Nonparametric volatility estimation in scalar diffusions: Optimality across observation frequencies

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The nonparametric volatility estimation problem of a scalar diffusion process observed at equidistant time points is addressed. Using the spectral representation of the volatility in terms of the invariant density and an eigenpair of the infinitesimal generator the first known estimator that attains the minimax optimal convergence rates for both high and low-frequency observations is constructed. The proofs are based on a posteriori error bounds for generalized eigenvalue problems as well as the path properties of scalar diffusions and stochastic analysis. The finite sample performance is illustrated by a numerical example.

Keywords: diffusion processes; nonparametric estimation; sampling frequency; spectral approximation

1. Introduction

Consider the problem of estimating the volatility of a diffusion process $(X_t, t \geq 0)$. The statistical properties depend, essentially, on the observation scheme. It is natural to assume discrete observations:

$$X_0, X_\Delta, \ldots, X_{N\Delta}, \qquad \Delta > 0, \qquad T = N\Delta.$$

The quality of an estimator is typically assessed by its asymptotic properties when the sample size N tends to infinity. The usual assumptions are either $\Delta \to 0$ or $T \to \infty$, which corresponds to high and low-frequency regimes, respectively. Different frequency assumptions require very different methods. Since the frequency regimes are a theoretical construct, for any given sample, we need to choose among high and low-frequency estimators. Therefore, it is of crucial interest to develop universal methods that will perform at optimal level regardless of the sampling frequency. In this paper, the first nonparametric estimator of the volatility that attains minimax optimal rates in both high and low-frequency regimes is introduced. In the parametric setting, the problem of the universal scale estimation was first raised in Jacobsen [15,16]. The constructed estimators were consistent and asymptotically Gaussian for all values of Δ, but nearly efficient for small values of Δ only. The estimation method, which relied on the use of the estimating functions, is different from the one applied in this paper.

It is a well-known consequence of the Girsanov theorem that when T is fixed, the drift coefficient is not identifiable. Since we are interested in a universal scale method, we focus on the volatility estimation and, henceforth, treat drift as a nuisance parameter.

The existing high-frequency estimators (see Florens-Zmirou [9], Hoffmann [13], Jacod [17], Bandi and Phillips [2]) are based on the interpretation of the squared volatility as the instantaneous conditional variance of the process. Consequently, the assumption $\Delta \to 0$ is crucial for the consistency of these estimators, see [8] and [27], Section 3. On the other hand, it has been conjectured that the minimax optimal low-frequency estimator introduced by Gobet, Hoffmann and Reiß (GHR) [11] also performs well in the high-frequency regime. This conjecture is based on the observation that the spectral representation of the volatility in terms of an eigenpair of the infinitesimal generator can be generalized by replacing the invariant density with the occupation density of the path $(X_t, t \leq T)$. While this generalization might be sufficient to obtain the consistency of the GHR estimator when applied to the high-frequency data, the numerical study reveals that the convergence rates are not optimal. The reason for this is that when the time horizon of the sample is fixed, the estimator inherits the poor regularity of the occupation density, which, contrary to the invariant density, is not linked to the regularity of the diffusion coefficients. As we show below, this difficulty can be solved with the appropriate averaging of the spectral estimator, which is the main motivation behind the Definition 6 of the universally optimal estimator. For more details, refer to Section 2.1.

Based on the spectral method, the low-frequency analysis of the universally optimal estimator is similar to [5,11]. The real difficulty is in the high-frequency analysis, where the universal estimator is compared to the benchmark high-frequency estimator introduced by Florens-Zmirou [9] (see Section 2.2). In particular, we develop the perturbation theory for bilinear coercive forms with Hölder regular coefficients (see Appendix B), which may be of independent interest.

In the next sections, we present the construction of the universal scale estimator and state the high and low-frequency convergence rates. In Section 2, we discuss the relation of the proposed estimator to the high and low-frequency benchmark estimators. Finite sample behaviour of the new estimator compared with the Florens-Zmirou and GHR estimators is illustrated in Section 2.3. In Section 2.4, we discuss the assumptions and possible extensions of the model. The proofs of the high and low-frequency convergence rates are shown in Sections 3 and 4, respectively.

1.1. Construction of the estimator

We follow the low-frequency literature [5,11,23] and consider a diffusion model on $[0, 1]$ with boundary reflection (see Section 2.4 for a discussion of the model). Let $\| \cdot \|_\infty$ denote the supremum norm on space $B([0, 1])$ of bounded measurable functions on $[0, 1]$. Finally, denote by

$$H^i = \left\{ f \in L^2\big([0, 1]\big) : f \text{ has } i \text{ weak derivatives with } f^{(j)} \in L^2\big([0, 1]\big), j \leq i \right\}$$

the L^2-Sobolev spaces on $[0, 1]$ of order $i = 1, 2$. H^i is a Hilbert space with the norm

$$\| f \|_{H^i} = \sum_{j \leq i} \big\| f^{(i)} \big\|_{L^2}.$$

Assumption 1. For given constants $0 < d < D$ suppose $(\sigma, b) \in \Theta$, where

$$\Theta := \Theta(d, D) = \left\{ (\sigma, b) \in H^1([0, 1]) \times B([0, 1]) : \|b\|_\infty \vee \big\|\sigma^2\big\|_{H^1} < D, \inf_{x \in [0,1]} \sigma^2(x) \geq d \right\}.$$

Let the process $(X_t, t \geq 0)$ be given by the following Skorokhod type stochastic differential equation:

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t + dK_t,$$
$$X_t \in [0, 1] \qquad \text{for every } t \geq 0, \tag{1}$$

where $(W_t, t \geq 0)$ is a standard Brownian motion and $(K_t, t \geq 0)$ is an adapted continuous process with finite variation, starting from 0, such that for every $t \geq 0$ we have $\int_0^t \mathbf{1}_{(0,1)}(X_s)\,dK_s = 0$. The Sobolev regularity of σ ensures that the SDE (1) has a unique strong solution, see [28], Theorem 4. As shown in [11], X admits an invariant measure with Lebesgue density.

Assumption 2. The initial condition x_0 is distributed with respect to the invariant measure μ on $[0, 1]$, independently of the driving Brownian motion W.

Under Assumption 2, the diffusion X is stationary and ergodic. We denote with $\mathbb{P}_{\sigma,b}$ the law of X on the canonical space Ω of continuous functions over the positive axis with values in $[0, 1]$, equipped with the topology of the uniform convergence on compact sets and endowed with its σ-field $\mathcal{F}$. We denote with $\mathbb{E}_{\sigma,b}$ the corresponding expectation operator.

Definition 3. Denote by $\widehat{\mu}_N$ the empirical measure associated to the observed sample:

$$\widehat{\mu}_N = \frac{1}{2N}\delta_{\{X_0\}} + \frac{1}{N}\sum_{n=1}^{N-1}\delta_{\{X_{n\Delta}\}} + \frac{1}{2N}\delta_{\{X_{N\Delta}\}}.$$

The underweighting of the first and the last observations is asymptotically negligible, but has meaningful finite sample interpretation both in the low and high-frequency regimes (see remarks before the equation (4) and after Definition 12). By ergodicity, when the time horizon T of the observed sample grows to infinity, the empirical measure $\widehat{\mu}_N(dx)$ converges weakly to the stationary distribution $\mu(dx)$. When T is fixed, but the observation frequency increases, the empirical measure tends to the occupation measure μ_T of the path $(X_t, 0 \leq t \leq T)$ (see Definition 7).

Definition 4. For $J \in \mathbb{N}_+$, $j = 1, \ldots, J$, let $\mathbf{1}_j(x) = \mathbf{1}(\frac{j-1}{J} \leq x < \frac{j}{J})$ be the indicator function of the jth sub-interval and

$$\psi_j(x) = \int_0^x \mathbf{1}_j(y)\,dy \qquad \text{for } j = 1, \ldots, J,$$
$$\psi_0(x) = 1.$$

Let $V_J = \text{span}\{\psi_j : j = 0, \ldots, J\}$ be the space of linear splines with knots at $\{0, \frac{1}{J}, \frac{2}{J}, \ldots, \frac{J-1}{J}, 1\}$ and $V_J^0 = \{v \in V_J : \int_0^1 v(x)\widehat{\mu}_N(dx) = 0\}$ be the subspace of functions $L^2(\widehat{\mu}_N)$-orthogonal to constants.

Consider the generalized symmetric eigenproblem:

Eigenproblem 5. Find $(\widehat{\gamma}, \widehat{u}) \in \mathbb{R} \times V_J$ with $\widehat{u} \neq 0$, such that

$$\widehat{l}(\widehat{u}, v) = \widehat{\gamma}\widehat{g}(\widehat{u}, v) \qquad \text{for all } v \in V_J,$$

where $\widehat{g}, \widehat{l} : V_J \times V_J \to \mathbb{R}$ are symmetric, bilinear forms defined by:

$$\widehat{g}(u, v) = \int_0^1 u(x)v(x)\widehat{\mu}_N(dx),$$

$$\widehat{l}(u, v) = \frac{1}{2T} \sum_{n=0}^{N-1} \big(u(X_{(n+1)\Delta}) - u(X_{n\Delta})\big)\big(v(X_{(n+1)\Delta}) - v(X_{n\Delta})\big).$$

When the observed sample visits at least twice every interval $[\frac{j-1}{J}, \frac{j}{J})$, the form $\widehat{g}$ is positive definite on V_J, while $\widehat{l}$ is positive semi-definite on V_J and positive definite on V_J^0. In such a case, Eigenproblem 5 has $\dim(V_J) = J + 1$ solutions $(\widehat{\gamma}_j, \widehat{u}_j)_{j=0,\dots,J}$, with non-negative eigenvalues $0 \leq \widehat{\gamma}_0 \leq \widehat{\gamma}_1 \leq \cdots \leq \widehat{\gamma}_J$ and $\widehat{g}$-orthogonal eigenfunctions. It is easy to check that $\widehat{\gamma}_0 = 0$ is an eigenvalue which corresponds to the constant function. Since the eigenfunctions are $\widehat{g}$-orthogonal, it follows that $\widehat{u}_j \in V_J^0$ for $1 \leq j \leq J$. Consequently, on the above event, which we prove in Lemma 18 and Corollary 45 to be of high probability in the high and low-frequency regimes respectively, it holds that $\widehat{\gamma}_1 > 0$.

Definition 6. Let

$$\widehat{\zeta}_1 = \frac{\log(1 - \Delta\widehat{\gamma}_1)}{\Delta}\mathbf{1}(\Delta\widehat{\gamma}_1 < 1) \quad \text{and} \quad \widehat{u}_1(x) = \sum_{j=0}^J \widehat{u}_{1,j}\psi_j(x).$$

When $\widehat{u}_{1,j} \neq 0$, we define the spectral estimator by

$$\widehat{\sigma}_{S,j}^2 = \frac{-2\widehat{\zeta}_1 \int_0^1 \psi_j(x)\widehat{u}_1(x)\widehat{\mu}_N(dx)}{\int_0^1 \psi_j'(x)\widehat{u}_{1,j}\widehat{\mu}_N(dx)},$$

$$\widehat{\sigma}_S^2(x) = \sum_{j=1}^J \widehat{\sigma}_{S,j}^2\mathbf{1}_j(x).$$

The condition $\mathbf{1}(\Delta\widehat{\gamma}_1 < 1)$ is a technical assumption which ensures that the estimator $\widehat{\zeta}_1$ is well defined. As explained in Section 2.1, $1 - \Delta\widehat{\gamma}_1$ is the estimator of the largest nontrivial eigenvalue of the transition operator. When $\Delta\widehat{\gamma}_1 \geq 1$, the estimated transition operator is negative definite on V_J^0, thus the spectral approach will not provide a reliable output. Proposition 20 and inequality (69) ensure that $\Delta\widehat{\gamma}_1 < 1$ with high probability, both in high and low-frequency regimes.

1.2. High-frequency convergence rate

The estimation of volatility at point x is possible only when the process spends enough time around x.

Definition 7. Set $T > 0$. Define the occupation density

$$\mu_T = \frac{L_T}{T\sigma^2}, \tag{2}$$

where L_T is the semimartingale local time of the path $(X_t : 0 \le t \le T)$.

For any bounded Borel measurable function f, the following occupation formula holds:

$$\frac{1}{T} \int_0^T f(X_s)\,ds = \int_0^1 f(x)\mu_T(x)\,dx. \tag{3}$$

In order to obtain the global rates of convergence, we must assume that the occupation density of the observed path is bounded from below. Therefore, for a fixed level v, we study the risk of the estimator conditioned to the event

$$\mathcal{L} = \mathcal{L}(v) = \Big\{ \inf_{x \in [0,1]} \mu_T(x) \ge v \Big\}.$$

Theorem 8. *Grant Assumptions* 1 *and* 2. *Fix* $T > 0$, $0 < a < b < 1$, *and* $c, v > 0$. *There exists a positive constant* $C = C(T, a, b, v, c, \epsilon, d, D)$ *such that for every* $\Delta > 0$ *and* $J \in \mathbb{N}$ *satisfying* $c^{-1}\Delta^{-1/3} \le J \le c\Delta^{-1/3}$ *there is an event* $\mathcal{R}_\Delta = \mathcal{R}_\Delta(T, a, b, v) \subseteq \mathcal{L}$ *such that*

$$\sup_{(\sigma,b) \in \Theta(d,D)} \mathbb{E}_{\sigma,b}\big[\mathbf{1}_{\mathcal{R}_{\epsilon,\Delta} \cap} \cdot \big\|\widehat{\sigma}_S^2 - \sigma^2\big\|_{L^1([a,b])}\big] \le C\Delta^{\frac{1}{3}},$$

and $\lim_{\Delta \to 0} \mathbb{P}_{\sigma,b}(\mathcal{L} \setminus \mathcal{R}_\Delta) = 0$.

Hoffmann [14], Proposition 2, shows that the rate $\Delta^{1/3}$ is optimal in the minimax sense even in the class of diffusions with Lipschitz volatility. To prove Theorem 8, we compare $\widehat{\sigma}_S^2$ with the benchmark Florens-Zmirou estimator, see Section 2.2. While the consistency of the spectral estimator can be obtained using the well-known path properties of diffusion processes, the proof of the exact convergence rate is rather demanding. As explained in Section 3.2, it is necessary to show the regularity properties of the estimated eigenfunction $\widehat{u}_1$, which requires rather sophisticated arguments from the perturbation theory of differential operators with non-smooth coefficients.

1.3. Low-frequency convergence rate

In the low-frequency regime, we need to threshold the estimator in order to ensure integrability and stability against large stochastic errors. As expected, $\widehat{\sigma}_S^2$ achieves the same mean L^2 rate as the original Gobet-Hoffmann and Reiß estimator. Furthermore, for $\sigma \in H^1$, this rate is minimax optimal, which can be obtained by the same proof as [11], Theorem 2.5.

Theorem 9. *Grant Assumptions* 1 *and* 2. *Fix* $\Delta > 0$, $0 < a < b < 1$ *and* $c > 0$. *Choose* J *such that* $c^{-1} N^{1/5} \leq J \leq c N^{1/5}$. *It holds*

$$\sup_{(\sigma,b)\in\Theta(d,D)} \mathbb{E}_{\sigma,b}\big[\big\|\widehat{\sigma}_S^2 \wedge D - \sigma^2\big\|_{L^2([a,b])}^2\big]^{\frac{1}{2}} \leq C N^{-\frac{1}{5}},$$

where $C = C(\Delta, a, b, c, d, D)$ *is a positive constant.*

The general idea of the proof is the same as in Gobet et al. [11] or [5]. We use the mixing property of the process X to control the approximation error of the stationary measure μ by the empirical measure $\widehat{\mu}_N$, see Corollary 45. Then, as discussed in Section 2.1, we bound the estimation error of (κ_1, u_1) – the first nontrivial eigenpair of the transition operator P_Δ, obtaining

$$|\widehat{\kappa}_1 - \kappa_1| + \|\widehat{u}_1 - u_1\|_{H^1} = O_{\mathbb{P}}\big(N^{-1/5}\big).$$

Finally, we bound the plug-in error of the spectral estimator $\widehat{\sigma}_S$. A tenuous point is in that the estimator $\widehat{u}_1$ converges to the eigenfunction u_1 in the sense of mean H^1 norm only, hence we can not postulate a uniform positive lower bound on $\inf_{x\in[a,b]}\widehat{u}_1'(x)$. Following Chorowski and Trabs [5], we are able to overcome this difficulty by applying the threshold $\widehat{\sigma}_S^2 \wedge D$.

2. Discussion

2.1. Connection to the GHR low-frequency estimator

In this section, we explain the relation between the defined estimator $\widehat{\sigma}_S$ above and the original spectral estimator introduced in [11], Section 3.2. First, let us review the construction of the GHR estimator.

Definition 10. As in Gobet et al. [11], Eq. (3.8), for $u, v \in V_J$ let

$$\widehat{p}(u, v) = \frac{1}{2N} \sum_{n=0}^{N-1} \big(u(X_{n\Delta})v(X_{(n+1)\Delta}) + v(X_{n\Delta})u(X_{(n+1)\Delta})\big).$$

A crucial observation is that, due to the appropriate weighting of the empirical measure, $\widehat{p}$ becomes a linear combination of $\widehat{l}$ and $\widehat{g}$. Indeed, using the summation by parts formula, we obtain

$$\widehat{l} = \frac{1}{\Delta}(\widehat{g} - \widehat{p}). \tag{4}$$

Hence, for $(\widehat{\gamma}_i, \widehat{u}_i)$ – any solution of the Eigenproblem 5, we have

$$\widehat{p}(\widehat{u}_i, v) = (1 - \Delta\widehat{\gamma}_i)\widehat{g}(\widehat{u}_i, v) \qquad \text{for every } v \in V_J. \tag{5}$$

Denote

$$\widehat{\kappa}_i = (1 - \Delta\widehat{\gamma}_i). \tag{6}$$

We conclude that the eigenpair $(\widehat{\kappa}_1, \widehat{u}_1)$ is equal to the estimator of the eigenpair of the transition operator which is defined in [11], Eq. (3.11). Taking into account that functions (ψ_j) are not orthonormal, following [11], Eq. (3.12) and Eq. (3.7), we define the GHR estimator as:

Definition 11.

$$\widehat{\sigma}^2_{\mathrm{GHR}}(x) = \frac{2\widehat{\zeta}_1 \int_0^x \widehat{u}_1(y)\widehat{\mu}_N(dy)}{\widehat{u}'_1(x)\widehat{\mu}(x)},$$

where

$$\widehat{\mu} = \sum_{j=0}^J \widehat{\mu}_j \psi_j \qquad \text{with } (\widehat{\mu}_j)_j = \left(\left[\int_0^1 \psi_i(y)\psi_j(y)\,dy \right]_{i,j} \right)^{-1} \left(\int_0^1 \psi_i(x)\widehat{\mu}_N(dx) \right)_i,$$

is an estimator of the stationary density.

Note that estimator $\widehat{\sigma}_S$ can be seen as a local average of $\widehat{\sigma}^2_{\mathrm{GHR}}$. Indeed, since $\mathbf{1}_j = \psi'_j$, integrating by parts gives us

$$\begin{aligned}
\widehat{\sigma}^2_{S,j} &= \frac{2\widehat{\zeta}_1 \int_0^1 \psi'_j(x)(\int_0^x \widehat{u}_1(y)\widehat{\mu}_N(dy))\,dx}{\int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{u}'_1(x)\widehat{\mu}_N(dx)} \\[2mm]
&= \frac{\int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\sigma}^2_{\mathrm{GHR}}(x)\widehat{u}'_1(x)\widehat{\mu}(x)\,dx}{\int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{u}'_1(x)\widehat{\mu}_N(dx)}.
\end{aligned} \tag{7}$$

Since we focus on volatility functions in H^1, the above averaging has no effect on the low-frequency convergence rate. On the other hand, there are multiple reasons why it is beneficial for optimality in the high-frequency regime. First, since $\widehat{u}'_1$ is constant on every interval $[\frac{j-1}{J}, \frac{j}{J}]$, after averaging we do not have to estimate the density of the occupation measure (which is not regular in the high-frequency setting), but the occupation measure of the intervals $[\frac{j-1}{J}, \frac{j}{J}]$. Furthermore, averaging reduces the variance of the estimator, which can be clearly seen in Figure 1. The intuitive explanation of this phenomenon is that while the original estimator $\widehat{\sigma}^2_{\mathrm{GHR}}$ inherits the rough behaviour of the occupation density (via the inverse of the derivative of the eigenfunction u_1 which has the same smoothness as the design density) this irregularity is removed by multiplication with $\widehat{u}'_1\widehat{\mu}$.

2.2. Connection to the Florens-Zmirou estimator

The general idea of the proof of the high-frequency convergence rate is to compare estimator $\widehat{\sigma}_S$ with the minimax optimal (see [14], Proposition 2) high-frequency estimator introduced in Florens-Zmirou [9]. In this section, we recall the definition of the Florens-Zmirou estimator and discuss its relation to $\widehat{\sigma}_S$.

Definition 12. Define the time-symmetric version of the well-known Nadaraya–Watson type estimator of the squared volatility coefficient, introduced in Florens-Zmirou [9], by

$$\widehat{\sigma}^2_{\mathrm{FZ},j} = \frac{\sum_{n=0}^{N-1}(\mathbf{1}_j(X_{n\Delta}) + \mathbf{1}_j(X_{(n+1)\Delta}))(X_{(n+1)\Delta} - X_{n\Delta})^2}{\Delta \sum_{n=0}^{N-1}(\mathbf{1}_j(X_{n\Delta}) + \mathbf{1}_j(X_{(n+1)\Delta}))},$$

$$\widehat{\sigma}^2_{\mathrm{FZ}}(x) = \sum_{j=1}^{J} \widehat{\sigma}^2_{\mathrm{FZ},j}\mathbf{1}_j(x).$$

Note that the underweighting of the first and last observation in the denominator of $\widehat{\sigma}^2_{\mathrm{FZ},j}$ appears naturally as an artifact of the time symmetry.

Remark 13. We call $\widehat{\sigma}^2_{\mathrm{FZ}}$ a time-symmetrized version of the Florens-Zmirou estimator, since it is an average of the standard Florens-Zmirou estimators (cf. [9], Eq. (1.1)) constructed for the process $(X_t, 0 \leq t \leq T)$ and the time reversed process $Y_t = X_{T-t}$. Indeed, let

$$\widehat{\sigma}^2_j(X_0, X_\Delta, \ldots, X_{N\Delta}) = \frac{\sum_{n=0}^{N-1}\mathbf{1}_j(X_{n\Delta})(X_{(n+1)\Delta} - X_{n\Delta})^2}{\Delta(\frac{1}{2}\mathbf{1}_j(X_0) + \sum_{n=1}^{N-1}\mathbf{1}_j(X_{n\Delta}) + \frac{1}{2}\mathbf{1}_j(X_{N\Delta}))}. \tag{8}$$

Then

$$\widehat{\sigma}^2_{\mathrm{FZ},j} = \frac{\widehat{\sigma}^2_j(X_0, X_\Delta, \ldots, X_{N\Delta}) + \widehat{\sigma}^2_j(Y_0, Y_\Delta, \ldots, Y_{N\Delta})}{2}.$$

Since stationary scalar diffusions are reversible, under the Assumption 2, the process $(Y_t, 0 \leq t \leq T)$ is identical in law to $(X_t, 0 \leq t \leq T)$. Hence, the $L^2(\mathbb{P}_{\sigma,b})$ bounds of estimator $\widehat{\sigma}^2_{\mathrm{FZ}}$ are of the same order as those of the classical Florens-Zmirou estimator.

Recall that $(\widehat{\gamma}_1, \widehat{u}_1)$ is an eigenpair of the Eigenproblem 5. From Definition 6 of the spectral estimator, it follows that

$$\widehat{\sigma}^2_{S,j} = \frac{-\widehat{\zeta}_1}{\widehat{\gamma}_1} \frac{2\widehat{l}(\widehat{u}_1, \psi_j)}{\widehat{u}_{1,j}\int_{\frac{j-1}{J}}^{\frac{j}{J}}\widehat{\mu}_N(dx)}. \tag{9}$$

A similar representation formula can be established for the time symmetric Florens-Zmirou estimator $\widehat{\sigma}^2_{\mathrm{FZ}}$.

