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Weak convergence of the empirical copula process with respect to weighted metrics

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The empirical copula process plays a central role in the asymptotic analysis of many statistical procedures which are based on copulas or ranks. Among other applications, results regarding its weak convergence can be used to develop asymptotic theory for estimators of dependence measures or copula densities, they allow to derive tests for stochastic independence or specific copula structures, or they may serve as a fundamental tool for the analysis of multivariate rank statistics. In the present paper, we establish weak convergence of the empirical copula process (for observations that are allowed to be serially dependent) with respect to weighted supremum distances. The usefulness of our results is illustrated by applications to general bivariate rank statistics and to estimation procedures for the Pickands dependence function arising in multivariate extreme-value theory.

Keywords: bivariate rank statistics; empirical copula process; Pickands dependence function; strongly mixing; weighted weak convergence

1. Introduction

The theory of weak convergence of empirical processes can be regarded as one of the most powerful tools in mathematical statistics. Through the continuous mapping theorem or the functional delta method, it greatly facilitates the development of asymptotic theory in a vast variety of situations [37].

For applying the continuous mapping theorem or the functional delta method, the course of action is often similar. Consider, for instance, the continuous mapping theorem: starting from some abstract weak convergence result, say $\mathbb{F}_n \rightsquigarrow \mathbb{F}$ in some metric space $(\mathcal{D}, d_{\mathcal{D}})$, one would like to deduce weak convergence of $\phi(\mathbb{F}_n) \rightsquigarrow \phi(\mathbb{F})$, where ϕ is some mapping defined on $(\mathcal{D}, d_{\mathcal{D}})$ with values in another metric space $(\mathcal{E}, d_{\mathcal{E}})$. This conclusion is possible provided ϕ is continuous at every point of a set which contains the limit $\mathbb{F}$, almost surely [37].

The continuity of ϕ is linked to the strength of the metric $d_{\mathcal{D}}$ – a stronger metric will make more functions continuous. For example, let $\mathcal{D} = \ell^\infty([0, 1])$ denote the space of bounded functions on $[0, 1]$ and consider the real-valued functional $\phi(f) := \int_{(0,1)} f(x)/x\, dx$ (with ϕ defined on a suitable subspace of $\mathcal{D}$). In Section 3.2 below, this functional will turn out to be of great interest for the estimation of Pickands dependence function and it is also closely related to the classical Anderson–Darling statistic. Now, if we equip $\mathcal{D}$ with the supremum distance, as is typically done in empirical process theory, the map ϕ is not continuous because $1/x$ is not integrable. Continuity of ϕ can be ensured by considering a weighted distance, such as, for instance, $\sup_{x\in[0,1]} |f_1(x) - f_2(x)|/g(x)$ for a positive weight function g such that $g(x)/x$ is integrable.

Similar phenomena arise with the functional delta method which is based on differentiability of statistical functionals. Here, the fact that stronger metrics than the supremum norm can be useful has been pointed out in [13,22], among others. In a series of papers, Richard Dudley proved that key statistical functionals actually are Fréchet differentiable if p-variation norms are used. A summary of related findings is provided in [14]. More recently, Beutner and Zähle [4] developed a modified functional delta method that applies to weighted supremum distances.

Summarizing the preceding paragraphs, it is desirable to establish weak convergence results with the metric $d_{\mathcal{D}}$ taken as strong as possible, with weighted supremum distances being of particular interest in many statistical applications. For classical empirical processes, corresponding weak convergence results are well known. For example, the standard d-dimensional empirical process $\mathbb{F}_n(\mathbf{x}) = \sqrt{n}\{F_n(\mathbf{x}) - F(\mathbf{x})\}$ with F having standard uniform marginals, converges weakly with respect to the metric induced by the weighted norm

$$\|G\|_\omega = \sup_{\mathbf{u}\in[0,1]^d} \left| \frac{G(\mathbf{u})}{\{g(\mathbf{u})\}^\omega} \right|, \qquad g(\mathbf{u}) = \Bigl(\min_{j=1}^{d} u_j\Bigr) \wedge \Bigl(1 - \min_{j=1}^{d} u_j\Bigr),$$

$\omega \in (0, 1/2)$. See, for example, [34] and [10] for the one-dimensional i.i.d.-case, [33] for the one-dimensional time series case or [20] for the bivariate i.i.d.-case. For $d = 2$, the graph of the function g is depicted in Figure 1.

