H}\leqslant\alpha n\}$ – see Theorem 4.3 and (4.19).

Finally, we examine the limit of $\mathscr{L}(T_n/\upsilon n^2|\mathscr{H}\in n+\mathscr{S})$ with $\mathscr{S}$ finite. Kesten (1986) proved its existence when $\mathscr{S}=\{0\}$ (i.e. $\mathscr{A}_n=\{\mathscr{H}=n\}$) without identifying its nature, and Karpenko and Nagaev (1994) gave a different proof in a more general context which yields its Laplace–Stieltjes transform. The resulting limit law is precisely that found from the case $\alpha=1$ in Theorem 3.5. (The corresponding limit law for the generation sizes is of course the point mass at zero.) We will show that this limit persists for any finite $\mathscr{S}$, and offer a modification of Kesten's argument which gives the limit law more easily than the Karpenko–Nagaev proof.

The total contribution of the limit theorems in Sections 3 and 5 is a unified description of the limit behaviour of generation sizes and total progeny given that extinction occurs in the immediate future ($\mathscr{H}<n+o(n)$), then grading into extinction in the intermediate future ($n<\mathscr{H}\leqslant\alpha n$ with $\alpha>1$), and then in the more remote future ($\mathscr{H}>\alpha n$), and finally, at the end of time ($\mathscr{H}=\infty$).

In Section 6 we point out that the essential content of Lemma 2.1, and Theorems 2.1, 2.2 and 2.4, extends to general Markov chains having a countable state space containing a proper irreducible and R-positive subset.

2. Discrete limit laws

Let $f_n(s)$ denote the n-fold functional iterate of $f(s)$ and $f_n = f_n(0)$. Then $E_i(s^{Z_n}) = (f_n(s))^i$. Let $\mathscr{M}(s)$ denote the probability generating function of the initial law μ. The basic conditional limit theorem for the subcritical branching process can be expressed as follows.

Proposition 2.1. *Suppose* $m < 1$ *and*

$$\mathscr{M}(s) = 1 - (1-s)^{\alpha} L_{\mu}((1-s))^{-1} \tag{2.1}$$

for some $0 < \alpha \leqslant 1$ *and slowly varying function* L_μ. *Then*

$$\bar{q}_j = \lim_{n\to\infty} P_\mu(Z_n = j | \mathscr{H} > n) \qquad (j = 1, 2, \ldots) \tag{2.2}$$

exists and defines a non-defective law. Its probability generating function is

$$\bar{Q}(s) = 1 - (1 - Q(s))^{\alpha},$$

where $Q(s) = \sum q_j s^j$ *is the probability generating function of the limit law when* $\mathscr{M}'(1-) < \infty$ *(and then* $\alpha = 1$*). The probability generating function* Q *is the unique such solution of*

$$1 - Q(f(s)) = m(1 - Q(s)). \tag{2.3}$$

Conversely, if for some initial law μ *the limit (2.2) exists and comprises a non-defective law, then* $\mathscr{M}$ *has the form (2.1) for some* $0 < \alpha \leqslant 1$ *and slowly varying* $\mathscr{L}_\mu$.

The direct assertion when $\mathscr{M}(s) = s^i$ and $f''(1) < \infty$ is the original form of Yaglom. The second-order moment condition was removed almost simultaneously by Heathcote *et al.* (1967) and by Joffe (1967); the dates of reception differ by several months. Joffe gives the result in the form (2.2), whereas the development of Heathcote *et al.* is entirely in terms of generating functions. The probabilistic interpretation of the latter contribution is exposed by Seneta and Vere-Jones (1966). The result is also attributed to Nagaev and Badalbaev (1967), but this reference is inaccessible to the author. Rubin and Vere-Jones (1968) contributed the last assertion of Proposition 1. A monograph account is given by Asmussen and Hering (1983). We refer to the limit law in (2.2) as a μ-limiting conditional law, and we refer to $\{q_j\}$ as the limiting conditional law.

The convergence parameter of the sub-stochastic matrix $[p_{ij}] = [P_i(Z_1 = j) : i, j \in \mathbb{N}]$ is $R = m^{-1}$ and this matrix is R-recurrent (for all m) and it is R-positive when $m < 1$ if and only if the LOG condition holds:

$$E_1(Z_1 \log^{+} Z_1) < \infty. \tag{2.4}$$

The existence of the limiting conditional law is, in the R-positive case, guaranteed by the general theory of quasi-stationary laws – see Seneta and Vere-Jones (1966) for these matters. The limiting conditional law is also a quasi-stationary law, which usually is understood in the sense of R-invariance, meaning that

$$R\sum_{i\geqslant 1} q_i p_{ij} = q_j \qquad (j \geqslant 1). \tag{2.5}$$

This system, together with the normalization $\sum q_j = 1$, is equivalent to (2.3). A much more satisfactory characterization is as follows. Write $P_Q(\cdot)$ for $P_\mu(\cdot)$ when $\mathscr{M}(s) = Q(s)$. Then (2.5) is equivalent to the probabilistic property

$$P_Q(Z_n = j|\mathscr{H} > n) = q_j \qquad (j,\, n \geqslant 1).$$

More generally,

$$P_{\bar{Q}}(Z_n = j|\mathscr{H} > n) = \bar{q}_j. \tag{2.6}$$

This relation is effectively that adopted by Kesten (1995) as his definition of a quasi-stationary law for countable-state Markov chains – see his (1.15).

We end this mini-review by recalling that

$$\mathrm{E}_i(Z_n|\mathscr{H} > n) = \frac{m^n}{1 - f_n} \to C = Q'(1-)$$

and that $C < \infty$ if and only if (2.4) holds. The convergence assertion with C finite is due originally to Kolmogorov under the assumption that $f''(1-) < \infty$. The full assertion is due to Heathcote *et al*. (1967), and an intermediate assertion was proved by Joffe (1967).

Let $\bar{m} = m^\alpha$ and let ξ_p denote a random variable having the shifted geometric law $P(\xi_p = j) = (1 - p)p^{j-1}$ $(j = 1, 2, \ldots)$. Next, let $\mathscr{S} \subset \mathbb{N}$. Using the fact that $(1 - f_{n+\nu})/(1 - f_n) \to m^\nu$ $(n \to \infty)$, we obtain the following lemma.

Lemma 2.1. *If* $m < 1$ *and (2.1) holds, then*

$$\lim_{n\to\infty} P_\mu(\mathscr{H} \in n + \mathscr{S}|\mathscr{H} > n) = P(\xi_{\bar{m}} \in \mathscr{S}). \tag{2.7}$$

This assertion appears in Pakes (1998) for the case $\mathscr{M}(s) = s^i$, and then $\bar{m} = m$. It is clear that

$$P_{\bar{Q}}(\mathscr{H} = n) = P(\xi_{\bar{m}} = n).$$

Our next result is a formal generalization of Proposition 2.1. Recall that $\mathscr{A}_n = \{\mathscr{H} \in n + \mathscr{S}\}$.

Theorem 2.1. *Suppose that* $\mathscr{S}$ *is fixed,* $m < 1$ *and (2.1) holds. Then*

$$\lim_{n\to\infty} P_\mu(Z_n = j|\mathscr{A}_n) = \bar{q}_j(\mathscr{S}) \equiv \bar{q}_j \frac{P_j(\mathscr{H} \in \mathscr{S})}{P(\xi_{\bar{m}} \in \mathscr{S})}. \tag{2.8}$$

Proof. This is substantially as in Pakes (1998), but we include it to emphasize that it follows from the direct assertion of Proposition 2.1. Simply observe that

$$P_\mu(Z_n = j|\mathscr{A}_n) = \frac{\sum_{\nu\in\mathscr{S}} P_\mu(Z_n = j;\, \mathscr{H} = n+\nu)}{P_\mu(\mathscr{H} \in n+\mathscr{S})}$$

$$= \frac{\sum_{\nu\in\mathscr{S}} P_\mu(Z_n = j)P_j(\mathscr{H} = \nu)}{P_\mu(\mathscr{H} \in n+\mathscr{S})}$$

$$= P_\mu(Z_n = j|\mathscr{H} > n)P_j(\mathscr{H} \in \mathscr{S})\frac{P_\mu(\mathscr{H} > n)}{P_\mu(\mathscr{H} \in n+\mathscr{S})}. \tag{2.9}$$

The first factor converges ($n \to \infty$) to $\bar{q}_j$ if and only if (2.1) holds, and the last factor converges to the denominator in (2.8) when (2.1) holds. The limit law is non-defective since $\bar{Q}(f_n) = 1 - \bar{m}^n$. □

Observe that the limit law in (2.8) has an invariance property dual to (2.6):

$$P_{\bar{Q}}(Z_n = j|\mathscr{A}_n) \equiv \bar{q}_j \frac{P_j(\mathscr{H} \in \mathscr{S})}{P(\xi_{\bar{m}} \in \mathscr{S})}.$$

The proof shows that (2.8) is equivalent to (2.2) when (2.1) holds. The existence of the limit in (2.8) would imply (2.1) if (2.7) were to imply (2.1). However, there appears to be insufficient structure to attain this implication. The convergence (2.7) is equivalent to

$$\lim_{n\to\infty} \frac{1 - \mathscr{M}(f_{n+\nu})}{1 - \mathscr{M}(f_n)} = m^{\alpha\nu} \qquad (\nu = 0, 1, \ldots).$$

Let $r(t) = (1 - \mathscr{M}(1-t))t^{-\alpha}$. By decomposing $r(t)$ into the monotone factors $(1 - \mathscr{M}(1-t))/t$ and $t^{1-\alpha}$ one can easily show for $0 < t < 1$ that the limit points of $r(t(1-f_n))/r(1-f_n)$ are contained in the interval $[m^\alpha, m^{-\alpha}]$ and hence $1 - \mathscr{M}(1 - 1/v)$ is O-regularly varying – see Bingham *et al.* (1987, p. 65). When $\alpha = 1$ the same approach shows that (2.1) follows from (2.7). We can deduce (2.1) from (2.7) under the additional hypothesis that $L_\mu(\cdot)$ is ultimately monotone. Of course none of this resolves the question of whether the existence of the limit (2.8) is equivalent to (2.1) since the n-dependent factors in the above proof could in principle oscillate in a self-compensating manner.