Definition 14. Define a bilinear form $\widehat{f} : V_J \times V_J \to \mathbb{R}$ by

$$\widehat{f}(u, v) = \frac{1}{2}\int_0^1 u'(x)v'(x)\widehat{\sigma}^2_{\mathrm{FZ}}(x)\widehat{\mu}_N(dx).$$

Consider vector $(v_j)_{j=1,...,J}$ such that $v_j \neq 0$ for every $j = 1, \ldots, J$ and the associated function $v \in V_J^0$. We have

$$\widehat{\sigma}^2_{\text{FZ},j} = \frac{2\widehat{f}(v, \psi_j)}{v_j \int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx)}. \tag{10}$$

As will be thoroughly explained in Section 3.2, when $\Delta \to 0$, the eigenvalue ratio $-\widehat{\zeta}_1/\widehat{\gamma}_1$ in (9) tends to 1. Consequently, the difference between estimators $\widehat{\sigma}^2_S$ and $\widehat{\sigma}^2_{\text{FZ}}$ is controlled by

$$\frac{2|\widehat{l}(\widehat{u}_1, \psi_j) - \widehat{f}(\widehat{u}_1, \psi_j)|}{\widehat{u}_{1,j} \int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx)}. \tag{11}$$

The main observation is that in the high-frequency analysis, we do not have to control the estimation error of the derivative $\widehat{u}_1'$. Indeed, to bound (11), we need only to show a uniform lower bound for $\widehat{u}_{1,j}$ and an upper bound for the difference $|\widehat{l}(\widehat{u}_1, \psi_j) - \widehat{f}(\widehat{u}_1, \psi_j)|$. Unfortunately, $|\widehat{l}(v, \psi_j) - \widehat{f}(v, \psi_j)|$ is not small enough for any bounded function v. To achieve the required upper bound for the estimated eigenfunction, we need to first obtain some regularity properties of $\widehat{u}_1$, which is the most difficult part of the high-frequency analysis.

2.3. A numerical example

In this section, we present the numerical results for the volatility estimation across different observation time scales. We compare three estimation methods: the time symmetric Florens-Zmirou estimator $\widehat{\sigma}^2_{\text{FZ}}$ (see Definition 12), the spectral estimator $\widehat{\sigma}^2_{\text{GHR}}$ (see Definition 11, cf. Gobet et al. [11], Section 3.2) with approximation space V_J of linear splines with equidistant knots, and finally, the locally averaged spectral estimator $\widehat{\sigma}^2_S$. We apply an oracle choice of the projection level J, minimizing the risk.

We compare the locally averaged spectral estimator $\widehat{\sigma}^2_S$ with benchmark estimators $\widehat{\sigma}^2_{\text{FZ}}$ and $\widehat{\sigma}^2_{\text{GHR}}$ in both high and low-frequency regimes. Following Chorowski and Trabs [5], Section 5, we consider diffusion process X with mean reverting drift $b(x) = 0.2 - 0.4x$, quadratic squared volatility function $\sigma^2(x) = 0.4 - (x - 0.5)^2$, and two reflecting barriers at 0 and 1. This choice of diffusion coefficients is supposed to minimize the reflection effect alongside with some variability in the volatility function. Nevertheless, the depicted behaviour is typical for other diffusion processes. The sample paths were generated using the Euler-Maruyama scheme with time step size $\Delta/100 \wedge 0.001$ with reflection after each step. All simulated paths were conditioned to have an occupation time density greater than $v = 0.2$. Table 1 presents the oracle mean $L^1([0.1, 0.9])$ estimation error of σ^2, obtained by a Monte Carlo simulation with 1000 iterations, in high ($T = 5$, $\Delta \to 0$) and low ($\Delta = 0.25$, $T \to \infty$) frequency regimes, respectively. The estimated volatility functions for 20 independent paths are depicted in Figure 1.

In the case of high-frequency observations, $\widehat{\sigma}^2_S$ performs similarly to the benchmark estimator $\widehat{\sigma}^2_{\text{FZ}}$. Relative to $\|\sigma^2\|_{L^1([0.1,0.9])} \approx 0.28$, the error decreases from approximately 6% for $\Delta = 10^{-3}$ to 3% for $\Delta = 10^{-4}$. The estimation error of spectral estimator $\widehat{\sigma}^2_{\text{GHR}}$ is almost twice

Table 1. Monte Carlo estimation errors in high and low-frequency regimes. The value of parameter J is given in the subscript

	High-Frequency Regime: $T = 5$					
	$\Delta = 0.001$	$\Delta = 0.00075$	$\Delta = 0.0005$	$\Delta = 0.00035$	$\Delta = 0.0002$	$\Delta = 0.0001$
$\widehat{\sigma}^2_{\text{GHR}}$	$0.0388_{(18)}$	$0.0353_{(23)}$	$0.0322_{(24)}$	$0.0292_{(32)}$	$0.0258_{(36)}$	$0.0220_{(49)}$
$\widehat{\sigma}^2_{S}$	$0.0195_{(9)}$	$0.0174_{(10)}$	$0.0149_{(10)}$	$0.0131_{(12)}$	$0.0108_{(13)}$	$0.0088_{(18)}$
$\widehat{\sigma}^2_{\text{FZ}}$	$0.0169_{(10)}$	$0.0153_{(11)}$	$0.0133_{(12)}$	$0.0119_{(12)}$	$0.0100_{(13)}$	$0.0080_{(20)}$
	Low-Frequency Regime: $\Delta = 0.25$					
	$T = 1\text{k}$	$T = 3\text{k}$	$T = 7\text{k}$	$T = 10\text{k}$	$T = 15\text{k}$	$T = 20\text{k}$
$\widehat{\sigma}^2_{\text{GHR}}$	$0.0386_{(5)}$	$0.0333_{(6)}$	$0.0256_{(11)}$	$0.0226_{(11)}$	$0.0198_{(11)}$	$0.0178_{(11)}$
$\widehat{\sigma}^2_{S}$	$0.0310_{(4)}$	$0.0245_{(6)}$	$0.0200_{(7)}$	$0.0182_{(8)}$	$0.0166_{(8)}$	$0.0155_{(9)}$
$\widehat{\sigma}^2_{\text{FZ}}$	$0.0821_{(5)}$	$0.0823_{(5)}$	$0.0823_{(5)}$	$0.0822_{(5)}$	$0.0823_{(5)}$	$0.0824_{(5)}$

as large, although the quality of the estimation improves when Δ decreases. It is important to note that the oracle values of space parameter J for $\widehat{\sigma}^2_{\text{GHR}}$ are much bigger than those for other estimation methods. When Δ is small, the eigenfunctions inherit the regularity of the local time;
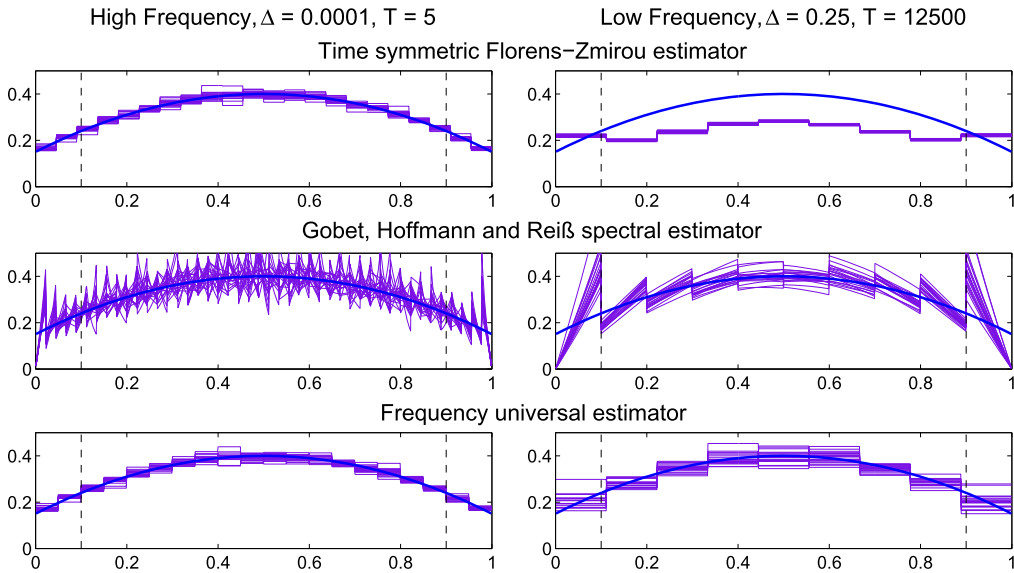

Figure 1. Estimated volatility functions for 20 independent trajectories.

the increase in dimension compensates for the projection error. Due to local averaging, this irregularity problem does not appear for $\widehat{\sigma}_S^2$, compare with Figure 1, where estimator $\widehat{\sigma}_{\mathrm{GHR}}^2$ oscillates heavily. Furthermore, there is no visible boundary effect, suggesting that the error rate of the spectral estimator does not deteriorate outside the fixed interval $[0.1, 0.9]$.

In the low-frequency regime, $\widehat{\sigma}_S^2$ performs slightly better than the original spectral estimator $\widehat{\sigma}_{\mathrm{GHR}}^2$. The boundary problem is visible, especially for $\widehat{\sigma}_{\mathrm{GHR}}^2$. The relative error decreases from 12% for $T = 1000$ to 5% for $T = 30\,000$. The Florens-Zmirou estimator $\widehat{\sigma}_{\mathrm{FZ}}^2$ underestimates the volatility and commits a relative error of 30%. This is expected and due mostly to the boundary reflection, which, for low-frequency observations, is not negligible in the interior of the state space. As found by unreported simulations, in the case of low-frequency observations, the locally averaged spectral estimator $\widehat{\sigma}_S^2$ will outperform the Florens-Zmirou estimator in the case of a highly varying volatility function σ^2, even when the sampling frequency is big enough to ignore the reflection effect.

2.4. Extensions and limitations

Stationarity of process X

In the high-frequency analysis, the stationarity assumption ensures that process X is time reversible. General initial distributions could be considered, but in order to preserve the performance of the estimation for the time reversed process, the coefficients of the backward process must belong to the nonparametric family Θ.

Due to the spectral gap of the generator, process X is geometrically ergodic. In particular, as $t \to \infty$, the one dimensional distributions of X_t converge exponentially fast to the invariant measure μ. It follows that, in the low-frequency regime, the assumption of stationarity can be made without loss of generality for asymptotic results.

Estimation at the boundaries

In the high-frequency regime, we prove the error bound in the interior of the state space. Restriction to the interval (a, b) allows us to obtain uniform lower bounds on the derivative of eigenfunction $\widehat{u}_1$, which, due to boundary conditions, are not valid in the entire state space. This restriction could be omitted by obtaining uniform bounds on the ratio of derivatives $\widehat{u}_{1,j+1}/\widehat{u}_{1,j}$. Unfortunately, since our proof relies on a posteriori error bounds on solutions for perturbed eigenvalue problems, we do not have sufficient tools to control the pointwise relative error of the eigenfunctions. Nevertheless, the numerical results suggest that the spectral estimation procedure also behaves well at the boundaries of the state space.

In the low-frequency regime, the spectral estimator is unstable at the boundary due to Neumann boundary conditions for the eigenfunctions of the infinitesimal generator. Refer to [11], Section 3.3.8, for a discussion of the boundary problem.

Boundary reflection

Following previous works on the spectral estimation in the low frequency setting, for example, [5,11,23], we consider an Itô diffusion model on the state space $[0, 1]$ with instantaneous reflec-

tion at the boundaries. The assumption of a compact state space makes the construction of the estimator easier and facilitates error analysis in the low-frequency setting, cf. Reiß [24]. We point out, here, that the reflection assumption is not restrictive in the high-frequency setting. Indeed, consider diffusion X defined on the entire real line with drift b and volatility σ. Let

$$A(t) = \int_0^t \mathbf{1}_{[0,1]}(X_s)\,ds$$

be the occupation time of interval $[0, 1]$. Assume that $\lim_{t \to \infty} A(t) = \infty$ and define the right-continuous inverse

$$C(t) = \inf\{s > 0 | A(s) > t\}.$$

Process $Y_t = X_{C(t)}$ follows the law of a reflected diffusion on $[0, 1]$ with drift b and volatility σ. Assume now that we are given observations $X_0, X_\Delta, \ldots, X_{N\Delta}$. The sub-sequence of the values that lie in $[0, 1]$ forms a chain of observation of Y. The sampling frequency is random (and depends on the path), but when Δ shrinks, it becomes close to equidistant. The difficulty in handling irregularities at the boundaries is similar to these found when considering the reflection effect. Unfortunately, while this reduction can be used under the assumption that Δ is small, it can't be applied in the low frequency setting, hence it is not practical in the context of scale invariant estimation.

Linear spline basis

The use of the linear spline basis is very convenient, as functions ψ_j appear naturally after applying integration by parts to the locally averaged GHR estimator, see (7). Nevertheless, unreported simulations suggest that the spectral estimation method performs as well with other bases. The Fourier cosines basis in $[0, 1]$ is especially efficient, consisting of the eigenfunctions of the reflected Brownian motion process.

Adaptivity

An important decision in the spectral estimation is the choice of the basis dimension J. The general problem is twofold: dimension J should adapt to the smoothness of the coefficients and simultaneously to the observation frequency. In [5], the authors applied Lepski's method to construct a data-driven version of the GHR estimator that adapts to the smoothness of the volatility. In the case of the low frequency data, the same selection rule can be applied for the universal estimator $\widehat{\sigma}_S$. The precise construction of a method that will adapt to the observation frequency remains open.

The numerical study shows that the proposed estimator $\widehat{\sigma}_S$ smoothly interpolates between the high and low-frequency estimators. The optimal convergence rates in both frequency regimes leave out the question of the paradigm to use when one has to consider data. The different convergence rates in high and low frequency regimes raise the question of bivariate asymptotics with respect to both Δ and T. Nevertheless, because of the structural differences of the high and low-frequency data, we believe that such an analysis would be particularly challenging.

3. High-frequency analysis

The proof of Theorem 8 is presented in Section 3.5 and is accomplished in several steps. In Section 3.3, we prove the convergence rate of the time-symmetric Florens-Zmirou estimator. Section 3.4 is devoted to the proof of Proposition 20 – the uniform bounds on the estimated eigenpair $(\widehat{\gamma}_1, \widehat{u}_1)$. In Section 3.6, we prove some technical results on the crossing intensity of the diffusion processes.

3.1. Preliminaries

From now on, we take the Assumptions 1 and 2 as granted. Fix $0 < a < b < 1$, the level $v > 0$ and constant $c > 0$. For simplicity, we set $T = 1$. Assume that $c^{-1}\Delta^{-1/3} \leq J \leq c\Delta^{-1/3}$. Furthermore, we will write $f \lesssim g$ (resp. $g \gtrsim f$) when $f \leq C \cdot g$ for some constant $C > 0$ that depends only on b, v, c, d, D. $f \approx g$ is equivalent to $f \lesssim g$ and $g \lesssim f$.

Sobolev regularity of the volatility implies that it is $1/2$-Hölder continuous. Indeed, by the Cauchy–Schwarz inequality it holds

$$\sup_{x,y \in [0,1]} \frac{|\sigma(x) - \sigma(y)|}{|x - y|^{1/2}} = \sup_{x,y \in [0,1]} \frac{|\int_x^y \sigma'(z)\,dz|}{|x - y|^{1/2}} \leq \|\sigma\|_{H^1}. \tag{12}$$

Recall Definition 7 of the occupation density μ_T. Formula (2), together with (12), imply that μ_T inherits the regularity properties of the local time. In particular, is the following theorem.

Theorem 15. *The function μ_1 is almost surely Hölder continuous of order α for every $\alpha < 1/2$. Moreover, for every $p \geq 1$, there exists a constant C_p such that*

$$\sup_{(\sigma,b) \in \Theta} \mathbb{E}_{\sigma,b}\left[\sup_{x \in [0,1]} \mu_1^p(x)\right] < \infty, \tag{13}$$

$$\sup_{(\sigma,b) \in \Theta} \mathbb{E}_{\sigma,b}\left[\left|\mu_1(x) - \mu_1(y)\right|^{2p}\right] \leq C_p|x - y|^p. \tag{14}$$

Proof. Since σ is uniformly bounded and $1/2$-Hölder continuous, the claim of the theorem can be deduced from the well-known properties of the family of the local times $(L_t, t \geq 0)$ of the semimartingale X, see the proof of [25], Chapter VI, Theorem 1.7, and the subsequent remark. $\square$

Definition 16. Denote by ω the modulus of continuity of the path $(X_t, 0 \leq t \leq 1)$, that is,

$$\omega(\delta) = \sup_{0 \leq s,t \leq 1, |t-s| \leq \delta} |X_t - X_s|.$$

Because of the ellipticity assumption $\sigma > 0$, the path $(X_t, 0 \leq t \leq 1)$ shares the properties of Brownian paths. In particular, we can apply the Brownian upper bounds (see Fischer and Nappo [7]) on the moments of ω.

Theorem 17. *For every $p \geq 1$, there exists a constant $C_p > 0$ such that*

$$\sup_{(\sigma,b) \in \Theta} \mathbb{E}_{\sigma,b}\big[\omega^p(\Delta)\big] \leq C_p \Delta^{p/2} \ln^p\big(\Delta^{-1}\big). \tag{15}$$

For the proof of Theorem 17 refer to Appendix A. Using (15), we can show that on $\mathcal{L}_v$ the occupation measure $\widehat{\mu}_N$ is spread uniformly on $[0,1]$ with high probability.

Lemma 18. *Let*

$$\mathcal{O}_\Delta = \mathcal{L} \cap \big\{\omega(\Delta)\|\mu_1\|_\infty \leq \Delta^{5/11} v\big\}. \tag{16}$$

For $\Delta < 1$ we have

$$\mathbb{P}_{\sigma,b}(\mathcal{L} \setminus \mathcal{O}_\Delta) \lesssim \Delta^{2/3}.$$

Furthermore, on the event $\mathcal{O}_\Delta$, for every $j = 1, \ldots, J$ it holds

$$v \lesssim J \int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx) \lesssim \|\mu_1\|_\infty.$$

The proof of Lemma 18 is postponed to Section 3.3.

As mentioned in Section 1.2, we want to compare the spectral estimator $\widehat{\sigma}_S^2$ with the benchmark high-frequency estimator $\widehat{\sigma}_{FZ}^2$. Before that, we have to prove a uniform upper bound on the mean L^2 error of the time symmetric Florens-Zmirou estimator. The result below is a generalization of [14], Proposition 2, where the same rate was obtained under the assumptions of smooth drift and Lipschitz volatility. As proved in [14], Proposition 2, the rate $\Delta^{1/3}$ is optimal in the minimax sense even on the class of diffusions with Lipschitz volatility.

Theorem 19. *We have*

$$\sup_{(\sigma,b) \in \Theta(d,D)} \mathbb{E}_{\sigma,b}\Big[\mathbf{1}_{\mathcal{O}_\Delta} \cdot \big\|\widehat{\sigma}_{FZ}^2 - \sigma^2\big\|_{L^2[1/J, 1-1/J]}^2\Big]^{\frac{1}{2}} \lesssim \Delta^{1/3}. \tag{17}$$

Because of the reflection, the rate deteriorates at the boundary. For $x \in [0, 1/J] \cup [1 - 1/J, 1]$

$$\sup_{(\sigma,b) \in \Theta(d,D)} \mathbb{E}_{\sigma,b}\Big[\mathbf{1}_{\mathcal{O}_\Delta} \cdot \big|\widehat{\sigma}_{FZ}^2(x) - \sigma^2(x)\big|^2\Big]^{\frac{1}{2}} \lesssim \Delta^{1/33}, \tag{18}$$

where the constant on the right-hand side does not depend on x.

The proof of Theorem 19 is postponed to Section 3.3. The main idea is the decomposition of the error into a martingale and deterministic approximation parts as in [14], Proposition 2. As expected, under the high-frequency assumption, the reflection has an effect only at the boundary. Inequalities (17) and (18) imply

$$\sup_{(\sigma,b) \in \Theta(d,D)} \mathbb{E}_{\sigma,b}\Big[\mathbf{1}_{\mathcal{O}_\Delta} \cdot \big\|\widehat{\sigma}_{FZ}^2(x) - \sigma^2(x)\big\|_{L^1[0,1]}\Big] \lesssim \Delta^{1/3},$$

but they are not sufficient to obtain the $\Delta^{1/3}$ rate for the root mean squared error on $[0,1]$.

3.2. Outline of the proof of the high-frequency convergence rate

Since by Theorem 19 the estimator $\widehat{\sigma}^2_{\mathrm{FZ}}$ attains the optimal rate $\Delta^{1/3}$, to prove Theorem 8 it is enough to upper bound the mean $L^1[0, 1]$ error between $\widehat{\sigma}^2_{\mathrm{FZ}}$ and $\widehat{\sigma}^2_S$. Using representations (9) and (10) $\widehat{\sigma}^2_{\mathrm{FZ}} - \widehat{\sigma}^2_S$ can be reduced to the difference of the forms $\widehat{f}$ and $\widehat{l}$ (cf. Lemma 35). First, we need however to list the properties of the eigenpair $(\widehat{\zeta}_1, \widehat{u}_1)$. The proof of the next Proposition is postponed to Section 3.4.

Proposition 20. *For every $\Delta > 0$ there exists an event $\mathcal{P}_\Delta$ satisfying $\lim_{\Delta \to 0} \mathbb{P}_{\sigma,b}(\mathcal{L} \setminus \mathcal{P}_\Delta) = 0$ such that*

$$\mathbf{1}_{\mathcal{P}_\Delta} \cdot |\widehat{\gamma}_1| \lesssim 1. \tag{19}$$

Furthermore, the eigenfunction $\widehat{u}_1$ can be chosen such that on $\mathcal{P}_\Delta$

$$\sum_{j=1}^J \widehat{u}_{1,j}^2 = J \quad and \quad \widehat{u}_{1,j} \sim 1, \quad and \quad \sum_{j=1}^J \widehat{u}_{1,j}^2 \mathbf{1}(\widehat{u}_{1,j} < 0) \lesssim 1$$

hold for any $j = \lfloor aJ \rfloor - 1, \ldots, \lceil bJ \rceil + 1$.

Remark 21. The normalization $\sum_{j=1}^J \widehat{u}_{1,j}^2 = J$ is natural, as it is equivalent to $\|\widehat{u}_1'\|_{L^2} = 1$. In short, Proposition 20 states the existence of uniform bounds on $\widehat{u}_1'|_{[a,b]}$. Because of the Neumann boundary conditions on the generator, the separation from the boundary is necessary for the existence of a lower bound.

Remark 22. From the general inequality

$$\left|1 + \log(1 - x)/x\right| \le x, \qquad 0 < x < 1/2,$$

together with the uniform bound (19) on the eigenvalue $\widehat{\gamma}_1$, we deduce that, on the high probability event $\mathcal{P}_\Delta$, $|1 + \widehat{\zeta}_1/\widehat{\gamma}_1| \lesssim \Delta$ holds. Consequently, the eigenvalue ratio $-\widehat{\zeta}_1/\widehat{\gamma}_1$ in (9) is of no importance in the high-frequency analysis.

Definition 23. Define

$$\widetilde{\sigma}^2_{S,j} = \frac{2\widehat{l}(\widehat{u}_1, \psi_j)}{\widehat{u}_{1,j} \int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx)},$$

$$\widetilde{\sigma}^2_S(x) = \sum_{j=1}^J \widetilde{\sigma}^2_{S,j} \mathbf{1}_j(x). \tag{20}$$

For simplicity, we will refer from now on to $\widetilde{\sigma}_S$ as to the spectral estimator. Comparing the representations (20) and (10) we obtain

$$\left|\widetilde{\sigma}_{S,j}^2 - \widehat{\sigma}_{\mathrm{FZ},j}^2\right| = \frac{2|\widehat{l}(\widehat{u}_1, \psi_j) - \widehat{f}(\widehat{u}_1, \psi_j)|}{\widehat{u}_{1,j} \int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx)}.$$

Since by Proposition 20 the derivative $\widehat{u}_{1,j}$ has a uniform lower bound, Lemma 18 implies that to show the convergence rate $\Delta^{1/3}$ we have to prove that

$$\left|\widehat{l}(\widehat{u}_1, \psi_j) - \widehat{f}(\widehat{u}_1, \psi_j)\right| = O_p\big(\Delta^{2/3}\big).$$

As argued in Proposition 36, for any function $v \in V_J$ with bounded derivative, it holds

$$\left|\widehat{l}(v, \psi_j) - \widehat{f}(v, \psi_j)\right| = O_p\big(\Delta^{1/2}\big),$$

which leads to a suboptimal rate $\Delta^{1/6}$. In order to achieve the optimal rate $\Delta^{1/3}$ we need to use the regularity of the first nontrivial eigenfunction $\widehat{u}_1$. By the means of the Perron–Frobenius theory, in Proposition 37, we define a high probability event $\mathcal{R}_\Delta$ such that

$$\mathbb{E}_{\sigma,b}\left[\mathbf{1}_{\mathcal{R}_\Delta} \cdot \left|\frac{\widehat{u}_1'(\frac{j}{J} \pm \frac{1}{J}) - \widehat{u}_1'(\frac{j}{J})}{J^{-1/2}}\right|^2\right]^{\frac{1}{2}} \lesssim 1$$

holds, which can be interpreted as the almost 1/2-Hölder regularity of $\widehat{u}_1'$ (see Remark 38). This regularity of the eigenfunction allows us to reduce the estimation error to an approximation problem of the occupation time, see decomposition (57) and Lemma 39.

3.3. Proof of Theorem 19

We begin with the proof of Lemma 18.

