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Macroscopic analysis of determinantal random balls

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We consider a collection of Euclidean random balls in $\mathbb{R}^d$ generated by a determinantal point process inducing inhibitory interaction into the balls. We study this model at a macroscopic level obtained by a zooming-out and three different regimes – Gaussian, Poissonian and stable – are exhibited as in the Poissonian model without interaction. This shows that the macroscopic behaviour erases the interactions induced by the determinantal point process.

Keywords: determinantal point processes; generalized random fields; limit theorem; point processes; stable fields

Introduction

A random balls model is a collection $\mathcal{B}$ of random Euclidean balls $B(x,r)=\{y\in\mathbb{R}^d : \|y-x\|\le r\}$ whose centers $x\in\mathbb{R}^d$ and radii $r\in\mathbb{R}_+$ are generated by a stationary point process Φ in $\mathbb{R}^d\times\mathbb{R}_+$. Such models are used to represent a variety of situations. Let us mention a few of them. In dimension one, $\mathcal{B}$ can represent the traffic in a communication network. In this case, the (half-)balls are intervals $[x,x+r]$ and represent sessions of connection to the network, x being the date of connection and r the duration of connection. Such a model is investigated in [19] in a Poissonian setting, see also [14]. In dimension two, $\mathcal{B}$ can represent a wireless network with x being the location of a base station emitting a signal with a range r so that $B(x,r)$ represents the covering area of the station x and the collection $\mathcal{B}$ gives the overall covering of the network, cf. [25]. The two-dimensional model is used also in imagery to represent Black and White pictures. In dimension three, such models are again used to represent porous media, for instance bones can be modeled in this way and an analysis of the model allows in this case to investigate anomalies such as osteoporosis, see [2]. Such random balls model is also known as germ-grain model with spherical grains in stochastic geometry, see the reference book [5].

In general in these models, one can think of at least two kinds of question. First, we can describe the geometrical – or morphological – aspect of the collection $\mathcal{B}$ of balls and the corresponding continuum percolation problem can be investigated; we refer to [18] for this line of work. The second question deals with scaling limits of aggregative functionals of the model and is the subject of this paper. Such aggregative functionals, that we shall call contributions in the

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sequel, can be for instance the number of balls covering a site $y \in \mathbb{R}^d$:

$$\#\{B(x,r) \in \mathcal{B} : y \in B(x,r)\} = \sum_{B \in \mathcal{B}} \delta_y(B) = \int_{\mathbb{R}^d \times \mathbb{R}_+} \delta_y\big(B(x,r)\big)\Phi(dx,dr), \tag{1}$$

where, for any set A, $\delta_y(A) = \mathbf{1}_A(y)$ defines a Dirac measure. Typically in the imagery setting ($d = 2$), such a quantity gives the level of grey of pixel $y \in \mathbb{R}^2$, see [2]. Another example of contribution is given by the sum of the volumes of the balls in restriction to some window W

$$\int_{\mathbb{R}^d \times \mathbb{R}_+} \big|B(x,r)\big|_W \Phi(dx,dr), \tag{2}$$

where $|\cdot|_W$ stands for the Lebesgue measure restricted to $W \in \mathcal{B}(\mathbb{R}^d)$. Typically in dimension $d = 1$, such a quantity represents the cumulative workload of some communication network, see [14,19] when Φ is a Poisson point process. More generally, replacing δ_y or $|\cdot|_W$ in the above integrals (1), (2) by a finite measure μ gives the so-called contribution of the model $\mathcal{B}$ into μ. This shot-noise type functional writes

$$M(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}_+} \mu\big(B(x,r)\big)\Phi(dx,dr), \tag{3}$$

and will be the basic object of interest of this paper.

So far, these models have been investigated with a Poissonian generating mechanism, that is, Φ is a (homogeneous) Poisson point process (PPP) with moreover center and radii behaviours being independent. In addition to the above references, let mention also [13] and [3] where the d-dimensional model is investigated, and [4] where weights are attached to the balls. A slight generalization is introduced in [9] where, still in a Poissonian paradigm, but non-homogeneous, the behaviours of the centers and of the radii are no more independent. Let us also mention [11, 15] and [16] for asymptotics in related model for shot-noise processes.

In the present paper, we go beyond the Poissonian setting and consider random balls generated by a stationary determinantal point process. As far as we know, except for the preliminary study [10] where Ginibre point process (a special case of determinantal point process) is considered to generate the collection $\mathcal{B}$ and which is the very origin of this paper, this article presents the first study of a random balls model generated by a determinantal point process, the so-called determinantal random balls. From a wireless network point of view, such a random mechanism is legitimate since it makes sense to install the stations not too close from one another. The repulsiveness of determinantal point processes justly realizes such a characteristic. From a modeling point of view, this choice has been recently explored in [7,20] or [17]. In particular, it is shown in [7] that a thinned Ginibre point process is capable of modeling many of the actual cellular networks. See also [21] for general determinantal point process used in this context.

Let us now be more specific about the macroscopic analysis provided in the sequel: we are interested in the behaviour of $M(\mu)$ in (3) when a zoom-out is performed in the model. This zooming-out scheme offers at the limit a distant view of the model, erasing the local specificities to make emerge only global characteristics. The scaling performed consists in $r \mapsto \rho r$ (with rate $\rho > 0$) changing the ball $B(x,r)$ into $B(x,\rho r)$ and the zooming-out is performed with $\rho \to 0$.

Obviously, for the model not to vanish under such a scaling, the intensity, say λ, of the point process Φ generating the balls has to be tuned accordingly into $\lambda(\rho) \to +\infty$. In the sequel, this is done by considering a family of point processes Φ_ρ, $\rho \in]0, 1]$, interpreted as a balls model with ρ-scaled radii and $\lambda(\rho)$-boosted centers, see details in Section 2. A first-level description of the resulting contribution $M_\rho(\mu)$ in μ is then given by its mean value

$$\mathbb{E}\big[M_\rho(\mu)\big] = \int_{\mathbb{R}^d \times \mathbb{R}_+} \mu\big(B(x,r)\big) n_\rho(dx, dr),$$

where n_ρ is the intensity measure of Φ_ρ. A finer analysis is given by the fluctuations of $M_\rho(\mu)$ with respect to its mean value, i.e. the limit of

$$\frac{M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)]}{n(\rho)} \tag{4}$$

for a proper normalization $n(\rho)$ when $\rho \to 0$. The limit above is investigated in distribution for each (suitable) measure μ, or, equivalently, because of the linear structure of (3) and thanks to the Cramér–Wold device, in the finite-dimensional distributions (fdd) sense. The relative behaviours of the scaling rate ρ and of the balls intensity $\lambda(\rho)$ will be responsible of the different possible macroscopic regimes. A similar study has been done for the Poissonian random balls model, in which three different regimes – Gaussian, Poissonian and stable – appear at the limit, see [3,13]. Our study will justify that these regimes prevail for the determinantal random balls model, exhibiting thus a kind of robustness of these regimes. Actually, since Poisson point processes are the universal limits of stationary and ergodic point processes undergoing standard operations (independent thinning, dilatation), it is not surprising to recover similar asymptotics as the ones for the Poissonian model. We can even expect for these limits to be, in some way, universal.

The article is organized as follows. Section 1 gives a detailed presentation of the model investigated. The main results with the macroscopic behaviours (Theorems 2.7, 2.12, 2.15) are stated and proved in Section 2. Several final comments are gathered in Section 3 on zoom-in asymptotics, α-determinantal/permanental processes and non-stationary random balls model. Finally, Appendix provides a very brief account on determinantal point processes with the required results for our analysis.

1. Determinantal random balls model

The model considered is a collection $\mathcal{B}$ of random (Euclidean) balls $B(x,r) = \{y \in \mathbb{R}^d : \|y - x\| \le r\}$ whose centers $x \in \mathbb{R}^d$ and radii $r \in \mathbb{R}_+$ are generated by a marked stationary determinantal point process (DPP) Φ on $\mathbb{R}^d \times \mathbb{R}_+$. In this section, we describe thoroughly the model and we refer to the Appendix for more details on DPPs, in particular see its definition in Def. A.2. First, consider a stationary DPP ϕ with a kernel K with respect to the Lebesgue measure $|\cdot|$ satisfying $K(x, y) = K(x - y)$ (for simplicity, we use the same letter K for two different functions), moreover we assume that the map $\mathbf{K}$ given for all $f \in L^2(\mathbb{R}^d, dx)$ and $x \in \mathbb{R}^d$

by

$$\mathbf{K}f(x) = \int_{\mathbb{R}^d} K(x,y)f(y)\,dy \tag{5}$$

satisfies the following hypothesis

Hypothesis 1. *The map* $\mathbf{K}$ *in* (5) *is a bounded symmetric integral operator* $\mathbf{K}$ *from* $L^2(\mathbb{R}^d, dx)$ *into* $L^2(\mathbb{R}^d, dx)$ *with spectrum included in* $[0, 1[$. *Moreover,* $\mathbf{K}$ *is locally trace-class, i.e. for all compact* $\Lambda \subset E$, *the restriction* $\mathbf{K}_\Lambda$ *of* $\mathbf{K}$ *on* $L^2(\Lambda, \lambda)$ *is of trace-class.*

This point process ϕ generates the centers of the balls and as a DPP exhibits repulsiveness between its particles. To obtain balls, we attach to each center x a (positive) mark interpreted as a radius r, this is done independently and these radii are identically distributed according to a distribution F, assumed to admit a probability density f. The collection of these marks and of the DPP ϕ forms a marked DPP Φ. According to Proposition A.7, Φ is still a DPP but on $\mathbb{R}^d \times \mathbb{R}_+$ and with kernel

$$\widehat{K}\big((x,r),(y,s)\big) = \sqrt{f(r)}K(x,y)\sqrt{f(s)},$$

with respect to the Lebesgue measure. In the sequel, we shall use the notation Φ both for the marked DPP (i.e. the random locally finite collection of points (X_i, R_i)) and for the associated random measure $\sum_{(X,R)\in\Phi} \delta_{(X,R)}$. We consider the contribution of the model in any suitable (signed) measure μ on $\mathbb{R}^d$ given by the following measure-indexed random field:

$$M(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}_+} \mu\big(B(x,r)\big)\Phi(dx,dr). \tag{6}$$

Note that from a mathematical point of view, it is not required for the measure μ to be positive and signed measures can be considered. However, in order to ensure that $M(\mu)$ in (6) is well defined, we restrain to measures μ with finite total variation (see below Proposition 1.1). In the sequel, $\mathcal{Z}(\mathbb{R}^d)$ stands the set of signed (Borelian) measures μ on $\mathbb{R}^d$ with finite total variation $\|\mu\|_{\mathrm{var}}(\mathbb{R}^d) < +\infty$. Moreover as in [13], assume the following assumption on the radius behaviour, for $d < \beta < 2d$,

$$f(r) \underset{r\to+\infty}{\sim} \frac{C_\beta}{r^{\beta+1}}, \qquad r^{\beta+1} f(r) \le C_0. \tag{7}$$

Since $\beta > d$, condition (7) implies that the mean volume of the random ball is finite:

$$v_d \int_0^{+\infty} r^d f(r)\,dr < +\infty, \tag{8}$$

where $v_d = |B(0,1)| = \pi^{d/2}/\,\Gamma(d/2+1)$ is the Lebesgue measure of the unit ball of $\mathbb{R}^d$. On the contrary, $\beta < 2d$ implies that F does not admit a moment of order $2d$ and the volume of the balls has an infinite variance. This is responsible of some kind of long-range dependence in the model, see [13], p. 530, and is in line with communication network models which exhibit interference. The asymptotics condition in (7) is of constant use in the following.

Proposition 1.1. *Assume* (7) *is in force. For all* $\mu \in \mathcal{Z}(\mathbb{R}^d)$, $\mathbb{E}[|M(\mu)|] < +\infty$. *As a consequence*, $M(\mu)$ *in* (6) *is almost surely well defined for all* $\mu \in \mathcal{Z}(\mathbb{R}^d)$.

Proof. Using properties of functionals of random measures (see Section 9.5 in [6]), we have:

$$\mathbb{E}\big[\big|M(\mu)\big|\big] = \int_{\mathbb{R}^d\times\mathbb{R}_+} \big|\mu\big(B(x,r)\big)\big|\widehat{K}\big((x,r),(x,r)\big)\,dx\,dr.$$

Since $\widehat{K}((x,r),(x,r)) = K(0)f(r)$, writing $\mu(B(x,r)) = \int_{\mathbb{R}^d} \mathbf{1}_{B(y,r)}(x)\mu(dy)$, we have

$$\begin{aligned}
\mathbb{E}\big[\big|M(\mu)\big|\big] &\le \int_{\mathbb{R}^d\times\mathbb{R}_+}\int_{\mathbb{R}^d} \mathbf{1}_{B(y,r)}(x)|\mu|(dy)K(0)f(r)\,dx\,dr \\
&\le K(0)\int_{\mathbb{R}^d}\int_0^{+\infty}\left(\int_{\mathbb{R}^d}\mathbf{1}_{B(y,r)}(x)\,dx\right)f(r)\,dr|\mu|(dy) \\
&\le K(0)\big|B(0,1)\big|\int_0^{+\infty} r^d f(r)\,dr\int_{\mathbb{R}^d}|\mu|(dy) \\
&\le v_d\|\mu\|_{\mathrm{var}}K(0)\left(\int_0^{+\infty} r^d f(r)\,dr\right).
\end{aligned}$$

This concludes the proof thanks to condition (8), due to (7). □

Example 1.2. Typical examples of DPPs are given by Bessel point processes and Ginibre point processes.

1. *Bessel process*. In our real framework, the Bessel-type process is a DPP ϕ^B with kernel

$$K^B(x,y) = \frac{\sqrt{\Gamma(d/2+1)}}{\pi^{d/4}}\frac{J_{d/2}(2\sqrt{\pi}\Gamma(d/2+1)^{1/d}\|x-y\|)}{\|x-y\|^{d/2}}, \qquad x,y\in\mathbb{R}^d, \tag{9}$$

with respect to the Lebesgue measure, where $J_{d/2}$ stands for the Bessel function of the first kind. For instance, for $d=1$ we have:

$$K^B(x,y) = \frac{\sin(\pi\|x-y\|)}{\pi\|x-y\|}.$$

2. *Ginibre process.* In our real framework, the Ginibre-type point process ϕ^G is a DPP with kernel

$$K^G(x,y) = \exp\left(-\frac{1}{2}\|x-y\|^2\right), \qquad x,y\in\mathbb{R}^d,$$

with respect to the Lebesgue measure. Such processes have been used recently to model wireless networks of communication, see [7,20].

