Bernoulli **26**(3), 2020, 2311–2318
https://doi.org/10.3150/20-BEJ1194

Weighted Lépingle inequality

PAVEL ZORIN-KRANICH

Mathematical Institute, University of Bonn, Bonn, Germany. E-mail: pzorin@uni-bonn.de

We prove an estimate for weighted pth moments of the pathwise r-variation of a martingale in terms of the A_p characteristic of the weight. The novelty of the proof is that we avoid real interpolation techniques.

Keywords: p-variation; Burkholder–Davis–Gundy inequality; Muckenhoupt A_p weight

1. Introduction

Lépingle's inequality [20] is a moment estimate for the pathwise r-variation of martingales. Finite r-variation is a parametrization-invariant version of Hölder continuity of order $1/r$ and plays a central role in Lyons's theory of rough paths [21].

Lépingle's inequality also found applications in ergodic theory [4] and harmonic analysis [24]; see [22] and [8,10] and references therein, respectively, for recent developments in these directions. Weighted inequalities in harmonic analysis go back to [23], and weighted variational inequalities have been studied since [6]. A major motivation of the weighted theory is the Rubio de Francia extrapolation theorem that allows to obtain vector-valued L^p inequalities for all $1 < p < \infty$ from scalar-valued weighted L^p inequalities for a single p; see [14], Section 3, for the most basic version of that result and [13], Theorem 8.1, for a version applicable to martingales.

In this article, we prove a weighted version of Lépingle's inequality for martingales with asymptotically sharp dependence on the A_p characteristic of the weight. For dyadic martingales, weighted variational inequalities were first obtained in [9], Lemma 6.1, using the real interpolation approach as in [4,17,22,27]. The argument in the dyadic case relied on the so-called open property of A_p classes; see, for example, [16], Theorem 1.2, that is in general false for martingale A_p classes, see the example in [3], Section 3, and [2]. Therefore, we use a new stopping time argument that is also simpler than the previous proofs of Lépingle's inequality even in the classical, unweighted, case.

1.1. Notation

Let $(\Omega, (\mathcal{F}_n)_{n=0}^{\infty}, \mu)$ be a filtered probability space and $\mathcal{F}_\infty := \bigvee_{n=0}^{\infty} \mathcal{F}_n$. A *weight* is a positive $\mathcal{F}_\infty$-measurable function $w : \Omega \to (0, \infty)$. The corresponding weighted L^p norm is given by $\|X\|_{L^p(\Omega,w)} := (\int_\Omega |X|^p w \,\mathrm{d}\mu)^{1/p}$. For $1 < p < \infty$, the *martingale A_p characteristic* of the weight w is defined by

$$Q_p(w) := \sup_\tau \big\| \mathbb{E}(w|\mathcal{F}_\tau)\mathbb{E}\big(w^{-1/(p-1)}|\mathcal{F}_\tau\big)^{p-1} \big\|_{L^\infty(w)},$$

1350-7265

where the supremum is taken over all adapted stopping times τ. For comparison of our main result with the unweighted case, note that for $w \equiv 1$ we have $Q_p(w) = 1$ for all $1 < p < \infty$.

For $0 < r < \infty$, a sequence of random variables $X = (X_n)_n$, and $\omega \in \Omega$, the r-variation of X at ω is defined by

$$V^r X(\omega) := V^r_n X_n(\omega) := \sup_{u_1 < u_2 < \cdots} \Big(\sum_j \big| X_{u_{j-1}}(\omega) - X_{u_j}(\omega) \big|^r \Big)^{1/r}, \tag{1.1}$$

where the supremum is taken over arbitrary increasing sequences.

1.2. Main result

For an integrable $\mathcal{F}_\infty$-measurable function $X : \Omega \to \mathbb{R}$, the associated martingale is defined by $X_n := \mathbb{E}(X|\mathcal{F}_n)$. We have the following weighted moment estimate for the pathwise r-variation of this martingale.