Observe that the probability generating function of the limit law is

$$\bar{Q}(s, \mathscr{S}) = \frac{\sum_{\nu\in\mathscr{S}} [\bar{Q}(sf_\nu) - \bar{Q}(sf_{\nu-1})]}{\sum_{\nu\in\mathscr{S}} [\bar{Q}(f_\nu) - \bar{Q}(f_{\nu-1})]}.$$

When $\mathscr{S} = \nu + \mathbb{N}$ we obtain the limiting probability generating function

$$\bar{Q}_\nu(s) = \frac{\bar{Q}(s) - \bar{Q}(sf_\nu)}{m^{\alpha\nu}} = \alpha(\zeta_\nu(s))^{\alpha-1}\frac{Q(s) - Q(s\nu)}{m^{\alpha\nu}}, \tag{2.10}$$

where $Q(sf_\nu) < \zeta_\nu(s) < Q(s)$. Now let $\nu \to \infty$. The limit is zero when $\alpha < 1$, but when $\alpha = 1$ we obtain (dropping the overbar)

$$\lim_{\nu\to\infty} Q_\nu(s) = sQ'(s)/C.$$

Summarizing, we have the following theorem.

Theorem 2.2. *Let $m < 1$ and (2.1) hold with $\alpha = 1$. Then*

$$\lim_{\nu\to\infty}\lim_{n\to\infty} P_\mu(Z_n = j|\mathscr{H} > n+\nu) = jq_j/C \tag{2.11}$$

and the limit comprises a non-defective law if and only if (2.4) holds.

The limit law in this theorem is usually seen in the case where the initial population size is fixed and the double limit is taken in *reverse* order, giving what we called the doubly-limiting conditional law. This appears first in Seneta and Vere-Jones (1966, p. 422), and subsequently in Papangelou (1968, p. 1476). If the limit law (2.11) is shifted one unit to the left, then the result is the limiting stationary law obtained from our branching process after augmenting it with independent arrivals of immigrants into each generation, and in numbers whose probability generating function is $f'(s)/m$. After allowing for this spatial shift, the n-step transition probabilities of the immigration process appear as the limits $\lim_{\nu\to\infty} P_i(Z_n = j|\mathscr{H} > n+\nu)$. This connection (valid for $m \leqslant 1$) was first noticed by Pakes (1971a) and independently by Khalili-Françon (1973). It was also noted much later by Sagitov (1986). A more formal development is given by Athreya and Ney (1972, p. 56) in terms of what they call the Q-process.

We observe that the relation between (2.11) and the more usual form giving the doubly-limiting conditional law is not completely symmetric as the the doubly-limiting conditional law requires a slightly stronger condition on μ for its existence. We show this in the next result, which also asserts a probabilistic invariance property of the doubly-limiting conditional law.

Theorem 2.3. *Suppose that $m < 1$ and (2.4) holds. Then*

$$\lambda_j(n) \equiv P_\mu(Z_n = j|\mathscr{H} = \infty)$$

(as defined by (1.2)) exists. The limit is identically zero if $\mathscr{M}'(1) = \infty$, but when

$$\mathscr{M}'(1) < \infty \tag{2.12}$$

the limit comprises a non-defective law having the probability generating function

$$\lambda_n(s) = s\frac{\mathscr{M}'(f_n(s))}{\mathscr{M}'(1)}\prod_{\ell=0}^{n-1} f'(f_\ell(s))/m. \tag{2.13}$$

The doubly-limiting conditional law exists if and only if (2.12) holds, and when it does,

$$\lim_{n\to\infty}\lim_{\nu\to\infty} P_\mu(Z_n = j|\mathscr{H} > n+\nu) = jq_j/C. \tag{2.14}$$

Finally,

$$P_Q(Z_n = j | \mathscr{H} = \infty) = jq_j/C \qquad (n = 1, 2, \ldots). \tag{2.15}$$

Proof. A two-stage application of the mean value theorem gives

$$\begin{aligned} \mathrm{E}_\mu(s^{Z_n} | \mathscr{H} > n + \nu) &= \frac{\mathscr{M}(f_n(s)) - \mathscr{M}(f_n(sf_\nu))}{1 - \mathscr{M}(f_{n+\nu})} \\ &= \frac{\mathscr{M}'(\xi_\nu(s))}{\mathscr{M}'(\zeta_\nu)} \cdot \frac{f_s(s) - f_n(sf_\nu)}{1 - f_{n+\nu}} \\ &= \frac{\mathscr{M}'(\xi_\nu(s))}{\mathscr{M}'(\zeta_\nu)} \cdot \frac{s(1 - f_\nu)}{1 - f_{n+\nu}} f'_n(s\zeta'_\nu), \end{aligned}$$

where $f_n(sf_\nu) < \xi_\nu(s) < f_n(s)$, $f_{n+\nu} < \zeta_\nu < 1$ and $f_\nu < \zeta'_\nu < 1$. Since $\xi_\nu(s) \to f_n(s)$ and ζ_ν, $\zeta'_\nu \to 1$ as $\nu \to \infty$, the above probability generating function converges, and the limit is identically zero when (2.12) fails. But when (2.12) holds the limit is given by (2.13) which is a probability generating function since $\lambda_n(1) = 1$. This completes the proof of the first assertion.

Differentiating (2.2) and iterating the result gives

$$Q'(s) = Q'(f_n(s)) \prod_{\ell=0}^{n-1} [f'(f_\ell(s))/m],$$

whence

$$\lambda_n(s) = \frac{\mathscr{M}'(f_n(s))}{\mathscr{M}'(s)} \frac{Q'(s)}{Q'(f_n(s))}.$$

Letting $n \to \infty$ yields (2.14). Finally, setting $\mathscr{M}(s) = Q(s)$ yields $\lambda_n(s) = Q'(s)/Q'(1)$ and (2.15) follows. $\square$

Observe that (2.13) is valid for $m \leqslant 1$. The coefficient of s in this equation is the probability generating function of the size at time n of the simple branching process defined above and having an initial law whose probability generating function is $\mathscr{M}'(s)/\mathscr{M}'(1)$. Indeed

$$\lambda_j(n) = jP_\mu(Z_n = j)/m^n \mathscr{M}'(1),$$

showing that the size biasing evident in (2.15) arises from the inner limiting operation in (2.14), i.e. in (1.2).

We end our discussion of the subcritical case by showing that the doubly-limiting conditional law can be obtained from a single limiting operation.

Theorem 2.4. *Let $m < 1$, (2.1) hold, $\mathscr{S} \in \mathbb{N}$, and n' be a sequence of integers tending to infinity. Then $\lim_{n\to\infty} P_\mu(Z_n = j | \mathscr{A}_{n+n'})$ exists. If $\alpha = 1$ and (2.4) holds, then*

$$\lim_{n\to\infty} P_\mu(Z_n = j | \mathscr{A}_{n+n'}) = jq_j/C,$$

and if either condition fails then the limit is zero.

Proof. Replace $\mathscr{S}$ in (2.9) with $n' + \mathscr{S}$. Then, using (2.2) and (2.7), we obtain

$$P_\mu(Z_n = j | \mathscr{A}_{n+n'}) \sim \bar{q}_j P(\xi_m \in \mathscr{S}) P_j(\mathscr{H} > n') \frac{P_\mu(\mathscr{H} > n)}{P(\xi_{\bar{m}} \in \mathscr{S}) P_\mu(\mathscr{H} > n + n')}$$

$$\sim j\,\bar{q}_j \frac{P(\xi_m \in \mathscr{S})}{P(\xi_{\bar{m}} \in \mathscr{S})} \cdot \frac{1 - f_{n'}}{m^{\alpha n'}}.$$

The last factor tends to C^{-1}, which is zero unless $\alpha = 1$ and (2.4) holds. □

We turn now to the critical case $m = 1$ and review some known results. Let $\{\pi_j; j \geqslant 1\}$ denote the stationary measure of the branching process. It is unique up to a constant multiplier and $\sum \pi_j = \infty$. Its generating function can be derived as in Seneta (1967, p. 492),

$$U(s) = \sum_{j \geqslant 1} \pi_j s^j = \lim_{n \to \infty} \frac{f_n(s) - f_n}{f_{n+1} - f_n}, \tag{2.16}$$

and this entails

$$U(f_n) = n. \tag{2.17}$$

The following result extends Proposition 3.2 of Pakes (1998) by allowing initial laws of type (2.1) and giving a sharper conclusion.