2. Asymptotics

We now detail our zooming-out procedure. This procedure acts accordingly both on the centers and on the radii (equivalently on the volume of the balls). First, a scaling $S_\rho : r \mapsto \rho r$ of rate $\rho \le 1$ changes balls $B(x,r)$ into $B(x,\rho r)$; this scaling changes the distribution F of the radius into $F_\rho = F \circ S_\rho^{-1}$. Second, the intensity of the centers is simultaneously adapted; to do this, we introduce actually a family of new kernels K_ρ, $\rho \in]0,1]$, that we shall refer to as scaled kernels, and we denote by ϕ_ρ the DPP with kernel K_ρ (with respect to the Lebesgue measure). In order to be in line with the scaling procedure investigated in the previous balls models (see [3,4,9,13]), we introduce $\lambda(\rho)$ given by

$$K_\rho(0) = \lambda(\rho) K(0) \tag{10}$$

with $\lim_{\rho\to 0} \lambda(\rho) = +\infty$. Using (41) in Prop. A.6, we have for any $\rho > 0$,

$$\int_{\mathbb{R}^d} \left|K_\rho(x)\right|^2 dx \underset{\rho\to 0}{=} \mathcal{O}\big(\lambda(\rho)\big). \tag{11}$$

Remark 2.1. The quantity $\lambda(\rho)$ introduced in (10) can be interpreted as the intensity of (centers of) the balls. Then $\lambda(\rho) \to +\infty$ indicates that there are more and more balls while the volumes of the balls are shrunk to ($\rho \to 0$), so that the zooming-out procedure consists of two competitive effects. The property (11) gives a control of $K_\rho(x)$ for $x \ne 0$ and, roughly speaking, means that the correlation of the centers of the balls is suitably controlled by the intensity of the centers.

In summary, the zoom-out procedure consists in considering a new marked DPP Φ_ρ on $\mathbb{R}^d \times \mathbb{R}_+$ with kernel:

$$\widehat{K}_\rho\big((x,r),(y,s)\big) = \sqrt{\frac{f(r/\rho)}{\rho}}\, K_\rho(x,y) \sqrt{\frac{f(s/\rho)}{\rho}},$$

with respect to the Lebesgue measure. The so-called scaled version of $M(\mu)$ is then the field

$$M_\rho(\mu) = \int_{\mathbb{R}^d\times\mathbb{R}_+} \mu\big(B(x,r)\big)\Phi_\rho(dx,dr).$$

In the sequel, we are interested in the fluctuations of $M_\rho(\mu)$ with respect to its expectation

$$\mathbb{E}\big[M_\rho(\mu)\big] = \int_{\mathbb{R}^d\times\mathbb{R}_+} \mu\big(B(x,r)\big) K_\rho(0) \frac{f(r/\rho)}{\rho}\, dx\, dr$$

and we introduce

$$\widetilde{M}_\rho(\mu) = M_\rho(\mu) - \mathbb{E}\big[M_\rho(\mu)\big] = \int_{\mathbb{R}^d\times\mathbb{R}_+} \mu\big(B(x,r)\big)\widetilde{\Phi}_\rho(dx,dr), \tag{12}$$

where $\widetilde{\Phi}_\rho$ stands for the compensated random measure associated to Φ_ρ.

Example 2.2. Continuing Example 1.2, we introduce the following family:

1. In the *Bessel* case, we consider the family of Bessel point processes ϕ_ρ^B, $\rho \in]0,1]$, with kernels:

$$K_\rho^B(x,y) = \frac{\sqrt{\lambda(\rho)\Gamma(d/2+1)}}{\pi^{d/4}} \frac{J_{d/2}(2\sqrt{\pi}\Gamma(d/2+1)^{1/d}\lambda(\rho)^{1/d}\|x-y\|)}{\|x-y\|^{d/2}} \tag{13}$$

with respect to the Lebesgue measure, where $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is a decreasing function with $\lim_{\rho\to 0}\lambda(\rho) = +\infty$. In this context, the property (11) easily follows from the following asymptotics of the Bessel functions of the first kind (see [1]):

$$J_\alpha(r) \underset{r\to 0}{\sim} \frac{1}{\Gamma(\alpha+1)}\left(\frac{r}{2}\right)^\alpha,$$
$$J_\alpha(r) \underset{r\to +\infty}{\sim} \sqrt{\frac{2}{\pi r}}\cos\left(r - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right).$$

2. In the *Ginibre* case, we consider the family of Ginibre point processes ϕ_ρ^G, $\rho \in]0,1]$, with kernels:

$$K_\rho^G(x,y) = \lambda(\rho)\exp\left(-\frac{\lambda(\rho)}{2}\|x-y\|^2\right), \qquad x, y \in \mathbb{R}^d, \tag{14}$$

with respect to the Lebesgue measure, where $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is a decreasing function with $\lim_{\rho\to 0}\lambda(\rho) = +\infty$, so that (11) is satisfied.

3. We can also consider the thinned and re-scaled Ginibre point process $\phi^{G,\alpha}$ (or α-Ginibre point process, see [20]) with kernel:

$$K^{G,\alpha}(x,y) = \exp\left(-\frac{\|x-y\|^2}{2\alpha}\right),$$

where $0 < \alpha \le 1$. Such a process is obtained by retaining independently each point of the Ginibre point process with probability α and then applying a scaling to conserve the density (mean number of points by volume unit) of the initial Ginibre point process. This so-called α-Ginibre point process bridges smoothly between the Ginibre point process ($\alpha = 1$) and the Poisson point process ($\alpha \to 0$). For the scaled version, replace (14) by

$$K_\rho^{G,\alpha}(x,y) = \lambda(\rho)\exp\left(-\frac{\lambda(\rho)}{2\alpha}\|x-y\|^2\right).$$

Heuristics

The asymptotic behaviour of $\widetilde{M}_\rho(\mu)$ when $\rho \to 0$ depends on how the scaling rate ρ and the intensity $\lambda(\rho)$ are tuned. Roughly speaking, three regimes appear according to $\rho \to 0$ faster, slower or well-balanced with respect to $\lambda(\rho) \to +\infty$. Heuristically, the key quantity ruling these regimes is the mean number of large balls, say balls of radii larger than 1 and, say, containing 0:

$$\begin{aligned}
&\mathbb{E}\big[\#\big\{(x,r) \in \Phi_\rho : 0 \in B(x,r), r > 1\big\}\big] \\
&= \int_{\{(x,r):0\in B(x,r),r>1\}} \widehat{K}_\rho\big((x,r),(x,r)\big)\,dx\,dr = \int_1^{+\infty}\int_{B(0,r)} K_\rho(x,x)\frac{f(r/\rho)}{\rho}\,dx\,dr \\
&= \int_{1/\rho}^{+\infty}\int_{B(0,\rho u)} \lambda(\rho)K(0)\,dx f(u)\,du \sim v_d K(0)\lambda(\rho)\rho^d \int_{1/\rho}^{+\infty} u^{-1-\beta+d}\,du \\
&\sim \frac{v_d K(0)}{\beta - d}\lambda(\rho)\rho^\beta
\end{aligned}$$

using both (10), (7). Thus, the balance between $\rho \to 0$ and $\lambda(\rho) \to +\infty$ is ruled by $\lambda(\rho)\rho^\beta$ and the three scaling regimes are the following when $\rho \to 0$:

- Large-balls scaling: $\lambda(\rho)\rho^\beta \to +\infty$. Roughly speaking, large balls prevail at the limit and they shape the limit according to some kind of central limit theorem (CLT). Moreover, since the large balls overlap, this regime yields dependence at the limit. In other words, the limit $\lambda(\rho)\rho^\beta \to +\infty$ acts as if $\lambda(\rho) \to +\infty$ first and $\rho \to 0$ next; the first limit ($\lambda(\rho) \to +\infty$) corresponds to the superposition of a large number of (overlapping) balls, which in line with a CLT argument, produces a Gaussian limit (with dependence), the second limit ($\rho \to 0$) only shapes the covariance of the Gaussian field. In this context, the proper normalization will be $n(\rho) = \sqrt{\lambda(\rho)\rho^\beta}$. See Section 2.1.
- Intermediate scaling: $\lambda(\rho)\rho^\beta \to a \in]0,+\infty[$. Roughly speaking, there is a proper balance between large and small balls and somehow the limit is incompletely taken and it only consists in an alteration of the generating point process with a dissolving of the interaction resulting in a Poisson point process. In this context, the proper normalization will just be a constant. See Section 2.2.
- Small-balls scaling: $\lambda(\rho)\rho^\beta \to 0$. Roughly speaking, small balls prevail. In other words, the limit $\lambda(\rho)\rho^\beta \to 0$ acts as if $\rho \to 0$ first and $\lambda(\rho) \to +\infty$ next. The first limit $\rho \to 0$ is a scaling killing the overlapping and thus producing independence at the limit. Next, with the second limit ($\lambda(\rho) \to +\infty$) the heavy-tails of F enter the picture: the contribution of the non-overlapping balls are in the domain of attraction of a stable distribution producing a stable regime. Moreover, the index of stability γ can be heuristically derived as follows: for a smooth measure μ, we have $\mu(B(x,r)) \asymp cr^d$ with (β/d)-regular tails under (7) and this is responsible for the index of stability $\gamma = \beta/d$. See Section 2.3.

General strategy

For the three regimes, the proofs will follow the same idea in Sections 2.1, 2.2, and 2.3 below, and the general strategy is presented. The main tool to study the so-called determinantal integrals (6) or (12) (integrals with respect to a determinantal random measure) is the Laplace transform given in Theorem A.4. However, this result applies for compactly supported integrands which is not the case in our setting with $(x,r) \mapsto \mu(B(x,r))$ (since when $r \to +\infty$, $\mu(B(x,r)) \to \mu(\mathbb{R}^d)$). As a consequence, we consider the following auxiliary truncated process:

$$M_\rho^R(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}_+} \mu\big(B(x,r)\big)\mathbf{1}_{\{r \le R\}} \Phi_\rho(dx, dr), \tag{15}$$

and the associated compensated determinantal integral $\widetilde{M}_\rho^R(\mu)$. Then, for a positive compactly supported measure μ, the application $(x,r) \mapsto \mu(B(x,r))\mathbf{1}_{\{r \le R\}}$ is indeed a compactly supported function. In the following, we thus restrain $\mathcal{Z}(\mathbb{R}^d)$ to $\mathcal{Z}_c^+(\mathbb{R}^d)$ the set of positive compactly supported Borelian measures on $\mathbb{R}^d$ with finite total variation. The relevance in introducing this auxiliary process appears in the following result:

Proposition 2.3. *Assume* (7) *and* (10)*. For all $\mu \in \mathcal{Z}_c^+(\mathbb{R}^d)$ and for all $\rho > 0$, $M_\rho^R(\mu)$ converges in L^1 when $R \to +\infty$ to $M_\rho(\mu)$. Moreover, in the intermediate and the small-balls scalings, there exists a constant $\rho_1 > 0$, independent of R, such that this convergence is uniform in ρ for $\rho \in]0, \rho_1[$.*

Proof. Let $\mu \in \mathcal{Z}_c^+(\mathbb{R}^d)$. By the monotone convergence theorem $M_\rho^R(\mu) \nearrow M_\rho(\mu)$ when $R \to +\infty$ and by the dominated convergence theorem $M_\rho^R(\mu) \to M_\rho(\mu)$ in L^1. Next, we have

$$M_\rho(\mu) - M_\rho^R(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}_+} \mu\big(B(x,r)\big)\mathbf{1}_{\{r > R\}} \Phi_\rho(dx, dr)$$

and

$$\begin{aligned}
\mathbb{E}\big[\big|M_\rho(\mu) - M_\rho^R(\mu)\big|\big] &= \mathbb{E}\bigg[\int_{\mathbb{R}^d \times \mathbb{R}_+} \mu\big(B(x,r)\big)\mathbf{1}_{\{r > R\}} \Phi_\rho(dx, dr)\bigg] \\
&= \int_{\mathbb{R}^d \times \mathbb{R}_+} \mu\big(B(x,r)\big)\mathbf{1}_{\{r > R\}} \widehat{K}_\rho\big((x,r),(x,r)\big)\, dx\, dr \\
&= \int_{\mathbb{R}^d} \int_R^{+\infty} \mu\big(B(x,r)\big) K_\rho(x,x) \frac{f(r/\rho)}{\rho}\, dx\, dr \\
&= \lambda(\rho) K(0) \int_{\mathbb{R}^d} \int_R^{+\infty} \mu\big(B(x,r)\big) \frac{f(r/\rho)}{\rho}\, dx\, dr.
\end{aligned}$$

But with Fubini theorem and a change of variables

$$\begin{aligned}
\int_{\mathbb{R}^d} \int_R^{+\infty} \int_{\mathbb{R}^d} \mathbf{1}_{B(x,r)}(y) \frac{f(r/\rho)}{\rho} \mu(dy)\, dx\, dr &= \int_{\mathbb{R}^d} \int_R^{+\infty} v_d r^d \frac{f(r/\rho)}{\rho}\, dr\, \mu(dy) \\
&= v_d \mu\big(\mathbb{R}^d\big) \rho^d \int_{R/\rho}^{+\infty} u^d f(u)\, du.
\end{aligned}$$

From (7), we have $f(u) \le C_0/u^{\beta+1}$ and when $\rho < 1$,

$$\rho^d \int_{R/\rho}^{+\infty} u^d f(u)\, du \le \rho^d \int_{R/\rho}^{+\infty} u^d \frac{C_0}{u^{1+\beta}}\, du = \frac{C_0}{\beta - d} R^{d-\beta} \rho^\beta$$

so that

$$\mathbb{E}\big[\big|M_\rho(\mu) - M_\rho^R(\mu)\big|\big] \le \frac{C_0}{\beta - d} R^{d-\beta} \lambda(\rho)\rho^\beta K(0) v_d \mu\big(\mathbb{R}^d\big),$$

which goes to 0 uniformly in $\rho \in]0,1[$ under the intermediate and small scalings since $\lambda(\rho)\rho^\beta$ is bounded in these case. □

This uniform convergence is crucial in order to interchange the limit in ρ and the limit in R whenever $\lim_{\rho\to 0} \widetilde{M}_\rho^R(\mu)$ exists:

$$\lim_{\rho\to 0} \mathcal{L}\big(\widetilde{M}_\rho(\mu)\big) = \lim_{\rho\to 0} \lim_{R\to+\infty} \mathcal{L}\big(\widetilde{M}_\rho^R(\mu)\big) = \lim_{R\to+\infty} \lim_{\rho\to 0} \mathcal{L}\big(\widetilde{M}_\rho^R(\mu)\big). \tag{16}$$

The strategy is now clear to obtain $\lim_{\rho\to 0} \widetilde{M}_\rho(\mu)$: (i) first, take $\lim_{\rho\to 0} \widetilde{M}_\rho^R(\mu)$ and (ii) next take the limit in $R \to +\infty$. In order to realize (i), we use the Laplace transform of a DPP (39) and the expansion (37) of the corresponding Fredholm determinant. In this expansion, the first term (for $n = 1$) is identified as a Poissonian term for which the asymptotics of the Poissonian model applies and the remaining terms ($n \ge 2$) are shown to be asymptotically negligible. Next, (ii) properly shapes the limit with $R \to +\infty$.

