Bernoulli **29**(3), 2023, 2392–2416
https://doi.org/10.3150/22-BEJ1546

Conditional quantiles: An operator-theoretical approach

LUCIANO DE CASTRO[1,a], BRUNO N. COSTA[3,4,c], ANTONIO F. GALVAO[2,b] and JORGE P. ZUBELLI[3,4,d]

[1]*Department of Economics, University of Iowa, Iowa City, USA,* [a]*decastro.luciano@gmail.com*
[2]*Department of Economics, Michigan State University, East Lansing, USA,* [b]*agalvao@msu.edu*
[3]*IMPA, Rio de Janeiro, Brazil,* [c]*brunonunes@impa.br*
[4]*Department of Mathematics, Khalifa University, Abu Dhabi, UAE,* [d]*jorge.zubelli@ku.ac.ae*

This paper derives several novel properties of conditional quantiles viewed as nonlinear operators. The results are organized in parallel to the usual properties of the expectation operator. We first define a τ-conditional quantile random set, relative to any sigma-algebra, as a set of solutions of an optimization problem. Then, well-known properties of unconditional quantiles, as translation invariance, comonotonicity, and equivariance to monotone transformations, are generalized to the conditional case. Moreover, a simple proof for Jensen's inequality for conditional quantiles is provided. We also investigate continuity of conditional quantiles as operators with respect to different topologies and obtain a novel Fatou's lemma for quantiles. Conditions for continuity in L^p and weak continuity are also derived. Then, the differentiability properties of quantiles are addressed. We demonstrate the validity of Leibniz's rule for conditional quantiles for the cases of monotone, as well as separable functions. Finally, although the law of iterated quantiles does not hold in general, we characterize the maximum set of random variables for which this law holds, and investigate its consequences for the infinite composition of conditional quantiles.

Keywords: Conditional quantiles; continuity for quantiles; Fatou's lemma for quantiles; Leibniz's rule

1. Introduction

Quantiles are of fundamental importance in several fields of theoretical and applied work such as statistics, biostatistics, economics, finance, and decision theory, among others.[1] Since the seminal paper [17], quantile regression has become an important tool in statistical analysis for estimating conditional quantile functions models (see, e.g., [16] and [19]). Quantile regression provides a systematic strategy for examining how covariates influence the location, scale, and shape of the entire response distribution. Quantiles are also important in decision theory. There has been increasing theoretical, empirical, and experimental interest in decision under uncertainty using quantile preferences (QP). This preference has been characterized in [22], where properties of a quantile model for individual's behavior were studied.[2] [23] introduced the concept of quantile-preserving spread, which is a notion of risk aversion for the quantile model that establishes a parallelism with mean-preserving spreads in the standard expected utility framework.

Although conditional quantiles have been largely employed in multiple fields, the literature still lacks a systematic investigation of their statistical and mathematical properties. This paper fulfills this gap.

[1]Quantiles have also been used in practical decision making in banking and investment (in the form of Value-at-Risk, see, e.g., [9] and [14]), goal-reaching problems and in mining, oil and gas industries (in the form of "probabilities of exceeding", see, e.g., [1] and [11]).
[2]More recently, QP have been formally axiomatized in [2,28], and [6].

1350-7265

We employ an operator theoretical view to define the τ-conditional quantiles, which enables us to enlarge the theory and rigorously establish results that have not been formalized before. To use this approach, we first define the τ-conditional quantile random set as the set of solutions of an optimization problem using the check function as the objective function, as proposed in [17]. This definition allows computation of conditional quantiles of any finite random variable conditional on any σ-algebra. By adjusting the argument in [33] for conditional medians, we provide the measurability of conditional quantile random sets. From this, we define the right and left-conditional quantiles as well as demonstrate their measurability.

Next, we show that, when restricted to L^p-spaces, conditional quantiles take value on a smaller L^p-space. Consequently, it is possible to view them as an one parameter family of non-linear operators mapping distinct L^p-spaces. Moreover, three equivalent definitions to conditional quantiles are offered. Then, after defining and establishing their measurability, basic properties enjoyed by conditional quantiles are investigated, such as monotonicity, idempotency, and independence.

We generalize several known properties of unconditional quantiles to the conditional case. First, we investigate invariance properties and provide conditions for additivity of quantiles, that is, the conditional quantile operator of a sum of random variables equals to the sum of the conditional quantile of each random variable. We show that the concept of conditional comonotonicity introduced in [15] may be useful to the conditional case. Under $\mathcal{G}$-comonotonicity, we show that conditional quantiles are additive. This result extends some findings from [10]. Furthermore, we show that positive homogeneity and translational invariance can be used to establish additivity for each quantile. Then, we generalize the property of invariance with respect to monotone transformation from the unconditional [see, e.g., 16] to the conditional case. Finally, we use the operator properties of conditional quantiles and the subdiffferentiability of concave and convex functions to present a simple proof for Jensen's inequality for conditional quantiles.[3]

The next natural aspect of a non-linear operator to be dissected is its continuity. We investigate the continuity of conditional quantiles as operators with respect to different topologies. We start by describing a novel Fatou's lemma for conditional quantiles, proving that it holds under less stringent assumptions than its conditional expected value counterpart. As a direct consequence of this Fatou's lemma, we obtain conditions for the continuity of conditional quantiles with respect to almost sure convergence. Moreover, we provide conditions for continuity of conditional quantiles in L^p spaces. We also revisit some of the main theorems regarding the continuity of quantiles with respect to weak convergence and enlarge it in the context of conditional weak convergence, as proposed in [31]. Overall, the results on continuity have important practical implications, as for instance, showing the convergence of quantiles under almost sure convergence of random variables.

We then investigate the differentiability properties of conditional quantiles. One of the most useful properties of the expected value is its ability of exchanging the order of the integration and differentiation, the well known Leibniz's rule.[4] The interchange of integration and differentiation has been extensively used in applications, for example, in deriving statistical properties of the maximum likelihood estimator [see, e.g., 12]. We extend Leibniz's rule to quantiles and establish a novel differentiability property that allows one to exchange the quantile and the derivative. In particular, we first show the validity of Leibniz' rule for monotone functions. Second, we extend this result to the case of separable functions.

[3]Reference [24] establishes an analogue of the Jensen's inequality for medians. Using a similar approach, [35] strengthens these inequalities.

[4]There are required conditions to achieve such a result. In the expectation case, interchanging a derivative with an expectation (an integral) can be established by applying the dominated convergence theorem. Intuitively, the conditions say that the derivative of the function of interest must be bounded by another function whose integral is finite.

Finally, we examine the analogue for the law of iterated expectations (LIE) for conditional quantiles.[5] We show that the law of iterated quantiles does not hold in general, that is the LIE does not extend to quantiles. Nevertheless, we characterize the maximum set of random variables for which this law holds, and investigate its consequences for the infinite composition of conditional quantiles.

The theory developed in this paper may have important developments and applications in practice. The results are theoretically important because they provide grounds for subsequent research on statistical and mathematical analysis of conditional quantiles. From a practical point of view, the results might be useful in decision theory studies with quantile preferences, as well as establishing properties of quantile regression models.

The remainder of the paper is organized as follows. Section 2 presents definitions and basic properties of quantiles. In Section 3, we study the invariance properties of conditional quantiles. Section 4 provides continuity results. Section 5 deals with differentiability of conditional quantiles and establishes a result that allows for interchanging the derivative and the operator. Section 6 investigates the composition of quantiles. Finally, Section 7 concludes. We relegate all proofs to the Supplementary Material [4].

2. Conditional quantiles: Definitions and basic properties

This section introduces the main definitions of this article, proves the measurability of the objects and establishes their basic properties. Section 2.1 specifies basic notation. In Section 2.2, we define conditional quantile random sets as well as right and left-quantiles. Besides, the measurability of each object is derived. Section 2.3 demonstrates how one can visualize conditional quantiles as an one-parameter family of operators acting on L^p-spaces. Finally, three equivalent ways to define conditional quantiles and a set of their basic properties are provided in Section 2.4.

2.1. Notation

Throughout this article, $(\Omega,\mathcal{F},\mathsf{P})$ will be a probability space and $\mathcal{G}$ and $\mathcal{H}$ will be σ-algebras satisfying $\mathcal{F}\supset\mathcal{G}\supset\mathcal{H}$. We denote by $\mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P};\mathbb{R}^n)$ the set of measurable maps $X:(\Omega,\mathcal{F})\to(\mathbb{R}^n,\mathcal{B}(\mathbb{R}^n))$, where $\mathcal{B}(\mathbb{R}^n)$ stands for the Borel σ-algebra of the Euclidean space $\mathbb{R}^n$. If $n=1$, we will simplify this notation to $\mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$. The set $\mathsf{L}^p(\Omega,\mathcal{F},\mathsf{P})\subset\mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$ corresponds to the random variables such that $\int_\Omega |f|^p d\mathsf{P}<\infty$. For all $A\subset\mathbb{R}^n$, we denote the set of continuous and bounded functions $f:A\to\mathbb{R}$ by $C_b(A)$. We write $a.s.$ for almost surely and, for all $X\in\mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$, supp X means the support of the probability measure P_X, where P_X is the measure on $\mathbb{R}$ induced by X. Given a pair $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$, if $x_i\le y_i$, for all $i\in\{1,\dots,n\}$, we say that x is smaller than y, denoting it by $x\le y$. If $(T,\mathcal{T})$ is a first-countable topological space, $(M,\mathcal{M})$ a topological space with a partial order $\le$, and $f:T\to M$ a function, then f is said to be lower (or upper) semicontinuous provided that, for all sequence $(x_n)_{n\in\mathbb{N}}\subset T$, such that $x_n\underset{\mathcal{T}}{\to}x$, then $f(x)\le\liminf_{n\in\mathbb{N}}f(x_n)$ (or $\limsup_{n\in\mathbb{N}}f(x_n)\le f(x)$). We say that a property holds *pointwise* if it is true for every $\omega\in\Omega$.

2.2. Definition and measurability

Given a random variable in a probability space $(\Omega,\mathcal{F},\mathsf{P})$ and a σ-algebra $\mathcal{G}\subset\mathcal{F}$, we want to define the conditional quantile map $\mathsf{Q}_\tau[X|\mathcal{G}]:(\Omega,\mathcal{G})\to(\mathbb{R},\mathcal{B}(\mathbb{R}))$. Intuitively, each realization of $\mathsf{Q}_\tau[X|\mathcal{G}]$

[5] The LIE is also known as the law of total expectation or the tower property of conditional expectations.

should give the smallest value y such that the conditional probability satisfies $\mathsf{P}[X \le y|\mathcal{G}](\omega) \ge \tau$, i.e.

$$\mathsf{Q}_\tau[X|\mathcal{G}](\omega) = \inf\{y \in \mathbb{R} : \mathsf{P}[X \le y|\mathcal{G}](\omega) \ge \tau\}. \tag{1}$$

If $\mathcal{G} = \sigma(Y)$, X and Y are simple random variables, $X = \sum_{j=1}^n x_i \mathbb{1}_{A_i}$ and $Y = \sum_{j=1}^n y_i \mathbb{1}_{B_i}$, with $\mathsf{P}(B_i) > 0$ for every i, then $\mathsf{P}[X \in \cdot\,|\mathcal{G}]$ is easily computed from Bayes' formula. Consequently, the definition of $\mathsf{Q}_\tau[X|\mathcal{G}]$ under this condition is trivial, as well as its measurability. However, for a general $\mathcal{G}$, this definition depends on transition kernels, $\mathsf{P}[X \in \cdot\,|\mathcal{G}]$, not easily computed, satisfying the following.

1. The map $\mathsf{P}[X \in \cdot\,|\mathcal{G}] : \Omega \times \mathcal{B}(\mathbb{R}) \to [0,1]$ is so that, for all $\omega \in \Omega$, then:
$$A \in \mathcal{B}(\mathbb{R}) \mapsto \mathsf{P}[X \in A|\mathcal{G}](\omega) \text{ is a probability measure.}$$
2. For all fixed $A \in \mathcal{B}(\mathbb{R})$, then:
$$\omega \in \Omega \mapsto \mathsf{P}[X \in A|\mathcal{G}](\omega) \text{ is } \mathcal{G}\text{-measurable.}$$
3. For all $G \in \mathcal{G}$ and $A \in \mathcal{B}(\mathbb{R})$, then:
$$\mathsf{P}[\{X \in A\} \cap G] = \int_G \mathsf{P}[X \in A|\mathcal{G}](\omega) d\mathsf{P}|_{\mathcal{G}}(\omega),$$
where $\mathsf{P}|_{\mathcal{G}} : \mathcal{G} \to [0,1]$ denotes the restriction of the probability measure to the sub-σ-algebra $\mathcal{G}$.

The existence of such kernel is guaranteed by the disintegration theorem — see [27, Theorem 9, p. 117]. As an immediate consequence of this definition, we obtain that $\mathsf{E}[f(X)|\mathcal{G}](\omega) = \int f(x)\mathsf{P}[X \in dx|\mathcal{G}](\omega)$ a.s., for all $f \in \mathcal{B}(\mathbb{R})$-measurable and bounded – see [21]. Even though $\mathsf{P}[X \in A|\mathcal{G}] = \mathsf{E}[\mathbb{1}_{X\in A}|\mathcal{G}]$ for all $A \in \mathcal{B}(\mathbb{R})$ a.s., $\mathsf{P}[X \in \cdot\,|\mathcal{G}](\omega)$ has the advantage of being a probability measure on $\mathcal{B}(\mathbb{R})$. Moreover, given two transition kernels, $\mathsf{P}[X \in \cdot\,|\mathcal{G}]$ and $\bar{\mathsf{P}}[X \in \cdot\,|\mathcal{G}]$, [21] shows that there is a set $\Omega' \subset \Omega$, with full measure, such that:

$$\mathsf{P}[X \in A|\mathcal{G}](\omega) = \bar{\mathsf{P}}[X \in A|\mathcal{G}](\omega), \text{ for all } A \in \mathcal{B}(\mathbb{R}) \text{ and } \omega \in \Omega'.$$

From now on, we assume that for all fixed X we are using the same version of $\mathsf{P}[X \in \cdot\,|\mathcal{G}]$, unless otherwise stated.

Instead of defining the τ-conditional quantile directly by $\mathsf{Q}_\tau[X|\mathcal{G}](\omega) = \inf\{y \in \mathbb{R} : \mathsf{P}[X \le y|\mathcal{G}](\omega) \ge \tau\}$, which is the most common in the literature, in this work, we adopt an optimization problem definition, similar to that in [17]. This method leads to conditional quantile random sets. By proving that these random sets are $\mathcal{G}$-measurable, we are able to define the left and right conditional quantiles simply as their composition with some specific measurable maps. An advantage of this approach is that it allows us to readily derive the measurability of both right and left conditional quantiles at once.