Theorem 2.5. *Suppose that $m = 1$ and (2.1) holds. Then*

$$\lim_{n \to \infty} P_\mu(Z_n = j | \mathscr{A}_n) = \begin{cases} \pi_j \dfrac{P_j(\mathscr{H} \in \mathscr{S})}{|\mathscr{S}|} & \text{if } |\mathscr{S}| < \infty \\ 0 & \text{if } |\mathscr{S}| = \infty. \end{cases}$$

Proof. We begin by observing that

$$\mathrm{E}_\mu(s^{Z_n} | \mathscr{H} = n + \nu) = \frac{\mathscr{M}(f_n(sf_\nu)) - \mathscr{M}(f_n(sf_{\nu-1}))}{\mathscr{M}(f_{n+\nu}) - \mathscr{M}(f_{n+\nu-1})}$$

$$= \frac{\mathscr{M}'(\zeta_{n,\nu})}{\mathscr{M}'(\xi_{n,\nu})} \cdot \frac{f_n(sf_\nu) - f_n(sf_{\nu-1})}{f_{n+\nu} - f_{n+\nu-1}}, \tag{2.18}$$

where $f_n(sf_{\nu-1}) < \zeta_{n,\nu} < f_n(sf_\nu)$ and $f_{n+\nu-1} < \xi_{n,\nu} < f_{n+\nu}$. The second factor is $\mathrm{E}_1(s^{Z_n} | \mathscr{H} = n + \nu)$, and this converges to $U(sf_\nu) - U(sf_{\nu-1})$ by virtue of (2.16). Observe that (2.16) and (2.17) yield

$$\frac{f_{n+\nu} - f_{n+\nu-1}}{f_{n+1} - f_n} \to U(f_\nu) - U(f_{\nu-1}) = 1.$$

Next, since $\mathscr{M}'(s)$ is monotone, it follows from (2.1) that $\mathscr{M}'(s) \sim \alpha(1 - s)^{\alpha-1} L_\mu((1 - s)^{-1})$. Then since

$$\frac{1 - f_n(sf_\nu)}{1 - f_n(sf_{\nu-1})} \leqslant \frac{1 - \xi_{n,\nu}}{1 - \zeta_{n,\nu}} \leqslant \frac{1 - f_n(sf_{\nu-1})}{1 - f_n(sf_\nu)}$$

and the left- and right-hand sides tend to 1, we conclude that $\mathscr{M}'(\zeta_{n,\nu})/\mathscr{M}'(\xi_{n,\nu}) \to 1$. Hence

$$P_\mu(Z_n = j | \mathscr{H} = n + \nu) \to \pi_j(\nu) \equiv \pi_j(f_\nu^j - f_{\nu-1}^j). \tag{2.19}$$

For the last step we rearrange the first equality in (2.9), obtaining

$$P_\mu(Z_n = j | \mathscr{A}_n) = \frac{\sum_{\nu \in \mathscr{S}} P_\mu(Z_n = j)(f_\nu^j - f_{\nu-1}^j)}{\sum_{\nu \in \mathscr{S}} P_\mu(\mathscr{H} = n + \nu)}. \tag{2.20}$$

Now (2.19) is equivalent to

$$P_\mu(Z_n = j)/P_\mu(\mathscr{H} = n) \to \pi_j,$$

and by working as above we obtain the ratio limit result

$$P_\mu(\mathscr{H} = n + \nu)/P_\mu(\mathscr{H} = n) \to 1.$$

The assertion for $|\mathscr{S}| < \infty$ follows directly.

When $|\mathscr{S}| = \infty$ we replace $\mathscr{S}$ in the numerator of (2.20) with $\mathbb{N}$ to obtain

$$P_\mu(Z_n = j | \mathscr{A}_n) \leqslant \frac{P_\mu(Z_n = j)}{P_\mu(\mathscr{H} = n)} \left(\sum_{\nu \in \mathscr{S}} P_\mu(\mathscr{H} = n) \right)^{-1}.$$

The first factor tends to π_j, and by Fatou's lemma the limsup of the second factor is less than or equal to $|\mathscr{S}|^{-1} = 0$. □

3. A joint limit law for the critical case with finite variance

Let $P_n(s, r) = \mathrm{E}_1(s^{Z_n} r^{T_n})$. Then, as shown by Pakes (1971a), $P_0(s, r) = sr$, $P_{n+1}(s, r) = rf(P_n(s, r))$ and $\mathrm{E}_\mu(s^{Z_n} r^{T_n}) = \mathscr{M}(P_n(s, r))$. We begin by stating the following conditional limit theorem for (Z_n, T_n).

Theorem 3.1. *Let $m = 1$ and $v = f''(1-)/2 < \infty$. If the initial law satisfies (2.1) then*

$$\mathscr{L}(Z_n/vn, T_n/vn^2 | \mathscr{H} > n) \Rightarrow (\varepsilon, A),$$

where

$$\mathrm{E}(\mathrm{e}^{-\zeta\varepsilon - \theta A}) = \phi(\zeta, \theta) \equiv \frac{\theta \operatorname{cosech} \sqrt{\theta}}{\sqrt{\theta} \cosh \sqrt{\theta} + \zeta \sinh \sqrt{\theta}} = \frac{2\sqrt{\theta} \operatorname{cosech}(2\sqrt{\theta})}{1 + \zeta \dfrac{\tanh \sqrt{\theta}}{\sqrt{\theta}}}. \tag{3.1}$$

In the case of the critical Markov branching process, Puri (1969) obtained the joint limit law for the population size at time t, and the number of deaths and the integral under the population trajectory up to time t. He assumes the offspring law has a finite third-order moment. Setting $u = \zeta$, $v = \theta$ and $w = 0$ in his Theorem 9 gives an expression that can be put into the form (3.1).

Before proving Theorem 3.1, we shall explore some of its consequences. Setting $\theta = 0$ in (3.1) yields the familiar exponential limit law for $\mathscr{L}(Z_n/vn|\mathscr{H} > n)$, and setting $\zeta = 0$ gives the limiting Laplace–Stieltjes transform for the total progeny found by Pakes (1971a). The density function of this law can be expressed in terms of a theta function, and we have an 'in law' representation as a sum of independent exponential random variables,

$$A \stackrel{d}{=} 4U = \frac{4}{\pi^2}\sum_{n=1}^{\infty} \varepsilon_n/n^2. \tag{3.2}$$

See Pakes (1997) for these.

We know $\phi(\zeta, \theta)$ is a bivariate Laplace–Stieltjes transform by virtue of its derivation. It is not obvious (to the author) how this could be demonstrated just from the analytical expression (3.1), or whether there is a random variable representation similar to (3.2). However, there is a representation in terms of functionals of a standard Brownian motion process (B_t), as follows.

The Laplace–Stieltjes transform of U in (3.2) is $\sqrt{\theta}\operatorname{cosech}\sqrt{\theta}$. Factoring this out of (3.1) gives

$$\phi(\zeta, \theta) = (\sqrt{\theta}\operatorname{cosech}\sqrt{\theta})\cdot\frac{\sqrt{\theta}}{\sqrt{\theta}\cosh\sqrt{\theta} + \zeta\sinh\sqrt{\theta}}. \tag{3.3}$$

Let $M_t = \max_{\tau \leqslant t} B_\tau$ and

$$\top = \inf\{t : M_t - B_t = 1\}.$$

The joint Laplace–Stieltjes transform of $(M_\top, \top)$ is the second factor in (3.3), i.e.

$$\mathrm{E}(\mathrm{e}^{-\zeta M_\top - \theta\top}) = \frac{\operatorname{sech}\sqrt{\theta}}{1 + \zeta\dfrac{\tanh\sqrt{\theta}}{\sqrt{\theta}}}. \tag{3.4}$$

This was first shown by Taylor (1975), and then more simply by Williams (1976). It follows that if U (defined in (3.2)) is chosen to be independent of $(M_\top, \top)$ then

$$\mathscr{L}(\varepsilon, A) = \mathscr{L}(M_\top, U + \top). \tag{3.5}$$

Proof of Theorem 3.1. It is clear that

$$\mathrm{E}_\mu(s^{Z_n} r^{T_n}|\mathscr{H} > n) = \frac{\mathscr{M}(P_n(s, r)) - \mathscr{M}(P_n(0, r))}{1 - \mathscr{M}(f_n)}. \tag{3.6}$$

Since critical processes are almost surely mortal, $T_n \uparrow T_\infty$ where T_∞ is the total number of individuals ever to have existed. Its law is non-defective and its probability generating function $g(\cdot)$ solves $g(r) = rf(g(r))$. Hence $P_n(s, r) \to g(r)$ as $n \to \infty$ and $0 \leqslant s \leqslant 1$ – see Pakes (1971a). So if $0 \leqslant s_n \leqslant 1$ and $r_n \to 1$, then $P_n(s_n, r_n) \to 1$, and it follows in the usual way that any limit law which might result from (3.6) will be independent of the form of μ provided (2.1) holds. Thus we can, and shall, take $\mathscr{M}(s) = s$.

We will use the following known weak convergence theorems for a sequence of simple branching processes. For each n let $\{Z_\nu^{(n)} : \nu = 0, 1, \ldots\}$ denote a simple critical branching process with offspring probability generating function $f(s)$ as above, $\upsilon < \infty$, and an initial size depending on n which satisfies $Z_0^{(n)} \to \infty$ $(n \to \infty)$. Define the continuous-time interpolated process $(\upsilon_t^{(n)} : t \geqslant 0)$ by

$$\upsilon_t^{(n)} = Z_{[nt]}^{(n)}/\upsilon n.$$

Building on earlier work, Lindvall (1974) has shown that if $\upsilon_0^{(n)} \to z > 0$ then

$$(\upsilon_t^{(n)}) \Rightarrow (\upsilon_t),$$

where the convergence is in the sense of weak convergence of random elements in the function space $D[0, \infty)$ of cadlag functions, and the limit process is Feller's continuous branching (CB) diffusion. The CB process solves the Itô equation

$$\mathrm{d}\upsilon_t = \sqrt{2\upsilon_t}\,\mathrm{d}B_t, \tag{3.7}$$

and its transition law is given via

$$\mathrm{E}_z(\mathrm{e}^{-\zeta\upsilon_t}) = \exp\left[-\frac{z\zeta}{1+\zeta t}\right]. \tag{3.8}$$

This law has an absolutely continuous component whose density is

$$p(x,\ t|z) = \frac{1}{t}\sqrt{\frac{z}{x}}\exp\left(-\frac{z+x}{t}\right) I_1\left(\frac{2\sqrt{zx}}{t}\right), \tag{3.9}$$

where $I_1(\cdot)$ is a modified Bessel function of the first kind. The origin is an accessible absorbing state. Let η denote its hitting time, i.e. the extinction time of the CB diffusion. Then

$$P_z(\upsilon_t = 0) = P_z(\eta \leqslant t) = \exp(-z/t). \tag{3.10}$$

Durrett (1978, Theorem 4.5) has shown that a conditional weak convergence theorem for $(\upsilon_t^{(n)})$ follows from Lindvall's theorem. It is more convenient for our purposes to express this as follows. Fix $\alpha > 0$. Then the conditioned process

$$(\upsilon_t^{(n)} : 0 \leqslant t \leqslant \alpha | \upsilon_\alpha^{(n)} > 0) \Rightarrow (V_t^+(\alpha) : 0 \leqslant t \leqslant \alpha | V_0^+(\alpha) = 0),$$

where the limiting process is an inhomogeneous Markov process for which

$$\mathrm{E}_0(\mathrm{e}^{-\zeta V_t^+(\alpha)}) = [(1+\zeta t)(1+\zeta t(1-t/\alpha))]^{-1}. \tag{3.11}$$

The zero state is entrance but not exit. The transition law has the following density function: for $0 \leqslant u < t \leqslant \alpha$,

$$P(V_t^+(\alpha) \in \mathrm{d}x | V_u^+(\alpha) = z) \equiv q(x,\ t|z,\ u;\ \alpha)\,\mathrm{d}x = p(x,\ t-u|z)\frac{P_x(\eta > \alpha - t)}{P_z(\eta > \alpha - u)}\,\mathrm{d}x. \tag{3.12}$$

This is the conditional probability that the CB diffusion occupies $(x, x + \mathrm{d}x)$ at t given that it starts from z at time u and does not hit zero before time α. The law defined by (3.11) is obtained from (3.12) by setting $u = 0$ and allowing $z \downarrow 0$. This algorithm provides the means for computing the law of functionals of $(V_t^+(\alpha))$ from the corresponding law for the same functional of (υ_t).