Theorem 1.1. *For every $1 < p < \infty$, there exists a constant $C_p < \infty$ such that, for every $r > 2$, every filtered probability space Ω, every weight w on Ω, and every integrable function $X : \Omega \to \mathbb{R}$, we have*

$$\big\| V^r X \big\|_{L^p(\Omega,w)} \le C_p \sqrt{\frac{r}{r-2}} Q_p(w)^{\max(1,1/(p-1))} \|X\|_{L^p(\Omega,w)}. \tag{1.2}$$

Remark 1.2. By the monotone convergence theorem, Theorem 1.1 extends to càdlàg martingales.

Remark 1.3. The example in [28], Theorem 2.1, shows that, for $p = 2$, the constant in (1.2) must diverge at least as

$$\sqrt{\log \frac{r}{r-2}} \quad \text{when } r \to 2. \tag{1.3}$$

Indeed, it is proved there that, if $(X_n)_{n=0}^N$ is a martingale with i.i.d. increments that are Gaussian random variables with zero expectation and unit variance, then $(V^2 X)^2 \ge cN \log\log N$ with probability converging to 1 as $N \to \infty$ for every $c < 1/12$. In this case, choosing r such that $r - 2 = 1/\log N$, by Hölder's inequality, we obtain

$$V^2 X \le N^{1/2-1/r} V^r X \le C V^r X.$$

This would lead to a contradiction if the constant in (1.2) diverges slower than stated in (1.3). The growth rate of the constant in (1.2) as $r \to 2$ is important, for example, in Bourgain's multi-frequency lemma, as explained in [31], Section 3.2.

Remark 1.4. The growth rate of the constant in (1.2) as $r \to 2$ is also related to endpoint estimates, in which the ℓ^r norm in (1.1) is replaced by an Orlicz space norm. The results of [29] for the Brownian motion suggest that it might be possible to use a Young function that decays as $x^2/\log\log x^{-1}$ when $x \to 0$. Such an estimate would imply an estimate of the form (1.3) for the constant in (1.2), and it would have useful consequences for rough differential equations; see [7], Remark 5. Our method allows to use Young functions that decay as $x^2/(\log x^{-1})^{1+\epsilon}$ when $x \to 0$.

Remark 1.5. A Fefferman–Stein type weighted estimate that substitutes (1.2) in the case $p = 1$ can be deduced from Corollary 2.4 and [25], Theorem 1.1.

2. Stopping times and a pathwise r-variation bound

In this section, we estimate the r-variation of an arbitrary adapted process pathwise by a linear combination of square functions. We consider an adapted process $(X_n)_n$ with values in an arbitrary metric space $(\mathcal{X}, d)$ and extend the definition of r-variation (1.1) by replacing the absolute value of the difference by the distance. We have the following metric spaces $\mathcal{X}$ in mind:

1. In Theorem 1.1, we will use $\mathcal{X} = \mathbb{R}$ (and $\rho = 2$ below).
2. In applications to the theory of rough paths, one takes $\mathcal{X}$ to be a free nilpotent group; see [15], Section 9.
3. When $\mathcal{X}$ is a Banach space with martingale cotype $\rho \in [2, \infty)$, Corollary 2.4 can be used to recover [27], Theorem 4.2.

Definition 2.1. Let $M_t := \sup_{t'' \le t' \le t} d(X_{t'}, X_{t''})$. For each $m \in \mathbb{N}$, define an increasing sequence of stopping times by

$$\tau_0^{(m)}(\omega) := 0, \qquad \tau_{j+1}^{(m)}(\omega) := \inf\bigl\{t \ge \tau_j^{(m)}(\omega) | d\bigl(X_t(\omega), X_{\tau_j(\omega)}(\omega)\bigr) \ge 2^{-m} M_t(\omega)\bigr\}. \tag{2.1}$$

Lemma 2.2. *Let $0 \le t' < t < \infty$ and $m \ge 2$. Suppose that*

$$2 < d\bigl(X_{t'}(\omega), X_t(\omega)\bigr) / \bigl(2^{-m} M_t(\omega)\bigr) \le 4. \tag{2.2}$$