[33] shows that it is possible to define the conditional median of a random variable with respect to some σ-algebra as a compact random set, proving its measurability with respect to the σ-algebra $\mathcal{B}(\mathcal{K})$ over the set of compact sets of the real line $\mathcal{K}$; more precisely,

$$\mathcal{B}(\mathcal{K}) = \sigma\left(\{K \in \mathcal{K} : K \cap G \neq \varnothing\} : \ G \subset \mathbb{R} \text{ open}\right),$$

see [25] for more details on $\mathcal{B}(\mathcal{K})$. Based on that, we define the τ-conditional quantile random set as the set of solutions of the following convex problem, where $\rho_\tau : \mathbb{R} \to \mathbb{R}$ stands for $\rho_\tau(x) := (\tau - 1)x\mathbb{1}_{[x<0]} + \tau x \mathbb{1}_{[x\ge 0]}$, also known as the *check function* – see [17].

Definition 2.1. Given $X \in \mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$ in a probability space $(\Omega,\mathcal{F},\mathsf{P})$, $\tau \in (0,1)$ and a σ-algebra $\mathcal{G} \subset \mathcal{F}$, such that the conditional law of X given $\mathcal{G}$ is $\mathsf{P}[X \in \cdot\,|\mathcal{G}] : \Omega \times \mathcal{B}(\mathbb{R}) \to [0,1]$, the τ-quantile random set of X conditional to $\mathcal{G}$ is a map $\Gamma_\tau[X|\mathcal{G}] : (\Omega,\mathcal{G}) \to (\mathcal{K},\mathcal{B}(\mathcal{K}))$ satisfying:

$$\Gamma_\tau[X|\mathcal{G}](\omega) = \operatorname*{argmin}_{y\in\mathbb{R}} \int \big(\rho_\tau(x-y) - \rho_\tau(x)\big)\,\mathsf{P}[X \in dx|\mathcal{G}](\omega),\ \forall\omega \in \Omega. \tag{2}$$

It is worth noting that if we choose another representative for the transition kernel $\mathsf{P}[X \in \cdot\,|\mathcal{G}]$, then the τ-quantile random set associated to each representative coincides in a set of full probability measure. Therefore, the above definition is unique up to a modification on a set of zero measure.

Our first result guarantees that the above map is well-defined as a compact random set, that is, takes compact values and is measurable. We adapt the proof given in [33] for $\tau = 1/2$.

Proposition 2.2. *Given $X \in \mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$ in a probability space $(\Omega,\mathcal{F},\mathsf{P})$, $\tau \in (0,1)$ and a σ-algebra $\mathcal{G} \subset \mathcal{F}$, such that its transition kernel is given by $\mathsf{P}[X \in \cdot\,|\mathcal{G}]$, then $\Gamma_\tau[X|\mathcal{G}](\omega)$ is nonempty and compact for all $\omega \in \Omega$. Moreover, the map $\Gamma_\tau[X|\mathcal{G}] : (\Omega,\mathcal{G}) \to (\mathcal{K},\mathcal{B}(\mathcal{K}))$ is measurable.*

It is trivial to show that the maps $\inf : (\mathcal{K},\mathcal{B}(\mathcal{K})) \to (\bar{\mathbb{R}},\mathcal{B}(\bar{\mathbb{R}}))$ and $\sup : (\mathcal{K},\mathcal{B}(\mathcal{K})) \to (\bar{\mathbb{R}},\mathcal{B}(\bar{\mathbb{R}}))$ are measurable.[6] Since $(\Gamma_\tau[X|\mathcal{G}])_{\tau\in(0,1)}$ is a family of measurable compact random sets, we can define the left and right conditional quantile as the composition of inf and sup with $\Gamma_\tau[X|\mathcal{G}]$:

Definition 2.3. Given $X \in \mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$ in a probability space $(\Omega,\mathcal{F},\mathsf{P})$, $\tau \in (0,1)$ and a σ-algebra $\mathcal{G} \subset \mathcal{F}$, such that the conditional law of X given $\mathcal{G}$ is $\mathsf{P}[X \in \cdot\,|\mathcal{G}] : \Omega \times \mathcal{B}(\mathbb{R}) \to [0,1]$, the τ-quantile, $\tau \in (0,1)$, of X conditional to $\mathcal{G}$ is:

$$\mathsf{Q}_\tau[X|\mathcal{G}](\omega) = \inf \Gamma_\tau[X|\mathcal{G}](\omega),\ \forall\omega \in \Omega. \tag{3}$$

We also define the right conditional quantile as:

$$\mathsf{Q}_{\tau+}[X|\mathcal{G}](\omega) = \sup \Gamma_\tau[X|\mathcal{G}](\omega),\ \forall\omega \in \Omega. \tag{4}$$

In the statistics literature, it is frequently assumed that both right and left conditional quantiles coincide a.s., i.e. $\mathsf{Q}_\tau[X|\mathcal{G}] = \mathsf{Q}_{\tau+}[X|\mathcal{G}]$ a.s. However, the conditions that assure this equality are not generally true for any $X \in \mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$ in a most abstract framework. To be precise, for a fixed $\tau \in (0,1)$, these random variables are indistinguishable if, and only if, there exists a $\mathcal{G}$-measurable set, with full probability, such that, on it, $x \in \mathbb{R} \mapsto \mathsf{P}[X \le x|\mathcal{G}]$ is strictly increasing in a neighborhood of $\mathsf{Q}_\tau[X|\mathcal{G}]$ – see Lemma 3.2 in the Supplementary Material [4] for more details.

Definition 2.3 includes the unconditional left and right-quantiles, when $\mathcal{G} = \{\varnothing,\Omega\}$. In this case, we omit the trivial σ-algebra and simply refer to $\mathsf{Q}_\tau[X|\mathcal{G}]$ and $\mathsf{Q}_{\tau+}[X|\mathcal{G}]$ as $\mathsf{Q}_\tau[X]$ and $\mathsf{Q}_{\tau+}[X]$, respectively.

Observe now that measurability of the right and left conditional quantile are easily derived from the fact that they are a composition of measurable maps.

Proposition 2.4. *Given $X \in \mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$ in a probability space $(\Omega,\mathcal{F},\mathsf{P})$ and a σ-algebra $\mathcal{G} \subset \mathcal{F}$, such that its transition kernel is given by $\mathsf{P}[X \in \cdot\,|\mathcal{G}]$, $\mathsf{Q}_\tau[X|\mathcal{G}] : \Omega \to \mathbb{R}$ and $\mathsf{Q}_{\tau+}[X|\mathcal{G}] : \Omega \to \mathbb{R}$ are well-defined and $\mathcal{G}$-measurable random variables.*

[6]Recall that, for $K \in \mathcal{K}$, $|\inf K| = |\sup K| = +\infty$ if, and only if, $K = \varnothing$. Since $\Gamma_\tau[X|\mathcal{G}]$ is non-empty, the composition of these maps generates an $\mathbb{R}$-valued random variable.

To illustrate the concepts defined along this section, we offer some examples in the Supplementary Material [4]. The first is designed to demonstrate how to determine the conditional quantile in a very simple framework, with auxiliary graphs. The second, on the other hand, characterizes a particular conditional quantile random set and, from it, obtains the associated left and right quantiles. Finally, the third shows a concrete example where the conditional quantile of the sum of two variables equals the sum of each individual quantile, which will be revisited in Section 3.1. See more details in the Supplementary Material [4].

Proposition 2.4 allows us to define maps, $\mathsf{Q}_\tau[\cdot|\mathcal{G}]$ and $\mathsf{Q}_{\tau+}[\cdot|\mathcal{G}]$, over $\mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$ and taking values on $\mathsf{L}^0(\Omega,\mathcal{G},\mathsf{P})$, which compute the τ left and right quantile of X conditional to $\mathcal{G}$, for all $\tau \in (0,1)$. Moreover, since, for any given $X \in \mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$, two transition kernels agree a.s.,we obtain that both maps are well-defined a.s. Therefore, from now on, we visualize $\mathsf{Q}_\tau[\cdot|\mathcal{G}] : \mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P}) \to \mathsf{L}^0(\Omega,\mathcal{G},\mathsf{P})$ and $\mathsf{Q}_{\tau+}[\cdot|\mathcal{G}] : \mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P}) \to \mathsf{L}^0(\Omega,\mathcal{G},\mathsf{P})$ as non-linear operators, and we derive their properties in the subsequent sections.

2.3. Conditional quantiles as operators

As we showed, it is possible to define a one parameter family of non-linear operators acting on $\mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$ and taking values on $\mathsf{L}^0(\Omega,\mathcal{G},\mathsf{P})$, $\mathsf{Q}_\tau[\cdot|\mathcal{G}] : \mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P}) \to \mathsf{L}^0(\Omega,\mathcal{G},\mathsf{P})$, for each $\tau \in (0,1)$. The next proposition investigates the properties of these operators when restricted to the space $\mathsf{L}^p(\Omega,\mathcal{F},\mathsf{P})$.

Proposition 2.5.

1. $X \in \mathsf{L}^p(\Omega,\mathcal{F},\mathsf{P})$, $p \in [1,+\infty)$, *if, and only if,* $\mathsf{Q}_\tau[X|\mathcal{G}] \in \mathsf{L}^p(\Omega,\mathcal{G},\mathsf{P})$, *for all* $\tau \in (0,1)$, $s \mapsto \mathsf{E}[|\mathsf{Q}_s[X|\mathcal{G}]|^p]$ *is left-continuous with right-limits and:*
$$\int_0^1 \mathsf{E}[|\mathsf{Q}_\tau[X|\mathcal{G}]|^p]d\tau < +\infty.$$
2. $X \in \mathsf{L}^\infty(\Omega,\mathcal{F},\mathsf{P})$ *if, and only if,* $\mathsf{Q}_\tau[X|\mathcal{G}] \in \mathsf{L}^\infty(\Omega,\mathcal{F},\mathsf{P})$, *for all* $\tau \in (0,1)$, *and:*
$$\sup_{\tau\in(0,1)} \|\mathsf{Q}_\tau[X|\mathcal{G}]\|_\infty < +\infty.$$
3. *If* $X \in \mathsf{L}^p(\Omega,\mathcal{F},\mathsf{P})$, *for* $p \in [1,+\infty)$, *then* $\tau \in (0,1) \mapsto \mathsf{Q}_\tau[X|\mathcal{G}] \in \mathsf{L}^p(\Omega,\mathcal{G},\mathsf{P})$ *is left-continuous with right-limits, as a curve in* $\mathsf{L}^p(\Omega,\mathcal{G},\mathsf{P})$.

It is worth noting that, as an immediate consequence of the Proposition 2.5, we obtain $\mathsf{Q}_\tau[\cdot|\mathcal{G}] : \mathsf{L}^p(\Omega,\mathcal{F},\mathsf{P}) \to \mathsf{L}^p(\Omega,\mathcal{G},\mathsf{P})$ for all $\tau \in (0,1)$ and $p \in [1,+\infty]$.

2.4. Basic properties

We now show that Definition 2.1 of the conditional τ-quantile operator coincides with the one considered by equation (1). In fact, the next result exhibits the equivalence of different definitions for conditional quantiles.

Theorem 2.6. *The following equalities hold:*

1. $\mathsf{Q}_\tau[X|\mathcal{G}] = \inf\{Y \in \mathsf{L}^0(\Omega,\mathcal{G},\mathsf{P}) : \mathsf{P}[X \le Y|\mathcal{G}] \ge \tau\}$ *pointwise.*
2. $\mathsf{Q}_\tau[X|\mathcal{G}] = \min\{\text{argmin}_{y\in\mathbb{R}} \mathsf{E}[\rho_\tau(X-y) - \rho_\tau(X)|\mathcal{G}]\}$ *a.s.*
3. $\mathsf{Q}_\tau[X|\mathcal{G}] = \inf\{y \in \mathbb{R} : \mathsf{P}[X \le y|\mathcal{G}] \ge \tau\}$ *pointwise.*

The result in item 1 of Theorem 2.6 means that if $Y \in \mathsf{L}^0(\Omega, \mathcal{G}, \mathsf{P})$, then we understand $\mathsf{P}[X \le Y|\mathcal{G}] \ge \tau$ as $\mathsf{P}[X \le Y(\omega)|\mathcal{G}](\omega) \ge \tau$, for all $\omega \in \Omega$, and the infimum is pointwise. Furthermore, item 2 of Theorem 2.6 assumes continuous sample paths of the objective function in the minimization problem. Notice that $(\mathsf{E}[\rho_\tau(X-y) - \rho_\tau(X)|\mathcal{G}])_{y\in\mathbb{R}}$ is a stochastic process satisfying:

$$\mathsf{E}[|\mathsf{E}[\rho_\tau(X-y) - \rho_\tau(X)|\mathcal{G}] - \mathsf{E}[\rho_\tau(X-z) - \rho_\tau(X)|\mathcal{G}]|^p] \le \left(\frac{1}{2} + \left|\frac{1}{2} - \tau\right|\right)^p |z-y|^p,$$

for all $p > 1$ and $y, z \in \mathbb{R}$. Therefore, Kolmogorov's theorem guarantees that exists a modification of this process with continuous sample paths, see [20]. Consequently, there is no loss of generality by imposing the continuity condition, and from now on, for each $X \in \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P})$ and $\mathcal{G} \subset \mathcal{F}$ given, we assume that the sample paths of $(\mathsf{E}[\rho_\tau(X-y) - \rho_\tau(X)|\mathcal{G}])_{y\in\mathbb{R}}$ are continuous.

We may combine Theorem 2.6 and Proposition 2.5 to obtain yet another equivalent characterization of conditional quantile as the minimal solution of an optimization problem in $\mathsf{L}^p(\Omega, \mathcal{F}, \mathsf{P})$.

Proposition 2.7. *For all $\tau \in (0,1)$ and $p \in [1, +\infty]$, the τ-conditional quantile operator, $\mathsf{Q}_\tau[\cdot|\mathcal{G}] : \mathsf{L}^p(\Omega, \mathcal{F}, \mathsf{P}) \to \mathsf{L}^p(\Omega, \mathcal{G}, \mathsf{P})$ satisfies:*

$$\mathsf{Q}_\tau[X|\mathcal{G}] = \inf\left\{Z \in \mathsf{L}^p(\Omega, \mathcal{G}, P), Z \in \underset{Y\in\mathsf{L}^p(\Omega,\mathcal{G},\mathsf{P})}{\operatorname{argmin}} \mathsf{E}[\rho_\tau(X-Y)]\right\} \text{ a.s.} \tag{5}$$

Moreover, the optimization problem in Theorem 2.6 item 1 can be restricted to $\mathsf{L}^p(\Omega, \mathcal{G}, \mathsf{P})$.

Remark 2.8. The infimum in expression (5) is understood as the essential infimum of a family of random variables as in [26].