Proof of Lemma 18. Note first that, on the event $\mathcal{L}$, we have

$$v \leq J \int_{\frac{j-1}{J}}^{\frac{j}{J}} \mu_1(dx) \leq \|\mu_1\|_\infty.$$

Using the occupation formula (3), we obtain that

$$\left|\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{1}_j(X_{n\Delta}) - \int_{\frac{j-1}{J}}^{\frac{j}{J}}\mu_1(x)\,dx\right|$$

$$\leq \sum_{n=0}^{N-1}\int_{n\Delta}^{(n+1)\Delta}\left|\mathbf{1}_j(X_{n\Delta}) - \mathbf{1}_j(X_s)\right|ds$$

$$\leq \sum_{n=0}^{N-1} \int_{n\Delta}^{(n+1)\Delta} \left(\mathbf{1}\left(\left| X_s - \frac{j-1}{J} \right| < \omega(\Delta) \right) ds + \mathbf{1}\left(\left| X_s - \frac{j}{J} \right| < \omega(\Delta) \right) \right) ds$$

$$= \int_{\frac{j-1}{J}-\omega(\Delta)}^{\frac{j-1}{J}+\omega(\Delta)} \mu_1(x)\,dx + \int_{\frac{j}{J}-\omega(\Delta)}^{\frac{j}{J}+\omega(\Delta)} \mu_1(x)\,dx \leq 4\omega(\Delta)\|\mu_1\|_\infty.$$

Hence, and since $J \sim \Delta^{-1/3}$, on the event $\mathcal{O}_\Delta$

$$\Delta^{\frac{1}{3}}v \lesssim \Delta^{\frac{1}{3}}v - 4\omega(\Delta)\|\mu_1\|_\infty \lesssim \int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx) \lesssim \left(\Delta^{\frac{1}{3}} + 4\omega(\Delta)\right)\|\mu_1\|_\infty \lesssim \Delta^{\frac{1}{3}}\|\mu_1\|_\infty,$$

holds for any $\Delta < 1$. Finally, to prove that $\mathcal{O}_\Delta$ is a high probability event, note that for any $p \geq 1$, Theorem 17 together with the inequality (13) imply

$$\mathbb{P}_{\sigma,b}(\mathcal{L} \setminus \mathcal{O}_\Delta) \lesssim \Delta^{-5p/11}\mathbb{E}_{\sigma,b}\left[\omega(\Delta)^p\|\mu_1\|_\infty^p\right]$$

$$\lesssim \Delta^{-5p/11}\mathbb{E}_{\sigma,b}\left[\omega(\Delta)^{2p}\right]^{1/2}\mathbb{E}_{\sigma,b}\left[\|\mu_1\|_\infty^{2p}\right]^{1/2}$$

$$\lesssim \Delta^{-5p/11}\Delta^{p/2}\ln^{p/2}\left(\Delta^{-1}\right).$$

We obtain the claim by choosing $p \geq 15$. $\qquad\square$

Now, we are ready to prove Theorem 19. The main ideas are as in [14], Proposition 2. The novelty consists on the direct treatment of the drift term and the analysis of the boundary behaviour, which is an artifact of the reflection.

Proof of Theorem 19. Recall the definition (8) and the discussion thereafter. It follows, that it is sufficient to prove the claim for $\widehat{\sigma}_j^2(X_0, X_\Delta, \ldots, X_{N\Delta})$.

Since

$$\Delta\left(\frac{1}{2}\mathbf{1}_j(X_0) + \sum_{n=1}^{N-1} \mathbf{1}_j(X_{n\Delta}) + \frac{1}{2}\mathbf{1}_j(X_{N\Delta}) \right) = \int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx),$$

by Lemma 18, on the event $\mathcal{O}_\Delta$, the denominator of $\widehat{\sigma}_j^2(X_0, X_\Delta, \ldots, X_{N\Delta})$ has a uniform lower bound of order $\Delta^{1/3}$. Hence, in order to prove (17), we have to show that, for any $j = 2, \ldots, J-1$ and $x \in [\frac{j-1}{J}, \frac{j}{J}]$, it holds

$$\mathbb{E}_{\sigma,b}\left[\mathbf{1}_{\mathcal{O}_\Delta} \cdot \left| \sum_{n=0}^{N-1} \mathbf{1}_j(X_{n\Delta})\left((X_{(n+1)\Delta} - X_{n\Delta})^2 - \Delta\sigma^2(x)\right) \right|^2 \right]^{\frac{1}{2}}$$

$$\lesssim \Delta^{1/2}\left(\int_{\frac{(j-3)}{J}\vee 0}^{\frac{(j+2)}{J}\wedge 1} \left[(\sigma^2)'(y)\right]^2 dy \right)^{\frac{1}{2}} + \Delta^{2/3}. \tag{21}$$

Indeed, (21) implies

$$\mathbb{E}_{\sigma,b}\big[\mathbf{1}_{\mathcal{O}_\Delta}\cdot\big\|\widehat{\sigma}^2_{\mathrm{FZ}}-\sigma^2\big\|^2_{L^2[1/J,1-1/J]}\big]\lesssim\sum_{j=2}^{J-1}\frac{1}{J\Delta^{\frac{2}{3}}}\bigg(\Delta\int_{\frac{(j-3)}{J}\vee 0}^{\frac{(j+2)}{J}\wedge 1}\big[(\sigma^2)'(y)\big]^2\,dy+\Delta^{\frac{4}{3}}\bigg)$$

$$\lesssim\Delta^{\frac{2}{3}}\big(\big\|\sigma^2\big\|^2_{H^1}+1\big).$$

Step 1. Error bound in the interior. Fix $2\le j\le J-1$ and $x\in[\frac{j-1}{J},\frac{j}{J}]$. Note that on the event $\mathcal{O}_\Delta$ the condition $\mathbf{1}_j(X_{n\Delta})=1$ implies that no reflection occurs for $t\in[n\Delta,(n+1)\Delta]$. Using Itô formula, we can decompose

$$\sum_{n=0}^{N-1}\mathbf{1}_j(X_{n\Delta})\big((X_{(n+1)\Delta}-X_{n\Delta})^2-\Delta\sigma^2(x)\big):=A_1+A_2+A_3+A_4,$$

where

$$A_1=\sum_{n=0}^{N-1}\mathbf{1}_j(X_{n\Delta})\bigg[\bigg(\int_{n\Delta}^{(n+1)\Delta}\sigma(X_s)\,dW_s\bigg)^2-\int_{n\Delta}^{(n+1)\Delta}\sigma^2(X_s)\,ds\bigg],$$

$$A_2=\sum_{n=0}^{N-1}\mathbf{1}_j(X_{n\Delta})\int_{n\Delta}^{(n+1)\Delta}\big(\sigma^2(X_s)-\sigma^2(x)\big)\,ds,$$

$$A_3=\sum_{n=0}^{N-1}\mathbf{1}_j(X_{n\Delta})\bigg(\int_{n\Delta}^{(n+1)\Delta}b(X_s)\,ds\bigg)^2,$$

$$A_4=-2\sum_{n=0}^{N-1}\mathbf{1}_j(X_{n\Delta})\int_{n\Delta}^{(n+1)\Delta}\sigma(X_s)\,dW_s\int_{n\Delta}^{(n+1)\Delta}b(X_s)\,ds.$$

We will bound the second moment of each of the terms $A_1,\dots,A_4$. First, note that arguing as in the proof of Lemma 18, we obtain

$$\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{1}_j(X_{n\Delta})\le\big(\Delta^{\frac{1}{3}}+4\omega(\Delta)\big)\|\mu_1\|_\infty. \tag{22}$$

Consequently, from the Cauchy–Schwarz inequality, together with Theorem 17 and the inequality (13) it follows that

$$\mathbb{E}_{\sigma,b}\Bigg[\bigg(\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{1}_j(X_{n\Delta})\bigg)^2\Bigg]^{\frac{1}{2}}\lesssim\Delta^{\frac{1}{3}}. \tag{23}$$

Denote by $\mathcal{F}_n$ the σ-field generated by $\{X_s : 0 \leq s \leq n\Delta\}$. Let

$$\eta_n = \left(\int_{n\Delta}^{(n+1)\Delta} \sigma(X_s)\,dW_s\right)^2 - \int_{n\Delta}^{(n+1)\Delta} \sigma^2(X_s)\,ds.$$

Since $(\eta_n)_n$ are $(\mathcal{F}_n)$-martingale increments, they are conditionally uncorrelated. Using the Burkholder–Davis–Gundy inequality, we obtain that $\mathbb{E}_{\sigma,b}[\eta_n^2 | \mathcal{F}_n] \lesssim \Delta^2$. Consequently,

$$\mathbb{E}_{\sigma,b}\big[A_1^2\big]^{\frac{1}{2}} = \left(\sum_{n=0}^{N-1} \mathbb{E}_{\sigma,b}\big[\mathbf{1}_j(X_{n\Delta})\eta_n^2\big]\right)^{\frac{1}{2}} \lesssim \Delta^{\frac{1}{2}}\mathbb{E}_{\sigma,b}\left[\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{1}_j(X_{n\Delta})\right]^{\frac{1}{2}} \lesssim \Delta^{\frac{2}{3}},$$

where we used (23) to obtain the last inequality. On the event $\mathcal{O}_\Delta$, when $\mathbf{1}_j(X_{n\Delta}) = 1$, we have

$$|X_s - x| \leq |X_s - X_{n\Delta}| + |X_{n\Delta} - x| \leq \omega(\Delta) + J^{-1} \leq \Delta^{5/11} + J^{-1} \leq 2J^{-1},$$

for $n\Delta \leq s \leq (n+1)\Delta$, $x \in [\frac{j-1}{J}, \frac{j}{J}]$ and Δ small enough. Hence,

$$A_2 \leq \sum_{n=0}^{N-1} \mathbf{1}_j(X_{n\Delta}) \int_{n\Delta}^{(n+1)\Delta} \left|\int_x^{X_s} (\sigma^2)'(y)\,dy\right|\,ds$$

$$\leq \sum_{n=0}^{N-1} \mathbf{1}_j(X_{n\Delta})\Delta \left|\int_{\frac{(j-3)}{J}\vee 0}^{\frac{(j+2)}{J}\wedge 1} (\sigma^2)'(y)\,dy\right|$$

$$\lesssim \frac{1}{N}\sum_{n=0}^{N-1} \mathbf{1}_j(X_{n\Delta})\Delta^{1/6}\left(\int_{\frac{(j-3)}{J}\vee 0}^{\frac{(j+2)}{J}\wedge 1} \big[(\sigma^2)'(y)\big]^2\,dy\right)^{1/2},$$

where we used the Cauchy–Schwarz inequality. The above implies that

$$\mathbb{E}_{\sigma,b}\big[\mathbf{1}_{\mathcal{O}_\Delta} \cdot A_2^2\big]^{\frac{1}{2}} \leq \Delta^{1/2}\left(\int_{\frac{(j-3)}{J}\vee 0}^{\frac{(j+2)}{J}\wedge 1} \big[(\sigma^2)'(y)\big]^2\,dy\right)^{1/2}.$$

The drift function b is uniformly bounded, hence $|A_3| \lesssim \Delta$. Denote

$$Y_t = \int_0^t \sigma(X_s)\,dW_s \quad \text{and} \quad \omega_Y(\Delta) = \sup_{\substack{0 \leq s,t \leq 1 \\ |t-s| \leq \Delta}} |Y_t - Y_s|.$$

The uniform bound on b, together with $|Y_{(n+1)\Delta} - Y_{n\Delta}| \leq \omega_Y(\Delta)$, and the inequality (22) imply

$$\mathbb{E}_{\sigma,b}\big[\mathbf{1}_{\mathcal{O}_\Delta} \cdot A_4^2\big]^{\frac{1}{2}} \lesssim \mathbb{E}_{\sigma,b}\left[\mathbf{1}_{\mathcal{O}_\Delta} \cdot \left(\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{1}_j(X_{n\Delta})\omega_Y(\Delta)\right)^2\right]^{\frac{1}{2}}$$

$$\lesssim \mathbb{E}_{\sigma,b}\big[\big(\Delta^{\frac{1}{3}}\|\mu_1\|_\infty\omega_Y(\Delta)\big)^2\big]^{\frac{1}{2}} \lesssim \Delta^{\frac{2}{3}},$$

where we used uniform bounds on the moments of modulus of continuity of semimartingales with bounded coefficients (see Theorem 17).

Step 2. Error bound at the boundaries. Set $j = 1$ (the case $j = J$ follows analogously) and $x \in [0, 1/J]$. On $\mathcal{O}_\Delta$, whenever $X_{n\Delta} \geq \Delta^{5/11}$, no reflection occurs for $t \in [n\Delta, (n+1)\Delta]$. Denote

$$\mathbf{1}_1(x) = \mathbf{1}(x < \Delta^{5/11}) + \mathbf{1}(\Delta^{5/11} \leq x < J^{-1}) := \mathbf{1}_{1,0}(x) + \mathbf{1}_{1,1}(x).$$

We decompose

$$\sum_{n=0}^{N-1} \mathbf{1}_1(X_{n\Delta})\big((X_{(n+1)\Delta} - X_{n\Delta})^2 - \Delta\sigma^2(x)\big) := E_1 + E_2,$$

with

$$E_1 = \sum_{n=0}^{N-1} \mathbf{1}_{1,0}(X_{n\Delta})\big((X_{(n+1)\Delta} - X_{n\Delta})^2 - \Delta\sigma^2(x)\big),$$

$$E_2 = \sum_{n=0}^{N-1} \mathbf{1}_{1,1}(X_{n\Delta})\big((X_{(n+1)\Delta} - X_{n\Delta})^2 - \Delta\sigma^2(x)\big).$$

On $\mathcal{O}_\Delta$ holds $|(X_{(n+1)\Delta} - X_{n\Delta})^2 - \Delta\sigma^2(x)| \lesssim \Delta^{10/11}$. Hence, arguing as in the proof of Lemma 18, we obtain that

$$\mathbb{E}_{\sigma,b}\big[\mathbf{1}_{\mathcal{O}_\Delta} \cdot E_1^2\big]^{\frac{1}{2}} \lesssim \Delta^{-\frac{1}{11}} \mathbb{E}_{\sigma,b}\left[\mathbf{1}_{\mathcal{O}_\Delta} \cdot \left(\frac{1}{N}\sum_{n=0}^{N-1} \mathbf{1}_{1,0}(X_{n\Delta})\right)^2\right]^{\frac{1}{2}}$$

$$\lesssim \Delta^{-\frac{1}{11}} \mathbb{E}_{\sigma,b}\left[\mathbf{1}_{\mathcal{O}_\Delta} \cdot \left(\int_0^1 \mathbf{1}_{1,0}(x)\mu_1(x)\,dx + 4\omega(\Delta)\|\mu_1\|_\infty\right)^2\right]^{\frac{1}{2}} \lesssim \Delta^{\frac{4}{11}}.$$

To bound the second moment of E_2, note that when $\mathbf{1}_{1,1}(X_{n\Delta}) = 1$ no reflection occurs for $t \in [n\Delta, (n+1)\Delta]$. Consequently, we can proceed as in Step 1, obtaining

$$\mathbb{E}_{\sigma,b}\big[\mathbf{1}_{\mathcal{O}_\Delta} \cdot E_2^2\big]^{\frac{1}{2}} \lesssim \Delta^{1/2}\left(\int_0^{\frac{3\wedge J}{J}} \big[(\sigma^2)'(y)\big]^2\,dy\right)^{\frac{1}{2}} + \Delta^{2/3} \lesssim \Delta^{1/2}\|\sigma^2\|_{H^1}.$$

We conclude that

$$\mathbb{E}_{\sigma,b}\big[\mathbf{1}_{\mathcal{O}_\Delta} \cdot |\widehat{\sigma}_{FZ}^2(x) - \sigma^2(x)|^2\big]^{\frac{1}{2}} \lesssim \Delta^{-1/3}\mathbb{E}_{\sigma,b}\big[\mathbf{1}_{\mathcal{O}_\Delta} \cdot (E_1 + E_2)^2\big]^{\frac{1}{2}} \lesssim \Delta^{1/33}. \qquad \square$$

Corollary 24. *For every Δ there exists an event $\widetilde{O}_\Delta$ satisfying $\lim_{\Delta \to 0} \mathbb{P}_{\sigma,b}(\mathcal{L} \setminus \widetilde{O}_\Delta) = 0$, such that on $\widetilde{O}_\Delta$ it holds*

$$(\forall x \in [0,1]) \qquad (1 - 2^{-1/2})d \leq \widehat{\sigma}_{FZ}^2(x) \leq (1 + 2^{-1/2})2D. \tag{24}$$

Proof. Since

$$\left\|\widehat{\sigma}_{FZ}^2 - \sigma^2\right\|_{L^2[0,1]}^2 = \left\|\widehat{\sigma}_{FZ}^2 - \sigma^2\right\|_{L^2[1/J,1-1/J]}^2 + \int_0^{\frac{1}{J}} \left|\sigma_{FZ}^2(x) - \sigma^2(x)\right|^2 dx$$

$$+ \int_{1-\frac{1}{J}}^1 \left|\sigma_{FZ}^2(x) - \sigma^2(x)\right|^2 dx,$$

using Theorem 19 we obtain that

$$\mathbb{E}_{\sigma,b}\left[\mathbf{1}_{\mathcal{O}_\Delta} \cdot \left\|\widehat{\sigma}_{FZ}^2 - \sigma^2\right\|_{L^2[0,1]}^2\right] \lesssim \Delta^{2/3} + J^{-1}\Delta^{2/33} \lesssim \Delta^{1/3+2/33}.$$

Define

$$\widetilde{O}_\Delta = \mathcal{O}_\Delta \cap \left\{\left\|\widehat{\sigma}_{FZ}^2 - \sigma^2\right\|_{L^2[0,1]}^2 \le (2J)^{-1} \inf_{x\in[0,1]} \sigma^4(x)\right\}.$$

From Markov's inequality, together with the lower bound on the probability of the event $\mathcal{O}_\Delta$, it follows that

$$\mathbb{P}_{\sigma,b}(\mathcal{L} \setminus \widetilde{O}_\Delta) \lesssim \Delta^{2/3} + \Delta^{2/33} \to 0 \qquad \text{as } \Delta \to 0.$$

Fix $j = 1, \ldots, J$. On the event $\widetilde{O}_\Delta$ it holds

$$\left|\widehat{\sigma}_{FZ,j}^2 - J\int_{\frac{j-1}{J}}^{\frac{j}{J}} \sigma^2(x)\,dx\right| \le J\int_{\frac{j-1}{J}}^{\frac{j}{J}} \left|\widehat{\sigma}_{FZ,j}^2 - \sigma^2(x)\right| dx \le J^{1/2}\left(\int_{\frac{j-1}{J}}^{\frac{j}{J}} \left|\sigma^2(x) - \widehat{\sigma}_{FZ,j}^2\right|^2 dx\right)^{\frac{1}{2}}$$

$$\le J^{1/2}\left\|\widehat{\sigma}_{FZ}^2 - \sigma^2\right\|_{L^2[0,1]} \le 2^{-1/2} \inf_{x\in[0,1]} \sigma^2(x)$$

$$\le 2^{-1/2} J \int_{\frac{j-1}{J}}^{\frac{j}{J}} \sigma^2(x)\,dx.$$

Since Assumption 1 implies $d \le J\int_{\frac{j-1}{J}}^{\frac{j}{J}} \sigma^2(x)\,dx \le 2D$, we conclude that the claim follows. $\square$

3.4. Properties of the eigenpair $(\widehat{\gamma}_1, \widehat{u}_1)$

In this section, we want to prove Proposition 20. Because of the tridiagonal structure of the form $\widehat{l}$, the direct analysis of the eigenfunction $\widehat{u}_1$ is difficult. Instead, we consider the generalized eigenvalue problem for forms $\widehat{f}$ (recall Definition 14) and $\widehat{g}$:

Eigenproblem 25. Find $(\widehat{\lambda}, \widehat{w}) \in \mathbb{R} \times V_J^0$, with $\widehat{w} \ne 0$, such that

$$\widehat{f}(\widehat{w}, v) = \widehat{\lambda}\widehat{g}(\widehat{w}, v) \qquad \text{for every function } v \in V_J^0.$$

On the high probability event $\widetilde{O}_\Delta$ such that $\widehat{\sigma}^2_{\text{FZ}} \sim 1$ (see Corollary 24), the form $\widehat{f}$ is positive-definite and symmetric. Consequently, on $\widetilde{O}_\Delta$ the Eigenproblem 25 has J solutions $(\widehat{\lambda}_j, \widehat{w}_j)_{j=1,\dots,J}$ with $0 < \widehat{\lambda}_1 \leq \widehat{\lambda}_2 \leq \cdots \leq \widehat{\lambda}_J$.

Definition 26. For $j = 1, \dots, J$ define $\psi_j^0 = \psi_j - \int_0^1 \psi_j(x)\widehat{\mu}_N(dx) \in V_J^0$. Let

$$\widehat{F}_{i,j} := \widehat{f}(\psi_i^0, \psi_j^0) = \widehat{f}(\psi_i, \psi_j) \quad \text{and} \quad \widehat{M}_{i,j} = \widehat{g}(\psi_i^0, \psi_j^0)$$

be the matrix representations of forms $\widehat{f}$ and $\widehat{g}$ on $V_J^0 \times V_J^0$ with respect to the algebraic basis $(\psi_j^0)_j$.

Arguing as in Gobet et al. [11], Lemma 6.1, we obtain that

$$\widehat{M}_{i,j} = \int_{\frac{i-1}{J}}^{\frac{i}{J}} \int_{\frac{j-1}{J}}^{\frac{j}{J}} \int_0^{y \wedge z} \widehat{\mu}_N(dx) \int_{y \vee z}^1 \widehat{\mu}_N(dx)\, dy\, dz. \tag{25}$$

$\widehat{F}$ is a diagonal matrix with strictly positive diagonal entries, hence it is invertible. Eigenproblem 25 is equivalent to

$$\widehat{F}^{-1}\widehat{M}(\widehat{w}_{i,j})_j = \widehat{\lambda}_i^{-1}(\widehat{w}_{i,j})_j,$$

where $(\widehat{w}_{i,j})_{j=1,\dots,J}$ indicates the coefficient vector associated to the eigenfunction $\widehat{w}_i$, i.e.

$$\widehat{w}_i = \sum_{j=1}^J \widehat{w}_{i,j}\psi_j^N = \sum_{j=1}^J \widehat{w}_{i,j}\psi_j + \widehat{w}_{i,0},$$

with some $\widehat{w}_{i,0}$ such that $\widehat{w}_i \in V_J^0$. On the event $\mathcal{O}_\Delta$ the matrix $\widehat{M}$ has all entries strictly positive, hence $\widehat{F}^{-1}\widehat{M}$ satisfies the conditions of the Perron–Frobenius theorem. Consequently, the eigenvector $(\widehat{w}_{1,j})_j$ can be chosen strictly positive, which corresponds to the monotonicity property of the eigenfunction $\widehat{w}_1$. In what follows, we will show that the Eigenproblem 25 is an approximation of the Eigenproblem 5 for forms $\widehat{l}$ and $\widehat{g}$, and deduce that the eigenfunction $\widehat{u}_1$ inherits the properties of $\widehat{w}_1$. Let $\| \cdot \|_{l^2}$ denote the standard Euclidean norm on $\mathbb{R}^J$.

Definition 27. Theorem 17, Corollary 24, Theorem 19 and the regularity properties of the occupation density μ_1 ensure that for $\alpha = 1/60$ the set $\mathcal{Q}_\Delta$ of paths contained in $\mathcal{L}$ that satisfy

(i) $\omega(\Delta) \leq \Delta^{1/2-\alpha}$;
(ii) for every $x \in (0, 1)$ holds $(1 - 2^{-1/2})d \leq \widehat{\sigma}^2_{\text{FZ}}(x) \leq (1 + 2^{-1/2})2D$;
(iii) $\|\widehat{\sigma}^2_{\text{FZ}} - \sigma^2\|_{L^2([1/J, 1-1/J])} \leq \Delta^{1/3-\alpha}$ and for $x \in [0, 1/J] \cup [1 - 1/J, 1]$ it holds $|\widehat{\sigma}^2_{\text{FZ}}(x) - \sigma^2(x)| \leq \Delta^{1/33-\alpha}$;
(iv) occupation density μ_1 is $1/2 - \alpha$ Hölder continuous with Hölder norm bounded by $\Delta^{-\alpha}$;

is of high-probability, more precisely

$$\lim_{\Delta \to 0} \mathbb{P}_{\sigma,b}(\mathcal{L} \setminus \mathcal{Q}_\Delta) = 0.$$

Remark 28. By the assumption (iv), on $\mathcal{Q}_\Delta$ we have $\|\mu_1\|_\infty \lesssim \Delta^{-\alpha}$. Furthermore, arguing as in the proof of Lemma 18, we obtain

$$\left| \int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx) - \int_{\frac{j-1}{J}}^{\frac{j}{J}} \mu_1(dx) \right| \lesssim \omega(\Delta) \|\mu_1\|_\infty \lesssim \Delta^{1/2 - 2\alpha}. \tag{26}$$

In particular, on $\mathcal{Q}_\Delta$

$$\int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx) \sim \Delta^{1/3} \qquad \text{holds for every } j = 1, \ldots, J. \tag{27}$$

Finally, the assumption (iii) and Hölder regularity of σ^2 (12) imply that on $\mathcal{Q}_\Delta$

$$\left| \widehat{\sigma}_{\text{FZ}}^2(x) - \sigma^2(x) \right| \lesssim \Delta^{1/6 - \alpha} \qquad \text{for all } x \in [1/J, 1 - 1/J]. \tag{28}$$

Proof of (28). Fix $j = 2, \ldots, J-1$ and $x \in [\frac{j-1}{J}, \frac{j}{J}]$. We have

$$\left| \widehat{\sigma}_{\text{FZ},j}^2 - \sigma^2(x) \right| \leq \left| \widehat{\sigma}_{\text{FZ},j}^2 - J \int_{\frac{j-1}{J}}^{\frac{j}{J}} \sigma^2(y)\,dy \right| + \left| \sigma^2(x) - J \int_{\frac{j-1}{J}}^{\frac{j}{J}} \sigma^2(y)\,dy \right|$$

$$\lesssim J \int_{\frac{j-1}{J}}^{\frac{j}{J}} \left| \widehat{\sigma}_{\text{FZ},j}^2 - \sigma^2(y) \right| dy + \Delta^{1/6}$$

$$\lesssim J^{1/2} \left(\int_{\frac{1}{J}}^{1 - \frac{1}{J}} \left| \widehat{\sigma}_{\text{FZ},j}^2 - \sigma^2(y) \right|^2 dy \right)^{\frac{1}{2}} + \Delta^{1/6}$$

$$\lesssim J^{1/2} \left\| \widehat{\sigma}_{\text{FZ}}^2 - \sigma^2 \right\|_{L^2([1/J, 1-1/J])} + \Delta^{1/6},$$

where we used the 1/2-Hölder regularity of σ^2 (12) and the Cauchy–Schwarz inequality to get the second and third inequalities respectively. Finally, since $J^{1/2} \sim \Delta^{-1/6}$, we conclude by Definition 27(iii) that the claim holds. $\qquad \square$

To bound the error between the solutions of the Eigenproblems 5 and 25 we need to establish uniform bounds on the spectral gap of the Eigenproblem 25.