However in order to realize (i), it is required to investigate the convergence of $\widetilde{M}_\rho^R(\mu)$ when $\rho \to 0$ on a restricted class of measures μ that we introduce now.

Definition 2.4. The set $\mathcal{M}_\beta^+$ consists of positive measures $\mu \in \mathcal{Z}_c^+(\mathbb{R}^d)$ such that there exist two real numbers p and q with $0 < p < \beta < q \le 2d$ and a positive constant C_μ such that

$$\int_{\mathbb{R}^d} \mu\big(B(x,r)\big)^2\, dx \le C_\mu\big(r^p \wedge r^q\big), \tag{17}$$

where $a \wedge b = \min(a, b)$.

The controls in (17) by both r^p and r^q are required to ensure that our quantities are well defined (see Proposition 2.5(i)); however in the sequel, only the control by r^q will be used. This definition is reminiscent of $\mathcal{M}_{2,\beta}$ in [4]. It is immediate that Dirac measures do not belong to $\mathcal{M}_\beta^+$. However absolutely continuous measures with respect to the Lebesgue measure, with density $\varphi \in L^2(\mathbb{R}^d)$ with compact support, do belong to $\mathcal{M}_\beta^+$ and will play an important role in the small-balls scaling. In this case, we shall abusively write $\mu \in L_c^2(\mathbb{R}^d)$ (here, the index c stands for compact support). Recall the following properties on $\mathcal{M}_\beta^+$ from Propositions 2.2 and 2.3 from [4].

Proposition 2.5. (i) *The set $\mathcal{M}_\beta^+$ is an affine subspace and, for all $\mu \in \mathcal{M}_\beta^+$,*

$$\int_{\mathbb{R}^d \times \mathbb{R}_+} \mu\big(B(x,r)\big)^2 r^{-\beta-1}\, dx\, dr < +\infty.$$

(ii) *If $d < \beta < 2d$, then $L^2_c(\mathbb{R}^d) \subset \mathcal{M}_\beta^+$ and for all $\mu \in L^2_c(\mathbb{R}^d)$:*

$$\int_{\mathbb{R}^d} \mu\big(B(x,r)\big)^2\, dx \le C_\mu\big(r^d \wedge r^{2d}\big).$$

Moreover, $\mathcal{M}_\beta^+$ is closed under translations, rotations and dilatations and is included in the subspace of diffuse measures, see Proposition 2.3 and Proposition 2.4 in [4] for details. See also [13], Section 2.2, for a sufficient condition to belong to $\mathcal{M}_\beta^+$ in terms of the Riesz energy of a measure.

Poissonian asymptotics

Since our strategy consists in identifying, in our functional, Poissonian terms to which well-known asymptotics are applied, we recall these Poissonian asymptotics from [13] but with our current notations, see also [3,4].

Theorem 2.6 (Poissonian asymptotics, [3,4] or [13]). *Let Φ be a marked PPP in (6) and (12) with compensator $K(0)\, dx\, F(dr)$ with F having density f satisfying (7) for $d < \beta < 2d$.*

(i) *Large-balls scaling: Assume $\lambda(\rho)\rho^\beta \to +\infty$. Then, for $n(\rho) = (\lambda(\rho)\rho^\beta)^{1/2}$, $\widetilde{M}_\rho(\cdot)/n(\rho)$ converges in the fdd sense on $\mathcal{M}_\beta^+$ to W where*

$$W(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}_+} \mu\big(B(x,r)\big) M_2(dx, dr)$$

and M_2 is a centered Gaussian random measure with control measure $K(0)C_\beta r^{-\beta-1}\, dx\, dr$.

(ii) *Intermediate scaling: Assume $\lambda(\rho)\rho^\beta \to a^{d-\beta} \in]0, +\infty[$. Then, for $n(\rho) = 1$, $\widetilde{M}_\rho(\cdot)/n(\rho)$ converges in the fdd sense on $\mathcal{M}_\beta^+$ to $\widetilde{P} \circ D_a$ where*

$$\widetilde{P}(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}_+} \mu\big(B(x,r)\big) \widetilde{\Pi}(dx, dr)$$

with $\widetilde{\Pi}$ a (compensated) PPP with compensator measure $K(0)C_\beta r^{-\beta-1}\, dx\, dr$ and D_a is the dilatation defined by $(D_a\mu)(B) = \mu(a^{-1}B)$.

(iii) *Small-balls scaling: Assume $\lambda(\rho)\rho^\beta \to 0$. Then, for $n(\rho) = (\lambda(\rho)\rho^\beta)^{1/\gamma}$ with $\gamma = \beta/d \in]1, 2[$, $\widetilde{M}_\rho(\cdot)/n(\rho)$ converges in the fdd sense in $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ to Z where*

$$Z(\mu) = \int_{\mathbb{R}^d} \phi(x) M_\gamma(dx) \qquad \textit{for } \mu(dx) = \phi(x)\, dx \textit{ with } \phi \in L^1\big(\mathbb{R}^d\big) \cap L^2\big(\mathbb{R}^d\big),$$

with M_γ a γ-stable measure with control measure $\sigma_\gamma dx$ where

$$\sigma_\gamma = \frac{K(0)C_\beta v_d^\gamma}{d} \int_0^{+\infty} \frac{1-\cos(r)}{r^{1+\gamma}}\, dr,$$

and with unit skewness.

Here, and in the sequel, we follow the notations of the standard reference [23] for stable random variables and integrals.

2.1. Large-balls scaling

In this section, we first investigate the behaviour of $\widetilde{M}_\rho^R(\mu)$ in (15) under the large-balls scaling $\lambda(\rho)\rho^\beta \to +\infty$ when $\rho \to 0$. In this section, set $n(\rho) = (\lambda(\rho)\rho^\beta)^{1/2}$.

As explained previously, the superposition due to $\lambda(\rho) \to +\infty$ acts firstly producing a Gaussian field W^R with a CLT type argument. Next, let $R \to +\infty$ to obtain the asymptotic behaviour of $\widetilde{M}_\rho(\mu)$ according to (16). The field obtained is given by a Gaussian integral similar to that of Theorem 2.6 (see also Theorem 2(i) in [13]).

Theorem 2.7 (Large-balls scaling asymptotics). *Assume (7) and the kernels K_ρ satisfy (10) and Hypothesis 1 for their associated operators $\mathbf{K}_\rho$ in (5). Suppose $\lambda(\rho)\rho^\beta \to +\infty$ when $\rho \to 0$, then the field $n(\rho)^{-1}\widetilde{M}_\rho(\cdot)$ converges in finite-dimensional distributions sense to $W(\cdot)$ in the space $\mathcal{M}_\beta^+$ where*

$$W(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}_+} \mu\big(B(x,r)\big) M_2(dx,dr),$$

with a centered Gaussian random measure M_2 with control measure $K(0)C_\beta r^{-\beta-1}\, dx\, dr$.

Following our strategy, we start with the asymptotics of $\widetilde{M}_\rho^R(\mu)$:

Proposition 2.8. *Suppose $\lambda(\rho)\rho^\beta \to +\infty$ when $\rho \to 0$. Then, for all fixed $R > 0$ and for all $\mu \in \mathcal{M}_\beta^+$, $n(\rho)^{-1}\widetilde{M}_\rho^R(\mu)$ converges in distribution when $\rho \to 0$ to*

$$W^R(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}_+} \mu\big(B(x,r)\big)\mathbf{1}_{\{r \le R\}} M_2(dx,dr),$$

uniformly in R, where M_2 is the same centered Gaussian random measure as in Theorem 2.7.

Proof. The convergence in distribution of $\widetilde{M}_\rho^R(\mu)$ for $\mu \in \mathcal{M}_\beta^+$ is shown by the convergence of its Laplace transform: for $\theta \ge 0$

$$\mathbb{E}\big[\exp\big(-\theta n(\rho)^{-1}\widetilde{M}_\rho^R(\mu)\big)\big] = \exp\big(\theta\mathbb{E}\big[n(\rho)^{-1}M_\rho^R(\mu)\big]\big)\mathbb{E}\big[\exp\big(-\theta n(\rho)^{-1}M_\rho^R(\mu)\big)\big]. \tag{18}$$

Since M_ρ^R given in (15) is a determinantal integral with a compactly supported (say in Λ_μ^R) integrand $g_\mu^R(x,r) := \mu(B(x,r))\mathbf{1}_{\{r\le R\}}$, and the kernel K_ρ satisfying Hypothesis 1, its Laplace transform is given by Theorem A.4:

$$\begin{aligned}\mathbb{E}\big[\exp\big(-\theta n(\rho)^{-1}M_\rho^R(\mu)\big)\big] &= \operatorname{Det}\big(I-\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1}g_\mu^R}\big]\big)\\ &= \exp\bigg(-\sum_{n\ge1}\frac{1}{n}\operatorname{Tr}\big(\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1}g_\mu^R}\big]^n\big)\bigg),\end{aligned}\tag{19}$$

where $\widehat{K}_\rho[1-e^{-\theta n(\rho)^{-1}g_\mu^R}]$ is the bounded operator of $L^2(\mathbb{R}^d\times\mathbb{R}_+)$ given in (40). We compute the first trace in the sum in (19) with Proposition A.5 applied with the DPP Φ_ρ with kernel $\widehat{K}_\rho$ on $\mathbb{R}^d\times\mathbb{R}_+$ restricted on the compact Λ_μ^R and the function $1-e^{-\theta n(\rho)^{-1}g_\mu^R}$ (see Proposition A.5):

$$\begin{aligned}\operatorname{Tr}\big(\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1}g_\mu^R}\big]\big) &= \mathbb{E}\bigg[\int_{\mathbb{R}^d\times\mathbb{R}_+}\big(1-e^{-\theta n(\rho)^{-1}g_\mu^R}\big)\Phi_\rho(dx,dr)\bigg]\\ &= \int_{\mathbb{R}^d\times\mathbb{R}_+}\big(1-e^{-\theta n(\rho)^{-1}\mu(B(x,r))\mathbf{1}_{\{r\le R\}}}\big)K_\rho(x,x)\frac{f(r/\rho)}{\rho}\,dx\,dr.\end{aligned}$$

With (10), this term for $n=1$ combines with the factor $\exp(\theta\mathbb{E}[n(\rho)^{-1}M_\rho^R(\mu)])$ of (18) into

$$\exp\bigg(\int_{\mathbb{R}^d\times\mathbb{R}_+}\psi\big(\theta n(\rho)^{-1}g_\mu^R\big)\lambda(\rho)K(0)\frac{f(r/\rho)}{\rho}\,dx\,dr\bigg),$$

with $\psi(u)=e^{-u}-1+u$. The Laplace transform of $n(\rho)^{-1}\widetilde{M}_\rho^R(\mu)$ in (18) thus rewrites

$$\begin{aligned}\mathbb{E}\big[\exp\big(-\theta n(\rho)^{-1}\widetilde{M}_\rho^R(\mu)\big)\big] &= \exp\bigg(\int_{\mathbb{R}^d\times\mathbb{R}_+}\psi\big(\theta n(\rho)^{-1}g_\mu^R\big)\lambda(\rho)K(0)\frac{f(r/\rho)}{\rho}\,dx\,dr\bigg)\\ &\quad\times\exp\bigg(-\sum_{n\ge2}\frac{1}{n}\operatorname{Tr}\big(\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1}g_\mu^R}\big]^n\big)\bigg).\end{aligned}\tag{20}$$

First, we deal with the first exponential term in (20): the key point is that this is the Laplace transform of $n(\rho)^{-1}\widetilde{P}_\rho^R(\mu)$ with

$$\widetilde{P}_\rho^R(\mu)=\int_{\mathbb{R}^d\times\mathbb{R}_+}\mu\big(B(x,r)\big)\mathbf{1}_{\{r\le R\}}\widetilde{\Pi}_\rho(dx,dr),\tag{21}$$

where $\widetilde{\Pi}_\rho$ is a compensated Poisson random measure on $\mathbb{R}^d\times\mathbb{R}_+$ with intensity

$$\lambda(\rho)K(0)\frac{f(r/\rho)}{\rho}\,dx\,dr.$$

From (i) in Theorem 2.6 (Theorem 2(i) in [13]), (21) converges in distribution when $\rho\to0$ to the Gaussian integral $W^R(\mu)$. We show now that this convergence is actually uniform in R, to that way, consider the difference of the log-Laplace transform of $n(\rho)^{-1}\widetilde{P}^R_\rho(\mu)$ and of $W^R(\mu)$:

$$\begin{aligned}
&\bigl|\log\bigl(\mathbb{E}\bigl[\exp\bigl(n(\rho)^{-1}\widetilde{P}^R_\rho(\mu)\bigr)\bigr]\bigr)-\log\bigl(\mathbb{E}\bigl[\exp\bigl(W^R(\mu)\bigr)\bigr]\bigr)\bigr| \\
&\quad\le\left|\int_{\mathbb{R}^d\times\mathbb{R}_+}\psi\bigl(n(\rho)^{-1}\mu\bigl(B(x,r)\bigr)\mathbf{1}_{\{r\le R\}}\bigr)\lambda(\rho)K(0)\frac{f(r/\rho)}{\rho}\right. \\
&\qquad\left.-\frac{\mu(B(x,r))^2}{2}\mathbf{1}_{\{r\le R\}}\frac{C_\beta K(0)}{r^{\beta+1}}\,dx\,dr\right| \qquad (22)\\
&\quad\le\int_{\mathbb{R}^d\times\mathbb{R}_+}\left|\psi\bigl(n(\rho)^{-1}\mu\bigl(B(x,r)\bigr)\mathbf{1}_{\{r\le R\}}\bigr)\lambda(\rho)K(0)\frac{f(r/\rho)}{\rho}\right. \\
&\qquad\left.-\frac{\mu(B(x,r))^2}{2}\mathbf{1}_{\{r\le R\}}\frac{C_\beta K(0)}{r^{\beta+1}}\right|\,dx\,dr.
\end{aligned}$$