Let $t = 1 \leqslant \alpha$ and observe that

$$\int_0^1 \upsilon_u^{(n)}\,\mathrm{d}u = (\upsilon n)^{-1}\int_0^1 Z_{[nu]}^{(n)}\,\mathrm{d}u = (\upsilon n^2)^{-1}\sum_{j=1}^{n} Z_j^{(n)}.$$

Since $\varphi_1(\upsilon_{(\cdot)}^{(n)}) = (\upsilon_1^{(n)}, \int_0^1 \upsilon_u^{(n)}\,\mathrm{d}u)$ is a continuous functional, the continuous mapping theorem applied to Durrett's theorem yields

$$(Z_n/\upsilon n,\ T_n/\upsilon n^2 | \mathscr{H} > n\alpha) \Rightarrow \varphi_1(V_{(\cdot)}^+(\alpha)). \tag{3.13}$$

We choose to compute the limit law by using the Feynman–Kac formula to calculate

$$\phi(t|z) \equiv \phi(\zeta, \theta, t|z) = \mathrm{E}_z\left[\exp\left(-\zeta\upsilon_t - \theta\int_0^t \upsilon_u\,\mathrm{d}u\right)\right].$$

The Feynman–Kac recipe applied to (3.7) (Karlin and Taylor 1981, p. 224) yields the following partial differential equation for $\phi(t|z)$:

$$\frac{\partial\phi}{\partial t} = z\frac{\partial^2\phi}{\partial z^2} - \theta z\phi, \qquad \phi(0|z) = \mathrm{e}^{-\zeta z}.$$

The branching property entails the existence of a function $w(t) \geqslant 0$ for which $\phi(t|z) = \exp(-zw(\mathrm{t}))$ and $w(0) = \zeta$. Substitution produces the first-order equation $w'(t) = \theta - w^2(t)$, and separation of variables leads to

$$w(t) \equiv w(\zeta, \theta, t) = \sqrt{\theta}\,\frac{\zeta + \sqrt{\theta}\tanh(t\sqrt{\theta})}{\zeta\tanh(t\sqrt{\theta}) + \sqrt{\theta}}.$$

Noting that $\phi(\infty, \theta, 1|z) = \mathrm{E}_z(\exp(-\theta\int_0^1 \upsilon_u\,\mathrm{d}u); \upsilon_1 = 0)$, the above computation algorithm yields an evaluation of the left-hand side of (3.1) as

$$\lim_{z\to 0}\frac{\phi(\zeta, \theta, 1|z) - \phi(\infty, \theta, 1|z)}{1 - \exp(-z)} = w(\infty, \theta, 1) - w(\zeta, \theta, 1).$$

Since $w(\infty, \theta, 1) = \sqrt{\theta}/\tanh\sqrt{\theta}$, (3.1) follows after some manipulation with hyperbolic function identities. □

An alternative direct way of proving Theorem 3.1 is to use existing bounds for the difference $g(r) - P_n(s, r)$, similar to the route followed by Pakes (1971a) for $\mathscr{L}(T_n/\upsilon n^2| \mathscr{H} > n)$. See also Pakes (1972). Again, the evaluation using the Feynman–Kac formula which follows (3.13) can be replaced by a direct evaluation of the limit for some tractable offspring law, namely, $f(s) = (2 + s)^{-1}$. Here $\upsilon = 1$ and $P_n(s, r)$ can be explicitly determined. However, the details for each of these are algebraically much more complicated.

Applying the computation algorithm as above, but for any $\alpha > 1$, shows that the joint Laplace–Stieltjes transform of the limit law in (3.13) is

$$\lim_{z\to 0}\frac{\phi(\zeta,\,\theta,\,1|z)-\phi(\zeta+(\alpha-1)^{-1},\,\theta,\,1|z)}{1-\exp(-z/a)}=\alpha[w(\zeta+(\alpha-1)^{-1},\,\theta,\,1)-w(\zeta,\,\theta,\,1)].$$

Carrying out the algebra leads to the following generalization of Theorem 3.1.

Theorem 3.2. *If the conditions of Theorem 3.1 hold and* $1 \leqslant \alpha < \infty$ *then*

$$\mathscr{L}(Z_n/\upsilon n,\,T_n/\upsilon n^2|\mathscr{H}>\alpha n)\Rightarrow(\varepsilon(\alpha),\,A(\alpha)),$$

where

$$\mathrm{E}(\mathrm{e}^{-\zeta\varepsilon(\alpha)-\theta A(\alpha)})\equiv\phi_\alpha(\zeta,\,\theta)=\frac{\operatorname{sech}^2\sqrt{\theta}}{\left[1+\zeta\dfrac{\tanh\sqrt{\theta}}{\sqrt{\theta}}\right]\left[1-\alpha^{-1}+(\alpha^{-1}+(1-\alpha^{-1})\zeta)\dfrac{\tanh\sqrt{\theta}}{\sqrt{\theta}}\right]}.\tag{3.14}$$

We will now use Theorem 3.1 to obtain a more robust version of Theorem 3.2 which allows a more general family of conditioning events. Observe that

$$\begin{aligned}\mathrm{E}_1(s^{Z_n}r^{T_n}|\mathscr{H}>n+\nu)&=\frac{P_n(s,\,r)-P_n(sf_\nu,\,r)}{1-f_{n+\nu}}\\&=\frac{1-f_n}{1-f_{n+\nu}}[\mathrm{E}_1(s^{Z_n}r^{T_n}|\mathscr{H}>n)-\mathrm{E}_1((sf_\nu)^{Z_n}r^{T_n}|\mathscr{H}>n)].\end{aligned}\tag{3.15}$$

We let $n\to\infty$ and $\nu/n\to\alpha-1\geqslant 0$. Using

$$1-f_n\sim(\upsilon n)^{-1}\tag{3.16}$$

(Athreya and Ney 1972) we obtain

$$\frac{1-f_n}{1-f_{n+\nu}}\to\begin{cases}\alpha & \text{if } \nu/n\to\alpha-1<\infty,\\ \infty & \text{if } \nu/n\to\infty.\end{cases}\tag{3.17}$$

Corollary 3.1. *Suppose the conditions of Theorem 3.1 hold and* $\nu/n\to\alpha-1$ *where* $1\leqslant\alpha<\infty$*. Then, in the notation of Theorem 3.2,*

$$\mathscr{L}(Z_n/\upsilon n,\,T_n/\upsilon n^2|\mathscr{H}>n+\nu)\Rightarrow(\varepsilon(\alpha),\,A(\alpha)).$$

Proof. When $\alpha>1$ the proof is a straightforward application of Theorem 3.1, (3.17) and the observation that if $s_n=\exp(-\zeta/\upsilon n)$ then

$$s_nf_\nu=\exp\left[-\frac{1}{\upsilon n}(\zeta+n/\nu)+o(n^{-1})\right]=\exp\left[-\frac{1}{\upsilon n}(\zeta+(\alpha-1)^{-1})+o(n^{-1})\right].$$

When $\alpha=1$ the proof is similar but easier, since

$$\mathrm{E}_1((sf_\nu)^{Z_n} r^{T_n} | \mathscr{H} > n) \leqslant \mathrm{E}(f_\nu^{Z_n} | \mathscr{H} > n) = P_1(\mathscr{H} \leqslant n + \nu | \mathscr{H} > n) \to 0. \qquad \square$$

We now show that the assertion of Corollary 3.1 remains true when $\alpha = \infty$. The proof of this assertion is a consequence of Theorem 3.1. It is rather more involved than for $\alpha < \infty$, and hence the following proof omits some analytical detail which the reader can supply.

Theorem 3.3. *Suppose the conditions of Theorem 3.1 hold and that $\nu/n \to \infty$. Then*

$$\mathscr{L}(Z_n/\upsilon n,\ T_n/\upsilon n^2 | \mathscr{H} > n + \nu) \Rightarrow (\varepsilon(\infty),\ A(\infty)),$$

where

$$\mathrm{E}(\mathrm{e}^{-\zeta\varepsilon(\infty) - \theta A(\infty)}) = \frac{\theta}{[\sqrt{\theta}\cosh\sqrt{\theta} + \zeta\sinh\sqrt{\theta}]^2}.$$

Proof. Observe first that

$$(1 - f_\nu)/(1 - f_{n+\nu}) \to 1. \tag{3.18}$$

Consequently, with s_n as in the previous proof we have $s_n f_\nu \sim s_n$. It follows from the first line of (3.15) that

$$\mathrm{E}_1\left(s_n^{Z_n} r_n^{T_n} | \mathscr{H} > n + \nu\right) \sim s_n \frac{1 - f_\nu}{1 - f_{n+\nu}} \cdot \frac{\partial}{\partial s} P_n(s,\ r)|_{s=\xi_n, r=r_n}. \tag{3.19}$$

Here $r_n = \exp(-\theta/\upsilon n^2)$ and $s_n f_\nu < \xi_n < s_n$. Clearly we can replace ξ_n with s_n.