Lemma 29. *On the event $\mathcal{Q}_\Delta$ the eigenvalue $\widehat{\lambda}_1$ is uniformly bounded by positive constants from below and above. Furthermore, the Eigenproblem 25 has a uniform spectral gap, i.e. $\widehat{\lambda}_1^{-1} - \widehat{\lambda}_2^{-1} \gtrsim 1$.*

Proof. Consider the generalized eigenvalue problem:

Eigenproblem 30. Find $(\lambda, w) \in \mathbb{R} \times V_J$ with $w \neq 0$ and $\int_0^1 w(x)\mu_1(x)\,dx = 0$ such that

$$\int_0^1 w'(x)v'(x)\sigma^2(x)\mu_1(x)\,dx = \lambda \int_0^1 w(x)v(x)\mu_1(x)\,dx$$

for all $v \in \left\{ v \in V_J : \int_0^1 v(x)\mu_1(x)\,dx = 0 \right\}$.

Eigenproblem 30 has J solutions, denoted by (λ_j, w_j) with $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_J$. By Proposition B.7 we have $\lambda_1 \approx 1$ and $\lambda_1^{-1} - \lambda_2^{-1} \gtrsim 1$. Let M, F be $J \times J$ matrices corresponding to the Eigenproblem 30 tested with functions $(\psi_j^1)_{j=1,\dots,J}$, where $\psi_j^1 = \psi_J - \int_0^1 \psi_j \mu_1(x)\,dx$. As in the case of the data driven Eigenproblem 25, we have

$$M_{i,j} = \int_{\frac{i-1}{J}}^{\frac{i}{J}} \int_{\frac{j-1}{J}}^{\frac{j}{J}} \int_0^{y \wedge z} \mu_1(x)\,dx \int_{y \vee z}^1 \mu_1(x)\,dx\,dy\,dz,$$

$$F_{i,j} = \begin{cases} 0, & i \neq j, \\ \int_{\frac{i-1}{J}}^{\frac{i}{J}} \sigma^2(x)\mu_1(x)\,dx, & i = j \end{cases}$$

and

$$F^{-1}M(w_{i,j})_j = \lambda_i^{-1}(w_{i,j})_j.$$

From Weyl's theorem for symmetric eigenvalue problems it follows that

$$\left| \widehat{\lambda}_i^{-1} - \lambda_i^{-1} \right| \leq \left\| F^{-1}M - \widehat{F}^{-1}\widehat{M} \right\|_{l^2}. \tag{29}$$

We will show that $\|F^{-1}M - \widehat{F}^{-1}\widehat{M}\|_{l^2} \lesssim \Delta^{1/6-2\alpha}$. Then, the uniform bound on the eigenvalue $\widehat{\lambda}_1$ and the lower bound on the spectral gap will follow from the properties of the Eigenproblem 30. First, let us observe that

$$|\widehat{F}_{j,j} - F_{j,j}| = \left| \int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\sigma}_{\mathrm{FZ},j}^2 \widehat{\mu}_N(dx) - \int_{\frac{j-1}{J}}^{\frac{j}{J}} \sigma^2(x)\mu_1(x)\,dx \right|$$

$$\lesssim \widehat{\sigma}_{\mathrm{FZ},j}^2 \left| \int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx) - \int_{\frac{j-1}{J}}^{\frac{j}{J}} \mu_1(x)\,dx \right| + \int_{\frac{j-1}{J}}^{\frac{j}{J}} |\widehat{\sigma}_{\mathrm{FZ}}^2(x) - \sigma^2(x)|\mu_1(x)\,dx.$$

On $\mathcal{Q}_\Delta$, the first term of the above display is by (26) of order $\Delta^{1/2-2\alpha}$. The second term is for $j = 2, \dots, J-1$ of order $\Delta^{1/2-\alpha}$ by (28) and for $j = 1$ or J of order $\Delta^{1/3+1/33-\alpha}$ by Definition 27(iii) directly. In any case, for Δ small enough it holds $\widehat{F}_{j,j} \approx F_{j,j} \approx \Delta^{1/3}$. Next, arguing as in the proof of Lemma 18, for any $i, j = 1, \dots, J$, we obtain

$$|M_{i,j} - \widehat{M}_{i,j}| \lesssim J^{-2}\omega(\Delta)\|\mu_1\|_\infty \lesssim \Delta^{7/6-2\alpha}.$$

Furthermore, $M_{i,j}, \widehat{M}_{i,j} \lesssim \Delta^{2/3}$. Since F and $\widehat{F}$ are diagonal matrices, it follows that

$$\left| \left(F^{-1}M - \widehat{F}^{-1}\widehat{M} \right)_{i,j} \right| \lesssim \Delta^{1/2 - 2\alpha}.$$

Hence, $\| F^{-1}M - \widehat{F}^{-1}\widehat{M} \|_{l^2}^2 \leq \sum_{i,j=1}^{J} (F^{-1}M - \widehat{F}^{-1}\widehat{M})_{i,j}^2 \lesssim \Delta^{1/3 - 4\alpha}.$ $\qquad\square$

Proposition 31. *Choose the eigenfunction $\widehat{w}_1$ increasing and normalized so that $\|(\widehat{w}_{1,j})_j\|_{l^2} = J^{1/2}$ (i.e. $\|\widehat{w}_1'\|_{L^2} = 1$). On the event $\mathcal{Q}_\Delta$, for any $\lfloor aJ \rfloor - 1 \leq j \leq \lceil bJ \rceil + 1$ and any $i = 1, \dots, J$ we have*

$$1 \vee \widehat{w}_{1,i} \lesssim \widehat{w}_{1,j} \wedge 1, \tag{30}$$

Furthermore for j s.t. $J^{1/2} \leq j \leq J - J^{1/2}$

$$\left| \frac{\widehat{w}_{1,j\pm 1}}{\widehat{w}_{1,j}} - 1 \right| \lesssim \Delta^{1/6 - 2\alpha}. \tag{31}$$

Proof. In the proof, we will use standard techniques from the Perron-Frobenius theory of non-negative matrices (cf. Minc [20], Chapter II). In particular, we shall repeatedly use the following inequality Minc [20], Chapter II, Section 2.1, Eq. (7): for any $q_1, q_2, \dots, q_n > 0$ and $p_1, p_2, \dots, p_n \in \mathbb{R}$

$$\min_{i=1,\dots,n} \frac{p_i}{q_i} \leq \frac{p_1 + p_2 + \cdots + p_n}{q_1 + q_2 + \cdots + q_n} \leq \max_{i=1,\dots,n} \frac{p_i}{q_i}. \tag{32}$$

Step 1: ($\widehat{w}_{1,i} \lesssim 1$). Fix $1 \leq i \leq J$. By Definition 27(ii), relation (27) and Lemma 29, on the event $\mathcal{Q}_\Delta$, we have

$$J^{-1}\widehat{w}_{1,i} \approx \widehat{w}_{1,i}\widehat{\sigma}_{\text{FZ},i}^2 \int_{\frac{i-1}{J}}^{\frac{i}{J}} \mu_N(dx) = \widehat{f}(\widehat{w}_1, \psi_i) = \widehat{\lambda}_1 \widehat{g}(\widehat{w}_1, \psi_i)$$

$$\approx \widehat{g}(\widehat{w}_1, \psi_i) = \sum_{m=1}^{J} \widehat{M}_{i,m}\widehat{w}_{1,m}. \tag{33}$$

Hence, by the Cauchy–Schwarz inequality

$$\widehat{w}_{1,i} \lesssim J \left(\sum_{m=1}^{J} M_{i,m}^2 \right)^{1/2} \left(\sum_{m=1}^{J} \widehat{w}_{1,m}^2 \right)^{1/2} \lesssim 1,$$

where we used $M_{i,m} \leq J^{-2}$ and the normalization of $(\widehat{w}_{1,j})$.

Step 2: ($\widehat{w}_{1,i} \lesssim \widehat{w}_{1,j}$). Fix $\lfloor aJ \rfloor - 1 \leq j \leq \lceil bJ \rceil + 1$. On the event $\mathcal{Q}_\Delta$, for any $1 \leq i \leq J$ the relation (33) together with the inequality (32) imply

$$\frac{\widehat{w}_{1,i}}{\widehat{w}_{1,j}} \approx \frac{\sum_{m=1}^{J} \widehat{M}_{i,m}\widehat{w}_{1,m}}{\sum_{m=1}^{J} \widehat{M}_{j,m}\widehat{w}_{1,m}} \lesssim \max_{m=1,\dots,J} \frac{\widehat{M}_{i,m}}{\widehat{M}_{j,m}}. \tag{34}$$

We need to show that for arbitrary m $\widehat{M}_{i,m}/\widehat{M}_{j,m} \lesssim 1$ holds. Consider first the case $i < j$. Then by (25)

$$\frac{\widehat{M}_{i,m}}{\widehat{M}_{j,m}} = \frac{\int_{\frac{i-1}{J}}^{\frac{i}{J}} \int_{\frac{m-1}{J}}^{\frac{m}{J}} \int_0^{y \wedge z} \widehat{\mu}_N(dx) \int_{y \vee z}^1 \widehat{\mu}_N(dx)\, dy\, dz}{\int_{\frac{j-1}{J}}^{\frac{j}{J}} \int_{\frac{m-1}{J}}^{\frac{m}{J}} \int_0^{y \wedge z} \widehat{\mu}_N(dx) \int_{y \vee z}^1 \widehat{\mu}_N(dx)\, dy\, dz}$$

$$\leq \frac{\int_{\frac{i-1}{J}}^{\frac{i}{J}} \int_{\frac{m-1}{J}}^{\frac{m}{J}} \int_0^{y \wedge \frac{i-1}{J}} \widehat{\mu}_N(dx) \int_{y \vee \frac{i-1}{J}}^1 \widehat{\mu}_N(dx)\, dy\, dz}{\int_{\frac{j-1}{J}}^{\frac{j}{J}} \int_{\frac{m-1}{J}}^{\frac{m}{J}} \int_0^{y \wedge \frac{i-1}{J}} \widehat{\mu}_N(dx) \int_{y \vee \frac{j}{J}}^1 \widehat{\mu}_N(dx)\, dy\, dz}$$

$$= \frac{\int_{\frac{m-1}{J}}^{\frac{m}{J}} f(y) \int_{y \vee \frac{i-1}{J}}^1 \widehat{\mu}_N(dx)\, dy}{\int_{\frac{m-1}{J}}^{\frac{m}{J}} f(y) \int_{y \vee \frac{j}{J}}^1 \widehat{\mu}_N(dx)\, dy},$$

where $f(y) = \int_0^{y \wedge \frac{i-1}{J}} \widehat{\mu}_N(dx)$. Consider $m > j$. For $y \in [\frac{m-1}{J}, \frac{m}{J}]$ holds $y = y \vee \frac{j}{J} = y \vee \frac{i-1}{J}$, hence the numerator and denominator are equal. Consider $m \leq j$. For $y \in [\frac{m-1}{J}, \frac{m}{J}]$ holds $y \vee \frac{j}{J} = \frac{j}{J}$. Hence, using (27), we obtain

$$\frac{\widehat{M}_{i,m}}{\widehat{M}_{j,m}} \leq \frac{\int_{\frac{m-1}{J}}^{\frac{m}{J}} f(y) \int_{y \vee \frac{i-1}{J}}^1 \widehat{\mu}_N(dx)\, dy}{\int_{\frac{m-1}{J}}^{\frac{m}{J}} f(y) \int_{y \vee \frac{j}{J}}^1 \widehat{\mu}_N(dx)\, dy}$$

$$\leq \frac{\int_{\frac{m-1}{J}}^{\frac{m}{J}} f(y)\, dy}{\int_{\frac{m-1}{J}}^{\frac{m}{J}} f(y)\, dy \int_{\frac{j}{J}}^1 \widehat{\mu}_N(dx)}$$

$$= \left(\int_{\frac{j}{J}}^1 \widehat{\mu}_N(dx) \right)^{-1} \approx \left(1 - \frac{j}{J} \right)^{-1} \lesssim 1.$$

We conclude that for $i < j$ and arbitrary m bound $\widehat{M}_{i,m}/\widehat{M}_{j,m} \lesssim 1$ holds. Proceeding analogously, we obtain the same claim for $i > j$. From (34), it follows that on the event $\mathcal{Q}_\Delta$, for $\lfloor aJ \rfloor - 1 \leq j \leq \lceil bJ \rceil + 1$ and any $1 \leq i \leq J$, we have

$$\widehat{w}_{1,i} \lesssim \widehat{w}_{1,j}. \tag{35}$$

Step 3: ($1 \lesssim \widehat{w}_{1,j}$). Let $\widehat{w}_{1,j_0} = \min_{\lfloor aJ \rfloor - 1 \leq j \leq \lceil bJ \rceil + 1} \widehat{w}_{1,j}$. Inequality (35) implies

$$1 = \frac{1}{J} \sum_{i=1}^J \widehat{w}_{1,i}^2 \lesssim \widehat{w}_{1,j_0}^2.$$

Step 4: (*proof of* (31)). We will only show $|\frac{\widehat{w}_{1,j+1}}{\widehat{w}_{1,j}} - 1| \lesssim \Delta^{1/6-2\alpha}$, the other bound can be obtained by a symmetric argument. First, note that from Definition 27(iv) together with the inequality (26) it follows that

$$
\left| \int_{\frac{j}{J}}^{\frac{j+1}{J}} \widehat{\mu}_N(dx) - \int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx) \right|
$$

$$
\leq \left| \int_{\frac{j}{J}}^{\frac{j+1}{J}} \widehat{\mu}_N(dx) - \int_{\frac{j}{J}}^{\frac{j+1}{J}} \mu_1(x)\, dx \right|
$$

$$
+ \left| \int_{\frac{j}{J}}^{\frac{j+1}{J}} \mu_1(x)\, dx - \int_{\frac{j-1}{J}}^{\frac{j}{J}} \mu_1(x)\, dx \right|
$$

$$
+ \left| \int_{\frac{j-1}{J}}^{\frac{j}{J}} \mu_1(x)\, dx - \int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx) \right|
$$

$$
\lesssim \Delta^{1/2-2\alpha} + \Delta^{1/3}\Delta^{-\alpha}\Delta^{(1/2-\alpha)/3} + \Delta^{1/2-2\alpha} \lesssim \Delta^{1/2-2\alpha}.
$$

Hence, by (27)

$$
\left| \frac{\int_{\frac{j}{J}}^{\frac{j+1}{J}} \widehat{\mu}_N(dx)}{\int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx)} - 1 \right| \lesssim \Delta^{1/6-2\alpha}.
$$

Similarly, by the $1/2$-Hölder regularity of σ^2 and (28) we have

$$
\left| \frac{\widehat{\sigma}^2_{\mathrm{FZ},j+1}}{\widehat{\sigma}^2_{\mathrm{FZ},j}} - 1 \right| \lesssim \Delta^{1/6-2\alpha}.
$$

Consequently, instead of $\frac{\widehat{w}_{1,j+1}}{\widehat{w}_{1,j}}$ we may consider

$$
\frac{\widehat{w}_{1,j+1}\widehat{\sigma}^2_{\mathrm{FZ},j+1} \int_{\frac{j}{J}}^{\frac{j+1}{J}} \widehat{\mu}_N(dx)}{\widehat{w}_{1,j}\widehat{\sigma}^2_{\mathrm{FZ},j} \int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx)} = \frac{\sum_{m=1}^{J} \widehat{M}_{j+1,m}\widehat{w}_{1,m}}{\sum_{m=1}^{J} \widehat{M}_{j,m}\widehat{w}_{1,m}}.
$$

By the inequality (32)

$$
\min_{m=1,\dots,J} \frac{\widehat{M}_{j+1,m}}{\widehat{M}_{j,m}} \leq \frac{\sum_{m=1}^{J} \widehat{M}_{j+1,m}\widehat{w}_{1,m}}{\sum_{m=1}^{J} \widehat{M}_{j,m}\widehat{w}_{1,m}} \leq \max_{m=1,\dots,J} \frac{\widehat{M}_{j+1,m}}{\widehat{M}_{j,m}}.
$$

Thus, it is enough to show, that for any $m = 1, \ldots, J$ bound $|\frac{\widehat{M}_{j+1,m}}{\widehat{M}_{j,m}} - 1| \lesssim \Delta^{1/6}$ holds.

$$
\begin{aligned}
\frac{\widehat{M}_{j+1,m}}{\widehat{M}_{j,m}} &= \frac{\int_{\frac{j}{J}}^{\frac{j+1}{J}} \int_{\frac{m-1}{J}}^{\frac{m}{J}} \int_0^{y \wedge z} \widehat{\mu}_N(dx) \int_{y \vee z}^1 \widehat{\mu}_N(dx) \, dy \, dz}{\int_{\frac{j-1}{J}}^{\frac{j}{J}} \int_{\frac{m-1}{J}}^{\frac{m}{J}} \int_0^{y \wedge z} \widehat{\mu}_N(dx) \int_{y \vee z}^1 \widehat{\mu}_N(dx) \, dy \, dz} \\[2mm]
&\leq \frac{\int_{\frac{j}{J}}^{\frac{j+1}{J}} \int_{\frac{m-1}{J}}^{\frac{m}{J}} \int_0^{y \wedge \frac{j+1}{J}} \widehat{\mu}_N(dx) \int_{y \vee \frac{j}{J}}^1 \widehat{\mu}_N(dx) \, dy \, dz}{\int_{\frac{j-1}{J}}^{\frac{j}{J}} \int_{\frac{m-1}{J}}^{\frac{m}{J}} \int_0^{y \wedge \frac{j-1}{J}} \widehat{\mu}_N(dx) \int_{y \vee \frac{j}{J}}^1 \widehat{\mu}_N(dx) \, dy \, dz} \\[2mm]
&\leq \frac{\int_{\frac{m-1}{J}}^{\frac{m}{J}} \int_0^{y \wedge \frac{j+1}{J}} \widehat{\mu}_N(dx) f(y) \, dy}{\int_{\frac{m-1}{J}}^{\frac{m}{J}} \int_0^{y \wedge \frac{j-1}{J}} \widehat{\mu}_N(dx) f(y) \, dy} = 1 + \frac{\int_{\frac{m-1}{J}}^{\frac{m}{J}} \int_{y \wedge \frac{j-1}{J}}^{y \wedge \frac{j+1}{J}} \widehat{\mu}_N(dx) f(y) \, dy}{\int_{\frac{m-1}{J}}^{\frac{m}{J}} \int_0^{y \wedge \frac{j-1}{J}} \widehat{\mu}_N(dx) f(y) \, dy},
\end{aligned}
$$

where $f(y) = \int_{y \vee \frac{j}{J}}^1 \widehat{\mu}_N(dx)$. Consider $m \leq j - 1$. For $y \in [\frac{m-1}{J}, \frac{m}{J}]$ we have $y = y \wedge \frac{j+1}{J} = y \wedge \frac{j-1}{J}$, hence the error term is zero. Consider $m \geq j$. For $y \in [\frac{m-1}{J}, \frac{m}{J}]$, we have $y \wedge \frac{j-1}{J} = \frac{j-1}{J}$. Consequently, using (27), we obtain that for $j \geq J^{1/2} \approx \Delta^{-1/6}$

$$
\frac{\int_{\frac{m-1}{J}}^{\frac{m}{J}} \int_{y \wedge \frac{j-1}{J}}^{y \wedge \frac{j+1}{J}} \widehat{\mu}_N(dx) f(y) \, dy}{\int_{\frac{m-1}{J}}^{\frac{m}{J}} \int_0^{y \wedge \frac{j-1}{J}} \widehat{\mu}_N(dx) f(y) \, dy} \leq \frac{\int_{\frac{m-1}{J}}^{\frac{m}{J}} \int_{\frac{j-1}{J}}^{\frac{j+1}{J}} \widehat{\mu}_N(dx) f(y) \, dy}{\int_{\frac{m-1}{J}}^{\frac{m}{J}} \int_0^{\frac{j-1}{J}} \widehat{\mu}_N(dx) f(y) \, dy}
$$

$$
\approx \frac{2 J^{-1}}{\frac{j-1}{J}} = \frac{2}{j-1} \lesssim \Delta^{1/6}.
$$

Finally, symmetric bound $1 - \frac{\widehat{M}_{j+1,m}}{\widehat{M}_{j,m}} \lesssim \Delta^{1/6}$ can be obtained by similar calculations. □

In previous proposition, we have established uniform bounds on the eigenfunction $\widehat{w}_1$. Next, we show that $\widehat{w}_1$ is a good approximation of $\widehat{u}_1$.