Since $\psi(u)\sim\frac{u^2}{2}$ when $u\to 0$ and since $n(\rho)=(\lambda(\rho)\rho^\beta)^{1/2}\to+\infty$ when $\rho\to 0$, using the tails behaviour (7), we have:

$$\begin{aligned}
\psi\bigl(n(\rho)^{-1}\mu\bigl(B(x,r)\bigr)\bigr)\lambda(\rho)K(0)\frac{f(r/\rho)}{\rho} &\underset{\rho\to 0}{\sim}\frac{\mu(B(x,r))^2}{2n(\rho)^2}\lambda(\rho)K(0)\frac{C_\beta\rho^\beta}{r^{\beta+1}} \\
&=\frac{\mu(B(x,r))^2}{2}K(0)\frac{C_\beta}{r^{\beta+1}},
\end{aligned}$$

proving that the integrand in (22) converges to 0. Moreover, using (7) and $\psi(x)\le x^2/2$, for all r and for all $\rho>0$, we have:

$$\begin{aligned}
&\left|\psi\bigl(n(\rho)^{-1}\mu\bigl(B(x,r)\bigr)\bigr)\lambda(\rho)K(0)\frac{f(r/\rho)}{\rho}-\frac{\mu(B(x,r))^2}{2}\frac{C_\beta K(0)}{r^{\beta+1}}\right| \\
&\quad\le\frac{\mu(B(x,r))^2}{2n(\rho)^2}\lambda(\rho)K(0)\frac{f(r/\rho)}{\rho}+\frac{\mu(B(x,r))^2}{2}\frac{C_\beta K(0)}{r^{\beta+1}} \\
&\quad\le K(0)(C_0+C_\beta)\frac{\mu(B(x,r))^2}{2r^{\beta+1}},
\end{aligned}$$

which is integrable over $\mathbb{R}^d\times\mathbb{R}_+$ according to Proposition 2.5. Then, the dominated convergence theorem ensures that (22) converges to 0 when $\rho\to 0$. Moreover, since it does not depend on R, the convergence of $n(\rho)^{-1}\widetilde{P}^R_\rho(\mu)$ to $W^R(\mu)$ is uniform in R.

Next, we deal with the other second exponential terms in (20) and show that they converge to 1 proving that for all $n\ge 2$,

$$\lim_{\rho\to 0}\operatorname{Tr}\bigl(\widehat{K}_\rho\bigl[1-e^{-\theta n(\rho)^{-1}g^R_\mu}\bigr]^n\bigr)=0.$$

More precisely, the convergence of (20) to 1 will derive from the following lemmas. Recall $g^R_\mu(x,r)=\mu(B(x,r))\mathbf{1}_{\{r\le R\}}$ and $\mu\in\mathcal{M}^+_\beta$; in particular g^R_μ is bounded with compact support.

Since K_ρ satisfies Hypothesis 1, Proposition A.8 first ensures $\widehat{K}_\rho$ satisfies also Hypothesis 1 and Proposition A.10 next ensures that $\widehat{K}_\rho[1-e^{-\theta n(\rho)^{-1}g_\mu^R}]$ is the kernel of an Hilbert–Schmidt operator in (5).

Lemma 2.9. *For all $n \geq 2$, we have*

$$\mathrm{Tr}\big(\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1}g_\mu^R}\big]^n\big) \leq \mathrm{Tr}\big(\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1}g_\mu^R}\big]^2\big)^{n/2}.$$

Lemma 2.10. *Assume Condition (8), and consider $\mu \in \mathcal{M}_\beta^+$. Then there is $\rho^* > 0$ and a constant $C_K \in]0,+\infty[$ such that for all $\rho \in]0,\rho^*[$, uniformly in R,*

$$\mathrm{Tr}\big(\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1}g_\mu^R}\big]^2\big) \leq C_K C_\mu C_f \theta^2 \frac{\lambda(\rho)\rho^q}{n(\rho)^2},$$

with $C_f = (\int_0^{+\infty} r^{q/2} f(r)\,dr)^2$.

As a consequence of both Lemmas 2.9 and 2.10, we have

$$\begin{aligned}\bigg|-\sum_{n\geq 2}\frac{1}{n}\mathrm{Tr}\big(\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1}g_\mu^R}\big]^n\big)\bigg| &\leq \sum_{n\geq 1}\frac{1}{n}\Big(\sqrt{\mathrm{Tr}\big(\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1}g_\mu^R}\big]^2\big)}\Big)^n \\ &= -\ln\Big(1-\sqrt{\mathrm{Tr}\big(\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1}g_\mu^R}\big]^2\big)}\Big).\end{aligned} \tag{23}$$

Next, since (8) holds true under (7), Lemma 2.10 applies and the bound (23) goes to 0 when $\rho \to 0$ since $\lambda(\rho)\rho^q/n(\rho)^2 = \rho^{q-\beta}$ with $q > \beta$. As a consequence,

$$\lim_{\rho\to 0}\exp\bigg(-\sum_{n\geq 2}\frac{1}{n}\mathrm{Tr}\big(\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1}g_\mu^R}\big]^n\big)\bigg) = 1,$$

and the limit in (20) writes

$$\lim_{\rho\to 0}\mathbb{E}\big[\exp\big(-\theta n(\rho)^{-1}\widetilde{M}_\rho^R(\mu)\big)\big] = \mathbb{E}\big[\exp\big(-\theta W^R(\mu)\big)\big],$$

achieving the proof of Proposition 2.8. □

It remains to prove Lemma 2.9 and Lemma 2.10.

Proof of Lemma 2.9. Recall that for a Hilbert–Schmidt operator T with operator norm $\|T\|$ and Hilbert–Schmidt norm $\|T\|_2$, we have $\|T\| \leq \|T\|_2$ (see for instance Theorem 1(ii) in [8] or [22] for details). Then, we have

$$\begin{aligned}\mathrm{Tr}\big(\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1}g_\mu^R}\big]^n\big) &\leq \big\|\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1}g_\mu^R}\big]\big\|^{n-2}\,\mathrm{Tr}\big(\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1}g_\mu^R}\big]^2\big) \\ &\leq \big\|\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1}g_\mu^R}\big]\big\|_2^{n-2}\,\mathrm{Tr}\big(\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1}g_\mu^R}\big]^2\big).\end{aligned}$$

Moreover, we have:

$$
\begin{aligned}
&\big\| \widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1} g_\mu^R}\big]\big\|_2^2 \\
&\quad = \int_{(\mathbb{R}^d\times\mathbb{R}_+)^2} \big|\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1} g_\mu^R}\big]\big((x,r),(y,s)\big)\big|^2\,dx\,dy\,dr\,ds \\
&\quad = \int_{(\mathbb{R}^d\times\mathbb{R}_+)^2} \big(1-e^{-\theta n(\rho)^{-1} g_\mu^R(x,r)}\big)\widehat{K}_\rho\big((x,r),(y,s)\big)^2 \\
&\qquad \times \big(1-e^{-\theta n(\rho)^{-1} g_\mu^R(y,s)}\big)\,dx\,dy\,dr\,ds \\
&\quad = \operatorname{Tr}\big(\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1} g_\mu^R}\big]^2\big),
\end{aligned}
$$

and thus, we obtain, for every $n \geq 2$:

$$
\operatorname{Tr}\big(\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1} g_\mu^R}\big]^n\big) \leq \operatorname{Tr}\big(\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1} g_\mu^R}\big]^2\big)^{n/2}. \qquad \square
$$

Proof of Lemma 2.10. The operator $\widehat{K}_\rho[1-e^{-\theta n(\rho)^{-1} g_\mu^R}]^2$ is an integral operator with kernel

$$
\begin{aligned}
&\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1} g_\mu^R}\big]^2\big((x,r),(y,s)\big) \\
&\quad = \sqrt{1-e^{-\theta n(\rho)^{-1} g_\mu^R(x,r)}}\sqrt{\frac{f(r/\rho)}{\rho}} \\
&\qquad \times \left(\int_{\mathbb{R}^d\times\mathbb{R}_+} \big(1-e^{-\theta n(\rho)^{-1} g_\mu^R(z,t)}\big)\frac{f(t/\rho)}{\rho} K_\rho(x,z)K_\rho(z,y)\,dz\,dt\right) \\
&\qquad \times \sqrt{1-e^{-\theta n(\rho)^{-1} g_\mu^R(y,s)}}\sqrt{\frac{f(s/\rho)}{\rho}}.
\end{aligned}
$$

Its trace is thus given by:

$$
\begin{aligned}
\operatorname{Tr}\big(\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1} g_\mu^R}\big]^2\big) &= \int_{\mathbb{R}^d\times\mathbb{R}_+} \widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1} g_\mu^R}\big]^2\big((x,r),(x,r)\big)\,dx\,dr \\
&= \int_{(\mathbb{R}^d\times\mathbb{R}_+)^2} \big(1-e^{-\theta n(\rho)^{-1} g_\mu^R(x,r)}\big)\big(1-e^{-\theta n(\rho)^{-1} g_\mu^R(z,t)}\big) \qquad (24) \\
&\quad \times \frac{f(r/\rho)}{\rho}\frac{f(t/\rho)}{\rho}\big|K_\rho(x,z)\big|^2\,dx\,dz\,dr\,dt.
\end{aligned}
$$

Since μ has a compact support, the function g_μ^R has also a compact support and $g_\mu^R(x,r) = 0$ for, say, $\|x\| \geq M$. Thus, the integrand in (24) is a positive function with compact support (for θ or ρ small enough). Dealing first with the integral over $\mathbb{R}^d \times \mathbb{R}^d$, since $1-e^{-\theta n(\rho)^{-1} g_\mu^R(x,r)} \leq$

$\theta n(\rho)^{-1}\mu(B(x,r))\mathbf{1}_{\{r\le R\}}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}^d}\big(1-e^{-\theta n(\rho)^{-1}g_\mu^R(x,r)}\big)\big(1-e^{-\theta n(\rho)^{-1}g_\mu^R(z,t)}\big)\big|K_\rho(x,z)\big|^2\,dx\,dz\\
&\quad=\int_{B(0,M)\times B(0,M)}\big(1-e^{-\theta n(\rho)^{-1}g_\mu^R(x,r)}\big)\big(1-e^{-\theta n(\rho)^{-1}g_\mu^R(z,t)}\big)\big|K_\rho(x,z)\big|^2\,dx\,dz\\
&\quad\le\int_{B(0,M)\times B(0,M)}\left(\frac{\theta}{n(\rho)}\right)^2\mu\big(B(x,r)\big)\mu\big(B(z,t)\big)\mathbf{1}_{\{r\le R\}}\mathbf{1}_{\{t\le R\}}K_\rho(x-z)^2\,dx\,dz \qquad (25)\\
&\quad\le\frac{\theta^2}{n(\rho)^2}\mathbf{1}_{\{r\le R\}}\mathbf{1}_{\{t\le R\}}\left(\int_{B(0,M)\times B(0,M)}\mu\big(B(x,r)\big)^2K_\rho(x-z)^2\,dx\,dz\right)^{1/2}\\
&\qquad\times\left(\int_{B(0,M)\times B(0,M)}\mu\big(B(z,t)\big)^2K_\rho(x-z)^2\,dx\,dz\right)^{1/2},
\end{aligned}$$

using the Cauchy–Schwarz inequality. But, with the Fubini theorem, we have

$$\begin{aligned}
\int_{\mathbb{R}^d\times\mathbb{R}^d}\mu\big(B(x,r)\big)^2K_\rho(x-z)^2\,dx\,dz&\le\int_{\mathbb{R}^d}\mu\big(B(x,r)\big)^2\left(\int_{\mathbb{R}^d}K_\rho(x-z)^2\,dz\right)dx\\
&\le C_K\lambda(\rho)C_\mu\big(r^p\wedge r^q\big),
\end{aligned}\qquad (26)$$

since $\mu\in\mathcal{M}_\beta^+$ and using condition (11). Plugging into (25), (26) and a similar bound for the second integral in (25), we have

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}^d}\big(1-e^{-\theta n(\rho)^{-1}g_\mu^R(x,r)}\big)\big(1-e^{-\theta n(\rho)^{-1}g_\mu^R(z,t)}\big)\big|K_\rho(x,z)\big|^2\,dx\,dz\\
&\quad\le C_K\theta^2\frac{\lambda(\rho)}{n(\rho)^2}\mathbf{1}_{\{r\le R\}}\mathbf{1}_{\{t\le R\}}C_\mu r^{q/2}t^{q/2}.
\end{aligned}$$

As a consequence, the bound (24) continues as follows

$$\begin{aligned}
&\operatorname{Tr}\big(\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1}g_\mu^R}\big]^2\big)\\
&\quad\le C_K\theta^2\frac{\lambda(\rho)}{n(\rho)^2}\int_{(\mathbb{R}^+)^2}\mathbf{1}_{\{r\le R\}}\mathbf{1}_{\{t\le R\}}C_\mu r^{q/2}t^{q/2}\frac{f(r/\rho)}{\rho}\frac{f(t/\rho)}{\rho}\,dr\,dt\\
&\quad=C_KC_\mu\theta^2\frac{\lambda(\rho)}{n(\rho)^2}\left(\int_0^R r^{q/2}\frac{f(r/\rho)}{\rho}\,dr\right)^2=C_KC_\mu\theta^2\frac{\lambda(\rho)\rho^q}{n(\rho)^2}\left(\int_0^{R/\rho}r^{q/2}f(r)\,dr\right)^2.
\end{aligned}$$

But since f is integrable and $q\le 2d$ (Definition 2.4) the finite volume condition (8) entails

$$\int_0^{R/\rho}r^{q/2}f(r)\,dr\le C_f:=\int_0^{+\infty}r^{q/2}f(r)\,dr<+\infty.$$

□

We continue following the strategy exposed page 1575. Since the convergence in ρ in Proposition 2.8 is uniform in R, the interchange (16) applies and we obtain:

$$\lim_{\rho\to 0}\mathcal{L}\big(n(\rho)^{-1}\widetilde{M}_\rho(\mu)\big)=\lim_{\rho\to 0}\lim_{R\to+\infty}\mathcal{L}\big(n(\rho)^{-1}\widetilde{M}_\rho^R(\mu)\big)=\lim_{R\to+\infty}\mathcal{L}\big(W^R(\mu)\big).$$

It remains now to identify $\lim_{R\to+\infty}W^R(\mu)$, this is done in the following proposition:

Proposition 2.11. *For all $\mu\in\mathcal{M}_\beta^+$, $W^R(\mu)$ converges in probability when $R\to+\infty$ to*

$$W(\mu)=\int_{\mathbb{R}^d\times\mathbb{R}_+}\mu\big(B(x,r)\big)M_2(dx,dr),$$

where M_2 is the same centered Gaussian random measure as in Theorem 2.7.