The results previously derived show that, for all $\tau \in (0,1)$ and $p \in [1, +\infty] \cup \{0\}$, the τ-conditional quantile is an invariant operator with respect to the regularity of the space, i.e. $\mathsf{Q}_\tau[\cdot|\mathcal{G}](\mathsf{L}^p(\Omega, \mathcal{F}, \mathsf{P})) = \mathsf{L}^p(\Omega, \mathcal{G}, \mathsf{P})$. Indeed, if $X \in \mathsf{L}^p(\Omega, \mathcal{G}, \mathsf{P})$, then $\mathsf{E}[\rho_\tau(X-X)] = 0$. Since, for all $Y \in \mathsf{L}^p(\Omega, \mathcal{G}, \mathsf{P})$, $\mathsf{E}[\rho_\tau(X-Y)] \ge 0$, with strict inequality when $X \ne Y$ in a non-negligible set, we conclude that X is the minimizer in Proposition 2.7. Therefore, $\mathsf{Q}_\tau[X|\mathcal{G}] = X$. Together with Proposition 2.5, we obtain $\mathsf{Q}_\tau[\cdot|\mathcal{G}](\mathsf{L}^p(\Omega, \mathcal{F}, \mathsf{P})) = \mathsf{L}^p(\Omega, \mathcal{G}, \mathsf{P})$. To further analyze its properties as an operator, next section investigates the conditions under which it is additive, as well as its other invariance properties.

The characterizations in Theorem 2.6 are the departing point for quantile regression. When estimating a linear or nonlinear quantile model, one simply replaces the population expectation in item 2 of the theorem with the corresponding sample average and uses linear programming to solve the optimization problem (see, e.g., [16] for details).

As an example of how each characterization may play an important role in the theory, we apply the results in Theorem 2.6 to provide a set of basic properties for conditional quantiles.

Proposition 2.9. *Let $X \in \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P})$ be fixed. We have the following:*

1. *For each $\omega \in \Omega$, the map $s \in (0,1) \mapsto \mathsf{Q}_s[X|\mathcal{G}](\omega)$ is non-decreasing, left-continuous with right-limits. Moreover, $\mathsf{Q}_{\tau+}[X|\mathcal{G}](\omega) = \lim_{s\downarrow\tau} \mathsf{Q}_s[X|\mathcal{G}](\omega)$ and it can be characterized as $\mathsf{Q}_{\tau+}[X|\mathcal{G}](\omega) = \sup\{y \in \mathbb{R} : \mathsf{P}[X \le y|\mathcal{G}](\omega) \le \tau\}$, for all $\omega \in \Omega$.*
2. *For every $\tau \in (0,1)$, then $\mathsf{Q}_\tau[X|\mathcal{G}] \in \operatorname{supp} X$ a.s.*
3. *(Monotonicity) If $X \le Y$ a.s., then, for all $\tau \in (0,1)$, $\mathsf{Q}_\tau[X|\mathcal{G}] \le \mathsf{Q}_\tau[Y|\mathcal{G}]$ a.s.*
4. *If Y is independent of $\mathcal{G}$, then $\mathsf{Q}_\tau[Y|\mathcal{G}] = \mathsf{Q}_\tau[Y]$ a.s., for all $\tau \in (0,1)$.*
5. *(Invariance) If X is $\mathcal{G}$-measurable, then $\mathsf{Q}_\tau[X|\mathcal{G}] = X$ a.s.*

6. *If* $g \in \mathsf{L}^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathsf{P}_X)$*, then the following holds a.s.:*[7]

$$\mathsf{E}[g(X)|\mathcal{G}] = \int_0^1 g(\mathsf{Q}_\tau[X|\mathcal{G}])d\tau.$$

3. Invariance properties of conditional quantiles

This section investigates the invariance properties of conditional quantiles, i.e. which transformations commute with the operator. As in the unconditional setting, we are able to demonstrate that there are conditions that guarantee additivity (Section 3.1) and monotone invariance for this family of operators (Section 3.2). In addition, in Section 3.3 we use these results to demonstrate a Jensen's inequality for conditional quantiles.

3.1. Conditions for additivity

We provide conditions under which the conditional quantile operator of a sum of random variables equals to the sum of the conditional quantile of each random variable. First, we show how the concept of $\mathcal{G}$-comonotonicity introduced in [15] may be used to obtain this result for the sum of $\mathcal{G}$-comonotonic random variables. This is similar to the works [8] and [10] for the unconditional quantile. Second, we generalize the concept of translational invariance and positive homogeneity for the conditional quantile operator.

We begin with an extension of the notion of comonotonicity of a random vector appropriated to the conditional case. Comonotonicity plays a key role in the additivity of quantiles, as it can be seen in [8] for the unconditional quantile. Equipped with the definition of conditional comonotonicity, we show that it is, in fact, a sufficient condition for the additivity of the conditional quantile of a sum of random variables. Hence, we first define what are comonotonic sets.

Definition 3.1. A set $A \subset \mathbb{R}^n$ is comonotonic if for all $x, y \in A$, then either $x \leq y$ or $y \leq x$.

[8] presents and discusses in details the concept of comonotonicity as well as its consequences to quantiles. For a clear visualization of closed comonotonic sets in $\mathbb{R}^n$, which are precisely the support of comonotonic random variables, we present in the Supplementary Material [4] a complete characterization of such sets. The result shows that any closed comonotonic set, $A \subset \mathbb{R}^n$, is the image of a map, $A = \psi((0,1))$, $\psi : (0,1) \to \mathbb{R}^n$, such that each component of ψ is a function that is non-decreasing, and left-continuous with right-limits.

In the conditional framework, we define $\mathcal{G}$-conditional random vector as:

Definition 3.2. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra, and $X \in \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P}; \mathbb{R}^n)$. X is a $\mathcal{G}$-comonotonic random vector, if supp $\mathsf{P}[X \in \cdot\, |\mathcal{G}](\omega)$ is comonotonic almost surely on Ω.

This definition is similar to one presented in [15] and [3], with the exception that we assume comonotonic support of the conditional law to be an almost surely property instead of pointwise, as they do. We adopted a different definition because changes in the conditional law representative would not alter the conditional comonotonicity property, since any representative would agree almost surely.

[7]Recall from Section 2.1 that P_X is the measure on $\mathbb{R}$ induced by X.

When $\mathcal{G} = \{\emptyset, \Omega\}$, we may see that Definition 3.2 coincides with the concept of comonotonic random variables as introduced (for $n = 2$) in [29], and further explored in [8]. Observe that, in this case, there is a set of full probability measure such that, for all $A \in \mathcal{B}(\mathbb{R}^n)$, then $\mathsf{P}[X \in A|\mathcal{G}](\omega) = \mathsf{P}[X \in A]$. Thus, $\text{supp}\, \mathsf{P}[X \in \cdot\, |\mathcal{G}](\omega) = \text{supp}\, \mathsf{P}[X \in \cdot\,]$ a.s., so that X is $\mathcal{G}$-comonotonic if, and only if, it is comonotonic, as it appears for instance in [8]. Furthermore, notice also that comonotonicity implies $\mathcal{G}$-comonotonicity, for all σ-algebras $\mathcal{G} \subset \mathcal{F}$, whereas the opposite implication is false – see a counterexample in the third example in Section 1 of Supplementary Material [4].

The first result of this section is a characterization of $\mathcal{G}$-comonotonic vectors similar to [3, Lemma 2]. We simply modified the results to hold a.s., agreeing to our definition of $\mathcal{G}$-comonotonicity.

Lemma 3.3. *Assume that* $(\Omega, \mathcal{F}, \mathsf{P})$ *is atomless. Let* $X = (X_1, \dots, X_n) \in \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P}; \mathbb{R}^n)$*. Then the following statements are equivalent:*

1. *X is $\mathcal{G}$-comonotonic.*
2. *There exists a set $\Omega' \in \mathcal{G}$, such that $\mathsf{P}[\Omega'] = 1$, and on Ω':*

$$\mathsf{P}[X \le x|\mathcal{G}](\omega) = \min_{i \in \{1,\dots,n\}} \mathsf{P}[X_i \le x_i|\mathcal{G}](\omega), \text{ for all } x \in \mathbb{R}^n.$$

3. *There exists a uniform random variable $U : \Omega \to (0,1)$ and $\Omega' \subset \Omega$, with $\mathsf{P}[\Omega'] = 1$, such that for all $\omega \in \Omega'$ and any set $A \in \mathcal{B}(\mathbb{R}^n)$:*

$$\mathsf{P}[X \in A|\mathcal{G}](\omega) = \mathsf{P}[(\mathsf{Q}_U[X_1|\mathcal{G}](\omega), \dots, \mathsf{Q}_U[X_n|\mathcal{G}](\omega)) \in A].$$

The assumption that $(\Omega, \mathcal{F}, \mathsf{P})$ is atomless is required for condition 3 above, but conditions 1 and 2 are still equivalent without it. This atomless assumption is equivalent to the probability space $(\Omega, \mathcal{F}, \mathsf{P})$ being rich enough to support a uniform random variable on $(0,1)$ – see Proposition A.27 in [13], which also provides other consequences of such assumption on the probability space. For instance, it is shown that this assumption is equivalent to the existence of an i.i.d sequence $(Y_i)_{i \in \mathbb{N}}$ of random variables such that $Y_1 \sim \mu$, for all probability measures $\mu \in \mathcal{P}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

As a consequence of the characterization obtained above, we derive the following generalization of [10, Proposition 3.1]:

Theorem 3.4. *Let $\psi : \mathcal{X} \subset \mathbb{R}^m \times \mathcal{Y} \subset \mathbb{R}^n \to \mathbb{R}$ be a function such that, for all $x \in \mathcal{X}$, then $y \in \mathcal{Y} \mapsto \psi(x, y)$ is non-decreasing, left-continuous with right-limits in each argument. Then, for any $\mathcal{G}$-comonotonic random vector $Y = (Y_1, \dots, Y_n)$, whose support lies in $\mathcal{Y}$, and $X \in \mathsf{L}^0(\Omega, \mathcal{G}, \mathsf{P}; \mathbb{R}^m)$, with support in $\mathcal{X}$, and $\tau \in (0,1)$ fixed:*

$$\mathsf{Q}_\tau[\psi(X, Y_1, \dots, Y_n)|\mathcal{G}] = \psi\left(X, \mathsf{Q}_\tau[Y_1|\mathcal{G}], \dots, \mathsf{Q}_\tau[Y_n|\mathcal{G}]\right), \text{ a.s.}$$

In the Supplementary Material [4], we show how the previous result may be employed in a quantile regression model to characterize the conditional distribution of the dependent variable, given the independent variables.

As a corollary of Theorem 3.4, we obtain that the τ-conditional quantile of the sum of the components of a $\mathcal{G}$-comonotonic random vector equals the sum of the individual quantiles a.s.

Corollary 3.5. *Let $X = (X_1, \dots, X_n)$ be a $\mathcal{G}$-comonotonic random vector. Then, for all $\tau \in (0,1)$:*

$$\mathsf{Q}_\tau\left[\sum_{i=1}^n X_i \middle| \mathcal{G}\right] = \sum_{i=1}^n \mathsf{Q}_\tau[X_i|\mathcal{G}], \text{ a.s.,}$$

and provided that $X_i \geq 0$ *a.s.,* $\forall i \in \{1,\ldots,n\}$*, then:*

$$\mathsf{Q}_\tau\left[\prod_{i=1}^{n} X_i \middle| \mathcal{G}\right] = \prod_{i=1}^{n} \mathsf{Q}_\tau\left[X_i|\mathcal{G}\right], \text{ a.s.}$$

Instead of using $\mathcal{G}$-comonotonicity, the following result provides a stronger version of positive homogeneity and translational invariance to establish additivity for each quantile.

Theorem 3.6. *If* $a, b \in \mathsf{L}^0(\Omega, \mathcal{G}, \mathsf{P})$*, then a.s.:*

$$\mathsf{Q}_\tau[a + bX|\mathcal{G}] = a + b\mathsf{Q}_\tau[X|\mathcal{G}]\mathbb{1}_{\{b\geq 0\}} + b\mathsf{Q}_{(1-\tau)+}[X|\mathcal{G}]\mathbb{1}_{\{b<0\}}.$$

Theorem 3.6 also holds for $\mathsf{Q}_{\tau+}[\cdot|\mathcal{G}]$, simply changing τ and $(1-\tau)+$ by $\tau+$ and $1-\tau$, respectively.

3.2. Equivariance to monotone transformations

Continuing the discussion on invariance properties of quantiles, we generalize their invariance to monotone transformation from the unconditional – given in [16] – to the conditional case. To accomplish this, we first provide a simple version of invariance for conditional quantiles with respect to monotone transformations. Then, we extend it to the case of monotone functions with measurable parameters.

Proposition 3.7. *Suppose* $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies that, for a given* $x \in \mathbb{R}$*,* $y \in \mathbb{R} \mapsto g(x, y)$ *is non-decreasing and left-continuous. Then, for all* $\tau \in (0,1)$ *fixed and* $Y \in \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P})$*,*

$$\mathsf{Q}_\tau[g(x, Y)|\mathcal{G}] = g\left(x, \mathsf{Q}_\tau[Y|\mathcal{G}]\right) \text{ a.s.}$$

If $y \in \mathbb{R} \mapsto g(x, y)$ *is non-increasing and left-continuous for a given* $x \in \mathbb{R}$*, then:*

$$\mathsf{Q}_\tau[g(x, Y)|\mathcal{G}] = g\left(x, \mathsf{Q}_{(1-\tau)+}[Y|\mathcal{G}]\right) \text{ a.s.}$$

Moving forward with the discussion on how $\mathcal{G}$-measurable parameters influence the invariance properties, we now extend Proposition 3.7 to monotone and left-continuous functions with measurable parameters.

Proposition 3.8. *Suppose that* $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies*

1. *For each* $x \in \mathbb{R}$*,* $y \in \mathbb{R} \mapsto g(x, y)$ *is non-decreasing and left-continuous.*
2. *For each* $y \in \mathbb{R}$*,* $x \in \mathbb{R} \mapsto g(x, y)$ *is* $\mathcal{B}(\mathbb{R})$*-measurable.*

Then, for all $X \in \mathsf{L}^0(\Omega, \mathcal{G}, \mathsf{P})$*,* $Y \in \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P})$ *there is a set* $\Omega' \in \mathcal{G}$*, with* $\mathsf{P}[\Omega'] = 1$*, such that for all* $\tau \in (0,1)$ *and* $\omega \in \Omega'$

$$\mathsf{Q}_\tau[g(X, Y)|\mathcal{G}](\omega) = g\left(X, \mathsf{Q}_\tau[Y|\mathcal{G}](\omega)\right),$$

$$\mathsf{Q}_{\tau+}[g(X, Y)|\mathcal{G}](\omega) = g\left(X, \mathsf{Q}_{\tau+}[Y|\mathcal{G}](\omega)\right).$$

If in item 1, the function g is non-increasing and left-continuous, and item 2 holds true, then:

$$\mathsf{Q}_\tau[g(X, Y)|\mathcal{G}](\omega) = g\left(X, \mathsf{Q}_{(1-\tau)+}[Y|\mathcal{G}](\omega)\right),$$

$$\mathsf{Q}_{\tau+}[g(X, Y)|\mathcal{G}](\omega) = g\left(X, \mathsf{Q}_{(1-\tau)}[Y|\mathcal{G}](\omega)\right).$$

We highlight that the invariance properties obtained along this section are generalizations of known results to conditional quantiles. Indeed, [18] presents Proposition 3.7 for the unconditional setup. We generalize it to the conditional case for all σ-algebras. Furthermore, Proposition 3.8 extends this invariance to functions with measurable parameters. Moreover, if $\mathsf{Q}_\tau[g(X,Y)|\mathcal{G}] = \mathsf{Q}_{\tau+}[g(X,Y)|\mathcal{G}]$, then it is possible to show that Proposition 3.8 is a direct consequence of Proposition 3.7 and Corollary 4.2, by proving the invariance along parameters which are simple random variables, approximating the real parameter by them and, then, using the continuity of g and Corollary 4.2.