Definition 32. Let $\widehat{L}$ be the matrix representation of the form $\widehat{l}$ with respect to the algebraic basis $(\psi_j^0)_j$ (see Definition 26), that is,

$$
\widehat{L}_{i,j} := \widehat{l}\big(\psi_i^0, \psi_j^0\big) = \widehat{l}(\psi_i, \psi_j).
$$

On the event $\mathcal{Q}_\Delta$, when Δ is sufficiently small, $\widehat{L}$ is symmetric tridiagonal. We want to bound the error between the solutions of the generalized eigenproblems:

$$
\widehat{M}(\widehat{w}_i) = \widehat{\lambda}_1^{-1} \widehat{F}(\widehat{w}_i) \quad \text{and} \quad \widehat{M}(\widehat{u}_i) = \widehat{\gamma}_1^{-1} \widehat{L}(\widehat{u}_i).
$$

Lemma 33. *On the event $\mathcal{Q}_\Delta$ holds*

$$\|\widehat{F} - \widehat{L}\|_{l^2} \lesssim \Delta^{1/2-4\alpha}. \tag{36}$$

Furthermore matrix $\widehat{L}$ is invertible and

$$\|\widehat{L}\|_{l^2}, \|\widehat{F}\|_{l^2}, \|\widehat{L}^{-1}\|_{l^2}^{-1}, \|\widehat{F}^{-1}\|_{l^2}^{-1} \approx \Delta^{1/3}.$$

Proof. Consider vector $(v_j)_j \in \mathbb{R}^J$ with $\|(v_j)_j\|_{l^2} = 1$ and the corresponding function $v = \sum_{j=1}^{J} v_j \psi_j^0(x) \in V_J^0$. Since

$$\left\|(\widehat{F} - \widehat{L})v\right\|_{l^2}^2 = \sum_{j=1}^{J} \left|\widehat{f}(v, \psi_j) - \widehat{l}(v, \psi_j)\right|^2$$

$$= \sum_{j=1}^{J} \left(v_{j-1}\widehat{L}_{j-1,j} + v_j(\widehat{F}_{j,j} - \widehat{L}_{j,j}) + v_{j+1}\widehat{L}_{j+1,j}\right)^2,$$

to obtain (36), we just have to argue that $\widehat{L}_{j-1,j}, |\widehat{F}_{j,j} - \widehat{L}_{j,j}|$ and $\widehat{L}_{j+1,j}$ are of order $\Delta^{1/2-4\alpha}$. By the definition of the forms $\widehat{l}$ and $\widehat{g}$ from the Eigenproblem 5

$$2|\widehat{L}_{j-1,j}| = \sum_{n=0}^{N-1} \left(\psi_{j-1}(X_{(n+1)\Delta}) - \psi_{j-1}(X_{n\Delta})\right)\left(\psi_j(X_{(n+1)\Delta}) - \psi_j(X_{n\Delta})\right)$$

$$= \sum_{n=0}^{N-1} \mathbf{1}\left(X_{n\Delta} < \frac{j-1}{J}\right)\mathbf{1}\left(X_{(n+1)\Delta} > \frac{j-1}{J}\right)$$

$$\cdot \left(\frac{j-1}{J} - X_{n\Delta}\right)\left(X_{(n+1)\Delta} - \frac{j-1}{J}\right)$$

$$+ \sum_{n=0}^{N-1} \mathbf{1}\left(X_{n\Delta} > \frac{j-1}{J}\right)\mathbf{1}\left(X_{(n+1)\Delta} < \frac{j-1}{J}\right)$$

$$\cdot \left(\frac{j-1}{J} - X_{(n+1)\Delta}\right)\left(X_{n\Delta} - \frac{j-1}{J}\right)$$

$$\lesssim \sum_{n=0}^{N-1} \mathbf{1}\left(X_{n\Delta} < \frac{j-1}{J}\right)\mathbf{1}\left(X_{(n+1)\Delta} > \frac{j-1}{J}\right)(X_{(n+1)\Delta} - X_{n\Delta})^2$$

$$+ \sum_{n=0}^{N-1} \mathbf{1}\left(X_{n\Delta} > \frac{j-1}{J}\right)\mathbf{1}\left(X_{(n+1)\Delta} < \frac{j-1}{J}\right)(X_{(n+1)\Delta} - X_{n\Delta})^2.$$

Moreover,

$$
|\widehat{F}_{j,j} - \widehat{L}_{j,j}| \le \frac{1}{2} \sum_{n=0}^{N-1} \left| \mathbf{1}\left(X_{n\Delta} < \frac{j-1}{J}\right) - \mathbf{1}\left(X_{(n+1)\Delta} < \frac{j-1}{J}\right) \right| (X_{(n+1)\Delta} - X_{n\Delta})^2
$$

$$
+ \frac{1}{2} \sum_{n=0}^{N-1} \left| \mathbf{1}\left(X_{n\Delta} < \frac{j}{J}\right) - \mathbf{1}\left(X_{(n+1)\Delta} < \frac{j}{J}\right) \right| (X_{(n+1)\Delta} - X_{n\Delta})^2.
$$

Hence, it suffices to show that for any $x \in (0,1)$

$$
\sum_{n=0}^{N-1} \mathbf{1}(X_{n\Delta} < x)\mathbf{1}(X_{(n+1)\Delta} > x)(X_{(n+1)\Delta} - X_{n\Delta})^2 \lesssim \Delta^{1/2-4\alpha}. \tag{37}
$$

By Definition 27(i), on the event $\mathcal{Q}_\Delta$, we have

$$
\sum_{n=0}^{N-1} \mathbf{1}(X_{n\Delta} < x)\mathbf{1}(X_{(n+1)\Delta} > x)(X_{(n+1)\Delta} - X_{n\Delta})^2
$$

$$
\le \Delta^{-2\alpha} \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{1}\left(|X_{n\Delta} - x| \le \Delta^{1/2-\alpha}\right).
$$

Arguing as in the proof of Lemma 18, we finally obtain

$$
\frac{1}{N} \sum_{n=0}^{N-1} \mathbf{1}\left(|X_{n\Delta} - x| \le \Delta^{1/2-\alpha}\right) \lesssim \int_{x-\Delta^{1/2-\alpha}}^{x+\Delta^{1/2-\alpha}} \mu_1(x)\,dx + \omega(\Delta)\|\mu_1\|_\infty \lesssim \Delta^{1/2-2\alpha}.
$$

Since $\widehat{F}$ is a diagonal matrix with diagonal entries of order $\Delta^{1/3}$, we have $\|\widehat{F}\|_{l^2}, \|\widehat{F}^{-1}\|_{l^2}^{-1} \approx \Delta^{1/3}$. As argued above, on $\mathcal{Q}_\Delta$, the upper and lower diagonal entries of $\widehat{L}$ are of order $\Delta^{1/2-4\alpha}$. Since for any $1 \le j \le J$ holds $|\widehat{L}_{j,j} - \widehat{F}_{j,j}| \lesssim \Delta^{1/2-4\alpha}$, matrix $\widehat{L}$ is diagonally dominant with diagonal entries of order $\Delta^{1/3}$. Hence, it is invertible and $\|\widehat{L}\|_{l^2}, \|\widehat{L}^{-1}\|_{l^2}^{-1} \approx \Delta^{1/3}$. $\qquad\square$

Lemma 34. *Eigenvectors* $(\widehat{w}_{1,j})$, $(\widehat{u}_{1,j})$, *normalized so that* $\|\widehat{w}_1\|_{l^2} = \|\widehat{u}_1\|_{l^2} = J^{1/2}$, *satisfy on* $\mathcal{Q}_\Delta$

$$
\left\|(\widehat{w}_{1,j}) - (\widehat{u}_{1,j})\right\|_{l^2} \lesssim \Delta^{-1/3}\left\|(\widehat{F} - \widehat{L})\widehat{w}_1\right\|_{l^2}.
$$

Proof. Recall that $(\widehat{\lambda}_j, \widehat{w}_j)_j$ are the eigenpairs of the Eigenproblem 25, with $\|(\widehat{w}_j)\|_{l^2} = \sqrt{J}$. [5], Theorem 26, implies that there exists an eigenpair $(\widehat{\lambda}_{j_0}, J^{-1/2}\widehat{w}_{j_0})$ such that

$$
\left|\widehat{\lambda}_{j_0}^{-1} - \widehat{\gamma}_1^{-1}\right| \lesssim J^{-1/2}\|\widehat{F}^{-1}\|_{l^2}\|(\widehat{F} - \widehat{L})\widehat{w}_1\|_{l^2} \lesssim \|\widehat{F}^{-1}\|_{l^2}\|\widehat{F} - \widehat{L}\|_{l^2},
$$

$$
\left\|(\widehat{w}_{j_0,j}) - (\widehat{u}_{1,j})\right\|_{l^2} \lesssim \delta^{-1}\left(\widehat{\lambda}_{j_0}^{-1}\right)\|\widehat{F}^{-1}\|_{l^2}^{3/2}\|\widehat{F}\|_{l^2}^{1/2}\|(\widehat{F} - \widehat{L})\widehat{w}_1\|_{l^2},
$$

where $\delta(\widehat{\lambda}_{j_0}^{-1})$ is the so called localizing distance, i.e. $\delta(\widehat{\lambda}_{j_0}^{-1}) = \min_{j \neq j_0} |\widehat{\lambda}_j^{-1} - \widehat{\gamma}_1^{-1}|$. From Lemma 33, we deduce

$$\left|\widehat{\lambda}_{j_0}^{-1} - \widehat{\gamma}_1^{-1}\right| \lesssim \Delta^{1/6-4\alpha}.$$

By Nakatsukasa [21], Theorem 8.3, for any $i = 1, \ldots, J$ we have

$$\left|\widehat{\lambda}_i^{-1} - \widehat{\gamma}_i^{-1}\right| \lesssim \|\widehat{L}^{-1}\|_{l^2} \|\widehat{\lambda}_i^{-1}(\widehat{F} - \widehat{L})\|_{l^2},$$

which together with Lemmas 29 and 33 imply

$$\left|\widehat{\lambda}_1^{-1} - \widehat{\gamma}_1^{-1}\right| \lesssim \Delta^{1/6-4\alpha}. \tag{38}$$

By Lemma 29 holds $|\widehat{\lambda}_1^{-1} - \widehat{\lambda}_2^{-1}| \gtrsim 1$, hence we must have $j_0 = 1$. Furthermore, from the same uniform lower bound on the spectral gap it follows

$$\delta(\widehat{\lambda}_{j_0}^{-1}) = \delta(\widehat{\lambda}_1^{-1}) \gtrsim 1.$$

Since by Lemma 33, we have $\|\widehat{F}^{-1}\|_{l^2}^{3/2} \|\widehat{F}\|_{l^2}^{1/2} \lesssim \Delta^{-1/3}$, we conclude that the claim holds. □

Proof of Proposition 20. Set

$$\mathcal{P}_\Delta = \mathcal{Q}_\Delta \cap \left\{\|\widehat{w}_1 - \widehat{u}_1\|_{l^2}^2 \leq \Delta^{1/7-8\alpha}\right\}.$$

Step 1. We will show that

$$\mathbb{E}_{\sigma,b}\left[\mathbf{1}_{\mathcal{Q}_\Delta} \cdot \|(\widehat{F} - \widehat{L})\widehat{w}_1\|_{l^2}^2\right]^{1/2} \lesssim \Delta^{5/12-4\alpha}. \tag{39}$$

In the proof of Lemma 33, we argued that for any $j = 1, \ldots, J$ it holds

$$\widehat{l}(\psi_j, \psi_{j-1}), \widehat{l}(\psi_j, \psi_{j+1}), \left|\widehat{l}(\psi_j, \psi_j) - \widehat{f}(\psi_j, \psi_j)\right| \lesssim \Delta^{1/2-4\alpha}. \tag{40}$$

Hence, using the uniform upper bound $\forall i = 1, \ldots, J \, \widehat{w}_{1,i} \lesssim 1$ from (30), we obtain that

$$\left|\widehat{l}(\widehat{w}_1, \psi_j) - \widehat{f}(\widehat{w}_1, \psi_j)\right| \lesssim \Delta^{1/2-4\alpha} \qquad \text{for any } j = 1, \ldots, J. \tag{41}$$

We will use the regularity of the eigenfunction $\widehat{w}_1$ to strengthen (41). Consider $J^{1/2} \leq j \leq J - J^{1/2}$. By the tridiagonal structure of the form $\widehat{l}$ it holds

$$\widehat{l}(\widehat{w}_1, \psi_j) - \widehat{f}(\widehat{w}_1, \psi_j)$$
$$= \widehat{l}(\psi_{j-1}, \psi_j)(\widehat{w}_{1,j-1} - \widehat{w}_{1,j}) + \widehat{w}_{1,j}\widehat{l}(I, \psi_j)$$
$$\quad - \widehat{w}_{1,j}\widehat{f}(\psi_j, \psi_j) + \widehat{l}(\psi_{j+1}, \psi_j)(\widehat{w}_{1,j+1} - \widehat{w}_{1,j})$$
$$= \widehat{w}_{1,j}\left[\widehat{l}(\psi_{j-1}, \psi_j)\left(\frac{\widehat{w}_{1,j-1}}{\widehat{w}_{1,j}} - 1\right) + \widehat{l}(I, \psi_j) - \widehat{f}(I, \psi_j) + \widehat{l}(\psi_{j+1}, \psi_j)\left(\frac{\widehat{w}_{1,j+1}}{\widehat{w}_{1,j}} - 1\right)\right].$$

Hence, using the upper bound $\widehat{w}_{1,j} \lesssim 1$ from (30), we obtain that

$$\left| \widehat{l}(\widehat{w}_1, \psi_j) - \widehat{f}(\widehat{w}_1, \psi_j) \right| \lesssim \widehat{l}(\psi_{j-1}, \psi_j) \left| \frac{\widehat{w}_{1,j-1}}{\widehat{w}_{1,j}} - 1 \right|$$
$$+ \left| \widehat{l}(I, \psi_j) - \widehat{f}(I, \psi_j) \right| + \widehat{l}(\psi_{j+1}, \psi_j) \left| \frac{\widehat{w}_{1,j+1}}{\widehat{w}_{1,j}} - 1 \right|.$$

Inequalities (40) and (31) imply

$$\widehat{l}(\psi_{j-1}, \psi_j) \left(\frac{\widehat{w}_{1,j-1}}{\widehat{w}_{1,j}} - 1 \right) + \widehat{l}(\psi_{j+1}, \psi_j) \left(\frac{\widehat{w}_{1,j+1}}{\widehat{w}_{1,j}} - 1 \right)) \lesssim \Delta^{2/3 - 6\alpha},$$

while, since $\mathcal{Q}_\Delta \subset \mathcal{O}_\Delta$, from Lemma 39 it follows that

$$\mathbb{E}_{\sigma,b} \left[\mathbf{1}_{\mathcal{Q}_\Delta} \cdot \left| \widehat{l}(I, \psi_j) - \widehat{f}(I, \psi_j) \right|^2 \right]^{\frac{1}{2}} \lesssim \Delta^{\frac{2}{3}}.$$

We conclude that for $J^{1/2} \leq j \leq J - J^{1/2}$

$$\mathbb{E}_{\sigma,b} \left[\mathbf{1}_{\mathcal{Q}_\Delta} \cdot \left| \widehat{l}(\widehat{w}_1, \psi_j) - \widehat{f}(\widehat{w}_1, \psi_j) \right|^2 \right]^{\frac{1}{2}} \lesssim \Delta^{\frac{2}{3} - 6\alpha}. \tag{42}$$

Since $\alpha < \frac{1}{24}$ inequalities (41) and (42) imply

$$\mathbb{E}_{\sigma,b} \left[\mathbf{1}_{\mathcal{Q}_\Delta} \cdot \left\| (\widehat{F} - \widehat{L}) \widehat{w}_1 \right\|_{l^2}^2 \right] = \sum_{j=1}^{J} \mathbb{E}_{\sigma,b} \left[\mathbf{1}_{\mathcal{Q}_\Delta} \cdot \left| \widehat{f}(\widehat{w}_1, \psi_j) - \widehat{l}(\widehat{w}_1, \psi_j) \right|^2 \right]$$
$$\lesssim J^{1/2} \Delta^{1 - 8\alpha} + J \Delta^{4/3 - 12\alpha} \lesssim \Delta^{5/6 - 8\alpha}.$$

Step 2. $\mathcal{P}_\Delta$ is a high probability event. Indeed, inequality (39) and Lemma 34 imply

$$\mathbb{E}_{\sigma,b} \left[\mathbf{1}_{\mathcal{Q}_\Delta} \cdot \left\| \widehat{w}_1 - \widehat{u}_1 \right\|_{l^2}^2 \right]^{1/2} \lesssim \Delta^{1/12 - 4\alpha}.$$

Hence, by Markov's inequality,

$$\mathbb{P}_{\sigma,b}(\mathcal{L} \setminus \mathcal{P}_\Delta) \leq \mathbb{P}_{\sigma,b}(\mathcal{L} \setminus \mathcal{Q}_\Delta) + \Delta^{-1/7 + 8\alpha} \mathbb{E}_{\sigma,b} \left[\mathbf{1}_{\mathcal{Q}_\Delta} \cdot \left\| \widehat{w}_1 - \widehat{u}_1 \right\|_{l^2}^2 \right]$$
$$\lesssim \mathbb{P}_{\sigma,b}(\mathcal{L} \setminus \mathcal{Q}_\Delta) + \Delta^{1/6 - 1/7} \to 0,$$

when $\Delta \to 0$.

Step 3. On the event $\mathcal{P}_\Delta$ it holds

$$\max_{i=1,\ldots,J} |\widehat{w}_{1,i} - \widehat{u}_{1,i}|^2 \leq \sum_{i=1}^{J} |\widehat{w}_{1,i} - \widehat{u}_{1,i}|^2 = \|\widehat{w}_1 - \widehat{u}_1\|_{l^2}^2 \lesssim \Delta^{1/7 - 8\alpha}.$$

Since $\alpha < 1/56$ the eigenvector $(\widehat{u}_{1,j})$ inherits the uniform bounds of the eigenvector $(\widehat{w}_{1,j})$. In particular, for any $j = \lfloor aJ \rfloor - 1, \ldots, \lceil bJ \rceil + 1$, we have

$$\widehat{u}_{1,j} \approx 1.$$

Moreover, since for any $j = 1, \ldots, J$ holds $\widehat{w}_{1,j} > 0$, we deduce that

$$\sum_{j=1}^{J} \widehat{u}_{1,j}^2 \mathbf{1}(\widehat{u}_{1,j} < 0) \leq \|\widehat{w}_1 - \widehat{u}_1\|_{l^2}^2 \lesssim 1.$$

Finally, note that on the event $\mathcal{P}_\Delta$ the eigenvalue $\widehat{\gamma}_1 \approx 1$ since on $\mathcal{Q}_\Delta$, by (38), holds $|\widehat{\lambda}_1^{-1} - \widehat{\gamma}_1^{-1}| \lesssim \Delta^{1/6-4\alpha} \lesssim 1$ and $\widehat{\lambda}_1^{-1} \approx 1$ by Lemma 29. $\qquad \square$

3.5. Proof of Theorem 8

As announced in Section 2.2, we will bound the approximation error of the spectral estimator and the time symmetric Florens-Zmirou estimator by the difference of forms $\widehat{f}$ and $\widehat{l}$.

Lemma 35. *On the high probability event $\mathcal{P}_\Delta$ from Proposition 20 it holds*

$$\big\|\widetilde{\sigma}_S^2 - \widehat{\sigma}_{\mathrm{FZ}}^2\big\|_{L^1([a,b])} \lesssim \sum_{j=\lfloor aJ \rfloor}^{\lceil bJ \rceil} \big|\widehat{l}(\widehat{u}_1, \psi_j) - \widehat{f}(\widehat{u}_1, \psi_j)\big|.$$

Proof. From representations (20) and (10), it follows that

$$\big\|\widetilde{\sigma}_S^2 - \widehat{\sigma}_{\mathrm{FZ}}^2\big\|_{L^1([a,b])} = \frac{1}{J} \sum_{j=\lfloor aJ \rfloor}^{\lceil bJ \rceil} \big|\widetilde{\sigma}_{S,j}^2 - \widehat{\sigma}_{\mathrm{FZ},j}^2\big| \lesssim \frac{1}{J} \sum_{j=\lfloor aJ \rfloor}^{\lceil bJ \rceil} \frac{|\widehat{l}(\widehat{u}_1, \psi_j) - \widehat{f}(\widehat{u}_1, \psi_j)|}{\widehat{u}_{1,j} \int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx)}.$$

By Proposition 20, for $j = \lfloor aJ \rfloor - 1 \leq j \leq \lceil bJ \rceil + 1$, we have $\widehat{u}_{1,j} \sim 1$. Since, by Lemma 18, $J \int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx) \approx 1$, we conclude that the claim holds. $\qquad \square$

Proposition 36. *For every function $v \in V_J^0$ and any $j = 1, \ldots, J$ we have*

$$\mathbb{E}_{\sigma,b}\big[\mathbf{1}_{\mathcal{O}_\Delta} \cdot \big|\widehat{f}(v, \psi_j) - \widehat{l}(v, \psi_j)\big|^2\big]^{\frac{1}{2}} \lesssim \big(|v_{j-1}|^2 + |v_j|^2 + |v_{j+1}|^2\big)^{\frac{1}{2}} \Delta^{\frac{1}{2}},$$

where v corresponds to the vector $(v_j)_{j=1,\ldots,J}$ and $v_0, v_{J+1} = 0$.

Proof. First, note that since for $i \neq j$ holds $\widehat{f}(\psi_i, \psi_j) = 0$ we have $\widehat{f}(v, \psi_j) = v_j \widehat{f}(\psi_j, \psi_j)$. Moreover, on the event $\mathcal{O}_\Delta$, for Δ sufficiently small, the increments of the process X are smaller than J^{-1}. Hence, for $|i - j| > 1$, holds $\widehat{l}(\psi_i, \psi_j) = 0$. Linearity implies

$$\widehat{l}(v, \psi_j) = v_{j-1}\widehat{l}(\psi_{j-1}, \psi_j) + v_j\widehat{l}(\psi_j, \psi_j) + v_{j+1}\widehat{l}(\psi_{j+1}, \psi_j). \tag{43}$$

Consequently, it is sufficient to show that

$$
\begin{aligned}
&\mathbb{E}_{\sigma,b}\big[\big|\widehat{l}(\psi_{j-1},\psi_j)\big|^2\big]^{\frac{1}{2}} + \mathbb{E}_{\sigma,b}\big[\big|\widehat{f}(\psi_j,\psi_j) - \widehat{l}(\psi_j,\psi_j)\big|^2\big]^{\frac{1}{2}} \\
&+ \mathbb{E}_{\sigma,b}\big[\big|\widehat{l}(\psi_j,\psi_{j-1})\big|^2\big]^{\frac{1}{2}} \lesssim \Delta^{\frac{1}{2}}.
\end{aligned}
\tag{44}
$$

Decomposing the terms above like in Lemma 33, we obtain that (44) follows from Theorem 41. $\qquad\square$

We are now able to prove the suboptimal rate $\Delta^{1/6}$ for the root mean squared $L^2([a,b])$ error of the spectral estimator $\widetilde{\sigma}_S$.

Proposition 37. *There exists an event $\mathcal{R}_\Delta \subseteq \mathcal{P}_\Delta$ satisfying $\lim_{\Delta\to 0}\mathbb{P}_{\sigma,b}(\mathcal{L}\setminus\mathcal{R}_\Delta)=0$, such that for every $x\in(a,b)$*

$$
\mathbb{E}_{\sigma,b}\big[\mathbf{1}_{\mathcal{R}_\Delta}\cdot\big|\widetilde{\sigma}_S^2(x)-\sigma^2(x)\big|^2\big]^{\frac{1}{2}} \lesssim \Delta^{\frac{1}{6}}.
\tag{45}
$$

Furthermore, on $\mathcal{R}_\Delta$, for every $\lfloor aJ\rfloor \le j \le \lceil bJ\rceil$ we have

$$
\widetilde{\sigma}_{S,j}^2 \approx 1,
\tag{46}
$$

$$
\mathbb{E}_{\sigma,b}\left[\mathbf{1}_{\mathcal{R}_\Delta}\cdot\left|\frac{\widehat{u}_{1,j\pm 1}}{\widehat{u}_{1,j}}-1\right|^2\right]^{\frac{1}{2}} \lesssim \Delta^{\frac{1}{6}}.
\tag{47}
$$

Remark 38. Given the uniform lower bound on the derivative $\widehat{u}_{1,j}$, and since $\Delta^{1/6}\approx J^{-1/2}$, inequality (47) can be reformulated as

$$
\mathbb{E}_{\sigma,b}\left[\mathbf{1}_{\mathcal{R}_\Delta}\cdot\left|\frac{\widehat{u}_1'(\frac{j}{J}\pm\frac{1}{J})-\widehat{u}_1'(\frac{j}{J})}{J^{-1/2}}\right|^2\right]^{\frac{1}{2}} \lesssim 1.
$$

By means of Markov's inequality the latter can be interpreted as almost $1/2$-Hölder regularity of $\widehat{u}_1'$. In that sense, Proposition 37 is a discrete time equivalent of Proposition B.4, which states that the derivatives of the eigenfunctions inherit the regularity of the design density, in the high-frequency case the regularity of the local time.

Proof of Proposition 37. Let $\mathcal{P}_\Delta$ be the high probability event introduced in Proposition 20. On $\mathcal{P}_\Delta$, we choose the eigenfunction $\widehat{u}_1$ s.t.

$$
\sum_{j=1}^{J}\widehat{u}_{1,j}^2 = J \quad\text{and}\quad \widehat{u}_{1,j}\approx 1 \qquad \text{for every } \lfloor aJ\rfloor-1 \le j \le \lceil bJ\rceil+1.
\tag{48}
$$

Step 1. Proof of (46). On the event $\mathcal{O}_\Delta$, for Δ sufficiently small, using the representation (20) together with (43) and (48) we obtain that

$$
\frac{\widehat{l}(\psi_j,\psi_j)}{\int_{\frac{j-1}{J}}^{\frac{j}{J}}\widehat{\mu}_N(dx)} \lesssim \widetilde{\sigma}_{S,j}^2 \lesssim \frac{\widehat{l}(\psi_{j-1}+\psi_j+\psi_{j+1},\psi_j)}{\int_{\frac{j-1}{J}}^{\frac{j}{J}}\widehat{\mu}_N(dx)}
\tag{49}
$$

holds for every $\lfloor aJ \rfloor \leq j \leq \lceil bJ \rceil$. Since

$$\widehat{l}(\psi_{j-1} + \psi_j + \psi_{j+1}, \psi_j) \lesssim \sum_{n=0}^{N-1} \big(\mathbf{1}_j(X_{n\Delta}) + \mathbf{1}_j(X_{(n+1)\Delta})\big)(X_{(n+1)\Delta} - X_{n\Delta})^2,$$

we deduce that $\widetilde{\sigma}_{S,j}^2 \lesssim \widehat{\sigma}_{\mathrm{FZ},j}^2$. Furthermore, since on $\mathcal{P}_\Delta$ holds

$$\widehat{l}(\psi_j, \psi_j) \geq \frac{1}{2} \sum_{n=0}^{N-1} \mathbf{1}\left(\frac{j-1}{J} + \Delta^{5/11} \leq X_{n\Delta} \leq \frac{j-1}{J} - \Delta^{5/11}\right)(X_{(n+1)\Delta} - X_{n\Delta})^2, \qquad (50)$$

the spectral estimator can be bounded from below by a time symmetric Florens-Zmirou estimator with bandwidth $\frac{1}{2}\Delta^{1/3} - \Delta^{5/11} \approx \Delta^{1/3}$. Arguing as in Corollary 24, we deduce that there exists a high probability event $\mathcal{R}_\Delta^{(1)}$, such that on $\mathcal{R}_\Delta^{(1)}$, bound $\widetilde{\sigma}_S^2(x) \gtrsim 1$ holds for any $x \in (a, b)$. Set

$$\mathcal{R}_\Delta = \mathcal{P}_\Delta \cap \mathcal{R}_\Delta^{(1)}.$$

Step 2. Proof of (45). Fix $x \in (a, b)$ and chose j s.t. $\frac{j-1}{J} \leq x < \frac{j}{J}$. Representations (20) and (10), together with Lemma 18, imply

$$\big|\widetilde{\sigma}_{S,j}^2 - \widehat{\sigma}_{\mathrm{FZ},j}^2\big| \lesssim \Delta^{-1/3}\big|\widehat{l}(\widehat{u}_1, \psi_j) - \widehat{f}(\widehat{u}_1, \psi_j)\big|.$$

Hence, from Proposition 36 and (48) it follows that

$$\mathbb{E}_{\sigma,b}\big[\mathbf{1}_{\mathcal{R}_\Delta} \cdot \big|\widetilde{\sigma}_S^2(x) - \widehat{\sigma}_{\mathrm{FZ}}^2(x)\big|^2\big]^{\frac{1}{2}} \lesssim \Delta^{1/6}.$$

By Theorem 19 and Hölder regularity of σ^2

$$\mathbb{E}_{\sigma,b}\big[\mathbf{1}_{\mathcal{R}_\Delta} \cdot \big\|\sigma^2 - \widehat{\sigma}_{\mathrm{FZ}}^2\big\|_\infty^2\big]^{\frac{1}{2}} \lesssim \Delta^{1/6}.$$

By the triangle inequality, we conclude that (45) holds.