The present paper is motivated by the apparent lack of such results for the empirical copula process $\hat{\mathbb{C}}_n$. This process, an element of $D([0, 1]^d)$ precisely defined in Section 2 below, plays a crucial role in the asymptotic analysis of statistical procedures which are based on copulas or ranks. Unweighted weak convergence of $\hat{\mathbb{C}}_n$ has been investigated by several authors under a variety of assumptions on the smoothness of the copula and on the temporal dependence of the underlying observations, see [6,7,16,17,32], among others. However, results regarding its weighted weak convergence are almost non-existent. To the best of our knowledge, the only

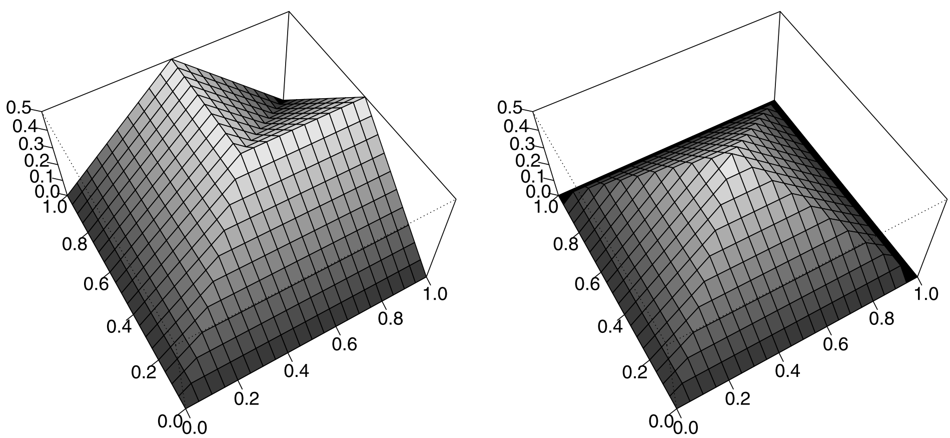

Figure 1. Graphs of $g(u, v) = \min\{u, v, 1 - \min(u, v)\}$ (left picture) and of $\tilde{g}(u, v) = \min\{u, v, (1 - u), (1 - v)\}$ (right picture).

reference appears to be [31], where, however, weight functions are only allowed to approach zero at the lower boundary of the unit cube. The restrictiveness of this condition becomes particularly visible in dimension $d = 2$ where it is known that the limit of the empirical copula process is zero on the entire boundary of the unit square [21]. This observation suggests that, for $d = 2$, it should be possible to maintain weak convergence of the empirical copula process when dividing by functions of the form $\{\tilde{g}(u, v)\}^{\omega}$ where

$$\tilde{g}(u, v) = u \wedge v \wedge (1 - u) \wedge (1 - v).$$

A picture of the graph of $\tilde{g}$ can be found in Figure 1, obviously, we have $\tilde{g} \le g$. The main result of this paper confirms the last-mentioned conjecture. More precisely, we establish weighted weak convergence of the empirical copula process in general dimension $d \ge 2$ with weight functions that approach zero wherever the potential limit approaches zero. We also do not require the observations to be i.i.d. and allow for exponential alpha mixing.

Potential applications of the new weighted weak convergence results are extensive. As a direct corollary, one can derive the asymptotic behavior of Anderson–Darling type goodness-of-fit statistics for copulas. The derivation of the asymptotic behavior of rank-based estimators for the Pickands dependence functions [20] can be greatly simplified and, moreover, can be simply extended to time series observations. Through a suitable partial integration formula, the results can also be exploited to derive weak convergence of multivariate rank statistics as, for instance, of certain scalar measures of (serial) dependence. The latter two applications are worked out in detail in Section 3 of this paper.

The remaining part of this paper is organized as follows. In Section 2, the empirical copula process is introduced and the main result of the paper, its weighted weak convergence, is stated. In Section 3, the main result is illustratively exploited to derive the asymptotics of multivariate rank statistics and of common estimators for extreme-value copulas. All proofs are deferred to Section 4, with some auxiliary results postponed to Section 5. Finally, an online supplement [3] contains some general results on (locally) bounded variation and integration for two-variate functions, as well as the proofs for two of the results from the main text.