As an application of these results, in Section 5 we will apply the invariance of this family of operators to derive important rules regarding the exchange in the order of derivative operator and conditional quantiles, reinforcing that the operator-theoretical approach to this family may lead to the understanding and enlargement of the properties enjoyed by them.

3.3. Jensen's inequality

This section derives Jensen's inequality for the one parameter family of operators. As a consequence of Theorem 3.6, we are able to establish a version of Jensen's inequality for quantiles using the same approach adopted in the proof of this inequality for conditional mean, Lemma 1.23 in [30].

Theorem 3.9 (Jensen's Inequality for Quantiles). *Let $u : \mathbb{R} \to \mathbb{R}$ be a function and $\tau \in (0,1)$.*

1. *If u is concave and $\tau \in (0, \frac{1}{2}]$, then:*

$$\mathsf{Q}_\tau[u(X)|\mathcal{G}] \le u\big(\mathsf{Q}_\tau[X|\mathcal{G}]\big), \text{ a.s.}$$

2. *If u is convex and $\tau \in (\frac{1}{2}, 1)$, then:*

$$u\big(\mathsf{Q}_\tau[X|\mathcal{G}]\big) \le \mathsf{Q}_\tau[u(X)|\mathcal{G}], \text{ a.s.}$$

3. *If u is convex and $\mathsf{Q}_{\frac{1}{2}}[X|\mathcal{G}] = \mathsf{Q}_{\frac{1}{2}+}[X|\mathcal{G}]$, then:*

$$u\big(\mathsf{Q}_{\frac{1}{2}}[X|\mathcal{G}]\big) \le \mathsf{Q}_{\frac{1}{2}}[u(X)|\mathcal{G}], \text{ a.s.} \tag{6}$$

Conversely, if (6) holds for all u convex, then $\mathsf{Q}_{\frac{1}{2}}[X|\mathcal{G}] = \mathsf{Q}_{\frac{1}{2}+}[X|\mathcal{G}]$ a.s.

We remark that [24] establishes an analogue of Jensen's inequality for medians. Recently, [35] strengthened these inequalities using a similar approach to the one used in [24]. Nevertheless, the results in Theorem 3.9 are obtained through a simple operator-like argument, providing a generalization for conditional medians and demonstrating the necessity and sufficiency of continuity of conditional quantiles at $\tau = \frac{1}{2}$ for Jensen's inequality. Besides that, it is trivial to see that Theorem 3.9 item 1 and 2 also hold for $\mathsf{Q}_{\tau+}[\cdot|\mathcal{G}]$ operator when $\tau \in (0, \frac{1}{2})$ and $\tau \in [\frac{1}{2}, 1)$, respectively.

Recall that if $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is regular enough, for example in $C_b(\mathbb{R}\times\mathbb{R})$, then, for all $X \in \mathsf{L}^0(\Omega,\mathcal{G},\mathsf{P})$ and $Y \in \mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$, the expected value of $g(X,Y)$, on a full probability set, with respect to $\mathcal{G}$ is equivalent to $\mathsf{E}[g(X,Y)|\mathcal{G}](\omega) = \int g(X(\omega), y)\mathsf{P}[Y \in dy|\mathcal{G}](\omega)$. In other words, we may interpret X as a parameter that does not affect the computation of the conditional expectation. Based on this, we generalize the former Jensen's inequality as following.

Corollary 3.10. *If $u : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ satisfies:*

1. *For each $x \in \mathbb{R}$, $y \in \mathbb{R} \mapsto u(x,y)$ is concave.*

2. *For each* $y \in \mathbb{R}$, $x \in \mathbb{R} \mapsto u(x,y)$ *is continuous.*
3. *The function* $u'_{2,+}\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, *defined by* $u'_{2,+}(x,y) = \lim_{h\downarrow 0} \frac{u(x,y+h)-u(x,y)}{h}$, *is* $\mathcal{B}(\mathbb{R}\times\mathbb{R})$*-measurable.*

Then, for all $\tau \in (0, \frac{1}{2}]$, $X \in \mathsf{L}^0(\Omega,\mathcal{G},\mathsf{P})$, *and* $Y \in \mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$ *fixed:*

$$\mathsf{Q}_\tau[u(X,Y)|\mathcal{G}] \leq u(X, \mathsf{Q}_\tau[Y|\mathcal{G}]), \text{ a.s.}$$

If $y \in \mathbb{R} \mapsto u(x,y)$ *is convex, for all* $x \in \mathbb{R}$, *and items 2 and 3 remain true, then for all* $\tau \in (\frac{1}{2}, 1)$*:*

$$\mathsf{Q}_\tau[u(X,Y)|\mathcal{G}] \geq u(X, \mathsf{Q}_\tau[Y|\mathcal{G}]), \text{ a.s.}$$

Finally, if $\tau = \frac{1}{2}$ *and* $\mathsf{Q}_{\frac{1}{2}+}[Y|\mathcal{G}] = \mathsf{Q}_{\frac{1}{2}}[Y|\mathcal{G}]$ a.s., *then:*

$$u(X, \mathsf{Q}_{\frac{1}{2}}[Y|\mathcal{G}]) \leq \mathsf{Q}_{\frac{1}{2}}[u(X,Y)|\mathcal{G}], \text{ a.s.}$$

4. Continuity

After establishing that conditional quantiles can be viewed as operators, and investigating conditions for their additivity, we now investigate their continuity properties. We start by describing a new Fatou's lemma for conditional quantiles, proving that it holds under less stringent assumptions than its conditional expected value counterpart. Then, as a direct consequence of our Fatou's lemma, we obtain conditions for the continuity of conditional quantiles with respect to almost sure convergence. Furthermore, since Proposition 2.5 guarantees that conditional quantiles are well-defined and invariant operators on L^p spaces, i.e. $\mathsf{Q}_\tau[\cdot|\mathcal{G}] : \mathsf{L}^p(\Omega,\mathcal{F},\mathsf{P}) \to \mathsf{L}^p(\Omega,\mathcal{G},\mathsf{P})$ for all $p \in [1,+\infty]$, we provide conditions for the continuity of these operators under the L^p-topology. Finishing this section, we revisit and enlarge some of the main theorems regarding the continuity of quantiles with respect to weak convergence, now in the context of conditional weak convergence a.s. and its implications to conditional quantiles.

4.1. Fatou's lemma and almost sure continuity

Fatou's lemma for conditional expectation states that given a sequence of non-negative random variables, $(X_n)_{n\in\mathbb{N}} \subset \mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$, then:

$$\mathsf{E}[\liminf_{n\in\mathbb{N}} X_n|\mathcal{G}] \leq \liminf_{n\in\mathbb{N}} \mathsf{E}[X_n|\mathcal{G}], \text{ a.s.} \tag{7}$$

As a consequence, if $\sup_{n\in\mathbb{N}} X_n < C$ a.s., then:

$$\mathsf{E}[\liminf_{n\in\mathbb{N}} X_n|\mathcal{G}] \leq \liminf_{n\in\mathbb{N}} \mathsf{E}[X_n|\mathcal{G}] \leq \limsup_{n\in\mathbb{N}} \mathsf{E}[X_n|\mathcal{G}] \leq \mathsf{E}[\limsup_{n\in\mathbb{N}} X_n|\mathcal{G}], \text{ a.s.} \tag{8}$$

In this section, we prove an analog to equation (7), substituting conditional expected value operator by conditional quantile. Moreover, we show that the requirement to be non-negative is not mandatory. However, to obtain an equation parallel to (8), we require the use of both $\mathsf{Q}_\tau[\cdot|\mathcal{G}]$ and $\mathsf{Q}_{\tau+}[\cdot|\mathcal{G}]$ operators. It is also possible to create an example where the last inequality in equation (8), with $\mathsf{Q}_\tau[\cdot|\mathcal{G}]$ replacing $\mathsf{E}[\cdot|\mathcal{G}]$, is false, i.e. $\limsup_{n\in\mathbb{N}} \mathsf{Q}_\tau[X_n|\mathcal{G}] > \mathsf{Q}_\tau[\limsup_{n\in\mathbb{N}} X_n|\mathcal{G}]$.

We begin with an analog of Fatou's lemma for conditional quantiles.

Theorem 4.1. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables in $\mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$, such that $\inf_{n\in\mathbb{N}} X_n$ and $\sup_{n\in\mathbb{N}} X_n$ are in $\mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$. For all $\tau\in(0,1)$ fixed, then:*

$$\mathsf{Q}_\tau[\liminf_{n\in\mathbb{N}} X_n|\mathcal{G}] \le \liminf_{n\in\mathbb{N}} \mathsf{Q}_\tau[X_n|\mathcal{G}] \le \limsup_{n\in\mathbb{N}} \mathsf{Q}_\tau[X_n|\mathcal{G}]$$

$$\le \limsup_{n\in\mathbb{N}} \mathsf{Q}_{\tau+}[X_n|\mathcal{G}] \le \mathsf{Q}_{\tau+}[\limsup_{n\in\mathbb{N}} X_n|\mathcal{G}], \text{ a.s.}$$

As we pointed out above, the previous result establishes a chain of inequalities similar to (8). Nevertheless, the following example shows that equation (8), with $\mathsf{Q}_\tau[\cdot|\mathcal{G}]$ replacing $\mathsf{E}[\cdot|\mathcal{G}]$, does not hold when using only the operator $\mathsf{Q}_\tau[\cdot|\mathcal{G}]$. Indeed, the inequality $\limsup_{n\in\mathbb{N}} \mathsf{Q}_\tau[X_n|\mathcal{G}] \le \mathsf{Q}_\tau[\limsup_{n\in\mathbb{N}} X_n|\mathcal{G}]$ is not necessarily true, except in the trivial case when $\mathsf{Q}_\tau[\limsup_{n\in\mathbb{N}} X_n|\mathcal{G}] = \mathsf{Q}_{\tau+}[\limsup_{n\in\mathbb{N}} X_n|\mathcal{G}]$.

Example. Let $\tau\in(0,1)$, and $U\in\mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$ be uniformly distributed in $(0,1)$ independently of $\mathcal{G}$. Define, for each $n\in\mathbb{N}$,

$$X_n(\omega) = \frac{1}{n}\mathbb{1}_{U\in[0,\tau)} - \big(n(\omega-\tau)\big)^{\frac{1}{n}}\mathbb{1}_{U\in[\tau,1]} \quad \text{and} \quad X = -\mathbb{1}_{U\in(\tau,1]}. \tag{9}$$

Then it is immediate to see that $X_n \underset{n\to\infty}{\longrightarrow} X$ pointwise, $\mathsf{Q}_\tau[X_n|\mathcal{G}]=0$ and $\mathsf{Q}_\tau[X|\mathcal{G}]=-1$. Finally, this implies that $\limsup_{n\in\mathbb{N}} \mathsf{Q}_\tau[X_n|\mathcal{G}] > \mathsf{Q}_\tau[\limsup_{n\in\mathbb{N}} X_n|\mathcal{G}]$.

This example also demonstrates that discontinuities of quantile sample paths play a crucial role in convergence theorems. Indeed, as operators on $\mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$, the previous result implies that conditional quantiles are continuous with respect to a.s. convergence, provided that $\tau\in(0,1)$ is a continuity point for the sample path of the conditional quantile of the limiting random variable, $s\mapsto\mathsf{Q}_s[X|\mathcal{G}]$, in a set of full probability measure.

Corollary 4.2. *Let $(X_n)_{n\in\mathbb{N}}\subset\mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$, such that $X_n\to X\in\mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$ a.s. Then, for each $\tau\in(0,1)$ such that $\mathsf{Q}_\tau[X|\mathcal{G}]=\mathsf{Q}_{\tau+}[X|\mathcal{G}]$ a.s., we have:*

$$\mathsf{Q}_\tau[X|\mathcal{G}] = \lim_{n\in\mathbb{N}} \mathsf{Q}_\tau[X_n|\mathcal{G}], \text{ a.s.}$$

Corollary 4.2 proves that, for each $\tau\in(0,1)$, $\mathsf{Q}_\tau[\cdot|\mathcal{G}]:\mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})\to\mathsf{L}^0(\Omega,\mathcal{G},\mathsf{P})$ is a continuous operator, if restricted to random variables such $x\in\mathbb{R}\mapsto\mathsf{P}[X\le x|\mathcal{G}]$ is strictly increasing a.s. and $\mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$ and $\mathsf{L}^0(\Omega,\mathcal{G},\mathsf{P})$ are considered with a.s. convergence of random variables. On the other hand, Theorem 4.1 shows that $\mathsf{Q}_\tau[\cdot|\mathcal{G}]$ and $\mathsf{Q}_{\tau+}[\cdot|\mathcal{G}]$ are, respectively, lower semicontinuous and upper semicontinuous operators on $\mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$, taking values in $\mathsf{L}^0(\Omega,\mathcal{G},\mathsf{P})$, when both spaces are considered with a.s. convergence. Next, we demonstrate that a similar phenomenon occurs when these operators are restricted to $\mathsf{L}^p(\Omega,\mathcal{F},\mathsf{P})$, $p\in[1,+\infty)$.

4.2. L^p continuity

As we proved in Proposition 2.5, the operator $\mathsf{Q}_\tau[\cdot|\mathcal{G}]$ maps $\mathsf{L}^p(\Omega,\mathcal{F},\mathsf{P})$ onto $\mathsf{L}^p(\Omega,\mathcal{G},\mathsf{P})$, for $p\in[1,+\infty]$. Thus, it is natural to study continuity properties for the family of conditional quantiles operators with respect to L^p-convergence. Our next proposition shows that $\mathsf{Q}_\tau[\cdot|\mathcal{G}]:\mathsf{L}^p(\Omega,\mathcal{F},\mathsf{P})\to\mathsf{L}^p(\Omega,\mathcal{G},\mathsf{P})$, for all $p\in[1,+\infty)$ and $\tau\in(0,1)$, is a lower semicontinuous operator with respect L^p-topology. Besides that, we show that the requirement for a.s. continuity of the sample paths of the conditional quantile of the limiting random variable, X, also guarantees L^p continuity of the operator $\mathsf{Q}_\tau[\cdot|\mathcal{G}]$ at X. We close this section demonstrating that $\mathsf{Q}_\tau[\cdot|\mathcal{G}]: \mathsf{L}^\infty(\Omega,\mathcal{F},\mathsf{P}) \to \mathsf{L}^\infty(\Omega,\mathcal{G},\mathsf{P})$ is 1-Lipschitz, whereas for all $p \in [1,+\infty)$ the operator $\mathsf{Q}_\tau[\cdot|\mathcal{G}]: \mathsf{L}^p(\Omega,\mathcal{F},\mathsf{P}) \to \mathsf{L}^p(\Omega,\mathcal{G},\mathsf{P})$ is not Lipschitz.