Step 3. Proof of (47). Set $\lfloor aJ \rfloor \leq j \leq \lceil bJ \rceil$. We will only prove

$$\mathbb{E}_{\sigma,b}\left[\mathbf{1}_{\mathcal{R}_\Delta} \cdot \left|\frac{\widehat{u}_{1,j+1}}{\widehat{u}_{1,j}} - 1\right|^2\right]^{\frac{1}{2}} \lesssim \Delta^{\frac{1}{6}}, \qquad (51)$$

as the symmetric bound on the second moment of $\mathbf{1}_{\mathcal{R}_\Delta} \cdot |\frac{\widehat{u}_{1,j-1}}{\widehat{u}_{1,j}} - 1|$ can be obtained analogously. The general idea of the proof is similar to the proof of (31) in Proposition 31. First, we will show that (51) follows from

$$\mathbb{E}_{\sigma,b}\left[\mathbf{1}_{\mathcal{R}_\Delta} \cdot \left|\frac{\widehat{u}_{1,j+1}\widetilde{\sigma}_{S,j+1}^2 \int_{\frac{j}{J}}^{\frac{j+1}{J}} \widehat{\mu}_N(dx)}{\widehat{u}_{1,j}\widetilde{\sigma}_{S,j}^2 \int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx)} - 1\right|^2\right]^{\frac{1}{2}} \lesssim \Delta^{\frac{1}{6}}. \qquad (52)$$

To that purpose, by the triangle inequality and since on $\mathcal{R}_\Delta$ the derivatives $\widehat{u}_{1,j}, \widehat{u}_{1,j+1} \approx 1$, we have to argue that

$$\mathbb{E}_{\sigma,b}\left[\mathbf{1}_{\mathcal{R}_\Delta} \cdot \left|\frac{\widetilde{\sigma}_{S,j+1}^2 \int_{\frac{j}{J}}^{\frac{j+1}{J}} \widehat{\mu}_N(dx)}{\widetilde{\sigma}_{S,j}^2 \int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx)} - 1\right|^2\right]^{\frac{1}{2}} \lesssim \Delta^{1/6}. \tag{53}$$

Step 3.1. Proof of (53). By Lemma 18 holds $J \int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx), J \int_{\frac{j}{J}}^{\frac{j+1}{J}} \widehat{\mu}_N(dx) \approx 1$. We defined above the event $\mathcal{R}_\Delta$ so that $\widetilde{\sigma}_{S,j}^2, \approx 1$. Hence, to prove (53), it suffices to show

$$\mathbb{E}_{\sigma,b}\left[\mathbf{1}_{\mathcal{R}_\Delta} \cdot |\widetilde{\sigma}_{S,j+1}^2 - \widetilde{\sigma}_{S,j}^2|^2\right]^{\frac{1}{2}} \lesssim \Delta^{1/6} \tag{54}$$

$$\mathbb{E}_{\sigma,b}\left[\mathbf{1}_{\mathcal{R}_\Delta} \cdot \left|\int_{\frac{j}{J}}^{\frac{j+1}{J}} \widehat{\mu}_N(dx) - \int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx)\right|^2\right]^{\frac{1}{2}} \lesssim \Delta^{1/2}. \tag{55}$$

(54) follows from (45) and $1/2$ Hölder regularity of σ^2. Indeed

$$\mathbb{E}_{\sigma,b}\left[\mathbf{1}_{\mathcal{R}_\Delta} \cdot |\widetilde{\sigma}_{S,j+1}^2 - \widetilde{\sigma}_{S,j}^2|^2\right]^{\frac{1}{2}}$$

$$\lesssim \mathbb{E}_{\sigma,b}\left[\mathbf{1}_{\mathcal{R}_\Delta} \cdot \left|\widetilde{\sigma}_{S,j+1}^2 - \sigma^2\left(\frac{j+1/2}{J}\right)\right|^2\right]^{\frac{1}{2}}$$

$$+ \mathbb{E}_{\sigma,b}\left[\mathbf{1}_{\mathcal{R}_\Delta} \cdot \left|\sigma^2\left(\frac{j+1/2}{J}\right) - \sigma^2\left(\frac{j-1/2}{J}\right)\right|^2\right]^{\frac{1}{2}}$$

$$+ \mathbb{E}_{\sigma,b}\left[\mathbf{1}_{\mathcal{R}_\Delta} \cdot \left|\sigma^2\left(\frac{j-1/2}{J}\right) - \widetilde{\sigma}_{S,j}^2\right|^2\right]^{\frac{1}{2}}$$

$$\lesssim \Delta^{1/6}.$$

To prove (55), let

$$\left|\int_{\frac{j}{J}}^{\frac{j+1}{J}} \widehat{\mu}_N(dx) - \int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx)\right| \leq \left|\int_{\frac{j}{J}}^{\frac{j+1}{J}} \widehat{\mu}_N(dx) - \int_{\frac{j}{J}}^{\frac{j+1}{J}} \mu_1(x)\,dx\right|$$

$$+ \left|\int_{\frac{j}{J}}^{\frac{j+1}{J}} \mu_1(x)\,dx - \int_{\frac{j-1}{J}}^{\frac{j}{J}} \mu_1(x)\,dx\right|$$

$$+ \left|\int_{\frac{j-1}{J}}^{\frac{j}{J}} \mu_1(x)\,dx - \int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx)\right|$$

$$:= E_1 + E_2 + E_3.$$

Arguing like in the proof of Lemma 18 we obtain

$$\mathbb{E}_{\sigma,b}\big[E_1^2 + E_3^2\big]^{\frac{1}{2}} \lesssim \Delta^{2/3},$$

while the Cauchy–Schwarz inequality, together with Theorem 15 yield

$$\mathbb{E}_{\sigma,b}[E_2^2]^{\frac{1}{2}} = \mathbb{E}_{\sigma,b}\Bigg[\bigg|\int_0^{J^{-1}} \mu_1\bigg(\frac{j}{J} + x\bigg) - \mu_1\bigg(\frac{j-1}{J} + x\bigg) dx\bigg|^2\Bigg]^{\frac{1}{2}}$$

$$\leq \Bigg[\frac{1}{J}\int_0^{J^{-1}} \mathbb{E}_{\sigma,b}\bigg[\bigg|\mu_1\bigg(\frac{j}{J} + x\bigg) - \mu_1\bigg(\frac{j-1}{J} + x\bigg)\bigg|^2\bigg] dx\Bigg]^{\frac{1}{2}} \lesssim \Delta^{\frac{1}{2}}.$$

Step 3.2. Proof of (52). The representation (20), together with the eigenpair property of $(\widehat{\gamma}_1, \widehat{u}_1)$, imply that

$$\frac{\widehat{u}_{1,j+1}\widetilde{\sigma}_{S,j+1}^2 \int_{\frac{j}{J}}^{\frac{j+1}{J}} \widehat{\mu}_N(dx)}{\widehat{u}_{1,j}\widetilde{\sigma}_{S,j}^2 \int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx)} = \frac{\widehat{l}(\widehat{u}_1, \psi_{j+1})}{\widehat{l}(\widehat{u}_1, \psi_j)} = \frac{\widehat{g}(\widehat{u}_1, \psi_{j+1})}{\widehat{g}(\widehat{u}_1, \psi_j)}.$$

In what follows, we want to apply methods from the Perron-Frobenius theory for nonnegative matrices. To that purpose, recall the definition of matrix $\widehat{M}$ from Section 3.4, Eq. (25). We have

$$\frac{\widehat{g}(\widehat{u}_1, \psi_{j+1})}{\widehat{g}(\widehat{u}_1, \psi_j)} = \frac{\sum_{m=1}^J \widehat{M}_{m,j+1}\widehat{u}_{1,m}}{\sum_{m=1}^J \widehat{M}_{m,j}\widehat{u}_{1,m}}.$$

To bound the above ratio, we would like to proceed as in the proof of inequality (31) in Proposition 31. Unfortunately, we can't, as we don't know if the vector of derivatives $(\widehat{u}_{1,j})$ is positive. Still, using the inequality (32) and arguing as in the proof of (31), we obtain that

$$\Bigg|\frac{\sum_{m=1}^J \widehat{M}_{m,j+1}\widehat{u}_{1,m}\mathbf{1}(\widehat{u}_{1,m} > 0)}{\sum_{m=1}^J \widehat{M}_{m,j}\widehat{u}_{1,m}\mathbf{1}(\widehat{u}_{1,m} > 0)} - 1\Bigg| \lesssim \Delta^{1/6}.$$

To finish the proof, we need to show that the possible error due to the negative derivative terms is small enough. On the event $\mathcal{R}_\Delta$ we have

$$\widehat{g}(\widehat{u}_1, \psi_j) = \widehat{\gamma}_1^{-1}\widehat{l}(\widehat{u}_1, \psi_j) \approx \widehat{l}(\widehat{u}_1, \psi_j) \geq \widehat{u}_{1,j}\widehat{l}(\psi_j, \psi_j) \sim \widehat{l}(\psi_j, \psi_j).$$

Furthermore, on the event $\mathcal{R}_\Delta$ we have $\widehat{l}(\psi_j, \psi_j) \gtrsim \int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx)$; indeed we defined $\mathcal{R}_\Delta$ such that the left hand side of (49) has a uniform lower bound. Thus, by Lemma 18

$$\sum_{m=1}^J \widehat{M}_{m,j}\widehat{u}_{1,m} = \widehat{g}(\widehat{u}_1, \psi_j) \gtrsim \int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx) \gtrsim \Delta^{1/3}.$$

Consequently, we need to show that

$$\sum_{m=1}^{J} (\widehat{M}_{m,j} + \widehat{M}_{m,j+1}) |\widehat{u}_{1,m}| \mathbf{1}(\widehat{u}_{1,m} \le 0) \lesssim \Delta^{\frac{1}{2}}.$$

From (25) it follows $\widehat{M}_{i,j} \lesssim J^{-2}$. By the Cauchy–Schwarz inequality and Proposition 20

$$\sum_{m=1}^{J} (\widehat{M}_{m,j} + \widehat{M}_{m,j+1}) |\widehat{u}_{1,m}| \mathbf{1}(\widehat{u}_{1,m} \le 0)$$

$$\lesssim J^{-3/2} \left(\sum_{m=1}^{J} |\widehat{u}_{1,m}|^2 \mathbf{1}(\widehat{u}_{1,m} \le 0) \right)^{\frac{1}{2}} \lesssim \Delta^{\frac{1}{2}}.$$

$\square$

To obtain the suboptimal rate $\Delta^{1/6}$ we only used uniform bounds on the derivatives vector $(\widehat{u}_{1,j})_j$ together with the general error bound from Proposition 36. Having established the regularity of the eigenfunction $\widehat{u}_1$, we are now able to argue that the error $\mathbb{E}_{\sigma,b}[\mathbf{1}_{\mathcal{R}_\Delta} \cdot |\widehat{l}(\widehat{u}_1, \psi_j) - \widehat{f}(\widehat{u}_1, \psi_j)|]$ is at most of order $\Delta^{2/3}$.

Lemma 39. *Denote* $I(x) = x - c_0$, *with* c_0 *such that* $I \in V_J^0$. *For* Δ *sufficiently small, for every* $j = 1, \ldots, J$, *it holds*

$$\mathbb{E}_{\sigma,b}\big[\mathbf{1}_{\mathcal{O}_\Delta} \cdot |\widehat{f}(I, \psi_j) - \widehat{l}(I, \psi_j)|^2\big]^{\frac{1}{2}} \lesssim \Delta^{2/3}. \tag{56}$$

Proof. We will reduce (56) to the term bounded in Theorem 42. By definition of the forms $\widehat{l}, \widehat{f}$ and the representation (10) it holds

$$\widehat{l}(I, \psi_j) = \frac{1}{2} \sum_{n=0}^{N-1} (X_{(n+1)\Delta} - X_{n\Delta})(\psi_j(X_{(n+1)\Delta}) - \psi_j(X_{n\Delta})),$$

$$\widehat{f}(I, \psi_j) = \frac{1}{4} \sum_{n=0}^{N-1} (\mathbf{1}_j(X_{n\Delta}) + \mathbf{1}_j(X_{(n+1)\Delta}))(X_{(n+1)\Delta} - X_{n\Delta})^2.$$

We will analyze the error contribution of a single summand. When $X_{n\Delta}, X_{(n+1)\Delta} \in [\frac{j-1}{J}, \frac{j}{J}]$ both forms contribute by $\frac{1}{2}(X_{(n+1)\Delta} - X_{n\Delta})^2$, hence cancel perfectly. When both $X_{n\Delta}$, $X_{(n+1)\Delta} \notin [\frac{j-1}{J}, \frac{j}{J}]$ neither of the forms contribute. Since on $\mathcal{O}_\Delta$, for Δ sufficiently small, the increment $|X_{(n+1)\Delta} - X_{n\Delta}| \le 1/J$ we deduce that the overall error $|\widehat{f}(I, \psi_j) - \widehat{l}(I, \psi_j)|$ is due only to summands with the increment $X_{n\Delta}, X_{(n+1)\Delta}$ crossing the boundary of $[\frac{j-1}{J}, \frac{j}{J}]$. In such case, the form $\widehat{f}$ contributes by $\frac{1}{4}(X_{(n+1)\Delta} - X_{n\Delta})^2$, while $\widehat{l}$ by $\frac{1}{2}(X_{(n+1)\Delta} - X_{n\Delta})\beta$, where

$$\beta = \text{sgn}(X_{(n+1)\Delta} - X_{n\Delta}) \cdot \text{length}\left([X_{n\Delta}, X_{(n+1)\Delta}] \cap \left[\frac{j-1}{J}, \frac{j}{J}\right]\right).$$

Let $\gamma = X_{(n+1)\Delta} - X_{n\Delta} - \beta$. The contribution of a single boundary crossing summand equals

$$\frac{1}{4}(X_{(n+1)\Delta} - X_{n\Delta})^2 - \frac{1}{2}(X_{(n+1)\Delta} - X_{n\Delta})\beta = \frac{1}{4}(\beta + \gamma)(\gamma - \beta) = \frac{\gamma^2 - \beta^2}{4}.$$

Considering all four possible crossing configurations, we obtain that

$$\widehat{f}(I, \psi_j) - \widehat{l}(I, \psi_j) = \sum_{n=0}^{N-1} \big(\mathbf{1}_{(\frac{j}{J}, 1]}(X_{(n+1)\Delta}) - \mathbf{1}_{(\frac{j}{J}, 1]}(X_{n\Delta}) \big)$$

$$\cdot \left(\left(X_{(n+1)\Delta} - \frac{j}{J} \right)^2 - \left(X_{n\Delta} - \frac{j}{J} \right)^2 \right)$$

$$+ \sum_{n=0}^{N-1} \big(\mathbf{1}_{(\frac{j-1}{J}, 1]}(X_{(n+1)\Delta}) - \mathbf{1}_{(\frac{j-1}{J}, 1]}(X_{n\Delta}) \big)$$

$$\cdot \left(\left(X_{(n+1)\Delta} - \frac{j-1}{J} \right)^2 - \left(X_{n\Delta} - \frac{j-1}{J} \right)^2 \right).$$

Thus, (56) indeed follows from Theorem 42. $\qquad\qquad\square$

Proof of Theorem 8. Let $\mathcal{R}_\Delta$ be the high probability event introduced in Proposition 37. In view of Remark 22, it is enough to prove the claim for the estimator $\widetilde{\sigma}_S^2$. By Lemma 35 and since $J \sim \Delta^{-1/3}$, it is sufficient to show that for any $\lfloor aJ \rfloor \leq j \leq \lceil bJ \rceil$ holds

$$\mathbb{E}_{\sigma,b} \big[\mathbf{1}_{\mathcal{R}_\Delta} \cdot |\widehat{l}(\widehat{u}_1, \psi_j) - \widehat{f}(\widehat{u}_1, \psi_j)| \big] \lesssim \Delta^{2/3}.$$

By Definition 14 holds $\widehat{f}(\widehat{u}_1, \psi_j) = \widehat{u}_{1,j} \widehat{f}(\psi_j, \psi_j) = \widehat{u}_{1,j} \widehat{f}(I, \psi_j)$. Since on the event $\mathcal{R}_\Delta$, for Δ sufficiently small, the increments $|X_{(n+1)\Delta} - X_{n\Delta}| \leq J^{-1}$, we have

$$\widehat{l}(\widehat{u}_1, \psi_j) = \widehat{u}_{1,j-1}\widehat{l}(\psi_j, \psi_{j-1}) + \widehat{u}_{1,j}\widehat{l}(\psi_j, \psi_j) + \widehat{u}_{1,j+1}\widehat{l}(\psi_j, \psi_{j+1}),$$
$$\widehat{l}(I, \psi_j) = \widehat{l}(\psi_j, \psi_{j-1}) + \widehat{f}(\psi_j, \psi_j) + \widehat{l}(\psi_j, \psi_{j+1}).$$

Consequently, since by Proposition 20 $\widehat{u}_{1,j} \approx 1$, we deduce that

$$\widehat{l}(\widehat{u}_1, \psi_j) - \widehat{f}(\widehat{u}_1, \psi_j) \approx \widehat{l}(\psi_j, \psi_{j-1}) \left(\frac{\widehat{u}_{1,j-1}}{\widehat{u}_{1,j}} - 1 \right) + \widehat{l}(I, \psi_j) - \widehat{f}(I, \psi_j)$$

$$+ \widehat{l}(\psi_j, \psi_{j+1}) \left(\frac{\widehat{u}_{1,j+1}}{\widehat{u}_{1,j}} - 1 \right). \tag{57}$$

By the Cauchy–Schwarz inequality together with Proposition 36 and the inequality (47) we can uniformly bound the mean absolute value of the first and third term by $\Delta^{2/3}$. Since $\mathcal{R}_\Delta \subset \mathcal{O}_\Delta$ the mean absolute value of the second term is bounded in Lemma 39. $\qquad\square$

3.6. Technical results

We devote this chapter to the proof of two technical results that provide us with control over, properly rescaled, mean number of crossing of a given level α.

Definition 40. For $\alpha \in (0, 1)$ and $n = 0, \ldots, N - 1$ define

$$\chi(n, \alpha) = \mathbf{1}_{[0,\alpha)}(X_{(n+1)\Delta}) - \mathbf{1}_{[0,\alpha)}(X_{n\Delta}).$$

The random variable χ codifies the event of the increment $X_{n\Delta}, X_{(n+1)\Delta}$ crossing the level α. The sign of χ contains information about the direction of the crossing. Since

$$\big|\chi(n, \alpha)\big| \leq \mathbf{1}\big(|X_{n\Delta} - \alpha| \leq \omega(\Delta)\big),$$

arguing as in the proof of Lemma (18) we can show that

$$\frac{1}{N} \sum_{n=0}^{N-1} \big|\chi(n, \alpha)\big| \leq 4\omega(\Delta)\mu_1.$$

Consequently, Theorem 17 implies that the mean number of crossings, rescaled by the sample size, can be upper bounded by $\Delta^{1/2} \log(\Delta)$. Keeping in mind that $(X_{(n+1)\Delta} - X_{n\Delta})^2$ is of the order $\Delta = 1/N$, the next result is a refinement of the bound above.

Theorem 41. *For every $\alpha \in (0, 1)$, we have*

$$\mathbb{E}_{\sigma,b}\left[\left(\sum_{n=0}^{N-1} \big|\chi(n, \alpha)\big|(X_{(n+1)\Delta} - X_{n\Delta})^2\right)^2\right]^{\frac{1}{2}} \lesssim \Delta^{1/2}.$$

Proof. Fix $\alpha \in (0, 1)$. Since $|\chi(n, \alpha)| = 1$ if and only if the increment $(X_{n\Delta}, X_{(n+1)\Delta})$ crosses the level α, the claim is equivalent to the inequalities:

$$\mathbb{E}_{\sigma,b}\left[\left(\sum_{n=0}^{N-1} \mathbf{1}(X_{n\Delta} < \alpha)\mathbf{1}(X_{(n+1)\Delta} > \alpha)(X_{(n+1)\Delta} - X_{n\Delta})^2\right)^2\right]^{\frac{1}{2}} \lesssim \Delta^{1/2},$$

$$\mathbb{E}_{\sigma,b}\left[\left(\sum_{n=0}^{N-1} \mathbf{1}(X_{n\Delta} > \alpha)\mathbf{1}(X_{(n+1)\Delta} < \alpha)(X_{(n+1)\Delta} - X_{n\Delta})^2\right)^2\right]^{\frac{1}{2}} \lesssim \Delta^{1/2}.$$

Below, we only prove the first inequality. The second one can be obtained in a similar way or by a time reversal argument. Denote

$$\eta_n = \mathbf{1}(X_{n\Delta} < \alpha)\mathbf{1}(X_{(n+1)\Delta} > \alpha)(X_{(n+1)\Delta} - X_{n\Delta})^2.$$

We have

$$
\mathbb{E}_{\sigma,b}\left[\left(\sum_{n=0}^{N-1} \mathbf{1}(X_{n\Delta} < \alpha)\mathbf{1}(X_{(n+1)\Delta} > \alpha)(X_{(n+1)\Delta} - X_{n\Delta})^2\right)^2\right]
$$

$$
= \sum_{n=0}^{N-1} \mathbb{E}_{\sigma,b}\left[\eta_n^2\right] + 2\sum_{0 \le n < m}^{N-1} \mathbb{E}_{\sigma,b}[\eta_n \eta_m].
$$

Denote by p_t the transition kernel of the diffusion X. Uniform bounds on diffusion coefficients imply that

$$
p_t(x, y) \le M_1 \frac{1}{\sqrt{t}} \exp\left(-\frac{(x-y)^2}{M_2 t}\right), \tag{58}
$$

with M_1, M_2 positive constants uniform on Θ, see [26], Lemma 2. From (58) and the inequality [1], Formula 7.1.13:

$$
\int_x^\infty e^{-z^2}\,dz \le \frac{e^{-x^2}}{x + \sqrt{x^2 + 4/\pi}} \le \frac{\sqrt{\pi}}{2} e^{-x^2}, \tag{59}
$$

it follows that

$$
\begin{aligned}
\int_0^\alpha \int_\alpha^1 p_\Delta(x, y)(y-x)^4\,dy\,dx &\lesssim \int_0^\alpha \int_\alpha^1 \frac{1}{\sqrt{\Delta}} e^{-\frac{(y-x)^2}{M_2\Delta}}(y-x)^4\,dy\,dx \\
&\lesssim \Delta^2 \int_0^\alpha \int_{\frac{\alpha-x}{\sqrt{\Delta M_2}}}^{\frac{1-x}{\sqrt{\Delta M_2}}} e^{-z^2} z^4\,dz\,dx \\
&\lesssim \Delta^2 \int_0^\alpha \int_{\frac{\alpha-x}{\sqrt{\Delta M_2}}}^{\frac{1-x}{\sqrt{\Delta M_2}}} e^{-\frac{z^2}{2}}\,dz\,dx \\
&\lesssim \Delta^2 \int_0^\alpha e^{-\frac{(\alpha-x)^2}{2M_2\Delta}}\,dx \lesssim \Delta^{5/2}.
\end{aligned} \tag{60}
$$

Similarly

$$
\int_0^\alpha \int_\alpha^1 p_\Delta(x, y)(y-x)^2\,dy\,dx \lesssim \Delta^{3/2}. \tag{61}
$$

For simplicity, we will use the stationarity of X, which is granted by Assumption 2. Using more elaborated arguments, the result could be obtained for an arbitrary initial condition. By stationarity, for any t, the one dimensional margin X_t is distributed with respect to the invariant measure $\mu(x)\,dx$. Conditioning on $X_{n\Delta}$, from (60) and uniform bounds on the density μ it follows

$$
\mathbb{E}_{\sigma,b}\left[\eta_n^2\right] = \int_0^\alpha \int_\alpha^1 p_\Delta(x, y)(y-x)^4\,dy\,\mu(x)\,dx \lesssim \Delta^{5/2}.
$$

Hence,

$$\sum_{n=0}^{N-1} \mathbb{E}_{\sigma,b}[\eta_n^2] \lesssim N\Delta^{\frac{5}{2}} = \Delta^{\frac{3}{2}}.$$

The Cauchy–Schwarz inequality implies

$$\sum_{n=0}^{N-2} \mathbb{E}_{\sigma,b}[\eta_n \eta_{n+1}] \lesssim \sum_{n=0}^{N-2} \mathbb{E}_{\sigma,b}[\eta_n^2]^{\frac{1}{2}} \mathbb{E}_{\sigma,b}[\eta_{n+1}^2]^{\frac{1}{2}} \lesssim N\Delta^{\frac{5}{2}} \lesssim \Delta^{\frac{3}{2}}.$$

Finally, using (61), for $m > n + 1$, we obtain

$$\begin{aligned}
\mathbb{E}_{\sigma,b}[\eta_n \eta_m] &= \int_0^\alpha \int_\alpha^1 \int_0^\alpha \int_\alpha^1 p_\Delta(x,y)(y-x)^2 p_{(m-n-1)\Delta}(z,x)(z-w)^2 \\
&\quad \cdot p_\Delta(w,z)\mu(w)\,dy\,dx\,dz\,dw \\
&\lesssim \int_0^\alpha \int_\alpha^1 p_\Delta(x,y)(y-x)^2\,dy\,dx \frac{1}{\sqrt{(m-n-1)\Delta}} \\
&\quad \cdot \int_0^\alpha \int_\alpha^1 (z-w)^2 p_\Delta(w,z)\,dz\,dw \\
&\lesssim \Delta^{5/2} \frac{1}{\sqrt{m-n-1}}.
\end{aligned}$$

Consequently

$$\sum_{n=0}^{N-3} \sum_{m=n+2}^{N-1} \mathbb{E}_{\sigma,b}[\eta_n \eta_m] \lesssim \Delta^{5/2} \sum_{n=0}^{N-3} \sum_{k=1}^{N-n-2} \frac{1}{\sqrt{k}} \lesssim \Delta^{5/2} \sum_{n=0}^{N-3} \sqrt{N-n-2}$$

$$= \Delta^{5/2} \sum_{n=1}^{N-2} \sqrt{n} \lesssim \Delta^{5/2} N^{3/2} = \Delta. \qquad \square$$

Note that the claim of Theorem 41 still holds when we replace $(X_{(n+1)\Delta} - X_{n\Delta})^2$ by $(X_{(n+1)\Delta} - \alpha)^2$ or $(X_{n\Delta} - \alpha)^2$. Next, we show that, when considering the direction of the crossings, cancellations occur that make the difference of $\sum_{n=0}^{N-1} \chi(n,\alpha)(X_{(n+1)\Delta} - \alpha)^2$ and $\sum_{n=0}^{N-1} \chi(n,\alpha)(X_{n\Delta} - \alpha)^2$ even smaller.