Then there exists j with $t' < \tau_j^{(m)}(\omega) \le t$ and

$$d\bigl(X_{t'}(\omega), X_t(\omega)\bigr) \le 8 d\bigl(X_{\tau_{j-1}^{(m)}(\omega)}(\omega), X_{\tau_j^{(m)}(\omega)}(\omega)\bigr). \tag{2.3}$$

Proof. We fix ω and omit it from the notation. Let j be the largest integer with $\tau' := \tau_j^{(m)} \le t$. We claim that $\tau' > t'$. Suppose for a contradiction that $\tau' < t'$ (the case $\tau' = t'$ is similar but easier). By the hypothesis (2.2) and the assumption that t, t' are not stopping times, we obtain

$$2 \cdot 2^{-m} M_t < d(X_{t'}, X_t) \le d(X_{\tau'}, X_{t'}) + d(X_{\tau'}, X_t) < 2^{-m} M_{t'} + 2^{-m} M_t \le 2 \cdot 2^{-m} M_t,$$

a contradiction. This shows $\tau' > t'$.

It remains to verify (2.3). Assume that $M_{\tau'} < M_t/2$. Then, for some $\tau' < \tau'' \leq t$, we have $d(X_{\tau'}, X_{\tau''}) \geq M_t/2 \geq 2^{-m} M_{\tau''}$, contradicting maximality of τ'. It follows that

$$d(X_{\tau_{j-1}^{(m)}}, X_{\tau_j^{(m)}}) \geq 2^{-m} M_{\tau'} \geq 2^{-m} M_t/2 \geq d(X_{t'}, X_t)/8. \qquad \square$$

Lemma 2.3. *For every* $0 < \rho < r < \infty$, *we have the pathwise inequality*

$$V_t^r \big(X_t(\omega)\big)^r \leq 8^\rho \sum_{m=2}^{\infty} \big(2^{-(m-2)} M_\infty(\omega)\big)^{r-\rho} \sum_{j=1}^{\infty} d\big(X_{\tau_{j-1}^{(m)}(\omega)}(\omega), X_{\tau_j^{(m)}(\omega)}(\omega)\big)^\rho. \tag{2.4}$$

Proof. We fix ω and omit it from the notation. Let (u_l) be any increasing sequence. For each l with $d(X_{u_l}, X_{u_{l+1}}) \neq 0$, let $m = m(l) \geq 2$ be such that

$$2 < d(X_{u_l}, X_{u_{l+1}})/\big(2^{-m} M_{u_{l+1}}\big) \leq 4.$$

Such m exists because the distance is bounded by $M_{u_{l+1}}$.

Let j be given by Lemma 2.2 with $t' = u_l$ and $t = u_{l+1}$. Then

$$d(X_{u_l}, X_{u_{l+1}})^r \leq 8^\rho d(X_{\tau_{j-1}^{(m)}}, X_{\tau_j^{(m)}})^\rho \cdot \big(4 \cdot 2^{-m} M_{u_{l+1}}\big)^{r-\rho}.$$

Since each pair (m, j) occurs for at most one l, this implies

$$\sum_l d(X_{u_l}, X_{u_{l+1}})^r \leq 8^\rho \sum_{m,j} d(X_{\tau_{j-1}^{(m)}}, X_{\tau_j^{(m)}})^\rho \cdot \big(2^{-(m-2)} M_\infty\big)^{r-\rho}.$$

Taking the supremum over all increasing sequences (u_l), we obtain (2.4). $\square$

Corollary 2.4. *For every* $0 < \rho < r < \infty$, *we have the pathwise inequality*

$$V_t^r \big(X_t(\omega)\big)^\rho \leq 8^\rho \sum_{m=2}^{\infty} 2^{-(m-2)(r-\rho)} \sum_{j=1}^{\infty} d\big(X_{\tau_{j-1}^{(m)}(\omega)}(\omega), X_{\tau_j^{(m)}(\omega)}(\omega)\big)^\rho. \tag{2.5}$$

Proof. By the monotone convergence theorem, we may assume that X_n becomes independent of n for sufficiently large n. In this case,

$$M_\infty(\omega) \leq V_t^r \big(X_t(\omega)\big) < \infty.$$

Substituting this inequality in (2.4) and canceling $V_t^r(X_t(\omega))^{r-2}$ on both sides, the claim follows. $\square$

3. Proof of the weighted Lépingle inequality

Estimates in weighted spaces $L^p(\Omega, w)$ for differentially subordinate martingales with sharp dependence on the characteristic $Q_p(w)$ were obtained in [30] in the discrete case (a simpler alternative proof is in [19]) and [12] in the continuous case (a simpler alternative proof is in [11]). By Khintchine's inequality, these results imply the following weighted estimate for the martingale square function.