Proof. Since $W^R(\mu)$ and $W(\mu)$ are both integral with respect to the same Gaussian measure M_2, we have:

$$W(\mu)-W^R(\mu)=\int_{\mathbb{R}^d\times\mathbb{R}_+}\mu\big(B(x,r)\big)\mathbf{1}_{\{r>R\}}M_2(dx,dr),$$

whose log-Laplace transform is

$$\log\big(\mathbb{E}\big[\exp\big(W(\mu)-W^R(\mu)\big)\big]\big)=\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}_+}\mu\big(B(x,r)\big)^2\mathbf{1}_{\{r>R\}}K(0)r^{-\beta-1}\,dx\,dr. \tag{27}$$

The integrand in (27) converges to 0 when $R\to+\infty$ and is bounded by

$$\mu\big(B(x,r)\big)^2K(0)r^{-\beta-1},$$

which, thanks to Proposition 2.5, is integrable for $\mu\in\mathcal{M}_\beta^+$. The dominated convergence theorem thus ensures that (27) converges to 0, i.e. $W(\mu)-W^R(\mu)\Rightarrow 0$ and $W^R(\mu)\xrightarrow{\mathbb{P}}W(\mu)$, $R\to+\infty$, which is Proposition 2.11. $\square$

So far, all the intermediate results are obtained to prove Theorem 2.7:

Proof of Theorem 2.7. The one-dimensional convergence is obtained by the combination of (16) with Proposition 2.3, Proposition 2.8 and Proposition 2.11. Now, remark that the fields $\widetilde{M}_\rho$ and W are both linear on $\mathcal{M}_\beta^+$. Thus, using the Cramér–Wold device and the linear structure of $\mathcal{M}_\beta$, we have immediately the convergence of the finite-dimensional distributions from the one-dimensional convergence. $\square$

2.2. Intermediate scaling

This section investigates the asymptotic behaviour of $\widetilde{M}_\rho$ in (12) under the intermediate scaling, when $\lim_{\rho\to+\infty}\lambda(\rho)\rho^\beta=a\in]0,+\infty[$. In this section, set $n(\rho)=1$.

Theorem 2.12 (Intermediate scaling asymptotics). *Assume* (7) *and the kernels* K_ρ *satisfy* (10) *and Hypothesis* 1 *for their associated operators* $\mathbf{K}_\rho$ *in* (5). *Suppose* $\lambda(\rho)\rho^\beta \to a^{d-\beta} \in]0, +\infty[$ *when* $\rho \to 0$, *then* $\widetilde{M}_\rho(\cdot)$ *converges in the finite-dimensional distributions sense to* $\widetilde{P} \circ D_a(\cdot)$ *in the space* $\mathcal{M}_\beta^+$, *where*

$$\widetilde{P}(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}_+} \mu\big(B(x,r)\big)\widetilde{\Pi}(dx, dr),$$

with $\widetilde{\Pi}$ *a compensated Poisson random measure on* $\mathbb{R}^d \times \mathbb{R}_+$ *with intensity measure* $K(0)C_\beta \times r^{-\beta-1}\,dx\,dr$ *and* D_a *standing for the dilatation defined by* $(D_a\mu)(B) = \mu(a^{-1}B)$.

Following the same strategy as previously (see page 1575), first investigate the asymptotic behaviour of $\widetilde{M}_\rho^R(\mu)$ in (15) when $\rho \to 0$ and next let $R \to +\infty$ in the obtained limit. Roughly speaking, as in the Poissonian case (see (ii) in Theorem 2.6, or Theorem 2(ii) in [13]), the limit corresponds to take the limit in the intensity of the underlying random measure. The result states as follows:

Proposition 2.13. *Suppose* $\lambda(\rho)\rho^\beta \to a \in]0, +\infty[$ *when* $\rho \to 0$. *Then, for all* $\mu \in \mathcal{M}_\beta^+$ *and* $R > 0$, $\widetilde{M}_\rho^R(\mu)$ *converges in distribution to*

$$\big(\widetilde{P}^R \circ D_a\big)(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}_+} (D_a\mu)\big(B(x,r)\big)\mathbf{1}_{\{r \le R\}}\widetilde{\Pi}(dx, dr),$$

where $\widetilde{\Pi}$ *is the same compensated Poisson random measure as in Theorem* 2.12.

Proof. The proof follows the same scheme as for Proposition 2.8. Recall that in this context, $n(\rho) = 1$ is set. The Laplace transform of $\widetilde{M}_\rho^R(\mu)$ is given by (20), that is,

$$\begin{aligned}\mathbb{E}\big[\exp\big(-\theta\widetilde{M}_\rho^R(\mu)\big)\big] &= \exp\bigg(\int_{\mathbb{R}^d \times \mathbb{R}_+} \psi\big(\theta\mu\big(B(x,r)\big)\mathbf{1}_{\{r \le R\}}\big)K_\rho(x,x)\frac{f(r/\rho)}{\rho}\,dx\,dr\bigg) \\ &\quad \times \exp\bigg(-\sum_{n \ge 2}\frac{1}{n}\operatorname{Tr}\big(\widehat{K}_\rho\big[1 - e^{-\theta g_\mu^R}\big]^n\big)\bigg).\end{aligned} \tag{28}$$

The first exponential in (28) is the Laplace transform of

$$\widetilde{P}_\rho^R(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}_+} \mu\big(B(x,r)\big)\mathbf{1}_{\{r \le R\}}\widetilde{\Pi}_\rho(dx, dr),$$

where $\widetilde{\Pi}_\rho$ is a compensated Poisson random measure on $\mathbb{R}^d \times \mathbb{R}_+$ with intensity measure $\lambda(\rho)K(0)\frac{f(r/\rho)}{\rho}\,dx\,dr$. From (ii) in Theorem 2.6 (see also Theorem 2(i) in [13]), under Condition (7), when $\lim_{\rho\to 0}\lambda(\rho)\rho^\beta = a^{d-\beta} \in]0, +\infty[$, this process converges to

$$\big(\widetilde{P}^R \circ D_a\big)(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}_+} (D_a\mu)\big(B(x,r)\big)\mathbf{1}_{\{r \le R\}}\widetilde{\Pi}(dx, dr),$$

where $\widetilde{\Pi}$ is a compensated Poisson random measure on $\mathbb{R}^d \times \mathbb{R}_+$ with intensity measure $K(0)r^{-\beta-1}\,dx\,dr$. In particular, we have:

$$\lim_{\rho\to 0}\exp\biggl(\int_{\mathbb{R}^d\times\mathbb{R}_+}\psi\bigl(\theta\mu\bigl(B(x,r)\bigr)\mathbf{1}_{\{r\le R\}}\bigr)K_\rho(x,x)\frac{f(r/\rho)}{\rho}\,dx\,dr\biggr)$$
$$= \mathbb{E}\bigl[\exp\bigl(-\theta\bigl(\widetilde{P}^R\circ D_a\bigr)(\mu)\bigr)\bigr].$$

The proof is completed by showing that the second exponential term in (28) converges to 1. Proceeding as in the proof of Proposition 2.8, with $n(\rho)=1$, Lemma 2.10 entails

$$\mathrm{Tr}\bigl(\widehat{K}_\rho\bigl[1-e^{-\theta g_\mu^R}\bigr]^2\bigr)\le C_K C_\mu C_f\theta^2\lambda(\rho)\rho^q,$$

which goes to 0 since $\lim_{\rho\to 0}\lambda(\rho)\rho^q=0$ for $q>\beta$. As a consequence

$$\lim_{\rho\to 0}\mathrm{Tr}\bigl(\widehat{K}_\rho\bigl[1-e^{-\theta g_\mu^R}\bigr]^2\bigr)=0.$$

Then, with Lemma 2.9, we still have for every $n\ge 2$

$$\mathrm{Tr}\bigl(\widehat{K}_\rho\bigl[1-e^{-\theta g_\mu^R}\bigr]^n\bigr)\le \mathrm{Tr}\bigl(\widehat{K}_\rho\bigl[1-e^{-\theta g_\mu^R}\bigr]^2\bigr)^{n/2},$$

and the second exponential term in (28) converges to 1, as in the proof of Proposition 2.8, page 1582, this concludes the proof of Proposition 2.13. □

Combining Proposition 2.13 with the interchange (16), we have:

$$\lim_{\rho\to 0}\mathcal{L}\bigl(\widetilde{M}_\rho(\mu)\bigr)=\lim_{R\to+\infty}\lim_{\rho\to 0}\mathcal{L}\bigl(\widetilde{M}_\rho^R(\mu)\bigr)=\lim_{R\to+\infty}\mathcal{L}\bigl(\widetilde{P}^R(\mu)\bigr).$$

It remains now to identify $\lim_{R\to+\infty}\widetilde{P}^R(\mu)$, this is done in the following proposition:

Proposition 2.14. *For all $\mu\in\mathcal{M}_\beta^+$, $\widetilde{P}^R(\mu)$ converges in L^1 when $R\to+\infty$ to*

$$\widetilde{P}(\mu)=\int_{\mathbb{R}^d\times\mathbb{R}_+}\mu\bigl(B(x,r)\bigr)\widetilde{\Pi}(dx,dr),$$

where $\widetilde{\Pi}$ is the same compensated Poisson random measure as in Theorem 2.12.

Proof. Since $\widetilde{P}^R(\mu)$ and $\widetilde{P}(\mu)$ are Poissonian integrals with respect to the same measure $\widetilde{\Pi}$, we have:

$$\bigl|\widetilde{P}^R(\mu)-\widetilde{P}(\mu)\bigr|$$
$$=\biggl|\int_{\mathbb{R}^d\times\mathbb{R}_+}\mu\bigl(B(x,r)\bigr)\mathbf{1}_{\{r>R\}}\widetilde{\Pi}(dx,dr)\biggr|,$$

and

$$\begin{aligned}\mathbb{E}\big[\big|\widetilde{P}^R(\mu)-\widetilde{P}(\mu)\big|\big] &\le 2\int_{\mathbb{R}^d\times\mathbb{R}_+}\mu\big(B(x,r)\big)\mathbf{1}_{\{r>R\}}K(0)r^{-\beta-1}\,dx\,dr\\ &\le 2v_d\mu\big(\mathbb{R}^d\big)K(0)\int_R^{+\infty}r^{d-\beta-1}\,dr\\ &=\frac{2v_d\mu(\mathbb{R}^d)K(0)}{(\beta-d)R^{\beta-d}}\longrightarrow 0,\qquad R\to+\infty.\end{aligned}$$

□

So far, all the intermediate results are obtained to prove Theorem 2.12:

Proof of Theorem 2.12. The one-dimensional convergence is obtained by the combination of (16) with Proposition 2.3, Proposition 2.13 and Proposition 2.14. Since the fields $\widetilde{M}_\rho$ and $\widetilde{P}$ are both linear on $\mathcal{M}_\beta^+$, using the Cramér–Wold device and the linear structure of $\mathcal{M}_\beta$, we have immediately the convergence of the finite-dimensional distributions from the one-dimensional convergence. □

2.3. Small-balls scaling

This section investigates the asymptotics of $\widetilde{M}_\rho^R(\mu)$ under the small-balls scaling, that is, when $\lim_{\rho\to 0}\lambda(\rho)\rho^\beta=0$. In this section, set $n(\rho)=(\lambda(\rho)\rho^\beta)^{1/\gamma}$ with $\gamma:=\beta/d\in]1,2[$. We deal first with the limit in ρ of the truncated field $\widetilde{M}_\rho^R(\mu)$. In this case, the obtained limit does not depend on R, roughly speaking this is due to the fast decreasing of the rescaled radii ρr since $\rho\to 0$ very fast in this regime. The limiting field thus obtained is a stable integral similar to the one obtained for the Poissonian model in (iii) of Theorem 2.6 (cf. also Theorem 2(iii) in [13] and cf. [23] for notations on stable integrals). In this case, the limit is driven by small balls and this requires to consider smooth measure $\mu(dx)=\varphi(x)\,dx$. Roughly speaking, if the measure μ were, for instance, atomic, there will be a possibility for the small balls driving the asymptotics to not charge μ and $M(\mu)$ would vanish.

Theorem 2.15. *Assume (7) and the kernels K_ρ satisfy (10) and Hypothesis 1 for their associated operators $\mathbf{K}_\rho$ in (5). Suppose $\lambda(\rho)\rho^\beta\to 0$ when $\rho\to 0$. Then, the field $n(\rho)^{-1}\widetilde{M}_\rho(\cdot)$ converges in the finite-dimensional distributions sense when $\rho\to 0$ to $Z(\cdot)$ in $L_c^2(\mathbb{R}^d)$ where*

$$Z(\mu)=\int_{\mathbb{R}^d}\varphi(x)M_\gamma(dx),\qquad \textit{for } \mu(dx)=\varphi(x)\,dx,$$

with M_γ a γ-stable measure with control measure $\sigma_\gamma\,dx$ where

$$\sigma_\gamma=\frac{K(0)C_\beta v_d^\gamma}{d}\int_0^{+\infty}\frac{1-\cos(r)}{r^{1+\gamma}}\,dr,$$

and constant unit skewness.