Proposition 4.3. *For each $p \in [1,+\infty)$ and $\tau \in (0,1)$, $\mathsf{Q}_\tau[\cdot|\mathcal{G}]: \mathsf{L}^p(\Omega,\mathcal{F},\mathsf{P}) \to \mathsf{L}^p(\Omega,\mathcal{G},\mathsf{P})$ is lower semicontinuous with respect to L^p-convergence, i.e. if $X_n \xrightarrow[n\to\infty]{\mathsf{L}^p} X$, then $\liminf_{n\in\mathbb{N}} \mathsf{Q}_\tau[X_n|\mathcal{G}] \in \mathsf{L}^p(\Omega,\mathcal{G},\mathsf{P})$ and $\limsup_{n\in\mathbb{N}} \mathsf{Q}_\tau[X_n|\mathcal{G}] \in \mathsf{L}^p(\Omega,\mathcal{G},\mathsf{P})$ with:*

$$\mathsf{Q}_\tau[X|\mathcal{G}] \le \liminf_{n\in\mathbb{N}} \mathsf{Q}_\tau[X_n|\mathcal{G}] \le \limsup_{n\in\mathbb{N}} \mathsf{Q}_\tau[X_n|\mathcal{G}] \le \mathsf{Q}_{\tau+}[X|\mathcal{G}], \text{ a.s.}$$

Furthermore, for all $\tau \in (0,1)$ where $s \mapsto \mathsf{E}[\mathsf{Q}_s[X|\mathcal{G}]]$ is continuous, then $\mathsf{Q}_\tau[X_n|\mathcal{G}] \xrightarrow{\mathsf{L}^p} \mathsf{Q}_\tau[X|\mathcal{G}]$.

When $p = +\infty$ the continuity condition for the family of conditional quantiles operator simplifies. Recall that, by Proposition 2.5 item 2, $\mathsf{Q}_\tau[\cdot|\mathcal{G}]: \mathsf{L}^\infty(\Omega,\mathcal{F},\mathsf{P}) \to \mathsf{L}^\infty(\Omega,\mathcal{G},\mathsf{P})$. Using Theorem 3.6 and the monotonicity of $\mathsf{Q}_\tau[\cdot|\mathcal{G}]$, Proposition 2.9 item 3, we are able to repeat the argument in Lemma 4.3 in [13] to derive that this family of operators are 1-Lipschitz and, hence, continuous.

Proposition 4.4. *For all $\tau \in (0,1)$, $\mathsf{Q}_\tau[\cdot|\mathcal{G}]: \mathsf{L}^\infty(\Omega,\mathcal{F},\mathsf{P}) \to \mathsf{L}^\infty(\Omega,\mathcal{G},\mathsf{P})$ is a continuous non-linear operator in L^∞-norm. Moreover, $\mathsf{Q}_\tau[\cdot|\mathcal{G}]$ is 1-Lipschitz, i.e. for all $X,Y \in \mathsf{L}^\infty(\Omega,\mathcal{F},\mathsf{P})$, then:*

$$\|\mathsf{Q}_\tau[X|\mathcal{G}] - \mathsf{Q}_\tau[Y|\mathcal{G}]\|_\infty \le \|X - Y\|_\infty.$$

As claimed before, we now show that $\mathsf{Q}_\tau[\cdot|\mathcal{G}]: \mathsf{L}^p(\Omega,\mathcal{F},\mathsf{P}) \to \mathsf{L}^p(\Omega,\mathcal{G},\mathsf{P})$ is not Lipschitz.

Example. For all $\tau \in (0,1)$ and $p \in [1,+\infty)$, let $(X_n)_{n\in\mathbb{N}} \subset \mathsf{L}^p(\Omega,\mathcal{F},\mathsf{P})$ and $X \in \mathsf{L}^p(\Omega,\mathcal{F},\mathsf{P})$ be as in (9). By the dominated convergence theorem, $X_n \to X$ in L^p. Therefore, given a $K \in (0,\infty)$, there exists a $n_0 \in \mathbb{N}$ such that, for all $n \ge n_0$, $\|X - X_n\|_{\mathsf{L}^p} < \frac{1}{K}$. However, $\|\mathsf{Q}_\tau[X_n|\mathcal{G}] - \mathsf{Q}_\tau[X|\mathcal{G}]\|_{\mathsf{L}^p} = 1$, for all $n \in \mathbb{N}$. Consequently, we conclude that, for all $n \ge n_0$,

$$\|\mathsf{Q}_\tau[X_n|\mathcal{G}] - \mathsf{Q}_\tau[X|\mathcal{G}]\|_{\mathsf{L}^p} > K\|X - X_n\|_{\mathsf{L}^p},$$

and $\mathsf{Q}_\tau[\cdot|\mathcal{G}]: \mathsf{L}^p(\Omega,\mathcal{F},\mathsf{P}) \to \mathsf{L}^p(\Omega,\mathcal{G},\mathsf{P})$ is not Lipschitz.

4.3. Weak continuity

Instead of considering $\mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$ with almost sure convergence, we may consider this space equipped with the convergence in distribution. Recall that a sequence of random variables, $(X_n)_{n\in\mathbb{N}} \subset \mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$, is said to converge in distribution (or weakly) to $X \in \mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$, $X_n \Rightarrow X$ (or $F_n \Rightarrow F$), if, and only if, the sequence of c.d.f.'s, $(F_n)_{n\in\mathbb{N}}$, converges pointwise at every continuity point of F, the c.d.f. of the limiting random variable. As we are dealing with a conditional framework, a suitable concept for conditional convergence in distribution is proposed in [31]:

Definition 4.5. A sequence of random variables, $(X_n)_{n\in\mathbb{N}} \subset \mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$, converges weakly to $X \in \mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$ conditional to $\mathcal{G}$ almost surely, $X_n \underset{\mathcal{G}}{\Rightarrow} X$ a.s., if there exists a set $\Omega' \in \mathcal{G}$, with full probability measure, such that:

$$\mathsf{P}[X_n \in \cdot|\mathcal{G}](\omega) \Rightarrow \mathsf{P}[X \in \cdot|\mathcal{G}](\omega), \text{ for all } \omega \in \Omega'.$$

Notice that, when restricted to $\mathcal{G} = \{\varnothing, \Omega\}$, $\mathcal{G}$-weak convergence a.s. reduces to weak convergence. Thus, the results derived in this section for the conditional framework might be immediately translated to the unconditional setup using weak convergence.

We begin by proving that conditional right and left-quantiles, when viewed as operators on $\mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P})$ and taking values on $\mathsf{L}^0(\Omega, \mathcal{G}, \mathsf{P})$, are, respectively, upper and lower semicontinuous with respect to $\mathcal{G}$-weak convergence a.s. In the unconditional framework, this result was initially proposed in [2]. Finally, we conclude with conditions for the convergence of the quantiles for a monotone sequence with respect to first order stochastic dominance in the conditional setup.

Theorem 4.6. *For all* $\tau \in (0,1)$*, the operators* $\mathsf{Q}_\tau[\cdot|\mathcal{G}] : \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P}) \to \mathsf{L}^0(\Omega, \mathcal{G}, \mathsf{P})$ *and* $\mathsf{Q}_{\tau+}[\cdot|\mathcal{G}] : \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P}) \to \mathsf{L}^0(\Omega, \mathcal{G}, \mathsf{P})$ *are weakly lower and upper semicontinuous, respectively. Moreover,* $X_n \underset{\mathcal{G}}{\Rightarrow} X$ *a.s. if, and only if, there exists a set* $\Omega' \in \mathcal{G}$*, such that* $\mathsf{P}(\Omega') = 1$ *and on it:*

$$\mathsf{Q}_\tau[X|\mathcal{G}] \le \liminf_{n\in\mathbb{N}} \mathsf{Q}_\tau[X_n|\mathcal{G}] \le \limsup_{n\in\mathbb{N}} \mathsf{Q}_\tau[X_n|\mathcal{G}] \le \limsup_{n\in\mathbb{N}} \mathsf{Q}_{\tau+}[X_n|\mathcal{G}] \le \mathsf{Q}_{\tau+}[X|\mathcal{G}],$$

for all $\tau \in (0,1)$*.*

We conclude this section discussing how Theorem 4.6 may be used to determine the convergence of quantiles along monotone sequences of random variables. In order to prove a monotone convergence theorem, we must first define what it means for a random variable to converge from above or below. To achieve this, we use the *first order stochastic dominance* concept adapted to the conditional setting.

Definition 4.7 (Conditional First Order Stochastic Dominance). Let $X, Y \in \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P})$, then $X \ge_{\mathcal{G}} Y$, if there exists $\Omega' \in \mathcal{G}$, with full probability measure, such that:

$$\mathsf{P}[X \le x|\mathcal{G}](\omega) \le \mathsf{P}[Y \le x|\mathcal{G}](\omega), \text{ for all } x \in \mathbb{R} \text{ and } \omega \in \Omega'.$$

Notice that, in particular, if $X \ge Y$ a.s., then $X \ge_{\mathcal{G}} Y$. Moreover, the definition of *conditional first order stochastic dominance* gives rise to a natural definition of monotone convergence for sequences of random variables. A sequence of random variables $(X_n)_{n\in\mathbb{N}} \subset \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P})$ is first order increasing if $X_{n+1} \ge_{\mathcal{G}} X_n$ for all $n \in \mathbb{N}$. If this happens and, additionally, $X_n \underset{\mathcal{G}}{\Rightarrow} X$ a.s., we write $X_n \uparrow_D X$. Similarly, a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ is first order decreasing if $X_{n+1} \le_{\mathcal{G}} X_n$ for all $n \in \mathbb{N}$. In this case, we write $X_n \downarrow_D X$ when $X_n \underset{\mathcal{G}}{\Rightarrow} X$ a.s.

Equipped with the concepts defined above, we are now able to provide sufficient and necessary conditions that assure convergence for a conditional first order monotone sequence of random variables.

Proposition 4.8. *Let* $\Pi = \{X_n\}_{n\in\mathbb{N}}$ *be a conditional first order monotone sequence of random variables. Then,* $(X_n)_{n\in\mathbb{N}}$ *is* $\mathcal{G}$*-weakly convergent a.s. if, and only if, there exists a* $\Omega' \in \mathcal{G}$*, with full probability, and, for all* $\epsilon > 0$*, there is a* $m(\epsilon, \omega) > 0$ *such that:*

$$\sup_{X\in\Pi\,\tau\in(\epsilon,1-\epsilon]} |\mathsf{Q}_\tau[X|\mathcal{G}](\omega)| \le m(\epsilon,\omega) \quad \text{and} \quad \sup_{X\in\Pi\,\tau\in[\epsilon,1-\epsilon)} |\mathsf{Q}_{\tau+}[X|\mathcal{G}](\omega)| \le m(\epsilon,\epsilon),$$

for all $\omega \in \Omega'$*.*

Furthermore, if it is convergent, then there is a $X \in \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P})$ *and* $\Omega' \in \mathcal{G}$*, with full measure, such that on it:*

$$\mathsf{Q}_\tau[X|\mathcal{G}] \le \lim_{n\in\mathbb{N}} \mathsf{Q}_\tau[X_n|\mathcal{G}] \le \mathsf{Q}_{\tau+}[X|\mathcal{G}], \text{ for all } \tau \in (0,1).$$

Finally, if $X_n \downarrow_D X$ *(or* $X_n \uparrow_D X$*) then* $\mathsf{Q}_\tau[X_n|\mathcal{G}] \downarrow \mathsf{Q}_\tau[X|\mathcal{G}]$ *a.s. (or* $\mathsf{Q}_\tau[X_n|\mathcal{G}] \uparrow \mathsf{Q}_\tau[X|\mathcal{G}]$ *a.s.) at every* $\tau \in (0,1)$ *such that* $\mathsf{Q}_\tau[X|\mathcal{G}] = \mathsf{Q}_{\tau+}[X|\mathcal{G}]$ *a.s.*

5. Interchanging quantiles and derivatives: Leibniz's rule

One of the most useful properties of the expected value is its ability of interchanging the order of integration and differentiation, also referred to as "Leibniz's rule." Interchanging a derivative with an expectation (an integral) can be established by applying the dominated convergence theorem. The interchange of integration and differentiation has been extensively used in applications; for example, in deriving statistical properties of the maximum likelihood estimator [see, e.g., 12].

Since quantiles are not linear, the commutation of derivatives and quantiles is not clear at a first glance. In fact, as we show below, it might fail. Nevertheless, making this interchange may be useful in optimization problems, where the quantile is either in the objective function or in the constraints. For instance, consider the problem $\max_x \mathsf{Q}_\tau[h(x,Y)|\mathcal{G}]$ for some differentiable function $(x,y) \in \mathbb{R}^2 \mapsto h(x,y)$. This problem might appear in statistics, decision theory, economics and operations research, see, e.g., [32] and [7]. The first order condition would invite one to calculate the derivative $\frac{d}{dx}\mathsf{Q}_\tau[h(x,Y)|\mathcal{G}]$. Interchanging quantile and derivative leads to the more familiar expression $\mathsf{Q}_\tau\left[\frac{\partial}{\partial x}h(x,Y)|\mathcal{G}\right]$, where the derivative of h can be directly used.

In this section, we show those invariance properties may be used extensively to prove differentiability of the quantile along a family of functions in the support of a random variable. Moreover, we are able to provide an example where the differentiation under the expectation sign fails, even though we may interchange the differentiation and the quantile functional for all $\tau \in (0,1)$. Section 5.1 deals with the Leibniz rule for monotone functions and it is related to Section 3.2. Section 5.2 uses Section 3.1 to separable functions to generate the same result.

5.1. Leibniz's rule for monotone functions

Recall that given a stochastic process $(X_t)_{t\in V}$, we call $(\bar{X}_t)_{t\in V}$ its modification if:

$$\mathsf{P}[X_t = \bar{X}_t] = 1, \text{ for all } t \in V.$$

Our first result is accomplished by applying Proposition 3.7 to obtain a differentiable modification of the process $(\mathsf{Q}_\tau[h(\bar{x},Y)|\mathcal{G}])_{\bar{x}\in V}$ in a neighbourhood V of x.