Let $D_n(s,\ r)$ denote the above derivative. It is easy to show that

$$D_n(s,\ r) = r^{n+1} \prod_{i=0}^{n-1} f'(P_i(s,\ r)).$$

Since $r_n^n \to 1$, we obtain

$$D_n(s_n,\ r_n) \sim [f'(g(r_n))]^n \prod_{i=0}^{n-1} \left[\frac{f'(P_i(s_n,\ r_n))}{f'(g(r_n))}\right].$$

It follows from Pakes (1971a, eq. (26)), that

$$[f'(g(r_n))]^n \to \mathrm{e}^{-2\sqrt{\theta}}. \tag{3.20}$$

Denote the product term by $Q_n(s,\ r)$. The mean value theorem yields

$$-\log Q_n(s_n,\ r_n) \sim 2\upsilon \sum_{i=0}^{n-1} [g(r_n) - P_i(s_n,\ r_n)].$$

Observe that since $T_\infty - T_n = 0$ on the event $\{Z_n = 0\}$,

$$\frac{g(r) - P_n(s, r)}{P_1(\mathscr{H} > n)} = \frac{g(r) - P_n(0, r)}{P_1(\mathscr{H} > n)} - \frac{P_n(s, r) - P_n(0, r)}{P_1(\mathscr{H} > n)}$$

$$= \mathrm{E}_1(rT_\infty | \mathscr{H} > n) - \mathrm{E}_1(s^{Z_n} r^{T_n} | \mathscr{H} > n).$$

Pakes (1971a, p. 188) has shown that

$$\mathrm{E}_1(r_n^{T_\infty} | \mathscr{H} > n) \to \beta(\theta) = (\sqrt{\theta}\,\mathrm{cosech}\,\sqrt{\theta})\,\mathrm{e}^{-\sqrt{\theta}},$$

the Laplace–Stieltjes transform of a continuous law. Theorem 3.1 and the discussion which follows it show that (ε, A) has a continuous law. We now invoke the fact that if a sequence of positive random vectors has a continuous and non-defective limit law, then their Laplace–Stieltjes transforms converge uniformly in the positive quadrant. For two dimensions this follows from the decomposition

$$1 - \mathrm{E}(\mathrm{e}^{-\zeta V - \theta W}) = \zeta \int_0^\infty \mathrm{e}^{-\zeta v} P(V > v)\,\mathrm{d}v + \theta \int_0^\infty \mathrm{e}^{-\theta w} P(W > w)\,\mathrm{d}w$$

$$- \zeta\theta \int_0^\infty \int_0^\infty \mathrm{e}^{-\zeta v - \theta w} P(V > v,\ W > w)\,\mathrm{d}v\,\mathrm{d}w.$$

Consequently,

$$\frac{g(r_n) - P_n(s_n, r_n)}{P_1(\mathscr{H} > n)} \to \psi(\zeta, \theta) \equiv \beta(\theta) - \phi(\zeta, \theta)$$

uniformly in $\{0 \leqslant \zeta, \theta < \infty\}$. It follows that as $i, n \to \infty$,

$$g(r_n) - P_i(s_n, r_n) = (vi)^{-1} \psi(\zeta i/n, \theta(i/n)^2)(1 + o(1)).$$

We can conclude then that

$$-\log Q_n(s_n, r_n) \sim (2/n) \sum_{i=1}^{n-1} \psi(\zeta i/n, \theta(i/n)^2)(n/i)$$

$$\to 2\int_0^1 \psi(\zeta u, \theta u^2)\,\mathrm{d}u/u.$$

The change of variables $v = \mathrm{e}^{-u\sqrt{\theta}/2}$ and algebraic reduction eventually yield

$$-\log Q_n(s_n, r_n) \to 2(\zeta - \sqrt{\theta}) \int_{\exp(-2\sqrt{\theta})}^1 \frac{\mathrm{d}v}{\zeta + \sqrt{\theta} - (\zeta - \sqrt{\theta})v}$$

$$= -2\log\left[\frac{\sqrt{\theta}\exp(\sqrt{\theta})}{\sqrt{\theta}\cosh\sqrt{\theta} + \zeta\sinh\sqrt{\theta}}\right].$$

This, together with (3.18)–(3.20) and further algebraic reduction, finishes the proof. □

Allowing $\nu \to \infty$ in the first line of (3.15) yields

$$\mathrm{E}_1(s^{Z_n} r^{T_n} | \mathscr{H} = \infty) = sD_n(s, r).$$

It follows that $\mathscr{L}(Z_n/\upsilon n, T_n/\upsilon n^2 | \mathscr{H} = \infty)$ has the same limit law as in Theorem 3.3. Furthermore, we note that the product term $\prod_{i=0}^{n-1} f'(P_i(s, r))$ is the joint probability generating function of the size and total progeny at time n of a simple branching process with immigration (BPI). Its immigration law has the probability generating function $f'(s)$.

Let X_n denote the size at time n of a BPI whose offspring law has the probability generating function $f(s)$ and whose immigration law has a general probability generating function $h(s)$, $h(0) < 1$. If $X_0 = 0$ and $Y_n = \sum_{i=1}^{n} X_i$ then

$$\mathrm{E}\big(s^{X_n} r^{Y_n}\big) = \prod_{i=0}^{n-1} h(P_i(s, r)).$$

Some simple alterations to the proof of Theorem 3.3 yield the following result.

Theorem 3.4. *Suppose* $m = 1$, $\upsilon < \infty$ *and let* $\beta = h'(1-) < \infty$. *Then*

$$(X_n/\upsilon n, Y_n/\upsilon n^2) \Rightarrow (\gamma, \mathrm{Y})$$

where (see (3.1))

$$\mathrm{E}(\mathrm{e}^{-\zeta\gamma - \theta \mathrm{Y}}) = \left[\frac{\theta}{\sqrt{\theta}\cosh\sqrt{\theta} + \zeta\sinh\sqrt{\theta}}\right]^{\beta/\upsilon}.$$

Setting $\theta = 0$ reproduces the well-known gamma limit law for $X_n/\upsilon n$ obtained independently by Foster (1969), Pakes (1971b) and Seneta (1970). Putting $\zeta = 0$ gives a result for the total progeny originally derived by Pakes (1972, Theorem 5).

The conditioning in Theorems 3.2 and 3.3 captures the essence of conditioning on extinction in the proximate to remote futures. The following result complements this idea by conditioning on extinction in the intermediate future. Results for the generation sizes conditioned by events of the form $\{n/\tau < \mathscr{H} < \alpha n/\tau\}$, where $\alpha > 1$ and $0 < \tau < \alpha$, were considered by Esty (1976). Here, and in the next section, we will find it notationally convenient to set $\rho = 1/\alpha$.

Theorem 3.5. *If the conditions of Theorem 3.1 hold and* $\nu \to \infty$ *and* $\nu/n \to \alpha - 1$, *where* $1 \leqslant \alpha \leqslant \infty$, *then*

$$\mathscr{L}(Z_n/\upsilon n, T_n/\upsilon n^2 | n < \mathscr{H} \leqslant n + \nu) \Rightarrow (\delta(\alpha), B(\alpha)),$$

where

$$\mathrm{E}(\mathrm{e}^{-\zeta\delta(\alpha) - \theta B(\alpha)}) \equiv \psi_\alpha(\zeta, \theta) = \frac{\theta\,\mathrm{cosech}\sqrt{\theta}}{(1-\rho)\sqrt{\theta}\cosh\sqrt{\theta} + (\rho + (1-\rho)\zeta)\sinh\sqrt{\theta}}. \tag{3.21}$$

Proof. When $\alpha > 1$ this is similar to Corollary 3.1, using the identity

$$\mathrm{E}_1(s^{Z_n} r^{T_n} | n < \mathscr{H} \leqslant n + \nu) = \mathrm{E}_1((sf_\nu)^{Z_n} r^{T_n} | \mathscr{H} > n) \cdot \frac{1 - f_n}{f_{n+\nu} - f_n}.$$

When $\alpha = 1$, it can be shown that $\mathrm{E}_1(Z_n/n | n < \mathscr{H} \leqslant n + \nu) = O(\nu/n) \to 0$, and hence from Markov's inequality that $(Z_n/n | n < \mathscr{H} \leqslant n + \nu) \xrightarrow{p} 0$. Consequently Z_n/n and T_n/n^2 are conditionally independent in the limit as $n \to \infty$, and hence we need only compute the marginal limit law for the total progeny. To do this, observe that

$$\mathrm{E}_1(r_n^{T_n} | n < \mathscr{H} \leqslant n + \nu) = (f_{n+\nu} - f_n)^{-1} \sum_{j=1}^{\nu} \mathrm{E}_1(r_n^{T_n} | \mathscr{H} = n + j)(f_{n+j} - f_{n+j-1}).$$

But Theorem 5.2 below implies that $\mathrm{E}_1(r_n^{T_n} | \mathscr{H} = n + j) \to \theta \operatorname{cosech}^2 \sqrt{\theta}$. □

The case $\alpha = 1$ can be interpreted as conditioning on extinction in the near future, $\{n < \mathscr{H} \leqslant n + o(n)\}$. Hence the conditioning in Theorem 3.5 interpolates this case and the Yaglom conditioning event $\{\mathscr{H} > n\}$. The limit law for the generation sizes is

$$\delta(\alpha) \overset{d}{=} (1 - \alpha^{-1})\varepsilon.$$

This degenerates when $\alpha = 1$, as observed in the proof, and it increases stochastically with α. It is always stochastically smaller than the weak limit (1.1).

4. Representations for the total progeny limit laws

Taking $\zeta = 0$ in (3.14) together with Corollaries 3.1 and 3.2 gives the following conditional limit theorem for the total progeny.