First, we have the following proposition.

Proposition 2.16. *Suppose $\lambda(\rho)\rho^\beta \to 0$ when $\rho \to 0$ and set $n(\rho) = (\lambda(\rho)\rho^\beta)^{1/\gamma}$. Then, for all $R > 0$ and for all $\mu \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, writing $\mu(dx) = \varphi(x)\,dx$, $n(\rho)^{-1}\widetilde{M}_\rho^R(\mu)$ converges in the finite-dimensional distributions sense when $\rho \to 0$ to*

$$Z(\mu) = \int_{\mathbb{R}^d} \varphi(x) M_\gamma(dx),$$

where M_γ is the same γ-stable measure as in Theorem 2.15.

Proof. Recall the Laplace transform of $\widetilde{M}_\rho^R(\mu)$ is given in (20):

$$\begin{aligned}\mathbb{E}\big[\exp\big(-\theta n(\rho)^{-1}\widetilde{M}_\rho^R(\mu)\big)\big] &= \exp\bigg(\int_{\mathbb{R}^d\times\mathbb{R}_+} \psi\big(\theta n(\rho)^{-1} g_\mu^R\big) K_\rho(x,x) \frac{f(r/\rho)}{\rho}\,dx\,dr\bigg)\\ &\quad\times \exp\bigg(-\sum_{n\geq 2}\frac{1}{n}\operatorname{Tr}\big(\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1} g_\mu^R}\big]^n\big)\bigg).\end{aligned}$$

The first exponential term is still the Laplace transform of $n(\rho)^{-1}\widetilde{P}_\rho(\mu)$ where $\widetilde{P}_\rho(\mu)$ is the compensated Poissonian integral (21). With the change of variable $r = n(\rho)^{1/d}s$, this log-Laplace transform becomes:

$$\begin{aligned}&\int_{\mathbb{R}^d\times\mathbb{R}_+} \psi\big(\theta n(\rho)^{-1}\mu\big(B\big(x, n(\rho)^{1/d}s\big)\big)\mathbf{1}_{\{s<n(\rho)^{-1/d}R\}}\big)\\ &\qquad\times \lambda(\rho)K(0)n(\rho)^{1/d}\,\frac{f(sn(\rho)^{1/d}/\rho)}{\rho}\,dx\,ds.\end{aligned} \tag{29}$$

For $\mu(dx) = \varphi(x)\,dx$ with $\varphi \in L_c^2(\mathbb{R}^d)$, then the following Lemma from [13] entails

$$\lim_{\rho\to 0}\theta n(\rho)^{-1}\mu\big(B\big(x, n(\rho)^{1/d}s\big)\big)\mathbf{1}_{\{s<n(\rho)^{-1/d}R\}} = \theta\varphi(x)v_d s^d,$$

dx-almost everywhere and

$$x \mapsto \sup_{r>0}\bigg(\frac{\mu(B(x,r))}{v_d r^d}\bigg) \in L^2\big(\mathbb{R}^d\big).$$

Lemma 2.17 (Lemma 4 in [13]). *Let C be a bounded Borelian set in $\mathbb{R}^d$ with Lebesgue measure $|C| = 1$.*

(i) *If $\varphi \in L^1$, then $\lim_{v\to 0} v^{-1}\int_{x+v^{1/d}C}\varphi(y)\,dy = \varphi(x)$ for dx-almost all x.*
(ii) *If $\varphi \in L^1$, then $\varphi_*(x) := \sup_{v>0} v^{-1}\int_{x+v^{1/d}C}|\varphi(y)|\,dy < +\infty$ for dx-almost all x.*
(iii) *Moreover if $\varphi \in L^p$ for some $p > 1$ then $\varphi_* \in L^p$.*

Then, using the very argument of the proof of Theorem 2 in [13] (see also the proof of Theorem 2.16 in [4])

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}_+} \psi\big(\theta n(\rho)^{-1}\mu\big(B\big(x, n(\rho)^{1/d} r\big)\big)\mathbf{1}_{\{r<n(\rho)^{-1/d}R\}}\big) \\
&\qquad \times \lambda(\rho)K(0)n(\rho)^{1/d}\frac{f(rn(\rho)^{1/d}/\rho)}{\rho}\,dx\,dr \qquad (30) \\
&\underset{\rho\to 0}{\sim} \lambda(\rho)K(0)\int_{\mathbb{R}^d\times\mathbb{R}_+} \psi\big(\theta\varphi(x)v_d r^d\big)n(\rho)^{1/d}\frac{f(rn(\rho)^{1/d}/\rho)}{\rho}\,dx\,dr.
\end{aligned}$$

Using now the proof of Theorem 2 in [13] under the small-ball scaling, the right-hand side in (29) converges to the Laplace transform of $Z(\mu)$. This implies that the random variable $n(\rho)^{-1}\widetilde{P}_\rho(\mu)$ converges in distribution to $Z(\mu)$.

The proof is completed by showing that the second exponential term in (29) converges to 1. Using the same conclusion as in the proof of Proposition 2.8 page 1582 with Lemma 2.9, it is enough to show that for this regime we still have

$$\lim_{\rho\to 0}\operatorname{Tr}\big(\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1}g_\mu^R}\big]^2\big)=0.$$

Since we consider $\mu\in L_c^2(\mathbb{R}^d)$, we have also $\mu\in L^1(\mathbb{R}^d)$ and Proposition 2.5(ii) ensures that we can take here $q=2d$ and then Lemma 2.10 writes with $n(\rho)=(\lambda(\rho)\rho^\beta)^{1/\gamma}$:

$$\operatorname{Tr}\big(\widehat{K}_\rho\big[1-e^{-\theta n(\rho)^{-1}g_\mu^R}\big]^2\big)\le C_K C_\mu C_f\theta^2\frac{\lambda(\rho)\rho^{2d}}{n(\rho)^2}=C_K C_\mu C_f\theta^2\lambda(\rho)^{(\beta-2d)/\beta},$$

which goes to 0 when $\rho\to 0$ since $\beta<2d$. □

So far, all the intermediate results are obtained to finish the proof of Theorem 2.15 as for Theorem 2.7 and Theorem 2.12.

3. Comments

3.1. Zoom-in asymptotics

For the Poisson random balls model, the study of the microscopic fluctuations obtained in [2] by zooming-in instead of zooming-out, leads to very similar results to those obtained in the macroscopic behaviour in [13] under the large-ball scaling and the intermediate scaling. This similarity is the origin of the unified approach for both types of scaling in [3], used also in the weighted model in [4]. In the microscopic point of view, this is the behaviour of small balls which matters and this is encapsulated in [3] in the following condition on small radii:

$$f(r)\sim_{r\to 0}\frac{1}{r^{\beta+1}}.$$

In this case, f cannot be a probability density nor be integrable. Consequently, we cannot study a determinantal random balls model under a zoom-in procedure. Indeed, even if we were to consider a marked DPP on $\mathbb{R}^d \times \mathbb{R}_+$ with kernel

$$\widehat{K}\big((x,r),(y,s)\big) = \sqrt{f(r)}K(x,y)\sqrt{f(s)}, \tag{31}$$

where K is a determinantal kernel on $\mathbb{R}^d$ and f is a function on $\mathbb{R}_+$ satisfying condition (7), this DPP would have no chance to satisfy Hypothesis 1 when f is not integrable.

3.2. α-determinantal and α-permanental processes

The DPPs actually belong to a larger class of point processes, the so-called α-determinantal/permanental processes. When $\alpha > 0$, such processes exhibit attraction between their particles, and when $\alpha < 0$, they exhibit repulsiveness. When $\alpha = -1$, the (usual) DPPs are recovered while the case $\alpha = 1$ corresponds to permanental processes. The definitions of α-determinantal/permanental processes follow the same lines as in Def. A.2 but with the determinant replaced by an α-determinant. Recall that for a matrix $A = (a_{i,j})_{1\le i,j\le n}$ and $\alpha \in \mathbb{R}$, its α-determinant is defined by

$$\det_\alpha A = \sum_{\sigma\in\mathfrak{S}_n} \alpha^{n-\nu(\sigma)} \prod_{i=1}^{n} a_{i,\sigma(i)}, \tag{32}$$

where $\mathfrak{S}_n$ is the symmetric group of permutation of $\{1,\dots,n\}$ and $\nu(\sigma)$ is the number of cycles in $\sigma \in \mathfrak{S}_n$. When $\alpha = -1$ (resp. $\alpha = 1$), (32) defines the (standard) determinant (resp. permanent) of A: $\det_{-1} A = \det A$, $\det_1 A = \operatorname{perm} A$.

The following result from [24] extends Theorem A.4 and proves the existence of such processes for some α's and it gives their Laplace transform:

Theorem 3.1 (Th. 1.2 in [24]). *Let E be a Polish space equipped with a diffuse Radon measure λ and K be a bounded symmetric integral operator on $L^2(E,\lambda)$ satisfying Hypothesis 1. Then for $\alpha \in \{2/m : m \in \mathbb{N}\} \cup \{-1/m : m \in \mathbb{N}\}$, there exists a unique point process ϕ such that*

$$\mathbb{E}\bigg[\exp\bigg(-\int f(x)\phi(dx)\bigg)\bigg] = \operatorname{Det}\big(I + \alpha K\big[1 - e^{-f}\big]\big)^{-1/\alpha}, \tag{33}$$

for each compactly supported measurable $f : E \to \mathbb{R}_+$ where $K[1 - e^{-f}]$ still stands for the kernel (40). Moreover, ϕ is a simple point process whose joint intensities are given by

$$\rho_{n,\alpha,K}(x_1, x_2, \dots, x_n) = \det_\alpha\big(\big(K(x_i,x_j)\big)_{1\le i,j\le n}\big).$$

Like for (38) below in the Appendix (for $\alpha = -1$), for a trace-class operator T with $\|\alpha T\| < 1$, the Fredholm determinant of $I - \alpha T$ expands in terms of α-determinant

$$\operatorname{Det}(I - \alpha T)^{-1/\alpha} = \sum_{n=0}^{+\infty} \frac{1}{n!} \int_{E^n} \det_\alpha\big(\big(T(x_i,x_j)\big)_{1\le i,j\le n}\big)\lambda^{\otimes n}(dx_1,\dots,dx_n).$$

Using the expansion (37) of the Fredholm determinant of the Laplace transform (33), our arguments can be carried out similarly for α-determinantal/permanental processes. Indeed, since $|\alpha| \leq 1$, the terms for $n \geq 2$ can be similarly bounded and are still asymptotically negligible while the term $n = 1$ is obviously the same Poissonian term. As a consequence, Theorems 2.7, 2.12, 2.15 have natural generalizations to α-determinantal/permanental processes.

3.3. Non-stationary determinantal random balls model

With slight modifications, our main results remain true for non-stationary determinantal random balls models. Consider a determinantal process ϕ with kernel $K(x, y)$ still satisfying Hypothesis 1 but also

$$x \longmapsto K(x, x) \in L^\infty(\mathbb{R}^d). \tag{34}$$

The zoom-out procedure consists now in introducing the family of DPPs ϕ_ρ, $\rho \in]0, 1]$, with kernels K_ρ with respect to the Lebesgue measure satisfying

$$K_\rho(x, x) \underset{\rho\to 0}{\sim} \lambda(\rho) K(x, x),$$

with $\lim_{\rho\to 0} \lambda(\rho) = +\infty$. We also replace (10) by

$$\sup_{x\in\mathbb{R}^d} K_\rho(x, x) \leq \lambda(\rho) \sup_{x\in\mathbb{R}^d} K(x, x), \tag{35}$$

and observe that with (34), (35) and (41), we can replace (11) by

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \left|K_\rho(x, y)\right|^2 dy \underset{\rho\to 0}{=} \mathcal{O}\big(\lambda(\rho)\big). \tag{36}$$

In this non-stationary context, Theorem 2.7, 2.12, 2.15 have the following counterparts:

Theorem 3.2. *Assume (7) and ϕ_ρ is a DPP with kernel satisfying (34), (35) and Hypothesis 1 for its associated operator $\mathbf{K}_\rho$ in (5).*

(i) *Large-balls scaling: Assume $\lambda(\rho)\rho^\beta \to +\infty$ and set $n(\rho) = (\lambda(\rho)\rho^\beta)^{1/2}$. Then, $\widetilde{M}_\rho(\cdot)/n(\rho)$ converges in the fdd sense on $\mathcal{M}_\beta^+$ to W where*

$$W(\mu) = \int_{\mathbb{R}^d\times\mathbb{R}_+} \mu\big(B(x, r)\big) M_2(dx, dr),$$

and M_2 is a centered Gaussian random measure with control measure $K(x, x)C_\beta r^{-\beta-1}\,dx\,dr$.

(ii) *Intermediate scaling: Assume $\lambda(\rho)\rho^\beta \to a^{d-\beta} \in]0, +\infty[$ and set $n(\rho) = 1$. Then, $\widetilde{M}_\rho(\cdot)/n(\rho)$ converges in the fdd sense on $\mathcal{M}_\beta^+$ to $\widetilde{P} \circ D_a$ where*

$$\widetilde{P}(\mu) = \int_{\mathbb{R}^d\times\mathbb{R}_+} \mu\big(B(x, r)\big) \widetilde{\Pi}(dx, dr),$$

with $\widetilde{\Pi}$ *a (compensated) PPP with compensator measure* $K(x,x)C_\beta r^{-\beta-1}\,dx\,dr$ *and* D_a *is the dilatation defined by* $(D_a\mu)(B) = \mu(a^{-1}B)$.

(iii) *Small-balls scaling: Suppose* $\lambda(\rho)\rho^\beta \to 0$ *when* $\rho \to 0$ *and set* $n(\rho) = (\lambda(\rho)\rho^\beta)^{1/\gamma}$ *with* $\gamma = \beta/d$. *Then, the field* $n(\rho)^{-1}\widetilde{M}_\rho(\cdot)$ *converges in the finite-dimensional distributions sense when* $\rho \to 0$ *to* $Z(\cdot)$ *in* $L^2_c(\mathbb{R}^d)$ *where*

$$Z(\mu) = \int_{\mathbb{R}^d} \varphi(x) M_\gamma(dx), \qquad \text{for } \mu(dx) = \varphi(x)\,dx,$$

with M_γ *a* γ*-stable measure with control measure* $\sigma_\gamma K(x,x)\,dx$ *where*

$$\sigma_\gamma = \frac{C_\beta v_d^\gamma}{d} \int_0^{+\infty} \frac{1-\cos(r)}{r^{1+\gamma}}\,dr,$$

and constant unit skewness.