2. Weighted empirical copula processes

Let $\mathbf{X} = (X_1, \ldots, X_d)'$ be a d-dimensional random vector with joint cumulative distribution function (c.d.f.) F and continuous marginal c.d.f.s $F_1, \ldots, F_d$. The copula C of F, or, equivalently, the copula of $\mathbf{X}$, is defined as the c.d.f. of the random vector $\mathbf{U} = (U_1, \ldots, U_d)'$ that arises from marginal application of the probability integral transform, that is, $U_j = F_j(X_j)$ for $j = 1, \ldots, d$. By construction, the marginal c.d.f.s of C are standard uniform on [0, 1]. By Sklar's theorem, C is the unique function for which we have

$$F(x_1, \ldots, x_d) = C\bigl\{F_1(x_1), \ldots, F_d(x_d)\bigr\}$$

for all $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

Let $\mathbf{X}_i, i = 1, \ldots, n$ be an observed stretch of a strictly stationary time series such that $\mathbf{X}_i$ is equal in distribution to $\mathbf{X}$. Set $\mathbf{U}_i = (U_{i1}, \ldots, U_{id}) \sim C$ with $U_{ij} = F_j(X_{ij})$. Define (observable) pseudo observations $\hat{\mathbf{U}}_i = (\hat{U}_{i1}, \ldots, \hat{U}_{id})$ of C through $\hat{U}_{ij} = nF_{nj}(X_{ij})(/n + 1)$ for $i = 1, \ldots, n$

and $j = 1, \ldots, d$. The empirical copula $\hat{C}_n$ of the sample $\mathbf{X}_1, \ldots, \mathbf{X}_n$ is defined as the empirical distribution function of $\hat{\mathbf{U}}_1, \ldots, \hat{\mathbf{U}}_n$, that is,

$$\hat{C}_n(\mathbf{u}) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(\hat{\mathbf{U}}_i \leq \mathbf{u}), \qquad \mathbf{u} \in [0,1]^d.$$

The corresponding empirical copula process is defined as

$$\mathbf{u} \mapsto \hat{\mathbb{C}}_n(\mathbf{u}) = \sqrt{n}\{\hat{C}_n(\mathbf{u}) - C(\mathbf{u})\}.$$

For $\omega \geq 0$, define a weight function

$$g_\omega(\mathbf{u}) = \min\left\{ \bigwedge_{j=1}^{d} u_j, \bigwedge_{j=1}^{d} \Bigl(1 - \min_{j' \neq j} u_{j'}\Bigr) \right\}^{\omega}.$$

For $d = 2$, the function is particularly nice and reduces to $g_\omega(u_1, u_2) = \min(u_1, u_2, 1 - u_1, 1 - u_2)^\omega$, see Figure 1. Note that for vectors $\mathbf{u} \in [0,1]^d$ such that at least one coordinate is equal to 0 or such that $d - 1$ coordinates are equal to 1, we have $g_\omega(\mathbf{u}) = 0$. As already mentioned in the Introduction for the case $d = 2$, these vectors are exactly the points where the limit of the empirical copula process is equal to 0, almost surely, whence one might hope to obtain a weak convergence result for $\mathbb{C}_n / g_\omega$. To prove such a result, a smoothness condition on C has to be imposed.

Condition 2.1. *For every $j \in \{1, \ldots, d\}$, the first order partial derivative $\dot{C}_j(\mathbf{u}) := \partial C(\mathbf{u})/\partial u_j$ exists and is continuous on $V_j = \{\mathbf{u} \in [0,1]^d : u_j \in (0,1)\}$. For every $j_2, j_2 \in \{1, \ldots, d\}$, the second order partial derivative $\ddot{C}_{j_1 j_2}(\mathbf{u}) := \partial^2 C(\mathbf{u})/\partial u_{j_1} \partial u_{j_2}$ exists and is continuous on $V_{j_1} \cap V_{j_2}$. Moreover, there exists a constant $K > 0$ such that*

$$\bigl|\ddot{C}_{j_1 j_2}(\mathbf{u})\bigr| \leq K \min\left\{ \frac{1}{u_{j_1}(1 - u_{j_1})}, \frac{1}{u_{j_2}(1 - u_{j_2})} \right\}, \qquad \forall \mathbf{u} \in V_{j_1} \cap V_{j_2}.$$

For completeness, define $\dot{C}_j(\mathbf{u}) = \limsup_{h \to 0}\{C(\mathbf{u} + h\mathbf{e}_j) - C(\mathbf{u})\}/h$ wherever it does not exist. Note, that Condition 2.1 coincides with Condition 2.1 and Condition 4.1 in [32], who used it to prove Stute's representation of an almost sure remainder term [35]. The condition is satisfied for many commonly occurring copulas [32].