Theorem 3.1 (cf. [11]). *Let $(X_j)_{j=0}^{\infty}$ be a martingale on a probability space Ω. Then, for every $1 < p < \infty$, we have*

$$\left\| \left(\sum_{j=1}^{\infty} |X_j - X_{j-1}|^2 \right)^{1/2} \right\|_{L^p(\Omega,w)} \le C_p Q_p(w)^{\max(1,1/(p-1))} \|X\|_{L^p(\Omega,w)}, \tag{3.1}$$

where the constant $C_p < \infty$ depends only on p, but not on the martingale X or the weight w.

An alternative proof that deals directly with the square function (3.1) appears in [1], but it is carried out only for continuous time martingales with continuous paths.

Proof of Theorem 1.1. By extrapolation (see [13], Theorem 8.1), it suffices to consider $p = 2$. We will in fact give a direct proof for $2 \le p < \infty$. A similar argument also works for $1 < p < 2$, but gives a poorer dependence on r than claimed in (1.2).

Let $\tau_j^{(m)}$ be the stopping times constructed in (2.1), and let

$$S_{(m)}(\omega) := \left(\sum_{j=1}^{\infty} \left| X_{\tau_{j-1}^{(m)}(\omega)}(\omega) - X_{\tau_j^{(m)}(\omega)}(\omega) \right|^2 \right)^{1/2}$$

denote the square function of the sampled martingale $(X_{\tau_j^{(m)}})_j$. Then Corollary 2.4 with $\mathcal{X} = \mathbb{R}$ and $\rho = 2$ gives

$$V^r X \le 8 \left(\sum_{m=2}^{\infty} 2^{-(m-2)(r-2)} S_{(m)}^2 \right)^{1/2}.$$

Since $2 \le p < \infty$, by Minkowski's inequality, this implies

$$\|V^r X\|_{L^p(\Omega,w)} \le 8 \left(\sum_{m=2}^{\infty} 2^{-(m-2)(r-2)} \|S_{(m)}\|_{L^p(\Omega,w)}^2 \right)^{1/2}.$$

Inserting the square function estimates (3.1) for the sampled martingales $(X_{\tau_j^{(m)}})_j$ on the right-hand side above, we obtain

$$\begin{aligned} \|V^r X\|_{L^p(\Omega,w)} &\le 8 C_p Q_p(w) \|X\|_{L^p(\Omega,w)} \left(\sum_{m=2}^{\infty} 2^{-(m-2)(r-2)} \right)^{1/2} \\ &= 8 C_p \left(1 - 2^{-(r-2)}\right)^{-1/2} Q_p(w) \|X\|_{L^p(\Omega,w)}. \end{aligned}$$

This implies (1.2). □

Remark 3.2. One can also directly apply Theorem 3.1 for $1 < p < 2$, without passing through the extrapolation theorem. But this seems to lead to a faster growth rate of the constant in (1.2) as $r \to 2$.

Remark 3.3. The unweighted Lépingle inequality (Theorem 1.1 with $w \equiv 1$) follows from Corollary 2.4 and the usual Burkholder–Davis–Gundy (BDG) inequality.

Remark 3.4. Corollary 2.4 can be used to recover the p-variation rough path BDG inequality [5], Theorem 4.7. For convex moderate functions $F(x) = x^p$ with $1 < p < \infty$, the required estimate for the square function appearing in (2.5) can be deduced from the usual BDG inequality and [18], Proposition 3.1. The latter result can be extended to arbitrary convex moderate functions F using the Davis martingale decomposition.