In this non-stationary case, the proof follows the same general strategy as in page 1575 but with technical details requiring (34), (35), (36). Roughly speaking, the limits are driven by the term $n = 1$ in (19) while the other terms ($n \geq 2$) are still negligible. Note that, in this non-stationary setting, the Poissonian limits for $n = 1$ come now from [9] (with $G = \delta_1$ therein) instead of [13]. Details are left to the interested readers.

Appendix: (Marked) determinantal point processes

In this section, we give a short presentation of determinantal point processes (DPPs). For a general reference on point processes, we refer to the two volumes book [6] and for a specific reference on DPPs, we refer to [12] and references therein. DPPs form a special class of point processes that exhibit repulsiveness between their points. Recall that, by definition, a point process ξ is a random locally finite collection of points. As it is customary done, we identify such random collection ξ with the corresponding random counting measure $\sum_{x\in\xi}\delta_x$. Below, we consider a point process ξ in, say, some Polish space E. In the sequel, to avoid any ambiguity, the points of the process are called particles. In the following, simple point processes, for which almost surely its particles are all distinct, are considered. Considering a reference Borel measure μ on E, the distribution law of ξ is, in general, characterized by its joint intensities.

Definition A.1. Let ξ be a point process on a Polish space E equipped with a measure μ. If there are functions $\rho_k : E \to [0, +\infty[$, $k \geq 1$, such that for any family of mutually disjoint Borelian subsets $D_1, \ldots, D_k$ of E:

$$\mathbb{E}\left[\prod_{i=1}^k \xi(D_i)\right] = \int_{\prod_{i=1}^k D_i} \rho_k(x_1,\ldots,x_k)\mu(dx_1)\cdots\mu(dx_k),$$

we call them joint intensities with respect to μ. Moreover, we require $\rho_k(x_1,\ldots,x_k) = 0$ whenever $x_i = x_j$ for some $i \neq j$.

Roughly speaking, $\rho_k(x_1, \dots, x_k)$ can be interpreted as the (infinitesimal) probability for ξ to have particles in each $x_1, \dots, x_k$. For example, for a homogeneous Poisson point process (PPP), the joint intensities are constant while for a general (but diffuse) PPP with intensity function λ, we have $\rho_k(x_1, \dots, x_k) = \lambda(x_1) \cdots \lambda(x_k)$. For a DPP, the joint intensities are given by a certain determinant of a measurable function $K : E^2 \to \mathbb{R}$, called its kernel and characterizing the process, hence its name.

Definition A.2. A point process ξ on E is said to be a determinantal point process with kernel K if it is simple and its joint intensities write for all $k \geq 1$ and all $x_1, \dots, x_k \in E$:

$$\rho_k(x_1, \dots, x_k) = \det\big(K(x_i, x_j)\big)_{1 \leq i,j \leq k} := \det[K](x_1, \dots, x_k).$$

See Theorem A.4 below for conditions ensuring the existence of such processes. Observe that the repulsiveness exhibited by a DPP can be read on its joint intensity of second order. Indeed, if K is continuous and $x_1, x_2 \in E$, the more they will be close to each other, the more the determinant of $(K(x_i, x_j))_{1 \leq i,j \leq 2}$ will be close to 0. Thus, $\rho_2(x_1, x_2) \approx 0$ whenever $x_1 \approx x_2$. This implies that, if there is a particle of the process in x_1, the probability that there is another particle in the close vicinity of x_1 is small. For a homogeneous PPP, the constant intensities show that the particles are independently drawn.

An important class of DPPs is the class of those whose kernel satisfies special properties (see Hypothesis 2 below). Note that Hypothesis 1 – the basic hypothesis of our setting – is a specialization of Hypothesis 2 in our setting ($E = \mathbb{R}^d$, $\mu := dx$). For that purpose, recall that, for a compact operator T on a separable Hilbert space H equipped with the scalar product $\langle \cdot, \cdot \rangle$, its trace is given by

$$\mathrm{Tr}(T) = \sum_{n=1}^{+\infty} \langle T e_n, e_n \rangle,$$

where $(e_n)_{n \geq 1}$ is (any) complete orthonormal (CONB in shorts) system of H. In particular, T is said to be a trace-class operator if

$$\|T\|_1 := \mathrm{Tr}\big(|T|\big) < +\infty,$$

where $|T| = \sqrt{T^* T}$. The hypothesis on the kernel K writes (see Assumption 4.2.3 in [12] or Condition A in [24]):

Hypothesis 2. *The Polish space E is equipped with a Radon σ-finite measure λ. The map $\mathbf{K}$ is an operator from $L^2(E, \lambda)$ into $L^2(E, \lambda)$ satisfying the following conditions*:

(i) $\mathbf{K}$ *is a bounded symmetric integral operator on $L^2(E, \lambda)$ with kernel K, i.e., for any $x \in E$ and any $f \in L^2(E, \lambda)$,*

$$\mathbf{K} f(x) = \int_E K(x, y) f(y) \lambda(dy).$$

(ii) *The spectrum of* $\mathbf{K}$ *is included in* $[0, 1[$.

(iii) *The map* $\mathbf{K}$ *is locally trace-class, i.e. for all compact* $\Lambda \subset E$, *the restriction* $\mathbf{K}_\Lambda$ *of* $\mathbf{K}$ *on* $L^2(\Lambda, \lambda)$ *is of trace-class.*

Remark A.3. If K is the kernel of a map $\mathbf{K}$ satisfying Hypothesis 2, then $x \mapsto K(x, x)$ is nonnegative.

In our argument, the limit in distribution of quantities (4) is investigated by considering the Laplace transform of a DPP. It is given in Theorem A.4 below from [24] and expressed in terms of Fredholm determinant. Recall that if T is a trace-class operator with $\|T\| < 1$, the Fredholm determinant of $I + T$ is given by

$$\mathrm{Det}(I + T) = \exp\left(\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} \mathrm{Tr}\big(T^n\big)\right) \tag{37}$$

(see [24], Lemma 2.1 iii)). Moreover, the following expansion ([24], Th. 2.4) in terms of determinants hold:

$$\mathrm{Det}(I + T) = \sum_{n=0}^{+\infty} \frac{1}{n!} \int_{E^n} \det\big(\big(T(x_i, x_j)\big)_{1 \le i, j \le n}\big) \lambda^{\otimes n}(dx_1, \ldots, dx_n). \tag{38}$$

Theorem A.4 (Th. 1.2 in [24]). *Let* E *be a Polish space equipped with a diffuse Radon measure* λ *and* K *be a bounded symmetric integral operator on* $L^2(E, \lambda)$ *satisfying Hypothesis 2. Then there exists a unique DPP* ϕ *as in Definition A.2 and its Laplace transform is given for each compactly supported measurable* $f : E \to \mathbb{R}_+$ *by*

$$\mathbb{E}\left[\exp\left(-\int f(x)\phi(dx)\right)\right] = \mathrm{Det}\big(I - K\big[1 - e^{-f}\big]\big), \tag{39}$$

where $K[1 - e^{-f}]$ *stands for the kernel*

$$K\big[1 - e^{-f}\big](x, y) = \sqrt{1 - \exp\big(-f(x)\big)}\, K(x, y) \sqrt{1 - \exp\big(-f(y)\big)}. \tag{40}$$

The following result is obtained by differentiation of the Laplace transform.

Proposition A.5. *Let* ϕ *be a DPP on a Polish space* E *with kernel* K *satisfying Hypothesis 2 with respect to a measure* λ *on* E. *For any compact set* Λ *of* E *and any non-negative function* f *defined on* E, *we have*

$$\mathbb{E}\left[\int_\Lambda f\, d\phi\right] = \int_\Lambda f(x) K(x, x) \lambda(dx) = \mathrm{Tr}\big(K_\Lambda[f]\big).$$

The following control of the kernel K has some importance in our setting (see (11) above). For the shake of completeness, we provide its proof (see also Lemma 3.2 in [21]).

Proposition A.6. *Let ϕ be a DPP with kernel K satisfying Hypothesis 2. Then, for all $x \in E$,*

$$\int_E \left|K(x,y)\right|^2 \lambda(dy) \leq K(x,x). \tag{41}$$

Proof. Let $x \in E$ be fixed and C a compact set containing x. The restriction of K_C of K on C has the following spectral expansion:

$$K_C(y,z) = \sum_{i=1}^{+\infty} \kappa_{C,i} \varphi_{C,i}(y) \varphi_{C,i}(z), \qquad y, z \in E,$$

where $\kappa_{C,i} \in [0, 1[$ and $\varphi_{C,i}$, $i \geq 1$, are the non-zero eigenvalues and corresponding orthonormal eigenfunctions of the trace-class operator $\mathbf{K}_C$ (Hyp. 2). Then, we have

$$\int_C \left|K(x,y)\right|^2 \lambda(dy) = \sum_{i=1}^{+\infty} \kappa_{C,i}^2 \varphi_{C,i}(x)^2 \leq \sum_{i=1}^{+\infty} \kappa_{C,i} \varphi_{C,i}(x)^2 = K_C(x,x) = K(x,x),$$

using both $\kappa_{C,i} \in [0, 1[$ and $x \in C$. The conclusion (41) follows by convergence monotone when $C \uparrow E$. □

In Section 1, marked determinantal point processes are considered and, for that purpose some useful results on marked DPPs are given in the rest of this section. First, the following classical result on PPPs (see for instance Lemma 6.4.VI in [6]) is easily extended: *If $\xi = \{X_i\}_{i \geq 1}$ is a PPP on a Polish space E with intensity $\lambda \in \mathbb{R}_+$ and $(R_i)_{i \geq 1}$ is a family of i.i.d. random variables with distribution F on a Polish space E' (independent of ξ), then $\xi' = (X_i, R_i)_{i \geq 1}$ is a PPP on $E \times E'$ with intensity $\lambda \otimes F$.* In the determinantal case, we have the following proposition.

Proposition A.7. *Let $\phi = (X_i)_{i \geq 1}$ be a determinantal point process on a Polish space E with kernel K, with respect to a Radon measure λ, and let $(R_i)_{i \geq 1}$ be a family of i.i.d. random variables on $\mathbb{R}_+$, independent of $(X_i)_{i \geq 1}$, with probability density f. Let $\Phi = \{(X_i, R_i)\}_{i \geq 1}$. Then, Φ is a determinantal point process on $E \times \mathbb{R}_+$ with kernel*

$$\widehat{K}\big((x,r),(y,s)\big) = \sqrt{f(r)} K(x,y) \sqrt{f(s)}, \tag{42}$$

with respect to the measure $\lambda(dx)\, dr$.

The result still holds true for marks with values in a Polish space but in the sequel, only positive marks are used (i.e., $R_i \in \mathbb{R}_+$).

Proof. To prove that Φ is a DPP with kernel $\widehat{K}$, the joint intensities are shown to write

$$\hat{\rho}_n\big((x_1,r_1), \dots, (x_n,r_n)\big) = \det\Big(\widehat{K}\big((x_i,r_i),(x_j,r_j)\big)_{1 \leq i,j \leq n}\Big).$$

For all $n \geq 1$ and all set A, the symbol $\sum_{a_1,\dots,a_n \in A}^{\neq}$ will stand for the sum over all n-tuples $(a_1, \dots, a_n) \in A$ with pairwise distinct a_i ($a_i \neq a_j$ for $i \neq j$ in $\{1, \dots, n\}$). Let $n \geq 1$ and h a

Borel function from $(E\times\mathbb{R}_+)^n$ to $\mathbb{R}_+$. We have:

$$
\begin{aligned}
&\mathbb{E}\Bigg[\sum_{(x_1,r_1),\dots,(x_n,r_n)\in\Phi}^{\neq} h\big((x_1,r_1),\dots,(x_n,r_n)\big)\Bigg]\\
&\quad=\mathbb{E}\Bigg[\sum_{x_1,\dots,x_n\in\phi}^{\neq} h\big((x_1,R_1),\dots,(x_n,R_n)\big)\Bigg]\\
&\quad=\mathbb{E}\Bigg[\mathbb{E}\Bigg[\sum_{x_1,\dots,x_n\in\phi}^{\neq} h\big((x_1,R_1),\dots,(x_{i_n},R_{i_n})\big)|\phi\Bigg]\Bigg]\\
&\quad=\mathbb{E}\Bigg[\int_{(\mathbb{R}_+)^n}\sum_{x_1,\dots,x_n\in\phi}^{\neq} h\big((x_1,r_1),\dots,(x_n,r_n)\big)\prod_{1\le i\le n} f(r_i)\,dr_i\Bigg]\\
&\quad=\mathbb{E}\Bigg[\sum_{x_1,\dots,x_n\in\phi}^{\neq}\int_{(\mathbb{R}_+)^n} h\big((x_1,r_1),\dots,(x_n,r_n)\big)\prod_{1\le i\le n} f(r_i)\,dr_i\Bigg]\\
&\quad=\int_{(E\times\mathbb{R}_+)^n} h\big((x_1,r_1),\dots,(x_n,r_n)\big)\prod_{1\le i\le n} f(r_i)\rho_n(x_1,\dots,x_n)\lambda(dx_1)\,dr_1\dots\lambda(dx_n)\,dr_n,
\end{aligned}
$$

where $\rho_n(x_1,\dots,x_n)=\mathrm{Det}[K](x_1,\dots,x_n)$ is the joint intensity of order n of the DPP ϕ. Now, note that

$$
\prod_{1\le i\le n} f(r_i)\,\mathrm{Det}[K](x_1,\dots,x_n)=\mathrm{Det}[\widehat{K}]\big((x_1,r_1),\dots,(x_n,r_n)\big),
$$

where $\widehat{K}$ is given in (42). Then

$$
\begin{aligned}
&\mathbb{E}\Bigg[\sum_{(x_1,r_1),\dots,(x_n,r_n)\in\Phi}^{\neq} h\big((x_1,r_1),\dots,(x_n,r_n)\big)\Bigg]\\
&\quad=\int_{(E\times\mathbb{R}_+)^n} h\big((x_1,r_1),\dots,(x_n,r_n)\big)\,\mathrm{Det}[\widehat{K}]\big((x_1,r_1),\dots,(x_n,r_n)\big)\prod_{1\le i\le n}\lambda(dx_i)\,dr_i,
\end{aligned}
$$

and, according to Definition A.1 and Definition A.2, Φ is a DPP on $E\times\mathbb{R}_+$ with kernel $\widehat{K}$ with respect to the measure $\lambda(dx)\,dr$. □

Next, in the case where $\mathbf{K}$ satisfies Hypothesis 2, the operator $\widehat{\mathbf{K}}$ associated to $\widehat{K}$ defined in (42) above inherits these properties.