Theorem 4.1. *Let $m = 1$, $\upsilon < \infty$ and $1 \leqslant \alpha \leqslant \infty$. If $\nu/n \to \alpha - 1$ then*

$$(T_n/\upsilon n^2 | \mathscr{H} > n + \nu) \Rightarrow A(\alpha),$$

where

$$a(\theta, \alpha) = \mathrm{E}(\mathrm{e}^{-\theta A(\alpha)}) = \frac{\operatorname{sech}^2 \sqrt{\theta}}{1 - \rho + \rho \dfrac{\tanh \sqrt{\theta}}{\sqrt{\theta}}} = \frac{\sqrt{\theta} \operatorname{sech} \sqrt{\theta}}{(1 - \rho)\sqrt{\theta} \cosh \sqrt{\theta} + \rho \sinh \sqrt{\theta}}. \tag{4.1}$$

Observe that

$$\mathrm{E}(A(\alpha)) = 1 - 1/3\alpha,$$
$$a(\theta, 1) = 2\sqrt{\theta} \operatorname{cosech}(2\sqrt{\theta}) \tag{4.2}$$

and that

$$a(\theta, \infty) = \operatorname{sech}^2 \sqrt{\theta}. \tag{4.3}$$

The limit law $\mathscr{L}(A(\alpha))$ can be represented as a kind of size-biased version of the limiting bivariate law (3.1) since

$$a(\theta,\,\alpha) = \mathrm{E}[(\exp(-(\alpha-1)^{-1}\varepsilon - \theta A)]/\mathrm{E}[\exp(-(\alpha-1)^{-1}\varepsilon)].$$

This, however, is not especially revealing.

Noting (4.2), rewrite (4.1) as

$$a(\theta,\,\alpha) = a(\theta,\,1)\kappa(\theta,\,\alpha), \tag{4.4}$$

where

$$\kappa(\theta,\,\alpha) = \frac{\kappa(\theta)}{1-\rho+\rho\kappa(\theta)}, \qquad \kappa(\theta) \equiv \kappa(\theta,\,\infty) = \theta^{-1/2}\tanh\sqrt{\theta}. \tag{4.5}$$

All of the above functions are Laplace–Stieltjes transforms of a functional of the standard Brownian motion process (B_t). Thus

$$a(\theta,\,1) = \mathrm{E}(\mathrm{e}^{-\theta C}|B_2 = 0),$$

where

$$C = \int_0^2 B_t^2\,\mathrm{d}t - \left(\int_0^2 B_t\,\mathrm{d}t\right)^2;$$

see Donati-Martin and Yor (1993, p. 577). Or, if $H = H(c,\,d)$ is the exit time of (B_t) from $(c,\,d)$, then taking $c = 0$ and $d = \sqrt{2}$ we have

$$a(\theta,\,1) = \mathrm{E}_0[\exp(-\theta H)|B_H = \sqrt{2})];$$

see entry 1.3.0.5(b) of Borodin and Salminen (1996).

Next, taking $d = -c = 1/\sqrt{2}$, we have

$$a(\theta,\,\infty) = [\mathrm{E}_0(\exp(-\theta H))]^2 = \mathrm{E}_0\left[\exp\left(-\theta\int_0^{\sqrt{2}}\beta_t^2(4)\,\mathrm{d}t\right)\right],$$

where $(\beta_t^2(\nu))$ is the ν-dimensional squared Bessel process; see entries 1.2.4.1 and 4.1.9.3 of Borodin and Salminen (1996) for these. Let $\gamma_\ell(2)$ ($\ell = 1,\,2,\,\ldots$) denote independent copies of $\varepsilon+\varepsilon'$, where ε' is an independent copy of ε. Then

$$A(\infty) \stackrel{d}{=} \pi^{-2}\sum_{\ell=1}^{\infty}\gamma_\ell(2)/b_\ell, \qquad b_\ell = (\ell-\tfrac{1}{2})^2;$$

see Pakes (1997).

Clearly $A(1)$ and $A(\infty)$ have infinitely divisible laws with Lévy measures which can be expressed in terms of theta functions. In an obvious sense the laws of the $A(\alpha)$ interpolate those of $A(1)$ and $A(\infty)$. We now procede to show that $\mathscr{L}(A(\alpha))$ too is infinitely divisible and expressible as an infinite sum of *randomly* weighted standard exponential random variables.

We begin by observing that $\kappa(\theta,\,\alpha)$ defined in (4.4) is the Laplace–Stieltjes transform of a probability law. Indeed, by taking $\alpha > 1$, $c = 0$ and $d = \alpha/\sqrt{2}$, we have

$$\kappa(\theta,\,\alpha) = \mathrm{E}_{1/\sqrt{2}}\left[\exp\left(-\theta\int_0^H 1_{[0,1/\sqrt{2}]}(B_t)\,\mathrm{d}t\right)|B_H = \alpha/\sqrt{2}\right];$$

see entry 1.3.4.5(b) of Borodin and Salminen (1996). Let K_α denote a random variable having this Laplace–Stieltjes transform.

Expressing $\kappa(\theta)$ (see (4.5)) in terms of $\kappa(\theta,\alpha)$ gives

$$\kappa(\theta) \equiv \kappa(\theta,\,\alpha)\frac{1-\rho}{1-\rho b(\theta,\,\alpha)} \qquad (1 \leqslant \alpha < \infty), \tag{4.6}$$

which also is a Laplace–Stieltjes transform. In addition $\kappa(\theta) = \mathrm{E}_0(\exp(-\theta K))$, where K is the last exit time from zero of the stopped process $B(t \wedge H(-1/\sqrt{2},\,1/\sqrt{2}))$; see Louchard (1984, p. 480). The density function of K is

$$k(x) = (4\sqrt{\pi x})^{-1}\sum_{j=-\infty}^{\infty}(-1)^j\exp(-j^2/x);$$

see Borodin and Salminen (1996, p. 451). The infinite product representation of $\tanh(\cdot)$ leads to the representation

$$K \overset{d}{=} \pi^{-2}\sum_{\ell=1}^{\infty} I_\ell\varepsilon_\ell/b_\ell, \tag{4.7}$$

where $\mathscr{L}(I_\ell) = \mathrm{Bern}(1-b_\ell/a_\ell)$, $a_\ell = \ell^2$, and all the I_ℓ and ε_ℓ are independent. Logarithmic differentiation of the second member of (4.4) eventually reveals that $\mathscr{L}(K)$ is infinitely divisible with canonical form:

$$\kappa(\theta) = \exp\left(-\int_0^\infty(1-\mathrm{e}^{-\theta x})n(\mathrm{d}x)\right),$$

where

$$n(\mathrm{d}x) = (2x)^{-1}P(A(1) > x)\,\mathrm{d}x.$$

In terms of random variables, (4.4) takes the form

$$A(\alpha) \overset{d}{=} A(1) + K_\alpha, \tag{4.8}$$

and from (4.5) this implies the following link between the extreme ends of our linear family of limit laws:

$$A(\infty) \overset{d}{=} A(1) + K,$$

where the summands are independent. (We follow this convention below.)

Observing that $1-\rho+\rho\kappa(\theta)$ is the Laplace–Stieltjes transform of $I(1/\alpha)K$, where $\mathscr{L}(I(\cdot)) = \mathrm{Bern}(\cdot)$, we have

$$A(\alpha) + I(1/\alpha)K \overset{d}{=} A(\infty).$$

This can be rewritten in various ways after observing that (4.6) is equivalent to

$$K \stackrel{d}{=} \sum_{\ell=1}^{1+N(\varepsilon/(\alpha-1))} K_\alpha^{(\ell)},$$

where $N(\cdot)$ is a unit-rate Poisson process, ε is standard exponential, and the $K_\alpha^{(\ell)}$ are independent copies of K_α.

Our principal representation theorems follow from the next result which expresses K_α as a randomly weighted sum of exponential random variables.

Theorem 4.2. *For $\alpha \geqslant 1$, the equation*

$$y^{-1}\tan y = -(\alpha - 1) \tag{4.9}$$

has real solutions $y_\ell = \pi t_{\pm\ell}(\alpha)$ ($\ell = 1, 2, \ldots$) where $t_\ell(\alpha) = -t_{-\ell}(\alpha)$ and

$$(\ell - 1/2) < t_\ell(\alpha) < \ell. \tag{4.10}$$

In addition, for fixed $\ell \geqslant 1$,

$$t_\ell(\alpha) \to \begin{cases} \ell & \text{as } \alpha \to 1 \\ \ell - \frac{1}{2} & \text{as } \alpha \to \infty, \end{cases} \tag{4.11}$$

and for any $\alpha > 1$,

$$t_\ell(\alpha) \sim \ell - \tfrac{1}{2} \qquad (\ell \to \infty). \tag{4.12}$$

Finally,

$$K_\alpha \stackrel{d}{=} \pi^{-2} \sum_{\ell=1}^{\infty} J_\ell \varepsilon_\ell / t_\ell(\alpha), \tag{4.13}$$

where $\mathscr{L}(J_\ell) = \mathrm{Bern}(1 - (t_\ell(\alpha)/\ell)^2)$ and the J_ℓ and ε_ℓ are independent.

Proof. We verify conditions allowing the application of Hadamard's product theorem (Veech 1967, Chapter 5) to the entire function $D(z) = 1/c(z^2, \alpha)$, where (see (4.1))

$$c(\theta, \alpha) = \frac{\sqrt{\theta}}{(1-\rho)\sqrt{\theta}\cosh\sqrt{\theta} + \rho\sinh\sqrt{\theta}}. \tag{4.14}$$

The function $D(z)$ has order 1 and its zeros coincide with the solutions of

$$z^{-1}\tanh z = -(\alpha - 1).$$

It can be shown that the left-hand side is real if and only if z is real or imaginary. In the former case the left-hand side is positive and hence there can be no solutions. When $z = \mathrm{i}y$, with y real, the above equation transforms to (4.9). Graphical considerations reveal the truth of (4.10) to (4.12).