For $-\infty \leq a < b \leq \infty$, let $\mathcal{F}_a^b$ denote the sigma-field generated by those $\mathbf{X}_i$ for which $i \in \{a, a+1, \ldots, b\}$ and define, for $k \geq 1$,

$$\alpha^{[\mathbf{X}]}(k) = \sup\bigl\{\bigl|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)\bigr| : A \in \mathcal{F}_{-\infty}^{i}, B \in \mathcal{F}_{i+k}^{\infty}, i \in \mathbb{Z}\bigr\}$$

as the alpha-mixing coefficient of the time series $(\mathbf{X}_i)_{i \in \mathbb{Z}}$. The sequence is called strongly mixing (or alpha-mixing) if $\alpha^{[\mathbf{X}]}(k) \to 0$ for $k \to \infty$. Finally,

$$\alpha_n(\mathbf{u}) = \sqrt{n}\{G_n(\mathbf{u}) - C(\mathbf{u})\}, \qquad G_n(\mathbf{u}) = n^{-1} \sum_{i=1}^{n} \mathbb{1}(\mathbf{U}_i \leq \mathbf{u}),$$

denotes the (unobservable) empirical process based on $\mathbf{U}_1, \ldots, \mathbf{U}_n$.

Theorem 2.2 (Weighted weak convergence of the empirical copula process). *Suppose that* $\mathbf{X}_1, \mathbf{X}_2, \dots$ *is a stationary, alpha-mixing sequence with* $\alpha^{[\mathbf{X}]}(k) = O(a^k)$, *as* $k \to \infty$, *for some* $a \in (0, 1)$. *If the marginals of the stationary distribution are continuous and if the corresponding copula* C *satisfies condition, then, for any* $c \in (0, 1)$ *and any* $\omega \in (0, 1/2)$,

$$\sup_{\mathbf{u}\in[c/n,1-c/n]^d} \left| \frac{\hat{\mathbb{C}}_n(\mathbf{u})}{g_\omega(\mathbf{u})} - \frac{\bar{\mathbb{C}}_n(\mathbf{u})}{g_\omega(\mathbf{u})} \right| = o_P(1), \tag{2.1}$$

where, for any $\mathbf{u} \in [0, 1]^d$,

$$\bar{\mathbb{C}}_n(\mathbf{u}) := \alpha_n(\mathbf{u}) - \sum_{j=1}^{d} \dot{C}_j(\mathbf{u})\alpha_n\big(\mathbf{u}^{(\mathbf{j})}\big),$$

with $\mathbf{u}^{(\mathbf{j})} = (1, \dots, 1, u_j, 1, \dots, 1)$. *Moreover, we have* $\bar{\mathbb{C}}_n/\tilde{g}_\omega \rightsquigarrow \mathbb{C}_C/\tilde{g}_\omega$ *in* $(\ell^\infty([0, 1]^d), \|\cdot\|_\infty)$, *where* $\tilde{g}_\omega(\mathbf{u}) = g_\omega(\mathbf{u}) + \mathbb{1}\{g_\omega(\mathbf{u}) = 0\}$, *where*

$$\mathbb{C}_C(\mathbf{u}) = \alpha_C(\mathbf{u}) - \sum_{j=1}^{d} \dot{C}_j(\mathbf{u})\alpha_C\big(\mathbf{u}^{(\mathbf{j})}\big),$$

and where α_C *denotes a tight, centered Gaussian process with covariance*

$$\operatorname{Cov}\big\{\alpha_C(\mathbf{u}), \alpha_C(\mathbf{v})\big\} = \sum_{i\in\mathbb{Z}} \operatorname{Cov}\big\{\mathbb{1}(\mathbf{U}_0 \le u), \mathbb{1}(\mathbf{U}_i \le v)\big\}.$$

The proof of theorem is given in Section 4.1 below. In fact, we state a more general result which is based on conditions on the usual empirical process α_n. These conditions are subsequently shown to be valid for exponentially alpha-mixing time series.

Remark 2.3. The supremum in (2.1) is taken over $[c/n, 1 - c/n]^d$, while it would be desirable to have a supremum over $(0, 1)^d$ or even $[0, 1]^d$. This, however, is not possible, as it can be easily seen that the function $\mathbf{u} \mapsto \hat{\mathbb{C}}_n(\mathbf{u})/g_\omega(\mathbf{u})$ is not even a bounded function on $(0, 1)^d$, in contrast to $\mathbf{u} \mapsto \bar{\mathbb{C}}_n(\mathbf{u})/g_\omega(\mathbf{u})$.

3. Applications

Theorem may be exploited in numerous ways. For instance, many of the most powerful goodness-of-fit tests for copulas are based on distances between the empirical copula and a parametric estimator for C [19]. The results of Theorem 2.2 can be exploited to validate tests for a richer class of distances, as for weighted Kolomogorov–Smirnov or L^2-distances. Second, estimators for extreme-value copulas can often be expressed through improper integrals involving the empirical copula (see [20], among others). Weighted weak convergence as in Theorem 2.2 facilitates the analysis of their asymptotic behavior and allows to extend the available results to time series observations. Details regarding the CFG- and the Pickands estimator are worked out in Section 3.2 below.