Proposition A.8. *Let* $\mathbf{K}$ *be an operator on* $L^2(E,\lambda)$ *satisfying Hypothesis* 2 *and* $\widehat{\mathbf{K}}$ *be the integral operator with kernel* (42) *with probability density* f. *Then,* $\widehat{\mathbf{K}}$ *satisfies Hypothesis* 2.

Proof. We show that each point of Hypothesis 2 is satisfied.

(i) $\widehat{\mathbf{K}}$ is obviously a symmetric integral operator and it is bounded since it is an Hilbert–Schmidt operator.

(ii) Let $\gamma \in [0, 1[$ be in the spectrum of $\widehat{\mathbf{K}}$ and g_γ an associated eigenfunction. Then,

$$\begin{aligned}\gamma g_\gamma(x,r) &= \widehat{\mathbf{K}} g_\gamma(x,r)\\ &= \int_{E\times\mathbb{R}_+} \sqrt{f(r)}K(x,y)\sqrt{f(s)}g_\gamma(y,s)\lambda(dy)\,ds\\ &= \sqrt{f(r)}\int_E K(x,y)\int_{\mathbb{R}_+}\sqrt{f(s)}g_\gamma(y,s)\,ds\lambda(dy)\\ &= \sqrt{f(r)}K\left(\int_{\mathbb{R}_+}\sqrt{f(s)}g_\gamma(\cdot,s)\,ds\right)(x).\end{aligned}$$

Thus, since f is a probability density,

$$\begin{aligned}\gamma\int_{\mathbb{R}_+}\sqrt{f(r)}g_\gamma(x,r)\,dr &= \int_{\mathbb{R}_+} f(r)K\left(\int_{\mathbb{R}_+}\sqrt{f(s)}g_\gamma(\cdot,s)\,ds\right)(x)\,dr\\ &= \int_{\mathbb{R}_+} f(r)\,dr\,K\left(\int_{\mathbb{R}_+}\sqrt{f(s)}g_\gamma(\cdot,s)\,ds\right)(x)\\ &= K\left(\int_{\mathbb{R}_+}\sqrt{f(s)}g_\gamma(\cdot,s)\,ds\right)(x),\end{aligned}$$

proving that γ is in the spectrum of K (associated to the eigenfunction $x \mapsto \int_{\mathbb{R}_+}\sqrt{f(r)}g_\gamma(x, r)\,dr$) and obviously $\gamma \in [0, 1[$.

(iii) First, let $\Lambda = \Lambda_E \times \Lambda_{\mathbb{R}_+}$ be a compact of $E \times \mathbb{R}_+$ and $\widehat{\mathbf{K}}_\Lambda$ be the restriction of $\widehat{\mathbf{K}}$ on Λ. In order to compute the trace of $\widehat{\mathbf{K}}_\Lambda$, consider a complete orthogonal basis (CONB) of $L^2(\Lambda, \lambda(dx)\,dr)$. Let $(e_n)_{n\geq 1}$, resp. $(b_n)_{n\geq 1}$, be a CONB of $L^2(\Lambda_E, \lambda)$, resp. of $L^2(\Lambda_{\mathbb{R}_+}, dr)$. Then $(h_{n,k})_{n,k\geq 1}$,with $h_{n,k}(x, r) = e_n(x)b_k(r)$ is a CONB of $L^2(\Lambda, \lambda(dx)\,dr)$ (see [22]) and

$$\mathrm{Tr}(\widehat{\mathbf{K}}_\Lambda) = \sum_{n,k\geq 1}\langle \widehat{\mathbf{K}}_\Lambda h_{n,k}, h_{n,k}\rangle_{L^2(\Lambda,\lambda(dx)\,dr)},$$

with for $n, k \geq 1$:

$$\begin{aligned}&\langle \widehat{\mathbf{K}}_\Lambda h_{n,k}, h_{n,k}\rangle_{L^2(\Lambda,\lambda(dx)\,dr)}\\ &\quad= \int_{\Lambda^2} h_{n,k}(x,r)\widehat{\mathbf{K}}_\Lambda h_{n,k}(x,r)\lambda(dx)\,dr\\ &\quad= \int_{\Lambda^2} e_n(x)b_k(r)\sqrt{f(r)}K(x,y)\sqrt{f(s)}e_n(y)b_k(s)\lambda(dy)\,ds\lambda(dx)\,dr\end{aligned}$$

$$= \left(\int_{\Lambda_{\mathbb{R}_+}} \sqrt{f(r)} b_k(r)\,dr\right)^2 \left(\int_{\Lambda_E^2} e_n(x) K(x,y) e_n(y) \lambda(dx)\lambda(dy)\right)$$
$$\leq \langle \sqrt{f}, b_k\rangle^2_{L^2(\mathbb{R}_+)} \langle K e_n, e_n\rangle_{L^2(\Lambda_E)},$$

with the Fubini theorem. As a consequence, with the Bessel inequality, $\mathrm{Tr}(\widehat{\mathbf{K}}_\Lambda) \leq \|\sqrt{f}\|^2_{L^2(\mathbb{R}_+)} \mathrm{Tr}(\mathbf{K}_{\Lambda_E}) < +\infty$, and $\widehat{\mathbf{K}}_\Lambda$ is locally trace-class. Note that it is still true for subset Λ of the form $\Lambda_E \times \mathbb{R}_+$.

Next, for a general compact set Λ of $E \times \mathbb{R}_+$, we have $\Lambda \subset \Lambda_E \times \Lambda_{\mathbb{R}_+}$ for compact sets Λ_E of E and $\Lambda_{\mathbb{R}_+}$ of $\mathbb{R}_+$. Using the reunion $(m_n)_{n\geq 1} = (c_n)_{n\geq 1} \cup (d_n)_{n\geq 1}$ of orthonormal basis $(c_n)_{n\geq 1}$ of $L^2(\Lambda, \lambda(dx)\,dr)$ and $(d_n)_{n\geq 1}$ of $L^2(\Lambda_E \times \Lambda_{\mathbb{R}_+} \setminus \Lambda, \lambda(dx)\,dr)$, we have an orthonormal basis of $L^2(\Lambda_E \times \Lambda_{\mathbb{R}_+}, \lambda(dx)\,dr)$ and by the first part

$$\begin{aligned}
\mathrm{Tr}(\widehat{\mathbf{K}}_{\Lambda_E \times \Lambda_{\mathbb{R}_+}}) &= \sum_{n\geq 1} \langle \widehat{\mathbf{K}} m_n, m_n\rangle_{L^2(\Lambda_E \times \Lambda_{\mathbb{R}_+}, \lambda(dx)\,dr)} \\
&= \sum_{n\geq 1} \langle \widehat{\mathbf{K}} c_n, c_n\rangle_{L^2(\Lambda, \lambda(dx)\,dr)} \\
&\quad + \sum_{n\geq 1} \langle \widehat{\mathbf{K}} d_n, d_n\rangle_{L^2(\Lambda_E \times \Lambda_{\mathbb{R}_+} \setminus \Lambda, \lambda(dx)\,dr)} \\
&= \mathrm{Tr}(\widehat{\mathbf{K}}_\Lambda) + \mathrm{Tr}(\widehat{\mathbf{K}}_{\Lambda_E \times \Lambda_{\mathbb{R}_+} \setminus \Lambda}).
\end{aligned}$$

Since all the summands are positive, we have $\mathrm{Tr}(\widehat{\mathbf{K}}_\Lambda) < +\infty$. □

Remark A.9. Straightforwardly, Proposition A.8 is still true for $f \in L^1(\mathbb{R}_+)$ but with condition (ii) replaced by: (ii') The spectrum of $\widehat{K}$ is included in $[0, \|f\|_1^{-1}[$.

Proposition A.10. *Let K be a kernel satisfying Hypothesis 2 and $g : E \to [0 + \infty[$ be a bounded function with compact support. Then $K[g]$ given by*

$$K[g](x,y) = \sqrt{g(x)} K(x,y) \sqrt{g(y)}$$

is the kernel of an Hilbert–Schmidt operator.

Proof. The Hilbert–Schmidt property is shown by proving

$$\int_{E\times E} K[g](x,y)^2\,dx\,dy < +\infty.$$

Let B be the compact support of g, using $\rho_2(x_1,x_2) = \det(K(x_i,x_j)_{1\leq i,j\leq 2}) \geq 0$, we have

$$\int_{E\times E} K[g](x,y)^2 \lambda(dx)\lambda(dy) = \int_{E\times E} g(x) K(x,y)^2 g(y) \lambda(dx)\lambda(dy)$$

$$\leq \|g\|_\infty^2 \int_{B\times B} K(x,x)K(y,y)\lambda(dx)\lambda(dy)$$
$$= \|g\|_\infty^2 \left(\int_B K(x,x)\lambda(dx)\right)^2,$$

which is finite since K is locally trace-class (Hypothesis 2). □

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References

[1] Arfken, G.B. and Weber, H.J. (2001). *Mathematical Methods for Physicists*, 5th ed. Burlington, MA: Harcourt/Academic Press. MR1810939
[2] Biermé, H. and Estrade, A. (2006). Poisson random balls: Self-similarity and x-ray images. *Adv. in Appl. Probab.* **38** 853–872. MR2285684
[3] Biermé, H., Estrade, A. and Kaj, I. (2010). Self-similar random fields and rescaled random balls models. *J. Theoret. Probab.* **23** 1110–1141. MR2735739
[4] Breton, J.-C. and Dombry, C. (2009). Rescaled weighted random ball models and stable self-similar random fields. *Stochastic Process. Appl.* **119** 3633–3652. MR2568289
[5] Chiu, S.N., Stoyan, D., Kendall, W.S. and Mecke, J. (2013). *Stochastic Geometry and Its Applications*, 3rd ed. Chichester: Wiley. MR3236788
[6] Daley, D.J. and Vere-Jones, D. (2002). *Introduction to Point Processes. Volumes* 1 *and* 2, 2nd ed.
[7] Deng, N., Zhou, W. and Haenggi, M. (2014). The Ginibre point process as a model for wireless networks with repulsion. Available at arXiv:1401.3677.
[8] Dragomir, S.S. (2017). Some trace inequalities for operators in Hilbert spaces. *Kragujevac J. Math.* **41** 33–55. MR3668251
[9] Gobard, R. (2015). Random balls model with dependence. *J. Math. Anal. Appl.* **423** 1284–1310. MR3278199
[10] Gobard, R. (2015). Fluctuations dans les modèles de boules aléatoires. Ph.D., Université de Rennes 1. Available at https://hal.inria.fr/IRMAR/tel-01167520v1.
[11] Heinrich, L. and Schmidt, V. (1985). Normal convergence of multidimensional shot noise and rates of this convergence. *Adv. in Appl. Probab.* **17** 709–730. MR0809427
[12] Hough, J.B., Krishnapur, M., Peres, Y. and Virág, B. (2009). *Zeros of Gaussian Analytic Functions and Determinantal Point Processes. University Lecture Series* **51**. Providence, RI: Amer. Math. Soc. MR2552864
[13] Kaj, I., Leskelä, L., Norros, I. and Schmidt, V. (2007). Scaling limits for random fields with long-range dependence. *Ann. Probab.* **35** 528–550. MR2308587
[14] Kaj, I. and Taqqu, M.S. (2008). Convergence to fractional Brownian motion and to the Telecom process: The integral representation approach. In *In and Out of Equilibrium*. 2. *Progress in Probability* **60** 383–427. Birkhäuser, Basel. MR2477392

[15] Klüppelberg, C. and Mikosch, T. (1995). Explosive Poisson shot noise processes with applications to risk reserves. *Bernoulli* **1** 125–147. MR1354458
[16] Lane, J.A. (1984). The central limit theorem for the Poisson shot-noise process. *J. Appl. Probab.* **21** 287–301. MR0741131
[17] Li, Y., Baccelli, F., Dhillon, H.S. and Andrews, J.G. (2015). Statistical modeling and probabilistic analysis of cellular networks model with determinantal point processes. *IEE Transactions on Communications* **63** 3405–3422.
[18] Meester, R. and Roy, R. (1996). *Continuum Percolation*. *Cambridge Tracts in Mathematics* **119**. Cambridge: Cambridge Univ. Press. MR1409145
[19] Mikosch, T., Resnick, S., Rootzén, H. and Stegeman, A. (2002). Is network traffic approximated by stable Lévy motion or fractional Brownian motion? *Ann. Appl. Probab.* **12** 23–68. MR1890056
[20] Miyoshi, N. and Shirai, T. (2014). A cellular network model with Ginibre configured base stations. *Adv. in Appl. Probab.* **46** 832–845. MR3254344
[21] Miyoshi, N. and Shirai, T. (2017). Tail asymptotics of signal-to-interference ratio distribution in spatial cellular network models. *Probab. Math. Statist.* **37** 431–453. MR3745394
[22] Reed, M. and Simon, B. (1972). *Methods of Modern Mathematical Physics*. *I. Functional Analysis*. New York: Academic Press. MR0493419
[23] Samorodnitsky, G. and Taqqu, M.S. (1994). *Stable Non-Gaussian Random Processes*: *Stochastic Models with Infinite Variance*. New York: Chapman & Hall. MR1280932
[24] Shirai, T. and Takahashi, Y. (2003). Random point fields associated with certain Fredholm determinants. I. Fermion, Poisson and boson point processes. *J. Funct. Anal.* **205** 414–463. MR2018415
[25] Yang, X. and Petropulu, A.P. (2003). Co-channel interference modeling and analysis in a Poisson field of interferers in wireless communications. *IEEE Trans. Signal Process.* **51** 64–76. MR1956093

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