Theorem 5.1. *Let* $\mathcal{X} \subset \mathbb{R}$, $\mathcal{Y} \subset \mathbb{R}$, $h : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ *and* $x \in \mathcal{X}$ *such that:*

1. *There exists an open neighbourhood of* $x \in V \subset \mathbb{R}$*, such that* $y \in \mathcal{Y} \mapsto h(\bar{x},y)$ *is non-decreasing and left-continuous for all* $\bar{x} \in V \cap \mathcal{X}$*.*
2. $\bar{x} \mapsto h(\bar{x},y)$ *is differentiable at* x*, for all* $y \in \mathcal{Y}$*.*

Then, for all $\tau \in (0,1)$ *and* $Y \in \mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$*, whose support lies in* $\mathcal{Y}$*, the stochastic process* $(\mathsf{Q}_\tau[h(\bar{x},Y)|\mathcal{G}])_{\bar{x}\in V\cap\mathcal{X}}$ *admits a modification differentiable a.s. at* x *so that:*

$$\frac{d}{dx}\mathsf{Q}_\tau[h(x,Y)|\mathcal{G}] = \frac{\partial h}{\partial x}(x,\mathsf{Q}_\tau[Y|\mathcal{G}]).$$

Moreover, if in condition 1 above h *is non-increasing and left-continuous, then, the stochastic process* $(\mathsf{Q}_\tau[h(\bar{x},Z)|\mathcal{G}])_{\bar{x}\in V\cap\mathcal{X}}$*, for all* $\tau \in (0,1)$*, admits a modification differentiable a.s. at* x *so that:*

$$\frac{d}{dx}\mathsf{Q}_\tau[h(x,Y)|\mathcal{G}] = \frac{\partial h}{\partial x}(x,\mathsf{Q}_{(1-\tau)+}[Y|\mathcal{G}]).$$

In many situations, we are interested in differentiating h under the Q_τ functional. Instead of exploiting the linearity, which is the key tool for the expectation, we may obtain such result just using again the invariance of conditional quantiles to monotone transformation.

Corollary 5.2. *Let $h : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be a function, $\tau \in (0,1)$ and $Y \in \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P})$, so that its support is in $\mathcal{Y}$. If h satisfies:*

1. *There exists an open neighbourhood of $x \in V \subset \mathbb{R}$, such that $y \in \mathcal{Y} \mapsto h(\bar{x}, y)$ is non-decreasing and left-continuous, for all $\bar{x} \in V \cap \mathcal{X}$.*
2. *$\bar{x} \in V \cap \mathcal{X} \mapsto h(\bar{x}, y)$ is differentiable at x, for all $y \in \mathcal{Y}$.*
3. *$y \in \mathcal{Y} \mapsto \frac{\partial h}{\partial x}(x, y)$ is non-decreasing and left-continuous.*

Then, there is a modification of the stochastic process $(\mathsf{Q}_\tau[h(\bar{x}, Y)|\mathcal{G}])_{\bar{x} \in V \cap \mathcal{X}}$ so that:

$$\frac{d}{dx}\mathsf{Q}_\tau[h(x,Y)|\mathcal{G}] = \mathsf{Q}_\tau\left[\frac{\partial h}{\partial x}(x,Y)\middle|\mathcal{G}\right], \text{ a.s.}$$

If in item 3 above $\frac{\partial h}{\partial x}(x, \cdot)$ is non-increasing and left-continuous, then there is a modification of the stochastic process $(\mathsf{Q}_\tau[h(\bar{x}, Y)|\mathcal{G}])_{\bar{x} \in V \cap \mathcal{X}}$ so that:

$$\frac{d}{dx}\mathsf{Q}_\tau[h(x,Y)|\mathcal{G}] = \mathsf{Q}_{(1-\tau)+}\left[\frac{\partial h}{\partial x}(x,Y)\middle|\mathcal{G}\right], \text{ a.s.}$$

Moreover, if in condition 1 h is non-increasing and left-continuous, then there is a modification of the stochastic process $(\mathsf{Q}_\tau[h(\bar{x}, Y)|\mathcal{G}])_{\bar{x} \in V \cap \mathcal{X}}$ so that:

$$\frac{d}{dx}\mathsf{Q}_\tau[h(x,Y)|\mathcal{G}] = \mathsf{Q}_{(1-\tau)+}\left[\frac{\partial h}{\partial x}(x,Y)\middle|\mathcal{G}\right], \text{ a.s.}$$

Finally, if in addition to the previous change in item 1, in item 3 above $\frac{\partial h}{\partial x}(x, \cdot)$ is non-increasing and left-continuous, then there is a modification of the process $(\mathsf{Q}_\tau[h(\bar{x}, Y)|\mathcal{G}])_{\bar{x} \in V \cap \mathcal{X}}$ so that:

$$\frac{d}{dx}\mathsf{Q}_\tau[h(x,Y)|\mathcal{G}] = \mathsf{Q}_\tau\left[\frac{\partial h}{\partial x}(x,Y)\middle|\mathcal{G}\right], \text{ a.s.}$$

5.2. Leibniz's rule for separable functions

Instead of exploring the monotonicity property, we may apply the translational invariance and homogeneity of quantiles to investigate the interchanging of differentiation and Q_τ. In the following result, assuming a separability condition on h, we are able to provide conditions for differentiability of the process $(\mathsf{Q}_\tau[h(\bar{x}, Y)|\mathcal{G}])_{\bar{x} \in V}$ with respect to the parameters $\bar{x}$, as well as for the interchange of derivative and quantiles in this setting.

Theorem 5.3. *Let $h : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be such that there are $\eta : \mathcal{Y} \to \mathbb{R}$, $\phi : \mathcal{X} \to \mathbb{R}$ and $\psi : \mathcal{X} \to \mathbb{R}$ with $h(x, y) = \phi(x) + \psi(x)\eta(y)$, for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Assume that $Y \in \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P})$, so that its support is in $\mathcal{Y}$, and both ψ and ϕ are differentiable at x.*

1. *If $\psi(\bar{x}) \geq 0$, for all $\bar{x} \in V \cap \mathcal{X}$ in an open neighbourhood V of x, then the stochastic process $(\mathsf{Q}_\tau[h(\bar{x}, Y)|\mathcal{G}])_{\bar{x} \in V \cap \mathcal{X}}$ admits a modification differentiable at x, so that:*

$$\frac{d}{dx}\mathsf{Q}_\tau[h(x,Y)|\mathcal{G}] = \phi'(x) + \psi'(x)\mathsf{Q}_\tau[\eta(Y)|\mathcal{G}].$$

Additionally, if $\psi'(x) \geq 0$, then the stochastic process $(\mathsf{Q}_\tau[h(\bar{x},Y)|\mathcal{G}])_{\bar{x}\in V\cap\mathcal{X}}$ has a modification differentiable at $x \in \mathcal{X}$ so that:

$$\frac{d}{dx}\mathsf{Q}_\tau[h(x,Y)|\mathcal{G}] = \mathsf{Q}_\tau\left[\frac{\partial h}{\partial x}(x,Y)\middle|\mathcal{G}\right], \text{ a.s.}$$

However, if $\psi'(x) < 0$, then the stochastic process $(\mathsf{Q}_\tau[h(\bar{x},Y)|\mathcal{G}])_{\bar{x}\in V\cap\mathcal{X}}$ has a modification differentiable at $x \in \mathcal{X}$ so that:

$$\frac{d}{dx}\mathsf{Q}_\tau[h(x,Y)|\mathcal{G}] = \mathsf{Q}_{(1-\tau)+}\left[\frac{\partial h}{\partial x}(x,Y)\middle|\mathcal{G}\right], \text{ a.s.}$$

2. *If $\psi(\bar{x}) \leq 0$, for all $\bar{x} \in V \cap \mathcal{X}$ in an open neighborhood V of x, then the stochastic $(\mathsf{Q}_\tau[h(\bar{x},Y)|\mathcal{G}])_{\bar{x}\in V\cap\mathcal{X}}$ admits a modification differentiable at x, such that:*

$$\frac{d}{dx}\mathsf{Q}_\tau[h(x,Y)|\mathcal{G}] = \phi'(x) + \psi'(x)\mathsf{Q}_{(1-\tau)+}[\eta(Y)|\mathcal{G}].$$

Moreover, if $\psi'(x) \geq 0$, then the process $(\mathsf{Q}_\tau[h(\bar{x},Y)|\mathcal{G}])_{\bar{x}\in V\cap\mathcal{X}}$ has a modification differentiable at $x \in \mathcal{X}$ so that:

$$\frac{d}{dx}\mathsf{Q}_\tau[h(x,Y)|\mathcal{G}] = \mathsf{Q}_{(1-\tau)+}\left[\frac{\partial h}{\partial x}(x,Y)\middle|\mathcal{G}\right], \text{ a.s.}$$

Nevertheless, if $\psi'(x) < 0$, then the process $(\mathsf{Q}_\tau[h(\bar{x},Y)|\mathcal{G}])_{\bar{x}\in V\cap\mathcal{X}}$ has a modification differentiable at $x \in \mathcal{X}$ so that:

$$\frac{d}{dx}\mathsf{Q}_\tau[h(x,Y)|\mathcal{G}] = \mathsf{Q}_\tau\left[\frac{\partial h}{\partial x}(x,Y)\middle|\mathcal{G}\right], \text{ a.s.}$$

3. *If $\psi(x) = 0$, $\bar{x} \in V \cap \mathcal{X} \mapsto \psi(\bar{x})$ is either non-decreasing or non-increasing in a neighborhood $V \subset \mathcal{X}$ of x, and $\psi'(x)\mathsf{Q}_\tau[\eta(Y)|\mathcal{G}] = \psi'(x)\mathsf{Q}_{(1-\tau)+}[\eta(Y)|\mathcal{G}]$ a.s., then the stochastic $(\mathsf{Q}_\tau[h(\bar{x}, Y)|\mathcal{G}])_{\bar{x}\in V\cap\mathcal{X}}$ admits a modification differentiable at x, such that:*

$$\frac{d}{dx}\mathsf{Q}_\tau[h(x,Y)|\mathcal{G}] = \phi'(x) + \psi'(x)\mathsf{Q}_\tau[\eta(Y)|\mathcal{G}], \text{ a.s.}$$

Beyond that, the process $(\mathsf{Q}_\tau[h(\bar{x},Y)|\mathcal{G}])_{\bar{x}\in V\cap\mathcal{X}}$ has a modification differentiable at $x \in \mathcal{X}$ so that:

$$\frac{d}{dx}\mathsf{Q}_\tau[h(x,Y)|\mathcal{G}] = \mathsf{Q}_\tau\left[\frac{\partial h}{\partial x}(x,Y)\middle|\mathcal{G}\right], \text{ a.s.}$$

We conclude with two examples, restricting ourselves to the unconditional framework. The first example shows explicitly when it is not possible to differentiate $\mathsf{Q}_\tau[h(x,Z)]$.

Example. Let Y be a random variable such that $\mathsf{P}[Y=1] = \mathsf{P}[Y=-1] = \frac{1}{2}$, and $h : \mathbb{R} \times \{-1,1\} \to \mathbb{R}$: $h(x,y) = xy$. Fixed $\tau \in (\frac{1}{2},1)$, then:

$$\begin{aligned}\mathsf{Q}_\tau[h(x,Y)] &= x\mathsf{Q}_\tau[Y]\mathbb{1}_{x\geq 0} + x\mathsf{Q}_{(1-\tau)+}[Y]\mathbb{1}_{x<0}\\ &= |x|,\end{aligned}$$

since $\mathsf{Q}_\tau[Y] = 1$ and $\mathsf{Q}_{(1-\tau)+}[Y] = -1$. It is clear that $x \mapsto \mathsf{Q}_\tau[h(x,Y)]$ is not differentiable at 0.

Our second example in this section exhibits that we may have $\mathsf{Q}_\tau[\frac{\partial h}{\partial x}(x,Y)] = \frac{d}{dx}\mathsf{Q}_\tau[h(x,Y)]$, though $\frac{d}{dx}\mathsf{E}[h(x,Y)] \neq \mathsf{E}[\frac{\partial h}{\partial x}(x,Y)]$.

Example. Let Y be a random variable whose support is supp $Y = (0,1)$ and its law is given by:

$$\mathsf{P}[Y \in A] = \int_A \frac{y^{-\frac{1}{2}}}{2} dy, \text{ for all } A \in \mathcal{B}((0,1)).$$

Suppose $h : \mathbb{R}_+ \times (0,1) \to \mathbb{R}$ is the function $h(x,y) = \ln(x+y) + 2\frac{y}{x+y}$. Then, $\mathsf{E}[h(x,Y)] = \ln(x+1)$ for all $x \geq 0$. In particular, $\frac{d}{dx}\mathsf{E}[h(x,Y)] = \frac{1}{x+1}$ for all $x \geq 0$ and, consequently, $\frac{d}{dx}\mathsf{E}[h(0,Y)] = 1$.

Now observe that $\frac{\partial h}{\partial x}(0,Y) = \frac{-1}{Y}$. This implies that:

$$\mathsf{E}\left[\frac{\partial h}{\partial x}(0,Y)\right] = -\infty \neq 1 = \frac{d}{dx}\mathsf{E}[h(x,Y)]\bigg|_{x=0}.$$

Therefore, it is not possible to differentiate under the integral sign. Nevertheless, for all τ-quantile, $\tau \in (0,1)$, one is able to do it.

Notice that $\frac{\partial h}{\partial y}(x,y) = \frac{1}{x+y} + \frac{2x}{(x+y)^2} > 0$ and, consequently, $y \in (0,1) \mapsto h(x,y)$ is strictly increasing and continuous for all $x \in \mathbb{R}_+$ fixed. Secondly, $x \mapsto h(x,y)$ is differentiable everywhere. Finally, $y \mapsto \frac{\partial h}{\partial x}(0,y)$ is strictly increasing, because $\frac{\partial^2 h}{\partial y \partial x}(0,y) = \frac{1}{y^2} > 0$. Therefore, Corollary 5.2 assures that:

$$\frac{d}{dx}\mathsf{Q}_\tau\left[h(x,Y)\right]\bigg|_{x=0} = \mathsf{Q}_\tau\left[\frac{\partial h}{\partial x}(0,Y)\right], \text{ for all } \tau \in (0,1).$$

6. Composition of conditional quantiles

Along this section, we investigate the behavior of the composition of conditional quantiles with respect to different σ-algebras. Firstly, Section 6.1 exhibits a general counterexample to the "law of iterated quantiles", which would be the analog of the "law of iterated expectations" to conditional quantiles. Besides that, we describe the properties of the domains where the "law of iterated quantiles" holds, building an analogy to the projection approach to both expected values and quantiles. Lastly, Section 6.2 analyzes the problem of countable compositions of conditional quantiles along a filtration. As we will show, this problem is much more complex than its expected value counterpart, since the "law of iterated quantiles" is generally false. We also provide two distinct conditions for the existence of the limit of infinitely many compositions of conditional quantiles.

6.1. The law of iterated quantiles

Recall that given σ-algebras $\mathcal{F} \supset \mathcal{G} \supset \mathcal{H}$ and a random variable $X \in \mathsf{L}^1(\Omega,\mathcal{F},\mathsf{P})$, the law of iterated expectations – also known as the law of total expectation or the tower property, [34] – is the following equality:

$$\mathsf{E}[\mathsf{E}[X|\mathcal{G}]|\mathcal{H}] = \mathsf{E}[X|\mathcal{H}] = \mathsf{E}[\mathsf{E}[X|\mathcal{H}]|\mathcal{G}], \text{ a.s.}$$

This equation can be interpreted as a commutative relation between the maps $\mathsf{E}[\cdot|\mathcal{H}] : \mathsf{L}^1(\Omega,\mathcal{F},\mathsf{P}) \to \mathsf{L}^1(\Omega,\mathcal{H},\mathsf{P})$ and $\mathsf{E}[\cdot|\mathcal{G}] : \mathsf{L}^1(\Omega,\mathcal{F},\mathsf{P}) \to \mathsf{L}^1(\Omega,\mathcal{G},\mathsf{P})$.