Remark 3.5. Let $\rho \in [2, \infty)$, and let $\mathcal{X}$ be a Banach space with martingale cotype ρ. Using Corollary 2.4 and the ρ-function bounds for $\mathcal{X}$-valued martingales in [26], Theorem 10.59, we see that, for every $1 < p < \infty$, $r > \rho$, every filtered probability space Ω, and every integrable function $X : \Omega \to \mathcal{X}$, we have

$$\left\| V^r X \right\|_{L^p(\Omega)} \le C_{\mathcal{X},p} \frac{r}{r - \rho} \|X\|_{L^p(\Omega)}. \tag{3.2}$$

In fact, it is possible to obtain a slightly better dependence on r, which we omit for simplicity. There is also an endpoint version of (3.2) at $p = 1$, in which X is replaced by the martingale maximal function on the right-hand side.

The vector-valued estimate (3.2) was first proved in [27], Theorem 4.2, with an unspecified dependence on r. The dependence on r stated in (3.2) can also be obtained using Theorem 1.3 and Lemma 2.17 in [22], as well as real interpolation, but this method does not work at the endpoint $p = 1$.

Acknowledgement

This work was partially supported by the Hausdorff Center for Mathematics (DFG EXC 2047).

References

[1] Bañuelos, R. and Osękowski, A. (2018). Weighted L^2 inequalities for square functions. *Trans. Amer. Math. Soc.* **370** 2391–2422. MR3748572 https://doi.org/10.1090/tran/7056
[2] Bekollé, D. and Bonami, A. (1978). Inégalités à poids pour le noyau de Bergman. *C. R. Acad. Sci. Paris Sér. A-B* **286** A775–A778. MR0497663
[3] Bonami, A. and Lépingle, D. (1979). Fonction maximale et variation quadratique des martingales en présence d'un poids. In *Séminaire de Probabilités, XIII* (*Univ. Strasbourg, Strasbourg*, 1977/78). *Lecture Notes in Math.* **721** 294–306. Berlin: Springer. MR0544802
[4] Bourgain, J. (1989). Pointwise ergodic theorems for arithmetic sets. *Inst. Hautes Études Sci. Publ. Math.* **69** 5–45. MR1019960

[5] Chevyrev, I. and Friz, P.K. (2019). Canonical RDEs and general semimartingales as rough paths. *Ann. Probab.* **47** 420–463. MR3909973 https://doi.org/10.1214/18-AOP1264

[6] Crescimbeni, R., Macías, R.A., Menárguez, T., Torrea, J.L. and Viviani, B. (2009). The ρ-variation as an operator between maximal operators and singular integrals. *J. Evol. Equ.* **9** 81–102. MR2501353 https://doi.org/10.1007/s00028-009-0003-0

[7] Davie, A.M. Differential equations driven by rough paths: An approach via discrete approximation. Art. ID abm009,40.

[8] Di Plinio, F., Do, Y.Q. and Uraltsev, G.N. (2018). Positive sparse domination of variational Carleson operators. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **18** 1443–1458. MR3829751

[9] Do, Y. and Lacey, M. (2012). Weighted bounds for variational Walsh–Fourier series. *J. Fourier Anal. Appl.* **18** 1318–1339. MR3000985 https://doi.org/10.1007/s00041-012-9231-8

[10] Do, Y., Oberlin, R. and Palsson, E.A. (2017). Variation-norm and fluctuation estimates for ergodic bilinear averages. *Indiana Univ. Math. J.* **66** 55–99. MR3623404 https://doi.org/10.1512/iumj.2017.66.5983

[11] Domelevo, K. and Petermichl, S. Continuous-time sparse domination. Preprint.