Theorem 2.2 may also be used outside the genuine copula framework, for instance, for proving asymptotic normality of multivariate rank statistics. The power of that approach lies in the fact that proofs for time series are essentially the same as for i.i.d. data sets. In Section 3.1, we derive a general weak convergence result for bivariate rank statistics.

3.1. Bivariate rank statistics

Bivariate rank statistics constitute an important class of real-valued statistics that can be written as

$$R_n = \frac{1}{n}\sum_{i=1}^{n} J(\hat{U}_{i1}, \hat{U}_{i2})$$

for some function $J : (0,1)^2 \to \mathbb{R}$, called score function. For $n \geq 2$, R_n can also be expressed as a Lebesgue–Stieltjes integral with respect to $\hat{C}_n$, that is,

$$R_n = \int_{(1/2n, 1-1/2n]^2} J(u,v)\,\mathrm{d}\hat{C}_n(u,v),$$

which offers the way to derive the asymptotic behavior of R_n from the asymptotic behavior of the empirical copula. This idea has already been exploited in [16]: however, in their Theorem 6, J has to be a bounded function which is not the case for many interesting examples. Also, the uniform central limit theorems for multivariate rank statistics in [38] require rather strong smoothness assumptions on J (which imply boundedness of J).

Example 3.1 (Rank autocorrelation coefficients). Suppose $Y_1, \dots, Y_n$ are drawn from a stationary, univariate time series $(Y_i)_{i\in\mathbb{Z}}$. Rank autocorrelation coefficients of lag $k \in \mathbb{N}$ are statistics of the form

$$r_{n,k} = \frac{1}{n-k}\sum_{i=k+1}^{n} J_1\left\{\frac{n}{n+1}F_n(Y_i)\right\} J_2\left\{\frac{n}{n+1}F_n(Y_{i-k})\right\},$$

where J_1, J_2 are real-valued functions on $(0,1)$ and F_n denotes the empirical c.d.f. of $Y_1, \dots, Y_n$. For example, the van der Waerden autocorrelation [25] is given by

$$r_{n,k,\mathrm{vdW}} = \frac{1}{n-k}\sum_{i=k+1}^{n} \Phi^{-1}\left\{\frac{n}{n+1}F_n(Y_i)\right\} \Phi^{-1}\left\{\frac{n}{n+1}F_n(Y_{i-k})\right\},$$

(with Φ and Φ^{-1} denoting the c.d.f. of the standard normal distribution and its inverse, resp.) and the Wilcoxon autocorrelation [25] is defined as

$$r_{n,k,W} = \frac{1}{n-k}\sum_{i=k+1}^{n} \left\{\frac{n}{n+1}F_n(Y_i) - \frac{1}{2}\right\} \log\left\{\frac{n/(n+1)F_n(Y_{i-k})}{1-n/(n+1)F_n(Y_{i-k})}\right\}.$$

Obviously, the corresponding score functions are unbounded. Asymptotic normality for these and similar rank statistics has been shown for i.i.d. observations and for ARMA-processes [24]. To the best of our knowledge, no general tool to handle the asymptotic behavior of such statistics for dependent observations seems to be available. Theorem 3.3 below aims at partially filling that gap.

Example 3.2 (The pseudo-maximum likelihood estimator). As a common practice in bivariate copula modeling one assumes to observe a sample $\mathbf{X}_1, \ldots, \mathbf{X}_n$ from a bivariate distribution whose copula belongs to a parametric copula family, parametrized by a finite-dimensional parameter $\theta \in \Theta \subset \mathbb{R}^p$. Except for the assumption of absolute continuity, the marginal distributions are often left unspecified in order to allow for maximal robustness with respect to potential miss-specification. In such a setting, the pseudo-maximum likelihood estimator (see [18] for a theoretical investigation) provides the most common estimator for the parameter θ. If c_θ denotes the corresponding copula density, the estimator is defined as

$$\hat{\theta}_n = \arg\max_{\theta \in \Theta} \sum_{i=1}^{n} \log\{c_\theta(\hat{U}_{i1}, \hat{U}_{i2})\}.$$