For the conditional quantiles, recall that $\mathsf{Q}_\tau[\mathsf{Q}_\tau[X|\mathcal{H}]|\mathcal{G}] = \mathsf{Q}_\tau[X|\mathcal{H}]$, due to Proposition 2.9 item 5 and $\mathsf{Q}_\tau[X|\mathcal{H}] \in \mathsf{L}^0(\Omega,\mathcal{H},\mathsf{P})$. However, as we pointed out and the following result shows, in general, the law of iterated quantiles, $\mathsf{Q}_\tau[\mathsf{Q}_\tau[X|\mathcal{G}]|\mathcal{H}] = \mathsf{Q}_\tau[X|\mathcal{G}]$, does not hold unrestrictedly in a wide class of probability spaces.

Proposition 6.1. *Suppose that $(\Omega,\mathcal{F},\mathsf{P})$ is a probability space, such that $\mathcal{F}$ is a non-trivial σ-algebra on Ω and $\tau \in (0,1)$. If there are disjoint sets $\{A_i\}_{i=1}^3 \subset \mathcal{F}$, with $\mathsf{P}[A_i] = p_i \in (0,1)$, $i = 1,2$ and 3, satisfying:*

1. $A_1 \cup A_2 \cup A_3 = \Omega$;
2. $0 < p_1 < \tau$;
3. $\tau - p_1 \le p_2 < \tau - \tau p_1$;

then, there are a random variable $X \in \mathsf{L}^0(\Omega,\mathcal{F},\mathsf{P})$ and sub-σ-algebras $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$, so that:

$$\mathsf{Q}_\tau[\mathsf{Q}_\tau[X|\mathcal{G}]|\mathcal{H}] \neq \mathsf{Q}_\tau[X|\mathcal{H}]. \tag{10}$$

Although Proposition 6.1 shows a negative result for a general law of iterated quantiles, we are able to characterize the maximal set where it holds by analyzing the family of conditional quantile operators as a family of non-linear projections.

Recall that, when restricted to $\mathsf{L}^2(\Omega,\mathcal{F},\mathsf{P})$ random variables, the conditional expectation may be computed from a specific class of optimization problems. According to this formulation, for all sub-σ-algebras $\mathcal{S} \subset \mathcal{F}$, then:

$$\mathsf{E}[X|\mathcal{S}] = \operatorname*{argmin}_{Y \in \mathsf{L}^2(\Omega,\mathcal{S},\mathsf{P})} \mathsf{E}[|X-Y|^2],$$

which is the minimization of the L^2 distance between a set and a point. In particular, $\mathsf{E}[\cdot|\mathcal{S}]$ may be seen as a linear projection from $\mathsf{L}^2(\Omega,\mathcal{F},\mathsf{P})$ onto $\mathsf{L}^2(\Omega,\mathcal{S},\mathsf{P})$. Denoting by $\pi_{\mathsf{L}^2(\Omega,\mathcal{S},\mathsf{P})} = \mathsf{E}[\cdot|\mathcal{S}]$ and $\pi^2_{\mathsf{L}^2(\Omega,\mathcal{S},\mathsf{P})}(X) = \pi_{\mathsf{L}^2(\Omega,\mathcal{S},\mathsf{P})}(X) \circ \pi_{\mathsf{L}^2(\Omega,\mathcal{S},\mathsf{P})}(X)$, this projection fulfills:

$$\pi^2_{\mathsf{L}^2(\Omega,\mathcal{S},\mathsf{P})}(X) = \pi_{\mathsf{L}^2(\Omega,\mathcal{S},\mathsf{P})}(X), \text{ for all } X \in \mathsf{L}^2(\Omega,\mathcal{F},\mathsf{P}),$$
$$\pi_{\mathsf{L}^2(\Omega,\mathcal{S},\mathsf{P})}(X) = X, \text{ for all } X \in \mathsf{L}^2(\Omega,\mathcal{S},\mathsf{P}).$$

Furthermore, restated in terms of projections, the law of iterated expectations is then just a commutative property enjoyed by projection operators,

$$\pi_{\mathsf{L}^2(\Omega,\mathcal{H},\mathsf{P})} \circ \pi_{\mathsf{L}^2(\Omega,\mathcal{G},\mathsf{P})}(X) = \pi_{\mathsf{L}^2(\Omega,\mathcal{G},\mathsf{P})}(X) = \pi_{\mathsf{L}^2(\Omega,\mathcal{G},\mathsf{P})} \circ \pi_{\mathsf{L}^2(\Omega,\mathcal{H},\mathsf{P})}(X).$$

Similarly, the conditional median may also be seen as a non-linear projection when restricted to $\mathsf{L}^1(\Omega,\mathcal{F},\mathsf{P})$, projecting it onto $\mathsf{L}^1(\Omega,\mathcal{S},\mathsf{P})$. Indeed, by Proposition 2.7 the conditional median satisfies:

$$\mathsf{Q}_{\frac{1}{2}}[X|\mathcal{S}] = \inf\left\{ Z \in \operatorname*{argmin}_{Y \in \mathsf{L}^1(\Omega,\mathcal{S},\mathsf{P})} \frac{1}{2}\mathsf{E}\left[|X-Y|\right] \right\}.$$

This characterization shows that conditional medians are the minimal random variables that minimize the L^1 distance between a closed convex subset, $\mathsf{L}^1(\Omega,\mathcal{S},\mathsf{P})$, and a point $X \in \mathsf{L}^1(\Omega,\mathcal{F},\mathsf{P})$. Moreover, defining $\mathsf{Q}_{\frac{1}{2}}[\cdot|\mathcal{S}] = \pi^{\frac{1}{2}}_{\mathsf{L}^1(\Omega,\mathcal{S},\mathsf{P})} : \mathsf{L}^1(\Omega,\mathcal{F},\mathsf{P}) \to \mathsf{L}^1(\Omega,\mathcal{S},\mathsf{P})$, then $\pi^{\frac{1}{2}}_{\mathsf{L}^1(\Omega,\mathcal{F},\mathsf{P})}$ is both an idempotent non-linear operator and invariant on $\mathsf{L}^1(\Omega,\mathcal{F},\mathsf{P})$ – see item 5 in Proposition 2.9.[8] Consequently,

$$\pi^{\frac{1}{2}}_{\mathsf{L}^1(\Omega,\mathcal{S},\mathsf{P})} \circ \pi^{\frac{1}{2}}_{\mathsf{L}^1(\Omega,\mathcal{S},\mathsf{P})}(X) = \pi^{\frac{1}{2}}_{\mathsf{L}^1(\Omega,\mathcal{S},\mathsf{P})}(X), \text{ for all } X \in \mathsf{L}^1(\Omega,\mathcal{F},\mathsf{P})$$
$$\pi^{\frac{1}{2}}_{\mathsf{L}^1(\Omega,\mathcal{S},\mathsf{P})}(X) = X, \text{ for all } X \in \mathsf{L}^1(\Omega,\mathcal{S},\mathsf{P}).$$

For the general case, i.e. $\tau \in (0,1)$, the conditional quantile operator may also be seen as the minimal minimizer of a quasimetric, which differs from a metric by not being symmetric. Indeed, fixed $\tau \in (0,1)$, define the quasimetric $d_\tau : \mathsf{L}^1(\Omega,\mathcal{F},\mathsf{P}) \times \mathsf{L}^1(\Omega,\mathcal{F},\mathsf{P}) \to \mathbb{R}_+$ by:

$$d_\tau(X,Y) = \mathsf{E}[\rho_\tau(X-Y)].$$

It is immediate to see that d_τ is, indeed, a quasimetric, since it satisfies the following items 1 and 3, though not 2 – except when $\tau = \frac{1}{2}$:

1. $d_\tau(X,Y) \geq 0$ and $d_\tau(X,Y) = 0$ if, and only if, $X = Y$ a.s.
2. $d_\tau(X,Y) = d_\tau(Y,X)$ for all $X,Y \in \mathsf{L}^1$.
3. $d_\tau(X,Y) \leq d_\tau(X,Y) + d_\tau(Y,Z)$, for all $X,Y,Z \in \mathsf{L}^1$.

Actually, item 2 above can be replaced by $d_\tau(X,Y) = d_{1-\tau}(Y,X)$, for the general case. Restated in this way, the conditional quantile is the minimal minimizer of the distance, measured by a quasimetric d_τ, between a point and a closed – in L^1-norm – convex set, $\mathsf{L}^1(\Omega,\mathcal{S},\mathsf{P})$:

$$\mathsf{Q}_\tau[X|\mathcal{S}] = \inf\left\{Z \in \underset{Y\in\mathsf{L}^1(\Omega,\mathcal{S},\mathsf{P})}{\operatorname{argmin}} \; d_\tau(X,Y)\right\}.$$

Because the τ-conditional quantile is both an idempotent operator, invariant on its image – Proposition 2.9 item 5 – and a minimizer of a quasimetric between a convex closed set and a point, we conclude that it may also be seen as a nonlinear projection. Indeed, abusing the terminology from the L^2 framework, we will call any pair (H,π_H), or simply π_H, a projection onto H, if $H \subset \mathsf{L}^1(\Omega,\mathcal{F},\mathsf{P})$ is a closed convex set, in L^1-norm, and $\pi_H : \mathsf{L}^1(\Omega,\mathcal{F},\mathsf{P}) \to H$ is an idempotent non-linear operator invariant on H. Thus, for all $\tau \in (0,1)$ the pair $(\mathsf{L}^1(\Omega,\mathcal{S},\mathsf{P}),\pi^\tau_{\mathsf{L}^1(\Omega,\mathcal{S},\mathsf{P})})$ defined by $\pi^\tau_{\mathsf{L}^1(\Omega,\mathcal{S},\mathsf{P})} = \mathsf{Q}_\tau[\cdot|\mathcal{S}]$ is, due to Propositions 2.9 item 5 and Proposition 2.5 item 1, a (generalized) projection onto $\mathsf{L}^1(\Omega,\mathcal{S},\mathsf{P})$.

Using the terminology of commutative algebra, the commutator of two maps A, B is known as $[A,B] = A \circ B - B \circ A$. Thus, the set of random variables such that the law of iterated conditional quantiles holds can be defined by:

$$C^\tau_{\mathcal{H},\mathcal{G}} = \left\{X \in \mathsf{L}^1(\Omega,\mathcal{F},\mathsf{P}) : \left[\pi^\tau_{\mathsf{L}^1(\Omega,\mathcal{H},\mathsf{P})},\pi^\tau_{\mathsf{L}^1(\Omega,\mathcal{G},\mathsf{P})}\right] = 0\right\}.$$

Fixed $\tau \in (0,1)$, the formulation of conditional quantiles as projections allows us to characterize the domains of $\mathsf{L}^1(\Omega,\mathcal{F},\mathsf{P})$ where the law of iterated quantiles holds as the set where two projections commute with the following properties.

Proposition 6.2. *Fixed $\tau \in (0,1)$ and $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ σ-algebras, then:*

1. $\mathsf{L}^1(\Omega,\mathcal{H},\mathsf{P}) \subset \mathsf{L}^1(\Omega,\mathcal{G},\mathsf{P}) \subset C^\tau_{\mathcal{H},\mathcal{G}}$. *This last inclusion is proper provided that there is at least one variable independent of $\mathcal{G}$, i.e.* $\mathsf{L}^1(\Omega,\mathcal{G},\mathsf{P}) \subsetneq C^\tau_{\mathcal{H},\mathcal{G}}$

[8] In fact, the invariance property, item 5 in Proposition 2.9, implies the idempotency property.

2. *Given any* $a \in \mathsf{L}^1(\Omega, \mathcal{H}, \mathsf{P})$, $b \in \mathsf{L}^\infty(\Omega, \mathcal{H}, \mathsf{P})$, *with* $b \geq 0$ *a.s., and* $X \in C^\tau_{\mathcal{H},\mathcal{G}}$, *then* $a + bX \in C^\tau_{\mathcal{H},\mathcal{G}}$. *In particular,* $C^\tau_{\mathcal{H},\mathcal{G}}$ *is a cone.*

6.2. Infinite composition of conditional quantiles

The non-validity of the law of iterated quantiles has further consequences on the dynamics of composed conditional quantiles. In finance and economics, the iteration of conditional quantiles along a filtration emerges naturally as way of modeling, in a consistent way, a dynamic decision process based on the quantiles of random outcomes – see e.g. [5]. Nevertheless, the behavior of the infinity composition of the conditional quantiles is still unknown. In this section, we show that this composition of conditional quantiles has a richer structure than the infinite composition of conditional expectations, being able to provide a concrete example where the infinite composition of conditional quantiles may lead to a random variable which is infinity a.s. Besides that, we derive conditions for the existence of a finite and measurable random variable which is the limit of the infinite composition of conditional quantiles either when $\mathsf{E}[X|\mathcal{F}_t] - \mathsf{E}[X|\mathcal{F}_{t-1}]$ is independent of $\mathcal{F}_{t-1}$, for all $t \in \mathbb{N}$, or when $X \in \mathsf{L}^\infty(\Omega, \mathcal{F}, \mathsf{P})$.

The law of iterated expectations is a powerful tool when considering the dynamics of conditional expectation of a random output variable, $X \in \mathsf{L}^1(\Omega, \mathcal{F}, \mathsf{P})$. Suppose, for instance, that we have a filtered probability space, $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{N}\cup\{0\}}, \mathsf{P}\right)$, with $\mathcal{F} = \mathcal{F}_\infty$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Thus, for any $n < m \in \mathbb{N}$, the law of iterated expectation implies that:

$$\mathsf{E}[\mathsf{E}[\ldots \mathsf{E}[X|\mathcal{F}_m] \ldots |\mathcal{F}_{n+1}]|\mathcal{F}_n] = \mathsf{E}[X|\mathcal{F}_n]. \tag{11}$$

Consequently, we obtain that $\mathsf{E}[X|\mathcal{F}_n] = \lim_{m\uparrow+\infty} \mathsf{E}[\mathsf{E}[\ldots \mathsf{E}[X|\mathcal{F}_m] \ldots |\mathcal{F}_{n+1}]|\mathcal{F}_n]$.

For the conditional quantile, however, even in the finite case, an equation similar to (11) does not have to hold. Beyond that, as the following example show, there exist a random variable and filtration such that $\lim_{m\uparrow+\infty} \mathsf{Q}_\tau[\mathsf{Q}_\tau[\ldots \mathsf{Q}_\tau[X|\mathcal{F}_m] \ldots |\mathcal{F}_{n+1}]|\mathcal{F}_n]$ diverges a.s. except when $\tau = \frac{1}{2}$.