[12] Domelevo, K. and Petermichl, S. (2019). Differential subordination under change of law. *Ann. Probab.* **47** 896–925. MR3916937 https://doi.org/10.1214/18-AOP1274

[13] Domelevo, K., Petermichl, S. and Wittwer, J. (2017). A linear dimensionless bound for the weighted Riesz vector. *Bull. Sci. Math.* **141** 385–407. MR3667592 https://doi.org/10.1016/j.bulsci.2017.05.004

[14] Duoandikoetxea, J. (2011). Extrapolation of weights revisited: New proofs and sharp bounds. *J. Funct. Anal.* **260** 1886–1901. MR2754896 https://doi.org/10.1016/j.jfa.2010.12.015

[15] Friz, P.K. and Victoir, N.B. (2010). *Multidimensional Stochastic Processes as Rough Paths: Theory and applications*. *Cambridge Studies in Advanced Mathematics* **120**. Cambridge: Cambridge Univ. Press. MR2604669 https://doi.org/10.1017/CBO9780511845079

[16] Hytönen, T., Pérez, C. and Rela, E. (2012). Sharp reverse Hölder property for A_∞ weights on spaces of homogeneous type. *J. Funct. Anal.* **263** 3883–3899. MR2990061 https://doi.org/10.1016/j.jfa.2012.09.013

[17] Jones, R.L., Seeger, A. and Wright, J. (2008). Strong variational and jump inequalities in harmonic analysis. *Trans. Amer. Math. Soc.* **360** 6711–6742. MR2434308 https://doi.org/10.1090/S0002-9947-08-04538-8

[18] Kovač, V. and Zorin-Kranich, P. (2019). Variational estimates for martingale paraproducts. *Electron. Commun. Probab.* **24** Paper No. 48, 14. MR4003122

[19] Lacey, M.T. (2017). An elementary proof of the A_2 bound. *Israel J. Math.* **217** 181–195. MR3625108 https://doi.org/10.1007/s11856-017-1442-x

[20] Lépingle, D. (1976). La variation d'ordre p des semi-martingales. *Z. Wahrsch. Verw. Gebiete* **36** 295–316. MR0420837 https://doi.org/10.1007/BF00532696

[21] Lyons, T.J. (1998). Differential equations driven by rough signals. *Rev. Mat. Iberoam.* **14** 215–310. MR1654527 https://doi.org/10.4171/RMI/240

[22] Mirek, M., Stein, E.M. and Zorin-Kranich, P. Jump inequalities via real interpolation. *Math. Ann.* To appear. https://doi.org/10.1007/s00208-019-01889-2

[23] Muckenhoupt, B. (1972). Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.* **165** 207–226. MR0293384 https://doi.org/10.2307/1995882

[24] Nazarov, F., Oberlin, R. and Thiele, C. (2010). A Calderón–Zygmund decomposition for multiple frequencies and an application to an extension of a lemma of Bourgain. *Math. Res. Lett.* **17** 529–545. MR2653686 https://doi.org/10.4310/MRL.2010.v17.n3.a11

[25] Osękowski, A. (2017). Weighted inequalities for the martingale square and maximal functions. *Statist. Probab. Lett.* **120** 95–100. MR3567926 https://doi.org/10.1016/j.spl.2016.09.020

[26] Pisier, G. (2016). *Martingales in Banach Spaces*. *Cambridge Studies in Advanced Mathematics* **155**. Cambridge: Cambridge Univ. Press. MR3617459

[27] Pisier, G. and Xu, Q.H. (1988). The strong p-variation of martingales and orthogonal series. *Probab. Theory Related Fields* **77** 497–514. MR0933985 https://doi.org/10.1007/BF00959613

[28] Qian, J. (1998). The p-variation of partial sum processes and the empirical process. *Ann. Probab.* **26** 1370–1383. MR1640349 https://doi.org/10.1214/aop/1022855756

[29] Taylor, S.J. (1972). Exact asymptotic estimates of Brownian path variation. *Duke Math. J.* **39** 219–241. MR0295434

[30] Thiele, C., Treil, S. and Volberg, A. (2015). Weighted martingale multipliers in the non-homogeneous setting and outer measure spaces. *Adv. Math.* **285** 1155–1188. MR3406523 https://doi.org/10.1016/j.aim.2015.08.019

[31] Zorin-Kranich, P. (2015). Variation estimates for averages along primes and polynomials. *J. Funct. Anal.* **268** 210–238. MR3280058 https://doi.org/10.1016/j.jfa.2014.10.018

Received August 2019