Using standard arguments from maximum-likelihood theory and imposing suitable regularity conditions, the asymptotic distribution of $\sqrt{n}(\hat{\theta}_n - \theta_0)$ can be derived from the asymptotic behavior of

$$R_n = \frac{1}{n} \sum_{i=1}^{n} J_{\theta_0}(\hat{U}_{i1}, \hat{U}_{i2}), \tag{3.1}$$

where θ_0 denotes the unknown true parameter and where $J_\theta = (\partial \log c_\theta)/(\partial\theta)$ denote the score function. Typically, this function is unbounded, as, for instance, in case of the bivariate Gaussian copula model where θ is the correlation coefficient and the score function takes the form

$$J_\theta(u, v) = \frac{\theta(1-\theta^2) - \theta\{\Phi^{-1}(u)^2 + \Phi^{-1}(v)^2\} + (1+\theta^2)\Phi^{-1}(u)\Phi^{-1}(v)}{1+\theta^2}.$$

Still, the conditions of Theorem 3.3 below can be shown to be valid.

Finally, note that pseudo-maximum likelihood estimators also arise in Markovian copula models [9] where copulas are used to model the serial dependence of a stationary time series at lag one. Again, their asymptotic distribution may be derived from rank statistics as in (3.1).

The following theorem is the central result of this section. It establishes weak convergence of bivariate rank-statistics by exploiting weighted weak convergence of the empirical copula process. For that purpose, we need a Lebesgue–Stieltjes integral with respect to right-continuous functions $J : (0, 1)^2 \to \mathbb{R}$ which are potentially unbounded *and* such that we can integrate by parts. The construction is standard, but despite extensive search, we were unable to find any reference where the details are worked out for the case of unbounded functions J. For convenience of the reader, we have collected all necessary details in the supplementary material (relying on [27] and [1]) and only give a very brief summary here: J must be of *locally bounded Hardy Krause variation*, $J \in \mathrm{BVHK}_{\mathrm{loc}}((0,1)^2)$, that is, for any sequences $0 < a_n < b_n < 1, a_n \to 0, b_n \to 1$, the restriction of J to the interval $[a_n, b_n]^2 \subset (0,1)^2$, is of (standard) bounded Hardy–Krause variation,[1] see, for example, Definition A.1 in the supplementary material [3]. The latter allows to define a sequence of unique signed Borel measure $\nu_n = \nu_n^+ - \nu_n^-$ on the Borel sets in $[a_n, b_n]^2$ such that $f(\mathbf{u}) = \nu([a_n, u_1] \times [a_n, u_2])$, see Theorem A.4 in the supplementary material [3] for details. By monotone convergence, we may define $[0,\infty]$-valued measures on the Borel sets in $(0,1)^2$ by

$$\nu^{\pm}(A) := \lim_{n\to\infty} \nu_n^{\pm}\big(A \cap (a_n, b_n]^2\big).$$

Proposition A.9 in the supplementary material [3] shows that this definition is independent of the choice of the sequences a_n and b_n and that

$$\nu\big((\mathbf{c}, \mathbf{d}]\big) := \nu^+\big((\mathbf{c}, \mathbf{d}]\big) - \nu^-\big((\mathbf{c}, \mathbf{d}]\big) = J(d_1, d_2) - J(c_1, d_2) - J(d_1, c_2) + J(c_1, c_2)$$

for any $\mathbf{c}, \mathbf{d} \in (0,1)^2$. Finally, for a measurable function $g : (0,1)^2 \to \mathbb{R}$ such that $\int |g| \,\mathrm{d}\nu^+ < \infty$ or $\int |g| \,\mathrm{d}\nu^- < \infty$, we may define the Lebesgue–Stieltjes integrals

$$\int_{(0,1)^2} g \,\mathrm{d}J := \int_{(0,1)^2} g \,\mathrm{d}\nu := \int_{(0,1)^2} g \,\mathrm{d}\nu^+ - \int_{(0,1)^2} g \,\mathrm{d}\nu^-,$$
$$\int_{(0,1)^2} g|\,\mathrm{d}J| := \int_{(0,1)^2} g|\,\mathrm{d}\nu| := \int_{(0,1)^2} g \,\mathrm{d}\nu^+ + \int_{(0,1)^2} g \,\mathrm{d}\nu^-.$$

For functions J that are two-times differentiable, these integrals can be expressed through the second order partial derivatives of J, see also Remark 3.4(i) below.

The proof of the following theorem is given in Section 4.3.