Example. Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}\cup\{0\}}, \mathsf{P}\right)$ be such that $(B_t)_{t\geq 0} : \Omega \longrightarrow \mathbb{R}$ is a Brownian Motion, $T > 0$ fixed, $t_0 = 0$, $t_n - t_{n-1} = \frac{6T}{\pi^2 n^2}$, for $n \in \mathbb{N}$, and $\mathcal{F}_n = \sigma(B_s : 0 \leq s \leq t_n)$. Define $X = B_T$, then, fixed $n \geq 0$, for all $m > n$ – see the details in the Supplementary Material [4]:

$$\mathsf{Q}_\tau[\mathsf{Q}_\tau[\ldots \mathsf{Q}_\tau[X|\mathcal{F}_m] \ldots |\mathcal{F}_{n+1}]|\mathcal{F}_n] = B_{t_n} + \left(\frac{\sqrt{6T}}{\pi} \sum_{j=1}^{m-n} \frac{1}{j} + \sqrt{T - t_m} \right) \mathsf{Q}_\tau[\mathrm{N}(0,1)].$$

Hence, because $\sum_{j=1}^{m-n} \frac{1}{j} \underset{m\to\infty}{\longrightarrow} +\infty$ and $\sqrt{T - t_m} \underset{m\to\infty}{\longrightarrow} 0$, we conclude that:

$$\lim_{m\to+\infty} \mathsf{Q}_\tau[\mathsf{Q}_\tau[\ldots \mathsf{Q}_\tau[X|\mathcal{F}_m] \ldots |\mathcal{F}_{n+1}]|\mathcal{F}_n] = \begin{cases} -\infty, & \text{if } \tau < \frac{1}{2} \\ B_{t_n}, & \text{if } \tau = \frac{1}{2} \\ +\infty, & \text{if } \tau > \frac{1}{2}. \end{cases}$$

The preceding computation on the example given above suggests that, for some particular cases, the series $\sum_{j\geq 1} \|\mathsf{E}[X|\mathcal{F}_j] - \mathsf{E}[X|\mathcal{F}_{j-1}]\|_{\mathsf{L}^1}$ may play an important role for the existence of

$$\lim_{m\to+\infty} \mathsf{Q}_\tau[\mathsf{Q}_\tau[\ldots \mathsf{Q}_\tau[X|\mathcal{F}_m] \ldots |\mathcal{F}_{n+1}]|\mathcal{F}_n].$$

Indeed, our next results exhibit this connection.

Proposition 6.3. *Let* $\left(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in\mathbb{N}\cup\{0\}},\mathsf{P}\right)$ *be a filtered probability space, with* $\mathcal{F}_\infty = \mathcal{F}$. *Define* $H \subset \mathsf{L}^1(\Omega,\mathcal{F},\mathsf{P})$*, such that if* $X \in H$*, then:*

1. $\sum_{j\geq 1} \|\mathsf{E}[X|\mathcal{F}_j] - \mathsf{E}[X|\mathcal{F}_{j-1}]\|_{\mathsf{L}^1} < +\infty$.
2. $\mathsf{E}[X|\mathcal{F}_j] - \mathsf{E}[X|\mathcal{F}_{j-1}]$ *independent of* $\mathcal{F}_{j-1}$, $\forall j \geq 1$.

Then, for all $X \in H$ *and* $\tau \in (0,1)$, $\lim_{m\to+\infty} \mathsf{Q}_\tau[\mathsf{Q}_\tau[\ldots \mathsf{Q}_\tau[X|\mathcal{F}_m]\ldots|\mathcal{F}_{n+1}]|\mathcal{F}_n] \in \mathsf{L}^1(\Omega,\mathcal{F}_n,\mathsf{P})$*, and:*

$$\lim_{m\to+\infty} \mathsf{Q}_\tau[\mathsf{Q}_\tau[\ldots \mathsf{Q}_\tau[X|\mathcal{F}_m]\ldots|\mathcal{F}_{n+1}]|\mathcal{F}_n] = \mathsf{E}[X|\mathcal{F}_n] + \sum_{j\geq 1} \mathsf{Q}_\tau[\mathsf{E}[X|\mathcal{F}_{n+j}] - \mathsf{E}[X|\mathcal{F}_{n+j-1}]], \text{ a.s.}$$

Although restrictive, the assumption of independent increments permits us to compute explicitly each iteration of the conditional quantile. Instead of explicitly calculating each iteration, we may bound them. In order to do so, we will restrict the random variables to $\mathsf{L}^\infty(\Omega,\mathcal{F},\mathsf{P})$. In this domain, we are able to show that, for the sequence of compositions $(\mathsf{Q}_\tau[\mathsf{Q}_\tau[\ldots \mathsf{Q}_\tau[X|\mathcal{F}_m]\ldots|\mathcal{F}_{n+1}]|\mathcal{F}_n])_{m>n}$, at least lim inf and lim sup exist and are finite a.s. This is a direct consequence of the fact that this sequence is bounded a.s. by $\pm\|X\|_{+\infty}$, due to Proposition 2.5 item 2. Moreover, we are also able to provide a subset where both lim inf and lim sup agree, leaving the question whether they agree on all $\mathsf{L}^\infty(\Omega,\mathcal{F},\mathsf{P})$.

Proposition 6.4. *Let* $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in\mathbb{N}\cup\{0\}},\mathsf{P})$ *be a filtered probability space, with* $\mathcal{F}_\infty = \mathcal{F}$. *Then, for each* $n \in \mathbb{N}\cup\{0\}$*, both operators* $\liminf_{m\in\mathbb{N}} \mathsf{Q}_\tau[\ldots \mathsf{Q}_\tau[\cdot|\mathcal{F}_m]\ldots|\mathcal{F}_n] : \mathsf{L}^\infty(\Omega,\mathcal{F},\mathsf{P}) \to \mathsf{L}^\infty(\Omega,\mathcal{F}_n,\mathsf{P})$ *and* $\limsup_{m\in\mathbb{N}} \mathsf{Q}_\tau[\ldots \mathsf{Q}_\tau[\cdot|\mathcal{F}_m]\ldots|\mathcal{F}_n] : \mathsf{L}^\infty(\Omega,\mathcal{F},\mathsf{P}) \to \mathsf{L}^\infty(\Omega,\mathcal{F}_n,\mathsf{P})$ *are well-defined non-linear operators. Moreover, they agree on* $\overline{\cup_{n\in\mathbb{N}\cup\{0\}}\mathsf{L}^\infty(\Omega,\mathcal{F}_n,\mathsf{P})}$*, where the closure is taken in* L^∞*-norm.*

7. Conclusion

This paper investigates the properties of conditional quantiles viewed as nonlinear operators. The results are organized in parallel to the usual properties of the expectation operator. We generalize well-known properties of unconditional quantiles to the conditional case, such as translation invariance, comonotonicity, and equivariance to monotone transformations. Moreover, we provide a simple proof for Jensen's inequality for conditional quantiles.

Continuity and differentiability of the conditional expectation operator are widely used in practice. Therefore, we extend these concepts to conditional quantiles. We investigate continuity of conditional quantiles as operators with respect to different topologies. We obtain a novel Fatou's lemma for quantiles, provide conditions for continuity in L^p, and also weak continuity. Moreover, we also investigate the differentiability properties of quantiles. We show the validity of the Leibniz's rule for conditional quantiles for the cases of monotone, as well as separable functions.

Finally, we investigate the validity of the law of iterated expectations – also known as law of total expectation or tower property – to the quantile case. We show that the law of iterated quantiles does not hold in general. Nevertheless, we characterize the maximum set of random variables for which this law holds, and investigate its consequences for the infinite composition of conditional quantiles.

The results in this paper are intended to shed new light and be useful for applications of quantiles, such as decision theory in statistics and economics, risk management in finance, and applications in quantile regression. To exemplify further prospects of some of the results derived in this paper, the Supplementary Material provides brief applications in analysis and quantile regression.

Acknowledgements

The authors would like to express their appreciation to Wenceslao Manteiga and the participants at Research in Options 2020 for helpful comments and discussions. All the remaining errors are ours. Jorge P. Zubelli and Bruno N. Costa acknowledge the support from the FSU-2020-09 grant from Khalifa University. Luciano de Castro acknowledges the support of the National Council for Scientific and Technological Development – CNPq.

Supplementary Material

Supplementary material (DOI: 10.3150/22-BEJ1546SUPP; .pdf). Supplement of "Conditional Quantiles: An Operator-Theoretical Approach" [4]. There is a the supplementary material associated to the present article. Proofs for each result stated in the paper are presented in details in the supplement.

References

[1] Apiwatcharoenkul, W., Lake, L.W. and Jensen, J.L. (2016). Uncertainty in proved reserves estimates by decline curve analysis. In *SPE/IAEE Hydrocarbon Economics and Evaluation Symposium* Society of Petroleum Engineers. https://doi.org/10.2118/179992-MS

[2] Chambers, C.P. (2009). An axiomatization of quantiles on the domain of distribution functions. *Math. Finance* **19** 335–342. MR2509291 https://doi.org/10.1111/j.1467-9965.2009.00369.x

[3] Cheung, K.C. (2007). Characterizations of conditional comonotonicity. *J. Appl. Probab.* **44** 607–617. MR2355579 https://doi.org/10.1239/jap/1189717532

[4] de Castro, L., Costa, B.N., Galvao, A.F. and Zubelli, J.P. (2023). Supplement to "Conditional Quantiles: An Operator-Theoretical Approach." https://doi.org/10.3150/22-BEJ1546SUPP

[5] de Castro, L. and Galvao, A.F. (2019). Dynamic quantile models of rational behavior. *Econometrica* **87** 1893–1939. MR4051682 https://doi.org/10.3982/ecta15146

[6] de Castro, L. and Galvao, A.F. (2022). Static and dynamic quantile preferences. *Econom. Theory* **73** 747–779. MR4414440 https://doi.org/10.1007/s00199-021-01355-8

[7] de Castro, L., Galvao, A.F., Montes-Rojas, G. and Olmo, J. (2022). Portfolio selection in quantile decision models. *Ann. Finance* **18** 133–181. MR4434243 https://doi.org/10.1007/s10436-021-00405-4

[8] Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R. and Vyncke, D. (2002). The concept of comonotonicity in actuarial science and finance: Theory *Insurance Math. Econom.* **31** 3–33. MR1956509 https://doi.org/10.1016/S0167-6687(02)00134-8

[9] Duffie, D. and Pan, J. (1997). An overview of value at risk. *J. of Derivatives* **4** 7–49.

[10] Embrechts, P., Höing, A. and Juri, A. (2003). Using copulae to bound the value-at-risk for functions of dependent risks. *Finance Stoch.* **7** 145–167. MR1968943 https://doi.org/10.1007/s007800200085

[11] Fanchi, J.R. and Christiansen, R.L. (2017). *Introduction to Petroleum Engineering*. New York: Wiley Online Library.

[12] Ferguson, T.S. (1996). *A Course in Large Sample Theory*. *Texts in Statistical Science Series*. London: CRC Press. MR1699953 https://doi.org/10.1007/978-1-4899-4549-5

[13] Föllmer, H. and Schied, A. (2002). *Stochastic Finance. An Introduction in Discrete Time*. *De Gruyter Studies in Mathematics* **27**. Berlin: de Gruyter. MR1925197 https://doi.org/10.1515/9783110198065

[14] Jorion, P. (2007). *Value at Risk. The New Benchmark for Managing Financial Risk*, 3rd ed. New York: McGraw-Hill.

[15] Jouini, E. and Napp, C. (2004). Conditional comonotonicity. *Decis. Econ. Finance* **27** 153–166. MR2104639 https://doi.org/10.1007/s10203-004-0049-y

[16] Koenker, R. (2005). *Quantile Regression*. *Econometric Society Monographs* **38**. Cambridge: Cambridge Univ. Press. MR2268657 https://doi.org/10.1017/CBO9780511754098

[17] Koenker, R. and Bassett, G. Jr. (1978). Regression quantiles. *Econometrica* **46** 33–50. MR0474644 https://doi.org/10.2307/1913643
[18] Koenker, R. and Bassett, G.W. Jr. (2010). March madness, quantile regression bracketology, and the Hayek hypothesis. *J. Bus. Econom. Statist.* **28** 26–35. MR2650598 https://doi.org/10.1198/jbes.2009.07093
[19] Koenker, R., Chernozhukov, V., He, X. and Peng, L. (2017). *Handbook of Quantile Regression*. New York, New York: CRC Press/CRC.
[20] Le Gall, J.-F. (2013). *Mouvement Brownien, Martingales et Calcul Stochastique*. *Mathématiques & Applications (Berlin) [Mathematics & Applications]* **71**. Heidelberg: Springer. MR3184878 https://doi.org/10.1007/978-3-642-31898-6
[21] Le Gall, J.-F. (2006). Intégration, Probabilités et Processus Aléatories. Mimeo, Département Mathématiques et Applications Ecole normale supérieure de Paris.
[22] Manski, C.F. (1988). Ordinal utility models of decision making under uncertainty. *Theory and Decision* **25** 79–104. MR0951879 https://doi.org/10.1007/BF00129169
[23] Mendelson, H. (1987). Quantile-preserving spread. *J. Econom. Theory* **42** 334–351. MR0898524 https://doi.org/10.1016/0022-0531(87)90091-3
[24] Merkle, M. (2005). Jensen's inequality for medians. *Statist. Probab. Lett.* **71** 277–281. MR2126412 https://doi.org/10.1016/j.spl.2004.11.010
[25] Molchanov, I. (2017). *Theory of Random Sets*, 2nd ed. London: Springer.
[26] Peskir, G. and Shiryaev, A. (2006). *Optimal Stopping and Free-Boundary Problems*. *Lectures in Mathematics ETH Zürich*. Basel: Birkhäuser. MR2256030
[27] Pollard, D. (2006). *A User's Guide to Measure Theoretic Probability*. Cambridge, MA: Cambridge Univ. Press.
[28] Rostek, M. (2010). Quantile maximization in decision theory. *Rev. Econ. Stud.* **77** 339–371. MR2599290 https://doi.org/10.1111/j.1467-937X.2009.00564.x
[29] Schmeidler, D. (1986). Integral representation without additivity. *Proc. Amer. Math. Soc.* **97** 255–261. MR0835875 https://doi.org/10.2307/2046508
[30] Shreve, S.E. (2004). *Stochastic Calculus for Finance. I: The Binomial Asset Pricing Model*. *Springer Finance*. New York: Springer. MR2049045
[31] Sweeting, T.J. (1989). On conditional weak convergence. *J. Theoret. Probab.* **2** 461–474. MR1011199 https://doi.org/10.1007/BF01051878
[32] Tasche, D. (2000). Conditional expectation as quantile derivative. Working Paper.
[33] Valadier, M. (1984). La multi-application médianes conditionnelles. *Probab. Theory Related Fields* **67** 279–282. MR0762082 https://doi.org/10.1007/BF00535005
[34] Williams, D. (1991). *Probability with Martingales*. *Cambridge Mathematical Textbooks*. Cambridge: Cambridge Univ. Press. MR1155402 https://doi.org/10.1017/CBO9780511813658
[35] Zhao, M., Yang, X., He, Q., Zhou, Z. and Ge, X. (2021). Quantile Jensen's inequalities. *J. Inequal. Appl.* **86** 1–8.

Received January 2022 and revised September 2022