Hadamard's theorem gives us the product representation

$$D(z) = \mathrm{e}^{c_1+c_2 z} \prod_{\ell \neq 0} [(1 - z/\mathrm{i}y_\ell)\, \mathrm{e}^{z/\mathrm{i}y_\ell}] = E^{c_1+c_2 z} \prod_{\ell=1}^{\infty} (1 + z^2/y_\ell^2),$$

where c_1 and c_2 are constants to be determined. We have $c_1 = 0$ since $D(0) = 1$, and expanding $\log D(x)$, $x \geqslant 0$, around the origin leads to $c_2 = 0$. We conclude that

$$c(\theta, \alpha) = \prod_{\ell=1}^{\infty} (1 + \theta/\pi^2 t_\ell^2(\alpha))^{-1}, \tag{4.15}$$

which shows that $c(\theta, \alpha)$ is the Laplace–Stieltjes transform of a probability law. In fact it has a representation very similar to that of $\kappa(\theta, \alpha)$ above.

The infinite product representation of $\sinh \sqrt{\theta}$ now yields

$$\kappa(\theta, \alpha) = c(\theta, \alpha) \sinh \sqrt{\theta} = \prod_{\ell=1}^{\infty} \frac{1 + \theta/\pi^2 \ell^2}{1 + \theta/\pi^2 t_\ell^2(\alpha)}.$$

The right-hand inequality of (4.10) allows us to conclude that each factor in the above product is the Laplace–Stieltjes transform of $\pi^{-2} J_\ell \varepsilon_\ell / t_\ell^2(\alpha)$. □

The decomposition (4.8) and the structure of $\mathscr{L}(K_\alpha)$ given by Theorem 4.2 together comprise our analogue of the relation (1.1) for the limit law for the generation sizes under the same family of conditioning events. When $\alpha = 1$ the random weights J_ℓ vanish almost surely, giving $K_1 = 0$. As $\alpha \to \infty$ the random weights $J_\ell \Rightarrow I_\ell$ – see (4.7). It follows then from (4.11) that $K_\alpha \Rightarrow K$, a fact we know from above considerations. Hence $A(\alpha)$ is the sum of $A(1)$ and an independent component which increases stochastically with α from zero to K.

A representation in terms of non-random weights follows from (4.1), (4.14) and (4.15) (compare the representation for $A(\infty)$):

$$A(\alpha) \stackrel{d}{=} \pi^{-2} \sum_{\ell=1}^{\infty} \varepsilon_\ell / t_\ell^2(\alpha) + \pi^{-2} \sum_{\ell=1}^{\infty} \varepsilon'_\ell / (\ell - \tfrac{1}{2})^2, \tag{4.16}$$

where the ε_ℓ and ε'_ℓ are independent with standard exponential laws.

Refering next to (3.21), we obtain the following theorem.

Theorem 4.3. *Let* $m = 1$, $\upsilon < \infty$ *and* $1 \leqslant \alpha \leqslant \infty$. *If* $\nu \to \infty$ *and* $\nu/n \to \alpha - 1$ *then*

$$(T_n/\upsilon n^2 | n < \mathscr{H} \leqslant n + \nu) \Rightarrow B(\alpha),$$

where

$$b(\theta, \alpha) = \mathrm{E}(\mathrm{e}^{-\theta B(\alpha)}) = \frac{\theta \operatorname{cosech} \sqrt{\theta}}{(1 - \rho)\sqrt{\theta} \cosh \sqrt{\theta} + \rho \sinh \sqrt{\theta}}. \tag{4.17}$$

This is similar in overall structure to $a(\theta, \alpha)$, and hence many of the above results apply here with suitable alteration. For later reference, note that

$$b(\theta,\, 1) = \theta \operatorname{cosech}^2 \sqrt{\theta}. \tag{4.18}$$

Algebra yields

$$\mathrm{E}(B(\alpha)) = (1 - \alpha^{-1})/3,$$

which increases as $\alpha \uparrow \infty$ to $\mathrm{E}(A(1))$. Analagous to (4.13), we see from (4.14), (4.17) and (4.18) that

$$B(\alpha) \overset{d}{=} B(1) + K_\alpha, \tag{4.19}$$

and hence that substituting ℓ for $\ell - \frac{1}{2}$ in (4.16) gives a representation for $B(\alpha)$.

5. The total progeny with extinction in the near future

As in Sections 3 and 4, we assume $m = 1$ and $\upsilon < \infty$. Karpenko and Nagaev (1994, p. 449), proved that

$$\mathrm{E}(T_n | \mathscr{H} = n) \sim \upsilon n^2/3. \tag{5.1}$$

They made a stronger moment assumption to obtain this result, but their assumption provides an error estimate. We can prove (5.1) under our weaker assumption by changing details of the proof of their Lemma 5 as follows. Let $u_n = \mathrm{E}_1(T_n;\; Z_n = 0)$. Karpenko and Nagaev (1994) show that

$$u_n = f_n + f'(f_{n-1})u_{n-1} = f_n + \sum_{i=1}^{n-1} f_i \sum_{\ell=i}^{n-1} f'(f_\ell).$$

Lemma 5.1. *If* $m = 1$ *and* $\upsilon < \infty$ *then*

$$u_n \sim n/3.$$

Proof. Let $\{n'\}$ be a subsequence of positive integers satisfying $n' < n$, $n' \to \infty$ and $n'/n \to 0$, both as $n \to \infty$. Write

$$f_n = 1 - (\upsilon n)^{-1}(1 + e(n)),$$

where $e(n) \to 0$. It follows that

$$u_n = O(n') + (1 + o(1)) \sum_{n' \leqslant i \leqslant n} \prod_{\ell=i}^{n-1} f'(f_\ell). \tag{5.2}$$

Following Karpenko and Nagaev, we see that the product equals

$$\exp\left\{-2\sum_{\ell=i}^{n-1} \ell^{-1}(1 - e(\ell))\right\} \cdot \exp\left\{\sum_{\ell=i}^{n-1} O(\ell^{-2})\right\}.$$

The second exponential factor contributes a factor $1+o(1)$ to the sum (5.2). Since $\sup_{n'\leqslant \ell\leqslant n} e(\ell)\to 0$, we can bound the sum at (5.2) with expressions of the form

$$(1+o(1))\sum_{i=n'}^{n}(i/n)^{2+\delta(n')},$$

where $\delta(n')\to 0$ as $n\to\infty$. Denoting the last sum by σ_n, integral test comparisons yield

$$\sigma_n \sim \int_{n'}^{n}(x/n)^{2+\delta(n')}\,\mathrm{d}x \sim n/3. \qquad \square$$

The asymptotic estimate (5.1) follows now from

$$\begin{aligned}\mathrm{E}_i(T_n|\mathscr{H}=n) = i\frac{u_n-u_{n-1}}{f_n^i-f_{n-1}^i} &\sim \frac{u_n-u_{n-1}}{f_n-f_{n-1}}\\ &= \frac{f_n-(1-f'(f_{n-1}))u_{n-1}}{f_n-f_{n-1}}\end{aligned}$$

and the estimate $1-f'(f_{n-1})\sim 2/n$.

Let $m_i(n,\nu)=\mathrm{E}_i(Z_n|\mathscr{H}=n+\nu)$.

Lemma 5.2. *If $m=1$ and ν is fixed then*

$$\lim_{n\to\infty} m_i(n,\nu)=\frac{f_\nu - f_{\nu-1}f'(f_{\nu-1})}{f_\nu-f_{\nu-1}}\mathscr{P}(\nu)$$

where $\mathscr{P}(\nu)$ is a constant which is defined in the proof.

Proof. We see from (2.18) that $m_i(n,\nu)\sim m_1(n,\nu)$ and

$$m_1(n,\nu)=\frac{f_\nu f'_n(f_\nu)-f_{\nu-1}f'_n(f_{\nu-1})}{f_{n+\nu}-f_{n+\nu-1}}.$$

But

$$\begin{aligned}\frac{f'_n(f_\nu)}{f_{n+\nu}-f_{n+\nu-1}} &= (f_\nu-f_{\nu-1})^{-1}f'_n(f_\nu)/f'_n(\zeta_\nu)\\ &= (f_\nu-f_{\nu-1})^{-1}\prod_{i=0}^{n-1}\frac{f'(f_{i+\nu})}{f'(f_i(\zeta_\nu))},\end{aligned}$$

where $f_{\nu-1}<\zeta_\nu<f_\nu$. Hence

$$1<f'(f_{i+\nu})/f'(f_i(\zeta_\nu))<f'(f_{i+\nu})/f'(f_{i+\nu-1}).$$

It follows that the product converges as $n\to\infty$ to

$$\mathscr{P}(\nu)=\prod_{i=0}^{\infty}f'(f_{i+\nu})/f'(f_i(\zeta_\nu))<\infty.$$

The proof is completed by further algebraic manipulation. □

It follows then that under the conditions of Lemma 5.1 and for fixed ν, we have

$$\mathrm{E}_i(T_n|\mathscr{H} = n+\nu) \sim \upsilon n^2/3,$$

whence the following theorem.

Theorem 5.1. *If* $m = 1$, $\upsilon < \infty$ *and* $\mathscr{S} \subset \mathbb{N}$ *has finite cardinality, then*

$$\mathrm{E}_i(T_n|\mathscr{H} \in n+\mathscr{S}) \sim \upsilon n^2/3.$$

The restriction on the cardinality of $\mathscr{S}$ is essential. For example, we can show from (3.6) that

$$\begin{aligned}\mathrm{E}_i(T_n|\mathscr{H} > n+\nu) &= i\frac{\mathrm{E}_1(T_n) - f_{n+\nu}^{i-1}\mathrm{E}_1(T_n;\ Z_{n+\nu} = 0)}{1 - f_{n+\nu}^i} \\ &= i\frac{\mathrm{E}_1(T_n) - \mathrm{E}_1(T_n;\ Z_{n+\nu} = 0)}{1 - f_{n+\nu}^i} + \frac{1 - f_{n+\nu}^{i-1}}{1 - f_{n+\nu}^i}\mathrm{E}_1(T_n;\ Z_{n+\nu} = 0) \\ &\sim 2\upsilon n^2/3 \qquad (n \to \infty\ \&\ \nu/n \to 0),\end{aligned}$$

because Lemma 5.1 entails $\mathrm{E}_1(T_n;\ Z_{n+\nu} = 0) = u_{n+\nu} + O(\nu)$.