Theorem 42. *For any $\alpha \in [\frac{1}{J}, 1 - \frac{1}{J}]$ we have*

$$\mathbb{E}_{\sigma,b}\left[\mathbf{1}_{\mathcal{O}_\Delta} \cdot \left|\sum_{n=0}^{N-1} \chi(n,\alpha)\big((X_{(n+1)\Delta} - \alpha)^2 - (X_{n\Delta} - \alpha)^2\big)\right|^2\right]^{\frac{1}{2}} \lesssim \Delta^{2/3}.$$

Due to the sign of the summands the proof of the next theorem cannot be done in a similar way as the previous result. In what follows, we show that on the event $\mathcal{O}_\Delta$

$$\sum_{n=0}^{N-1} \chi(n,\alpha)\big((X_{(n+1)\Delta}-\alpha)^2 - (X_{n\Delta}-\alpha)^2\big) = \int_0^1 \mathbf{1}(X_s < \alpha)\,ds - \frac{1}{N}\sum_{n=0}^{N-1}\mathbf{1}(X_{n\Delta}<\alpha) + R,$$

where the remainder term is of the right order. Thus we are left with showing that

$$\mathbb{E}_{\sigma,b}\left[\left|\int_0^1 \mathbf{1}(X_s < \alpha)\,ds - \frac{1}{N}\sum_{n=0}^{N-1}\mathbf{1}(X_{n\Delta}<\alpha)\right|^2\right]^{\frac{1}{2}} \lesssim \Delta^{2/3}. \tag{62}$$

Note that $\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{1}(X_{n\Delta}<\alpha)$ is a Riemann type estimator of the occupation time of the interval $[0,\alpha)$. The problem of establishing the rate of convergence was recently considered in [19,22]. Although obtained results do not apply, as they require higher smoothness of the coefficients, they suggest that an even better rate $\Delta^{3/4}$ holds. Indeed, in the case of a reflected diffusion with bounded coefficients, we can show that

$$\mathbb{E}_{\sigma,b}\left[\left|\int_0^1 f(X_s)\,ds - \frac{1}{N}\sum_{n=0}^{N-1} f(X_{n\Delta})\right|^2\right]^{\frac{1}{2}} \lesssim \Delta^{\frac{1+s}{2}}\|f\|_{H^s},$$

for any cadlag function f with Sobolev regularity $0 \le s \le 1$, see [4].

Proof. Fix $\alpha \in [\frac{1}{J}, 1-\frac{1}{J}]$. On the event $\mathcal{O}_\Delta$, whenever $\mathbf{1}_{[0,\alpha)}(X_{(n+1)\Delta}) - \mathbf{1}_{[0,\alpha)}(X_{n\Delta}) \neq 0$ we must have $|X_{n\Delta}-\alpha|, |X_{(n+1)\Delta}-\alpha| \le \omega(\Delta) < \Delta^{4/9}$. Consider function $d:[0,1]\to\mathbb{R}$ given by

$$d(x) = (x-\alpha)^2\mathbf{1}\big(|x-\alpha| \le \Delta^{4/9}\big) + \Delta^{8/9}\mathbf{1}\big(|x-\alpha| > \Delta^{4/9}\big).$$

We have

$$\big(\mathbf{1}_{[0,\alpha)}(X_{(n+1)\Delta}) - \mathbf{1}_{[0,\alpha)}(X_{n\Delta})\big)\big((X_{(n+1)\Delta}-\alpha)^2 - (X_{n\Delta}-\alpha)^2\big)$$
$$= \big(\mathbf{1}_{[0,\alpha)}(X_{(n+1)\Delta}) - \mathbf{1}_{[0,\alpha)}(X_{n\Delta})\big)\big(d(X_{(n+1)\Delta}) - d(X_{n\Delta})\big).$$

Step 1. We will first show that

$$\mathbb{E}_{\sigma,b}\left[\mathbf{1}_{\mathcal{O}_\Delta}\cdot\left|\sum_{n=0}^{N-1}\mathbf{1}_{[0,\alpha)}(X_{n\Delta})\big(d(X_{(n+1)\Delta}) - d(X_{n\Delta})\big)\right|^2\right]^{\frac{1}{2}} \lesssim \Delta^{2/3}. \tag{63}$$

Note that

$$d'(x) = 2(x-\alpha)\mathbf{1}\big(|x-\alpha| \le \Delta^{4/9}\big),$$

$$\frac{1}{2}d''(x) = -\Delta^{4/9}\delta_{\{\alpha-\Delta^{4/9}\}} + \mathbf{1}\big(|x-\alpha| \le \Delta^{4/9}\big) - \Delta^{4/9}\delta_{\{\alpha+\Delta^{4/9}\}},$$

where the second derivative must be understood in the distributional sense. Since we fixed α separated from the boundaries, $d'(0) = d'(1) = 0$ for Δ small enough. Denote by

$$L^x_{s,t} := L^x_t - L^x_s,$$

the local time of the path fragment $(X_u, s \le u \le t)$. From the Itô–Tanaka formula [25], Chapter VI, Theorem 1.5, it follows that

$$
\begin{aligned}
d(X_{(n+1)\Delta}) - d(X_{n\Delta}) &= \int_{n\Delta}^{(n+1)\Delta} d'(X_s)\sigma(X_s)\,dW_t + \int_{n\Delta}^{(n+1)\Delta} d'(X_s)b(X_s)\,ds \\
&\quad + \int_{n\Delta}^{(n+1)\Delta} \sigma^2(X_s)\mathbf{1}\big(|X_s - \alpha| \le \Delta^{4/9}\big)\,ds \\
&\quad - \Delta^{4/9} L^{\alpha-\Delta^{4/9}}_{n\Delta,(n+1)\Delta} - \Delta^{4/9} L^{\alpha+\Delta^{4/9}}_{n\Delta,(n+1)\Delta} \\
&:= \int_{n\Delta}^{(n+1)\Delta} d'(X_s)\sigma(X_s)\,dW_t + D_n.
\end{aligned}
$$

First, we will bound the sum of the martingale terms. Since martingale increments are uncorrelated, using Itô isometry, we obtain that

$$
\begin{aligned}
&\mathbb{E}_{\sigma,b}\left[\left|\sum_{n=0}^{N-1} \mathbf{1}_{[0,\alpha)}(X_{n\Delta}) \int_{n\Delta}^{(n+1)\Delta} d'(X_s)\sigma(X_s)\,dW_t\right|^2\right] \\
&= \sum_{n=0}^{N-1} \mathbb{E}_{\sigma,b}\left[\mathbf{1}_{[0,\alpha)}(X_{n\Delta}) \int_{n\Delta}^{(n+1)\Delta} \big(d'(X_s)\sigma(X_s)\big)^2\,ds\right] \\
&\lesssim \Delta^{\frac{8}{9}} \mathbb{E}_{\sigma,b}\left[\int_0^1 \mathbf{1}\big(|X_s - \alpha| \le \Delta^{\frac{4}{9}}\big)\,ds\right] \\
&= \Delta^{\frac{8}{9}} \int_{\alpha-\Delta^{\frac{4}{9}}}^{\alpha+\Delta^{\frac{4}{9}}} \mathbb{E}_{\sigma,b}\big[\mu_1(x)\big]\,dx \lesssim \Delta^{\frac{4}{3}},
\end{aligned}
$$

where the last inequality follows from (13). Now, we will bound the sum of the finite variation terms: $\sum_{n=0}^{N-1} \mathbf{1}_{[0,\alpha)}(X_{n\Delta})D_n$. Note first that since b is uniformly bounded, we have

$$
\begin{aligned}
\sum_{n=0}^{N-1} \mathbf{1}_{[0,\alpha)}(X_{n\Delta})\left|\int_{n\Delta}^{(n+1)\Delta} d'(X_s)b(X_s)\,ds\right| &\lesssim \Delta^{4/9} \int_0^1 \mathbf{1}\big(|x - \alpha| \le \Delta^{4/9}\big)\mu_1(x)\,dx \\
&\lesssim \Delta^{8/9}\|\mu_1\|_\infty.
\end{aligned}
$$

Since by the inequality (13) $\|\mu_1\|_\infty$ has all moments finite, the root mean squared value of this sum is of smaller order than $\Delta^{2/3}$. Now, note that since on the event $\mathcal{O}_\Delta$ we have $\omega(\Delta) < \Delta^{4/9}$,

condition $X_{n\Delta} < \alpha$ implies that $L_{n\Delta,(n+1)\Delta}^{\alpha+\Delta^{4/9}} = 0$. On the other hand, whenever $L_{n\Delta,(n+1)\Delta}^{\alpha-\Delta^{4/9}} \neq 0$ we must have $X_{n\Delta} < \alpha$. Hence,

$$\sum_{n=0}^{N-1} \mathbf{1}_{[0,\alpha)}(X_{n\Delta})\big(\Delta^{4/9}L_{n\Delta,(n+1)\Delta}^{\alpha-\Delta^{4/9}} + \Delta^{4/9}L_{n\Delta,(n+1)\Delta}^{\alpha+\Delta^{4/9}}\big) = \Delta^{4/9}L_1^{\alpha-\Delta^{4/9}}.$$

Using first the Cauchy–Schwarz inequality and then the regularity of the local time (see [25], Chapter VI, Corollary 1.8 and the remark before), we obtain

$$\mathbb{E}_{\sigma,b}\bigg[\bigg|\Delta^{4/9}L_1^{\alpha-\Delta^{4/9}} - \int_{\alpha-\Delta^{4/9}}^{\alpha} L_1^x \, dx\bigg|^2\bigg] \leq \Delta^{4/9}\int_{\alpha-\Delta^{4/9}}^{\alpha} \mathbb{E}_{\sigma,b}\big[\big|L_1^x - L_1^{\alpha-\Delta^{4/9}}\big|^2\big]\,dx$$

$$\lesssim \Delta^{4/9}\int_{\alpha-\Delta^{4/9}}^{\alpha}\big|x - (\alpha-\Delta^{4/9})\big|\,dx \lesssim \Delta^{4/3}.$$

Consequently, to prove (63), we just have to argue that the root mean squared error of

$$\int_{\alpha-\Delta^{4/9}}^{\alpha} L_1^x \, dx - \sum_{n=0}^{N-1} \mathbf{1}_{[0,\alpha)}(X_{n\Delta})\int_{n\Delta}^{(n+1)\Delta} \sigma^2(X_s)\mathbf{1}\big(|X_s - \alpha| \leq \Delta^{4/9}\big)\,ds$$

$$= \sum_{n=0}^{N-1}\int_{n\Delta}^{(n+1)\Delta} \big(\mathbf{1}(X_s < \alpha) - \mathbf{1}(X_{n\Delta} < \alpha)\big)\sigma^2(X_s)\mathbf{1}\big(|X_s - \alpha| \leq \Delta^{4/9}\big)\,ds \qquad (64)$$

$$= \sum_{n=0}^{N-1}\int_{n\Delta}^{(n+1)\Delta} \big(\mathbf{1}(X_s < \alpha) - \mathbf{1}(X_{n\Delta} < \alpha)\big)\sigma^2(X_s)\,ds$$

is of order $\Delta^{2/3}$. From the $1/2$-Hölder property of σ^2 it follows that

$$\bigg|\sum_{n=0}^{N-1}\int_{n\Delta}^{(n+1)\Delta} \big(\mathbf{1}(X_s < \alpha) - \mathbf{1}(X_{n\Delta} < \alpha)\big)\big(\sigma^2(X_s) - \sigma^2(\alpha)\big)\,ds\bigg|$$

$$\lesssim \sum_{n=0}^{N-1}\int_{n\Delta}^{(n+1)\Delta} \mathbf{1}\big(|X_s - \alpha| \leq \Delta^{\frac{4}{9}}\big)\Delta^{\frac{2}{9}}\,ds \lesssim \Delta^{\frac{2}{9}}\int_{\alpha-\Delta^{4/9}}^{\alpha+\Delta^{4/9}} \mu_1(dx) \lesssim \Delta^{\frac{2}{3}}\|\mu_1\|_\infty.$$

Thus, by (13), we reduced (64) to

$$\int_0^1 \mathbf{1}(X_s < \alpha)\,ds - \frac{1}{N}\sum_{n=0}^{N-1}\mathbf{1}(X_{n\Delta} < \alpha),$$

which is of the right order by (62). We conclude that (63) holds.

Step 2. Consider the time reversed process $Y_t = X_{1-t}$. Since X is reversible, the process Y, under the measure $\mathbb{P}_{\sigma,b}$, has the same law as X. Furthermore, the occupation density and the

modulus of continuity of processes Y and X are identical, hence $\mathcal{O}_\Delta$ is a "good" event also for Y. Inequality (63) is equivalent to

$$\mathbb{E}_{\sigma,b}\left[\mathbf{1}_{\mathcal{O}_\Delta}\cdot\left|\sum_{m=0}^{N-1}\mathbf{1}_{[0,\alpha)}(Y_{m\Delta})\big(d(Y_{(m+1)\Delta})-d(Y_{m\Delta})\big)\right|^2\right]^{\frac{1}{2}}\lesssim\Delta^{\frac{2}{3}}.$$

Substituting $n=N-m$, we obtain

$$\sum_{m=0}^{N-1}\mathbf{1}_{[0,\alpha)}(Y_{m\Delta})\big(d(Y_{(m+1)\Delta})-d(Y_{m\Delta})\big)$$

$$=-\sum_{n=0}^{N-1}\mathbf{1}_{[0,\alpha)}(X_{(n+1)\Delta})\big(d(X_{(n+1)\Delta})-d(X_{n\Delta})\big).$$

$\square$

4. Low-frequency analysis

4.1. Spectral estimation method

In 1998, Hansen et al. [12] explained how the coefficients of a diffusion process are related to the spectral properties of its infinitesimal generator. In this section, we want to shortly introduce the main idea of their method.

The generator L of the reflected diffusion X is an unbounded operator on $L^2([0,1])$ with

$$\mathrm{dom}(L)=\big\{f\in H^2:f'(0)=f'(1)=0\big\},$$

$$Lf(x)=\mu^{-1}(x)\left(\frac{1}{2}\sigma^2(x)\mu(x)f'(x)\right)'\qquad\text{for }f\in\mathrm{dom}(T).$$

Spectral properties of L are discussed in Appendix B. Seen as an operator on the equivalent Hilbert space $L^2([0,1],\mu)$, generator L is elliptic, self-adjoint and has a compact resolvent operator. Consequently, the eigenproblem

Eigenproblem 43. Find $(\zeta,u)\in\mathbb{R}\times L^2$, with $u\neq0$, such that

$$Lu=\zeta u$$

has countably many non-positive eigenvalues $0=\zeta_0>\zeta_1>\zeta_2\geq\cdots$, with μ-orthogonal eigenfunctions $(u_i)_{i=0,\ldots}$. The eigenvalue ζ_1 is simple and the corresponding eigenfunction u_1 is strictly monotone, see Proposition B.4. The main idea of the spectral estimation method is that the diffusion coefficient σ^2 can be expressed in terms of the invariant density μ and the eigenpair (ζ_1,u_1) (cf. Hansen et al. [12], Eq. (5.2)):

$$\sigma^2(x)=\frac{2\zeta_1\int_0^x u_1(y)\mu(y)\,dy}{u_1'(x)\mu(x)}.\tag{65}$$

4.2. Estimation error of the invariant measure

From now on, we take the Assumptions 2 and 1 as granted. Fix $\Delta > 0$ and $0 < a < b < 1$. Set $J \sim N^{1/5}$. Since the generator L has a spectral gap, diffusion X is geometrically ergodic. Below, we state general bounds on the variance of integrals with respect to the empirical measure $\widehat{\mu}_N$, which are due to the mixing property of the observed sample $(X_{n\Delta})_{n=0,\ldots,N}$. For the proof, refer to Chorowski and Trabs [5], Lemma 10.

Lemma 44. *For any $v, u \in L^2([0, 1])$, we have*

$$\operatorname{Var}_{\sigma,b}\left[\int_0^1 v(x)\widehat{\mu}_N(dx)\right] \lesssim N^{-1}\|v\|_{L^2}^2,$$

$$\operatorname{Var}_{\sigma,b}\left[\frac{1}{N}\sum_{n=0}^{N-1} v(X_{n\Delta})u(X_{(n+1)\Delta})\right] \lesssim N^{-1}\|v \cdot P_\Delta u\|_{L^2}^2.$$

Corollary 45. *There exists a high probability event $\mathcal{T}_1$, with $\mathbb{P}_{\sigma,b}(\Omega \setminus \mathcal{T}_1) \lesssim N^{-1}J^2$, such that, for any $1 \le j \le J$, on $\mathcal{T}_1$ we have*

$$J\int_{\frac{j-1}{J}}^{\frac{j}{J}} \widehat{\mu}_N(dx) \sim 1.$$

Proof. Since the invariant density μ is uniformly bounded on Θ, there exist constants $0 < c < C$ such that $c \le J\int_{\frac{j-1}{J}}^{\frac{j}{J}} \mu(x)\,dx \le C$. Let

$$\mathcal{T}_1 = \left\{\forall j = 1, \ldots, J \text{ it holds } \left|\int_0^1 \psi_j'(x)\widehat{\mu}_N(dx) - \int_0^1 \psi_j'(x)\mu(x)\,dx\right| \le \frac{c}{2J}\right\}.$$

Using first the Markov inequality and then $\|\psi_j'\|_{L^2}^2 = J^{-1}$ with $\|\psi_j'\|_{L^2}^2 = J^{-1}$ we conclude that the claim holds. $\qquad\square$

4.3. Proof of Theorem 9

First, we state the approximation properties of the spaces V_J.

Definition 46. Denote by π_J and π_J^μ the L^2 and $L^2(\mu)$-orthogonal projections on V_J respectively.

Since V_J is the space of linear spline functions with regular knots at $\{0, \frac{1}{J}, \frac{2}{J}, \ldots, \frac{J-1}{J}, 1\}$, it satisfies the following Jackson and Bernstein type inequalities:

$$\|(I - \pi_J)f\|_{H^k} \lesssim J^{-(2-k)\alpha}\|f\|_{C^{1,\alpha}} \qquad \text{for } f \in C^{1,\alpha}([0, 1]) \text{ and } k = 0, 1, \tag{66}$$

$$\|v\|_{H^1} \lesssim J\|v\|_{L^2} \qquad \text{for } v \in V_J. \tag{67}$$

Definition 47. Denote by $(\phi_j)_{j=0,\dots,J}$ the Franklin system on $[0,1]$, i.e. the L^2-orthogonal basis of V_J, obtained from the Schauder algebraic basis by the Gram–Schmidt orthonormalization procedure.

For construction and properties of the Franklin system refer to Ciesielski [6]. In particular, basis functions $(\phi_j)_j$ satisfy the following uniform bound (cf. Ciesielski [6], Theorem 5):

$$\left\| \sum_{j=0}^{J} \phi_j^2 \right\|_{\infty} \lesssim J. \tag{68}$$

Proof of Theorem 9. As noted in Section 2.1, the estimator $(\widehat{\zeta}_1, \widehat{u}_1)$ is constructed in the exactly same way as the eigenpair estimator in Gobet et al. [11], Chorowski and Trabs [5]. Given the properties of the Franklin system, arguing as in Chorowski and Trabs [5], Corollary 19, we obtain that there exists a high probability event $\mathcal{T}_2$, with $\mathbb{P}_{\sigma,b}(\Omega \setminus \mathcal{T}_2) \lesssim N^{-2/5}$, such that

$$\mathbb{E}_{\sigma,b}\big[\mathbf{1}_{\mathcal{T}_2} \cdot \big(|\gamma_1 - \widehat{\zeta}_1|^2 + \|u_1 - \widehat{u}_1\|_{H^1}^2\big)\big]^{\frac{1}{2}} \lesssim N^{-1/5}. \tag{69}$$

Furthermore, on the event $\mathcal{T}_2$, we have $|\widehat{v}_1| \sim 1$ and $\|\widehat{u}_1\|_{H^1} \lesssim 1$.

Before we can prove the upper bound on the estimation error, we need to face one more technical difficulty. Since the estimator $\widehat{u}_1$ converges to the eigenfunction u_1 in the sense of the mean H^1 norm, we can't postulate a uniform positive lower bound on $\inf_{x \in [a,b]} \widehat{u}_1'(x)$. Following Chorowski and Trabs [5], Lemma 20, this difficulty can be overcome by applying the threshold $\widehat{\sigma}_S^2 \wedge D$. We conclude that there exists a high probability event $\mathcal{T}_3 \subset \mathcal{T}_2 \cap \mathcal{T}_1$, with $\mathbb{P}_{\sigma,b}(\Omega \setminus \mathcal{T}_3) \lesssim N^{-2/5}$, such that on $\mathcal{T}_3$, for j s.t. $[\frac{j-1}{J}, \frac{j}{J}] \subset (a,b)$, we have

$$\widehat{\sigma}_{S,j}^2 \wedge D = \frac{-2\widehat{\zeta}_1 \int_0^1 \psi_j(x)\widehat{u}_1(x)\widehat{\mu}_N(dx)}{(\widehat{u}_{1,j} \vee c_{a,b}) \int_0^1 \psi_j'(x)\widehat{\mu}_N(dx)} \wedge D, \tag{70}$$

for a deterministic constant $c_{a,b} > 0$ satisfying $c_{a,b} \leq \inf_{x \in [a,b]} u_1'(x)$.

Having established (69) and (70), the plug-in error can be controlled by similar considerations as in the proof of Chorowski and Trabs [5], Theorem 7. $\qquad \square$

Appendix A: Construction and properties of a scalar diffusion with two reflecting barriers

In this section, we construct a weak solution of the SDE (1). Later, we will use presented construction to generalize properties of scalar diffusions to reflected processes. The main idea of the following reasoning is to extend the diffusion coefficients b and σ to the whole real line, apply general SDEs theory to obtain a solution on $\mathbb{R}$ and finally project this solution to the interval $[0,1]$ in a way that corresponds to the instantaneous reflection. We refer the reader to [10], I.23, for a very similar construction of a diffusion on $[-1,1]$ with two reflecting barriers.