Theorem 3.3. *Suppose the conditions of Theorem 2.2 are met. Let* $J \in \mathrm{BVHK}_{\mathrm{loc}}((0,1)^2)$ *be right-continuous and assume that there exists* $\omega \in (0, 1/2)$ *such that* $|J(\mathbf{u})| \le \mathrm{const} \times g_\omega(\mathbf{u})^{-1}$ *and such that*

$$\int_{(0,1)^2} g_\omega(\mathbf{u}) \big| \mathrm{d}J(\mathbf{u})\big| < \infty. \tag{3.2}$$

Moreover, for $\delta \to 0$, *suppose that*

$$\int_{(\delta,1-\delta]} \big|J(\mathrm{d}u, \delta)\big| = O\big(\delta^{-\omega}\big) \quad \text{and} \quad \int_{(\delta,1-\delta]} \big|J(\mathrm{d}u, 1-\delta)\big| = O\big(\delta^{-\omega}\big), \tag{3.3}$$

$$\int_{(\delta,1-\delta]} \big|J(\delta, \mathrm{d}v)\big| = O\big(\delta^{-\omega}\big) \quad \text{and} \quad \int_{(\delta,1-\delta]} \big|J(1-\delta, \mathrm{d}v)\big| = O\big(\delta^{-\omega}\big). \tag{3.4}$$

[1]The literature contains various notions of bounded variation for multidimensional functions (see [27] and [39] for additional details). To the best of our knowledge, only the notion of bounded Hardy–Krause variation allows for a definition of a Lebesgue–Stieltjes integral and the validity of a formula for integration by parts.

Then, as $n \to \infty$,

$$\sqrt{n}\{R_n - \mathbb{E}[J(\mathbf{U})]\} \rightsquigarrow \int_{(0,1)^2} \mathbb{C}_C(\mathbf{u})\,\mathrm{d}J(\mathbf{u}).$$

The weak limit is normally distributed with mean 0 *and variance*

$$\sigma^2 = \int_{(0,1)^2}\int_{(0,1)^2} \mathbb{E}[\mathbb{C}_C(\mathbf{u})\mathbb{C}_C(\mathbf{v})]\,\mathrm{d}J(\mathbf{u})\,\mathrm{d}J(\mathbf{v}).$$

Remark 3.4. (i) Provided the second order partial derivative $\ddot{J}_{12}(u,v) := \partial^2 J(u,v)/\partial u\,\partial v$ exists, then the conditions (3.2)–(3.4) are equivalent to $\int_{(0,1)^2} g_\omega(u,v)|\ddot{J}_{12}(u,v)|\,\mathrm{d}(u,v) < \infty$ and, as $\delta \to 0$,

$$\int_\delta^{1-\delta} |\dot{J}_1(u,\delta)|\,\mathrm{d}u = O(\delta^{-\omega}) \quad \text{and} \quad \int_\delta^{1-\delta} |\dot{J}_1(u,1-\delta)|\,\mathrm{d}u = O(\delta^{-\omega}),$$
$$\int_\delta^{1-\delta} |\dot{J}_2(\delta,v)|\,\mathrm{d}v = O(\delta^{-\omega}) \quad \text{and} \quad \int_\delta^{1-\delta} |\dot{J}_2(1-\delta,v)|\,\mathrm{d}v = O(\delta^{-\omega}),$$

where $\dot{J}_1(u,v) := \partial J(u,v)/\partial u$, $\dot{J}_2(u,v) := \partial J(u,v)/\partial v$.

(ii) A careful check of the proof of Theorem 3.3 shows that the theorem actually remains valid under the more general conditions of Theorem 4.5 below, with $\omega \in (0,1/2)$ replaced by $\omega \in (0, \frac{\theta_1}{2(1-\theta_1)} \wedge \frac{\theta_2}{2(1-\theta_2)} \wedge (\theta_3 - 1/2))$.

As a simple application of Theorem 3.3 let us return to the autocorrelation coefficients from Example 3.1. It can easily be shown that both $J_{\mathrm{vdW}}(u,v) = \Phi^{-1}(u)\Phi^{-1}(v)$ and $J_W(u,v) = (u - \frac{1}{2})\log(\frac{v}{1-v})$ satisfy the conditions of Theorem 3.3. To prove this for J_{vdW} use that $|\Phi^{-1}(u)| \le \{u(1-u)\}^{-\varepsilon}$ for any $\varepsilon > 0$ and that $\frac{1}{\phi\{\Phi^{-1}(u)\}} \le \{u(1-u)\}^{-1}$, with ϕ denoting the density of the standard normal distribution. Therefore, both coefficients are asymptotically normally distributed for any stationary, exponentially alpha-mixing time series provided that the copula of (Y_t, Y_{t-k}) satisfies Condition 2.1. This broadens results from [24], which may be further extended along the lines of Remark 3.4(ii) by a more thorough investigation of Conditions 4.1–4.3. Details are omitted for the sake of brevity.