One may ask whether it is possible to interpolate the constants $\frac{1}{3}$ and $\frac{2}{3}$ which appear in these asymptotic estimates by allowing $\mathscr{S}$ to expand unboundedly as $n \to \infty$.

It follows from Theorem 5 of Karpenko and Nagaev (1994) that (cf. (4.18))

$$\mathrm{E}_1(\mathrm{e}^{-\theta T_n/\upsilon n^2}|\mathscr{H} = n) \to b(\theta,\ 1). \tag{5.3}$$

They obtained this limiting Laplace–Stieltjes transform for the more general paracritical case where m may differ from unity but $m \to 1$ as $n \to \infty$. Their proof requires a stronger moment assumption than we are making here. As we mentioned in Section 1, Kesten (1986) proved the existence of a limit law under our moment conditions, but he did not identify it. We offer the following variation of Kesten's approach which yields (5.3). It rests on Theorem 3.4 and hence, like Kesten's proof, on Durrett's functional limit theorem.

Let n be large and $0 \ll a < 1$. In the following we will understand the product an to mean $[an]$ wherever it occurs in time indices. Now, with r_n as above,

$$\begin{aligned}
\mathrm{E}_1(r_n^{T_{an}}|\mathscr{H}=n) &= (f_n - f_{n-1})^{-1}\sum_{j=1}^{\infty}\mathrm{E}_1(r_n^{T_{an}};\ Z_{an}=j)[f^j_{(1-a)n} - f^j_{(1-a)n-1}] \\
&\sim (1-a)^{-2}\sum_{j=1}^{\infty}\mathrm{E}_1(r_n^{T_{an}};\ Z_{an}=j) f^{j-1}_{(1-a)n} \\
&= (1-a)^{-2}\frac{\partial}{\partial s}P_{an}(s,\ r_n)|_{s=f_{(1-a)n}} \\
&= (1-a)^{-2} r_n^{an+1} D_{an}(f_{(1-a)n},\ r_n) \\
&\sim (1-a)^{-2}\prod_{i=0}^{an-1} f'(P_i(f_{(1-a)n},\ r_n)) \\
&= (1-a)^{-2}\mathrm{E}_0((f_{(1-a)n})^{X_n} r_n^{Y_{an}}),
\end{aligned}$$

where (X_n) is a branching process with offspring probability generating function f and immigration probability generating function $f'(s)$.

Since

$$f_{(1-a)n} = (1+o(1))\exp[-((1-a)vn)^{-1}],$$

we obtain

$$\begin{aligned}
\mathrm{E}_1(r_n^{T_{an}}|\mathscr{H}=n) &\sim (1-a)^{-2}\mathrm{E}_0[\exp(-(a/(1-a))X_{an}/van - (a^2\theta)Y_{an}/van^2)] \\
&\to (1-a)^{-2}\frac{a^2\theta}{\left[a\sqrt{\theta}\cosh(a\sqrt{\theta}) + \dfrac{a}{1-a}\sinh(a\sqrt{\theta})\right]^2},
\end{aligned}$$

on using Theorem 3.4. As shown by Kesten (1986), we thus obtain

$$\lim_{n\to\infty}\mathrm{E}_1\left(r_n^{T_n}|\mathscr{H}=n\right) = \lim_{a\to 1}\lim_{n\to\infty}\mathrm{E}_1\left(r_n^{T_{an}}|\mathscr{H}=n\right) = \theta/(\sinh\sqrt{\theta})^2.$$

Recalling the representation (3.2) and the discussion which precedes it, we see that the limit law represented by (5.3) can be expressed in random variable terms as

$$(T_n/vn^2|\mathscr{H}=n) \Rightarrow V = \pi^{-2}\sum_{\ell=1}^{\infty}\gamma_l(2)/\ell^2. \tag{5.4}$$

It is now an easy step to the main result of this section, and complement to Theorem 4.3.

Theorem 5.2. *Let the conditions of Theorem 5.1 hold. Then*

$$(T_n/vn^2|\mathscr{H}\in n+\mathscr{S}) \Rightarrow V.$$

Proof. It suffices to prove the assertion with $\mathscr{S}=\{\nu\}$. However, it follows from Theorem 5.1 that $n^{-2}\mathrm{E}_1[(T_{n+\nu}-T_n)|\mathscr{H}=n+\nu]\to 0$, and hence Markov's inequality implies that

$$n^{-2}(T_{n+\nu} - T_n | \mathscr{H} = n + \nu) \xrightarrow{p} 0.$$

The assertion follows now from (5.4) after replacing n with $n + \nu$. □

6. Countable-state Markov chains

In this section we suppose that (Z_n) is a Markov chain on a countable state space containing an irreducible proper subset $\mathscr{T}$. Let $\mathscr{H}$ be the hitting time of the complement set $\bar{\mathscr{T}}$, and suppose $r_i = P_i(\mathscr{H} < \infty) > 0$ for each $i \in \mathscr{T}$.

The essential structure of branching processes exploited in Lemma 2.1 and Theorem 2.1 is preserved by assuming that

$$\mathscr{T} \text{ is } R\text{-positive for some } R > 1. \tag{A}$$

Let $p_{ij}^{(n)} = P_i(Z_n = j)$. Then $p_{ij}^{(n)} \to 0$ $(n \to \infty;\ i, j \in \mathscr{T})$ and the irreducibility of $\mathscr{T}$ implies that the power series $\sum p_{ij}^{(n)} z^n$ have a common radius of convergence $R \geqslant 1$. We say $\mathscr{T}$ is R-positive if $p_{ij}^{(n)} \sim R^{-n} x_i m_j$, where $\{x_i\}$ and $\{m_i\}$ are positive sequences which are right- and left-invariant, respectively, for the matrix $[Rp_{ij} : i, j \in \mathscr{T}]$ and normalized so that $\sum x_i m_i = 1$. More specifically, $\{m_i\}$ is an R-invariant measure:

$$R \sum_{i \in \mathscr{T}} m_i p_{ij} = m_j \qquad (j \in \mathscr{T}). \tag{6.1}$$

Making the further assumption

$$M = \sum_{i \in \mathscr{T}} r_i m_i < \infty, \tag{B}$$

Seneta and Vere-Jones (1966) show that

$$P_i(Z_n = j | n < \mathscr{H} < \infty) \to r_j m_j / M. \tag{6.2}$$

The summability condition (B) is equivalent to

$$\lim_{n \to \infty} R^n \sum_{j \in \mathscr{T}} p_{ij}^{(n)} r_j = \lim_{n \to \infty} R^n P_i(n < \mathscr{H} < \infty) = M x_i. \tag{6.3}$$

Seneta and Vere-Jones give conditions under which the limits (6.2) and (6.3) are preserved when (Z_n) has an initial law $(\mu(i))$. These are that either

$$\sum r_i m_i < \infty \text{ and there exists } A > 0 \text{ such that } \mu(i) < A r_i m_i; \tag{C}$$

or

$$\inf_{i \in \mathscr{T}} x_i > 0 \text{ and } \sum \mu(i) x_i < \infty. \tag{D}$$

For the subcritical branching process we have $x_i = i$, whence condition (D) is just that the initial law has a finite first moment.

When (A), (B) and (C) or (D) hold, then details of the proofs of results in Section 2 can be altered to give the following assertions. Let $r_\mu(n) = P_\mu(\mathscr{H} = n)$. First,

$$R^n P_\mu(n < \mathscr{H} < \infty) \to Mx_i, \tag{6.4}$$

whence

$$R^n r_\mu(n) \to (R-1)Mx_i.$$

Consequently, if we set $\bar{m} = 1/R$ then (2.7) holds in our present setting. Again, setting $\bar{\mu}(i) = r_i m_i / M$, then

$$P_{\bar{\mu}}(\mathscr{H} \in n + \mathscr{S}) = P(\xi_{\bar{m}} \in \mathscr{S}).$$

Observing that $\mathscr{H} \in n + \mathscr{S}$ entails $\mathscr{H} < \infty$, and alterating some of the algebraic steps in (2.9), we obtain

$$P_\mu(Z_n = j | \mathscr{A}_n) = P_\mu(Z_n = j | n < \mathscr{H} < \infty) \frac{P_j(\mathscr{H} \in \mathscr{S} | \mathscr{H} < \infty)}{P_\mu(\mathscr{A}_n | \mathscr{H} < \infty)}$$

and hence we obtain a generalized version of (2.8):

$$\lim_{n\to\infty} P_\mu(Z_n = j | \mathscr{H} \in n + \mathscr{S}) = \bar{\mu}(j) \frac{P_j(\mathscr{H} \in \mathscr{S} | \mathscr{H} < \infty)}{P(\xi_{\bar{m}} \in \mathscr{S})}.$$

This reduces to (6.2) when $\mathscr{S} = \mathbb{N}$ since the numerator conditional probability becomes unity. Observe that $\bar{\mu}$ is quasi-stationary in the sense that

$$P_{\bar{\mu}}(Z_n = j | \mathscr{H} \in n + \mathscr{S}) \equiv \bar{\mu}(j) \frac{P_j(\mathscr{H} \in \mathscr{S} | \mathscr{H} < \infty)}{P(\xi_{\bar{m}} \in \mathscr{S})}.$$

Seneta and Vere-Jones (1966) show that under (A), (B) and (C) or (D) the doubly-limiting conditional law exists:

$$\lim_{n\to\infty} \lim_{\nu\to\infty} P_\mu(Z_n = j | n + \nu < \mathscr{H} < \infty) = x_j m_j.$$

It is easy to show the limits can be taken in reverse order, giving an extension of Theorem 2.2. Finally, Theorem 2.4 holds in the form: if $n' \to \infty$ as $n \to \infty$, then

$$\lim_{n\to\infty} P_\mu(Z_n = j | \mathscr{H} \in n + n' + \mathscr{S}) = x_j m_j.$$

Under certain moment conditions, Pakes (1974) proves conditional laws of large numbers and central limit theorems for $T_n = \sum_{\ell=1}^n Z_\ell$. It seems very likely that the limit laws there will subsist under the conditioning events examined in this section.

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