Definition A.1. Define $f : \mathbb{R} \to [0, 1]$ by

$$f(x) = \begin{cases} x - 2n, & 2n \le x < 2n + 1, \\ 2(n + 1) - x, & 2n + 1 \le x < 2n + 2 \end{cases} \quad \text{for } n \in \mathbb{N}.$$

Function f is almost everywhere differentiable with the derivative

$$f'(x) = \begin{cases} 1, & 2n < x \le 2n + 1, \\ -1, & 2n + 1 < x \le 2n + 2. \end{cases}$$

For $\sigma, b : [0, 1] \to \mathbb{R}$ define the extended coefficients $\widetilde{\sigma}, \widetilde{b} : \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{b}(x) = f'(x) \cdot b \circ f(x),$$

$$\widetilde{\sigma}(x) = \sigma \circ f(x).$$

Theorem A.2. *Grant Assumption* 1. *For every initial condition* $x_0 \in [0, 1]$ *that is independent of the driving Brownian motion* W *the SDE*

$$dY_t = \widetilde{b}(Y_t) \, dt + \widetilde{\sigma}(Y_t) \, dW_t,$$
$$Y_0 = x_0,$$

(71)

has a non-exploding unique strong solution. Define

$$X_t = f(Y_t).$$

The process $(X_t, t \ge 0)$ *is a weak solution of the SDE* (1).

Proof. $\widetilde{b}$ is bounded and $\widetilde{\sigma}' \in L^2_{\text{loc}}(\mathbb{R})$. Hence, the existence of a unique strong solution $(Y_t, t \ge 0)$ of the SDE (71) follows from [28], Theorem 4. As discussed in the proof of [18], Chapter 5, Proposition 5.17, the boundedness of $\widetilde{b}$ prevents the explosion of the solution. Process Y is a continuous semimartingale, hence by [25], Chapter VI, Theorem 1.2, it admits a local time process $(L_t^Y, t \ge 0)$. By the Itô–Tanaka formula ([25], Chapter VI, Theorem 1.5) process X satisfies

$$X_t = x_0 + \int_0^t \widetilde{b}(Y_s) f'(Y_s) \, ds + \int_0^t \widetilde{\sigma}(Y_s) f'(Y_s) \, dW_s + \sum_{n \in \mathbb{Z}} L_t^Y(2n) - \sum_{n \in \mathbb{Z}} L_t^Y(2n + 1)$$

$$= x_0 + \int_0^t b(X_s) \, ds + \int_0^t \sigma(X_s) \, dB_s + K_t,$$

where $B_t = \int_0^t f'(Y_s) \, dW_s$ and $K_t = \sum_{n \in \mathbb{Z}} L_t^Y(2n) - \sum_{n \in \mathbb{Z}} L_t^Y(2n + 1)$. Note that for any $T > 0$ the path $(X_t, 0 \le t \le T)$ is bounded, hence K is well defined. Process B is a continuous martingale with quadratic variation

$$\langle B \rangle_t = \int_0^t \left(f'(Y_s) \right)^2 ds = t.$$

Hence, Lévy's characterization theorem implies that B is a standard Brownian motion. From the properties of the local time L_t^Y follows that K is an adapted continuous process with finite variation, starting from zero and varying on the set $\bigcup_{n \in \mathbb{Z}} \{Y_t = 2n\} \cup \{Y_t = 2n + 1\} \subseteq \{X_t \in \{0, 1\}\}$. Consequently, X satisfies the SDE (1). □

Next, we use the above construction of a reflected diffusion process to prove Brownian bounds on the moments of the modulus of continuity of X.

Proof of Theorem 17. Fischer and Nappo [7] proved claimed upper bound for the standard Brownian motion. We will now generalize their result to diffusions with boundary reflection.

Step 1. Consider a martingale M satisfying $dM_t = \sigma(X_t) dW_t$. By the Dambis, Dubins–Schwarz theorem, $M_t = B_{\int_0^t \sigma^2(X_u) du}$ for some Brownian motion B. Consequently,

$$|M_t - M_s| = |B_{\int_0^t \sigma^2(X_u) du} - B_{\int_0^s \sigma^2(X_u) du}| \leq \omega^B \big(|t - s| \|\sigma^2\|_\infty \big),$$

where ω^B is the modulus of continuity of B. Thus, (15) holds for the martingale M, with a constant that depends only on the uniform upper bound on the volatility σ.

Step 2. Consider a semimartingale Y satisfying $dY_t = b(X_t) dt + dM_t$. Then

$$|Y_t - Y_s| \leq \left| \int_0^t b(Y_u) du - \int_0^s b(Y_u) du \right| + |M_t - M_s| \leq |t - s| \|b\|_\infty + \omega^M \big(|t - s| \big).$$

Consequently, (15) holds for the semimartingale Y, with a constant that depends only on the upper bounds on σ and b.

Step 3. For $(\sigma, b) \in \Theta$ consider the reflected diffusion process X satisfying the SDE (1). Let

$$dY_t = \widetilde{b}(Y_t) dt + \widetilde{\sigma}(Y_t) dW_t,$$
$$X_t = f(Y_t),$$

where $\widetilde{b}, \widetilde{\sigma}$ and f are as in Definition A.1. From Step 2, it follows that (15) holds for the semimartingale Y with a uniform constant on Θ. By the construction of the reflected process X, we have $|X_s - X_t| \leq |Y_s - Y_t|$. We conclude that $\omega^X \leq \omega^Y$, hence the claim holds for the reflected diffusion X. □

Appendix B: Bilinear coercive form

Recall that H^1, H^2 denote the L^2-Sobolev spaces on $[0, 1]$ of orders 1 and 2, respectively. For differentiable, strictly positive functions σ and μ consider an elliptic operator T on $L^2([0, 1])$, with Neumann type domain $\text{dom}(T) = \{v \in H^2 : v'(0) = v'(1) = 0\}$, given in the divergence form by

$$Tv(x) = -\frac{(\sigma^2(x)\mu(x)v'(x))'}{2\mu(x)} \qquad \text{for } v \in \text{dom}(T). \tag{72}$$

Note that the operator $-T$ is an infinitesimal generator of the diffusion process on $[0, 1]$ with instantaneous reflection at the boundaries, volatility function σ and an invariant measure with density μ. We want to analyze the eigenvalue problem for T, that is,

Eigenproblem B.1. Find $(\lambda, w) \in \mathbb{R} \times \mathrm{dom}(T)$, with $w \neq 0$, such that

$$Tw = \lambda w.$$

Integrating by parts, one can check, that the eigenpairs of the Eigenproblem B.1 solve

Eigenproblem B.2. Find $(\lambda, w) \in \mathbb{R} \times H^1$, with $w \neq 0$, such that

$$\int_0^1 w'(x)v'(x)\sigma^2(x)\mu(x)\,dx = 2\lambda \int_0^1 w(x)v(x)\mu(x)\,dx \qquad \text{for all } v \in H^1. \tag{73}$$

Eigenproblem B.2 is a weak formulation of the Eigenproblem B.1 for the associated Dirichlet form $l(u, v) = \langle Tu, v \rangle_\mu$. The biggest advantage of the weak formulation is that the Eigenproblem B.2 makes sense for any, not necessarily regular, functions μ. When μ is not differentiable, the Eigenproblem B.1 has no longer probabilistic interpretation in terms of the infinitesimal generator. Nevertheless, such problems arise naturally when one considers spectral estimation method with fixed time horizon, when the role of the invariant measure is taken by the non differentiable occupation density.

In what follows, we want to generalize the results of [11] on the spectral properties of an infinitesimal generator, to the solutions of the Eigenproblem B.2 with a Hölder regular function μ. For $0 < \alpha \leq 1$ denote by C^α the space of α-Hölder regular functions on $[0, 1]$. Furthermore, for $k \in \mathbb{N}$ let $C^{k,\alpha}$ be the space of k-times differentiable functions with kth derivative in C^α.

Definition B.3. For any given $0 < d < D$ let

$$\Theta_\alpha := \left\{ (\sigma, \mu) \in H^1\big([0, 1]\big) \times C^\alpha\big([0, 1]\big) : \|\sigma\|_{H^1}, \|\mu\|_{C^\alpha} \leq D, \right.$$

$$\left. \inf_{x \in [0, 1]} \big(\sigma(x) \wedge \mu(x)\big) \geq d, \int_0^1 \mu(x)\,dx = 1 \right\}$$

Eigenproblem B.2 is a conforming eigenvalue problem for a bilinear coercive form on the Hilbert space $L^2(\mu)$. [3] is a standard reference.

Proposition B.4. *Let $(\sigma, \mu) \in \Theta_\alpha$. The Eigenproblem B.2 has countably many solutions $(\lambda_i, w_i)_i$, with real nonnegative eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ and μ-orthogonal eigenfunctions, satisfying Neumann boundary conditions $w_i'(0) = w_i'(1) = 0$. The smallest positive eigenvalue λ_1 is simple, the derivative w_1' of the corresponding eigenfunction is $1/2 \wedge \alpha$ Hölder continuous and strictly monotone.*

Proof. It is easy to check that for any (σ, μ) $\lambda_0 = 0$ and $w_0 \equiv 1$ form an eigenpair. Let $L_0^2(\mu) = \{v \in L^2(\mu) : \int_0^1 v(x)\mu(x)\,dx = 0\}$ and $H_0^1(\mu) = L_0^2(\mu) \cap H^1$. $L_0^2(\mu)$ with the $L^2(\mu)$ inner product and $H_0^1(\mu)$ with $\langle u, v \rangle_{H^1(\mu)} = \langle u, v \rangle_{L_0^2(\mu)} + \int_0^1 u'(x)v'(x)\mu(x)\,dx$ are Hilbert spaces. The identity embedding $I : H_0^1(\mu) \to L_0^2(\mu)$ is compact.

For $u, v \in H_0^1(\mu)$ let

$$l(u, v) = \int_0^1 u'(x)v'(x)\sigma^2(x)\mu(x)\,dx.$$

l is a symmetric positive-definite bilinear form on $H_0^1(\mu) \times H_0^1(\mu)$. Furthermore, for any $u \in H_0^1(\mu)$ holds

$$c\|u\|_{H_0^1(\mu)}^2 \leq l(u, u) \leq C\|u\|_{H_0^1(\mu)}^2, \tag{74}$$

for some constants $0 < c < C$ that depend only on d, D. Indeed, since σ and μ are uniformly bounded, we only have to show that $\int_0^1 u^2(x)\,dx \leq \int_0^1 (u'(x))^2\,dx$. Consider $u \in C^1([0, 1]) \cap H_0^1(\mu)$. Since u is continuous and integrates to zero, there exists $x_0 \in [0, 1]$ s.t. $u(x_0) = 0$. Since $u(x) = \int_{x_0}^x u'(y)\,dy$, the upper bound $\|u\|_{L^2} \leq \|u'\|_{L^2}$ follows from the Cauchy–Schwarz inequality. As Lipschitz functions are dense in H^1, we conclude that (74) holds.

l is the Dirichlet form of an unbounded operator T on $L_0^2(\mu)$. Define $D = \mathrm{dom}(T)$ as these $u \in H_0^1(\mu)$, that the functional $v \mapsto l(u, v)$ is continuous on $H_0^1(\mu)$ with norm $\|\cdot\|_{L^2(\mu)}$. By the definition of the weak differentiability, domain $D = \{u : H_0^1(\mu) : u'\sigma^2\mu \in H^1\}$. Furthermore, D is dense in $L_0^2(\mu)$ (see [3], Exercise 4.51). For $u \in D$, we define Tu via the Riesz representation theorem by $l(u, v) = \langle Tu, v \rangle_{L^2(\mu)}$. Such defined T is an elliptic, densely defined, self-adjoint operator with compact resolvent (see [3], Proposition 4.17). Consequently, T has a discrete spectrum $(\lambda_i)_{i=1,\dots}$, with all eigenvalues positive and corresponding eigenfunctions μ-orthogonal.

Integrating by parts the right-hand side of (73), we obtain

$$\int_0^1 w_i'(x)\sigma^2(x)\mu(x)v'(x)\,dx = -2\lambda_i \int_0^1 \int_0^x w_i(y)\mu(y)\,dy\,v'(x)\,dx \qquad \text{for all } v \in H^1.$$

Since $\{v' : v \in H^1\}$ is dense in L^2, it follows that

$$w_i'(x) = \frac{2\lambda_i \int_0^x w_i(y)\mu(y)\,dy}{\sigma^2(x)\mu(x)}. \tag{75}$$

By Sobolev embedding σ^2 is $1/2$-Hölder regular. Consequently w_i' lies in $C^{1/2 \wedge \alpha}$. Since the eigenfunctions μ-integrate to zero, we deduce that $w_i'(0) = w_i'(1) = 0$.

Finally, we need to show that λ_1 is simple and that w_1 is strictly monotone. By the variational formula for the eigenpairs of a self-adjoint operator

$$2\lambda_1 = \inf_{u \in H_0^1(\mu)} \frac{\int_0^1 (u'(x))^2 \sigma^2(x)\mu(x)\,dx}{\int_0^1 u^2(x)\mu(x)\,dx}. \tag{76}$$

Arguing as in [11], Lemma 6.1, we obtain that $\int_0^1 u^2(x)\mu(x)\,dx = \int_0^1 \int_0^1 m(y,z)u'(y)u'(z)\,dy\,dz$ with $m(y,z) = \int_0^{y\wedge z} \mu(x)\,dx \int_{y\vee z}^1 \mu(x)\,dx$. We deduce that the derivative of the eigenfunction w_1 must have a constant sign, otherwise we could reduce the ratio in (76) by considering

$$\widetilde{w}_1 = w_1 \mathbf{1}\big(w_1' \geq 0\big) - w_1 \mathbf{1}\big(w_1' \leq 0\big).$$

Hence, the set $\{x : w_1'(x) = 0\}$ has zero Lebesgue measure. From (75), it follows that $w_1'(x) = 0$ only for $x = 0, 1$, meaning that w_1 is strictly monotone on $(0,1)$. Consequently, for any two eigenfunctions w_1 and $\bar{w}_1$, which correspond to λ_1, the scalar product

$$\int_0^1 w_1(x)\bar{w}_1(x)\mu(x)\,dx = \int_0^1 \int_0^1 m(y,z)w_1'(y)\bar{w}_1'(z)\,dy\,dz \neq 0,$$

hence the eigenspace corresponding to λ_1 is one dimensional. $\qquad\square$

Proposition B.5. *The eigenvalues λ_1, λ_2 and the norm ratio $\|w_1\|_{C^{1,1/2\wedge\alpha}}/\|w_1\|_{L^2(\mu)}$ are uniformly bounded for all $(\sigma,\mu) \in \Theta_\alpha$. Furthermore, for every $0 < a < b < 1$, $\inf_{x\in[a,b]} |w_1'(x)|$ and the spectral gap $\lambda_2 - \lambda_1$ have uniform lower bounds on Θ_α.*

Proof. We adapt the notation from the proof of Proposition B.4. Choose w_1 normalized s.t. $\|w_1\|_{L^2(\mu)} = 1$. We will first argue that λ_1, λ_2 and $\|w_1\|_{C^{1,1/2\wedge\alpha}}$ are uniformly bounded on Θ_α. From (74), we imply that

$$2\lambda_1 = l(w_1, w_1) \geq c\|w_1\|_{H^1(\mu)}^2 \geq c,$$

with $c > 0$ depending only on the bounds on σ and μ. It follows that the eigenvalues are uniformly separated from zero. By the variational formula

$$2\lambda_2 = \inf_{\substack{S\subset H^1 \\ \dim(S)=3}} \sup_{u\in S} \frac{\int_0^1 (u'(x))^2\sigma^2(x)\mu(x)\,dx}{\int_0^1 u^2(x)\mu(x)\,dx} \leq \inf_{\substack{S\subset H^1 \\ \dim(S)=3}} \sup_{u\in S} \frac{D^3 \int_0^1 (u'(x))^2\,dx}{d \int_0^1 u^2(x)\,dx} \leq 4\pi^2 \frac{D^3}{d},$$

since $4\pi^2$ is the third eigenvalue of the negative Laplace operator on $L^2([0,1])$ with Neumann boundary conditions. We conclude that the eigenvalues λ_1 and λ_2 are uniformly bounded. The uniform bound on $\|w_1\|_{C^{1,1/2\wedge\alpha}}$ follows from the representation (75).

   We will now prove a uniform lower bound on the spectral gap $\lambda_2 - \lambda_1$. Assume by contradiction that for some sequence of coefficients $(\sigma_n, \mu_n) \in \Theta_\alpha$ the corresponding spectral gaps $(\lambda_{n,2} - \lambda_{n,1})$ converge to zero. Since Θ_α is compact in the uniform convergence metric, we can assume that (σ_n, μ_n) converges uniformly to some $(\sigma, \mu) \in \Theta_\alpha$. We will argue that the uniform convergence of the coefficients leads to convergence of the eigenvalues, hence contradicts Proposition B.4 (cf. [11], proof of Proposition 6.5). However, since the function μ is embedded in the definition of spaces $L_0^2(\mu)$ and $H_0^1(\mu)$, we need first to reduce the Eigenproblem B.2 to a universal function space.

Let $U(x) = \int_0^x \mu(y)\, dy$ be the distribution function of μ. Substituting $U(x) = y$, we find that the Eigenproblem B.2 is equivalent to

$$\int_0^1 \widetilde{w}'(x)\widetilde{v}'(x)\widetilde{\sigma}^2\, dx = 2\lambda \int_0^1 \widetilde{w}'(x)\widetilde{v}'(x)\, dx \qquad \text{for all } \widetilde{v} \in H^1,$$

$$\widetilde{w} = w \circ U^{-1},$$

with $\widetilde{\sigma} = (\sigma\mu) \circ U^{-1}$. Consider $(\widetilde{\sigma}_n)_n$ and $\widetilde{\sigma}$ corresponding to (σ_n, μ_n) and (σ, μ) respectively. Note that $\widetilde{\sigma}_n$ converges to $\widetilde{\sigma}$ in the uniform norm. Denote $L_0^2 = L_0^2(1)$ and $H_0^1 := H_0^1(1)$. For $u, v \in H_0^1$ denote

$$\widetilde{l}_n(u, v) = \int_0^1 u'(x)v'(x)\widetilde{\sigma}_n(x)^2\, dx$$

and by $\widetilde{T}_n$ the corresponding operators on L_0^2. Recall that the operators $\widetilde{T}_n$ are unbounded and self-adjoint on L_0^2, with dense domains $\widetilde{D}_n$. Domains $\widetilde{D}_n$ do not have to possess a common core, which is needed to study the convergence of the sequence $(\widetilde{T}_n)_n$. We circumvent this difficulty by introducing inverse operators $\widetilde{R}_n = \widetilde{T}_n^{-1}$. Using the divergence formula (72) for $\widetilde{T}_n$, we check that for $u \in L_0^2$

$$\widetilde{R}_n u(x) = -2 \int_0^x \widetilde{\sigma}_n^{-2}(y) \int_0^y u(z)\, dz + c_n(u), \tag{77}$$

where $c_n(u) \in \mathbb{R}$ is such that $\int_0^1 \widetilde{R}_n u(x)\, dx = 0$. The convergence $\widetilde{\sigma}_n \to \widetilde{\sigma}$ in $C^1[(0, 1)]$ implies that operators $\widetilde{R}_n$ converge to $\widetilde{R}$ in the operator norm on L_0^2. By [3], Proposition 5.28, this entails the regular convergence, which, by [3], Theorem 5.20, is equivalent to the strongly stable convergence. Finally, [3], Proposition 5.6, ensures the convergence of the eigenvalues with preservation of their multiplicities.

Set $0 < a < b < 1$. We finally have to prove the uniform lower bound on $\inf_{x \in [a,b]} |w_1'(x)|$. We will use the same indirect arguments as when bounding the spectral gap. Assume that for some sequence $(\sigma_n, \mu_n) \in \Theta_\alpha$, with (σ_n, μ_n) converging in the uniform norm to $(\sigma, \mu) \in \Theta_\alpha$, the corresponding eigenfunctions $w_{1,n}$ satisfy $\inf_n \inf_{x \in [a,b]} |w_{1,n}'(x)| = 0$. Arguing as for the spectral gap, we reduce the problem to bounded operators $(\widetilde{R}_n)_n$ and $\widetilde{R}$. From formula (77), we deduce that the uniform convergence of coefficients implies $\widetilde{R}_n \to \widetilde{R}$ in the operator norm on $C([0, 1])$. We conclude, that the eigenfunctions converge in the uniform norm, which contradicts Proposition B.4. □

Eigenproblem B.6. Let V_J be a finite dimensional subspace of L^2. Find $(\lambda_J, w_J) \in \mathbb{R} \times V_J$, with $w_J \neq 0$ such that

$$\int_0^1 w'(x)v'(x)\sigma^2(x)\mu(x)\, dx = \lambda \int_0^1 w(x)v(x)\mu(x)\, dx \qquad \text{for any } v \in V_J.$$

Proposition B.7. *Let* $(V_J)_{J=1,\dots}$ *be a sequence of approximation spaces satisfying the following Jackson's type inequality*:

$$\left\|(I - \pi_J)v\right\|_{H^1} \le C J^{-\alpha} \|v\|_{C^{1,\alpha}} \qquad \text{for } v \in C^{1,\alpha},$$

where π_J *is the* L^2*-orthogonal projection on* V_J *and* $C > 0$ *some universal constant. Furthermore, assume that every* V_J *contains constant functions.*

For $(\sigma, \mu) \in \Theta_\alpha$ *the Eigenproblem* B.6 *has* $\dim(V_J)$ *solutions* $(\lambda_{J,i}, w_{J,i})_i$ *with real eigenvalues* $0 = \lambda_{J,0} < \lambda_{J,1} < \lambda_{J,2} \le \cdots \le \lambda_{J,\dim(V_J)-1}$. *For* J *big enough, the eigenvalue* $\lambda_{J,1}$ *and the spectral gap* $\lambda_{J,2} - \lambda_{J,1}$ *are uniformly bounded on* Θ_α.

Proof. We adapt the notation from the proof of Proposition B.4. By the Lax–Milgram theorem, there exists an isomorphism $S_l : H_0^1(\mu) \to H_0^1(\mu)$ such that

$$l(S_l v, u) = \langle v, u \rangle_{H^1(\mu)} \qquad \text{for all } v, u \in H_0^1(\mu).$$

Note that since for any $v \in L_0^2(\mu)$ the functional $H_0^1(\mu) \ni u \longmapsto \langle v, u \rangle_{L^2(\mu)} \in \mathbb{R}$ is continuous on $H_0^1(\mu)$, by the Riesz representation theorem there exists a continuous operator $K : L_0^2(\mu) \to H_0^1(\mu)$ such that

$$\langle v, u \rangle_{L^2(\mu)} = \langle K v, u \rangle_{H^1(\mu)}.$$

Define the operator $B_l = S_l \circ K \circ I$, where I is the identity embedding of $H_0^1(\mu)$ into $L_0^2(\mu)$. By (74), the form l defines an equivalent norm on $H_0^1(\mu)$. Note that B_l is a self-adjoint and compact operator on the Hilbert space $H_0^1(\mu)$ with l-induced inner product. Consider (λ_i, w_i), a solution of the Eigenproblem B.2. For any $v \in H_0^1(\mu)$ we have

$$l(w_i, v) = \lambda_i \langle w_i, v \rangle_{L^2(\mu)} = \lambda_i \langle K w_i, v \rangle_{H^1(\mu)} = \lambda_i l(S_l K w_i, v) = l(\lambda_i B_l w_i, v),$$

hence (λ_i^{-1}, w_i) is an eigenpair of the operator B_l. In particular, Proposition B.4 implies that the biggest eigenvalue λ_1^{-1} is simple.

Denote by π_J^l the l-orthogonal projection on the subspace V_J. Define the operator $B_{l,J} = \pi_J^l B_l \pi_J^l$. Since $B_{l,J}$ is a self-adjoint operator on V_J, with the l-induced inner product, it has $\dim(V_J) - 1$ solutions $(\lambda_{J,i}^{-1}, w_{J,i})_i$, with the eigenvalues $\lambda_{J,1}^{-1} \ge \lambda_{J,2}^{-1} \ge \cdots \ge \lambda_{J,\dim(V_J)-1}^{-1}$. Analogously as for the operator B_l, we check that $(\lambda_{J,i}, w_{J,i})$ are solutions of the finite dimensional Eigenproblem B.6. From (74) together with the uniform bound on μ it follows that

$$\left\|(I - \pi_n^l)w_1\right\|_l \le \left\|(I - \pi_n^l)(I - \pi_J)w_1\right\|_l \le 2\left\|(I - \pi_J)w_1\right\|_l \le C\left\|(I - \pi_J)w_1\right\|_{H^1},$$

for some, uniform on Θ_α, constant C. Using Jackson's inequality, the uniform bound on the Hölder norm of w_1 and uniform bounds on the eigenvalues λ_1, λ_2, we conclude that, for J large enough,

$$\left\|(I - \pi_n^l)w_1\right\|_l < \frac{\lambda_1^{-1} - \lambda_2^{-1}}{6\lambda_1^{-1}}.$$

The claim follows from [5], Theorem 25. $\square$

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