3.2. Nonparametric estimation of pickands dependence function

Theorem 2.2 can be used to extend recent results for the estimation of Pickands dependence functions. Recall that C is a multivariate extreme-value copula if and only if C has a representation of the form

$$C(\mathbf{u}) = \exp\left\{\left(\sum_{j=1}^d \log u_j\right) A\left(\frac{\log u_1}{\sum_{j=1}^d \log u_j}, \dots, \frac{\log u_{d-1}}{\sum_{j=1}^d \log u_j}\right)\right\}, \qquad \mathbf{u} \in (0,1)^d,$$

for some function $A : \Delta_{d-1} \to [1/d, 1]$, where Δ_{d-1} denotes the unit simplex $\Delta_{d-1} = \{\mathbf{w} = (w_1, \dots, w_{d-1}) \in [0,1]^{d-1} : \sum_{j=1}^{d-1} w_j \le 1\}$. In that case, A is necessarily convex and satisfies the relationship

$$\max(w_1,\dots,w_d) \le A(w_1,\dots,w_{d-1}) \le 1 \qquad \Biggl(w_d = 1 - \sum_{j=1}^{d-1} w_j\Biggr),$$

for all $\mathbf{w} \in \Delta_{d-1}$. By reference to [28], A is called Pickands dependence function. Nonparametric estimation methods for A in the i.i.d. case and under the additional assumption that the marginal distributions are known have been considered in [8,12,28,30], among others. In the more realistic case of unknown marginal distribution, rank-based estimators have, for instance, been investigated in [2,5,20,23], among others. For illustrative purposes, we restrict attention to the rank-based versions of the Pickands estimator in [23] in the following, even though the results easily carry over to, for instance, the CFG-estimator. The Pickands-estimator is defined as

$$\hat{A}_n^P(\mathbf{w}) = \Biggl[\frac{1}{n}\sum_{i=1}^n \min\biggl\{\frac{-\log(\hat{U}_{i1})}{w_1},\dots,\frac{-\log(\hat{U}_{id})}{w_d}\biggr\}\Biggr]^{-1}$$

and it follows by simple algebra (see Lemma 1 in [23]) that $\mathbb{A}_n^P := \sqrt{n}(\hat{A}_n^P - A) = -A^2\mathbb{B}_n^P/(1 + n^{1/2}\mathbb{B}_n^P)$, where

$$\mathbb{B}_n^P(\mathbf{w}) = \int_0^1 \hat{\mathbb{C}}_n\bigl(u^{w_1},\dots,u^{w_d}\bigr)\frac{\mathrm{d}u}{u}.$$

Note that $\int_0^1 u^{-1}\,\mathrm{d}u$ does not converge, which hinders a direct application of the continuous mapping theorem to deduce weak convergence of $\mathbb{B}_n^P$ (and hence of $\mathbb{A}_n^P$) in $\ell^\infty(\Delta_{d-1})$ just on the basis of (unweighted) weak convergence of $\hat{\mathbb{C}}_n$. Deeper results are necessary and in fact, [20] and [23] deduce weak convergence of $\mathbb{B}_n^P$ by using Stute's representation for the empirical copula process based on i.i.d. observations (see [35,36]) and by exploiting a weighted weak convergence result for α_n.

With Theorem 2.2, we can give a much simpler proof. Write

$$\mathbb{B}_n^P(\mathbf{w}) = \int_0^1 \frac{\hat{\mathbb{C}}_n(u^{w_1},\dots,u^{w_d})}{\min(u^{w_1},\dots,u^{w_d})^\omega}\,\frac{\min(u^{w_1},\dots,u^{w_d})^\omega}{u}\,\mathrm{d}u.$$

Then, since $\int_0^1 \min(u^{w_1},\dots,u^{w_d})^\omega \frac{\mathrm{d}u}{u} \le \int_0^1 u^{\omega/d-1}\,\mathrm{d}u$ exists for any $\omega > 0$, weak convergence of $\mathbb{B}_n^P$ is a direct consequence of the continuous mapping theorem and Theorem 2.2. Note that this method of proof is not restricted to the i.i.d. case.

4. Proofs

4.1. Proof of Theorem 2.2

Theorem 2.2 will be proved by an application of a more general result on the empirical copula process. For its formulation, we need a couple of additional conditions which, subsequently, will be shown to be satisfied for exponentially alpha-mixing time series.

Condition 4.1. *There exists some $\theta_1 \in (0, 1/2]$ such that, for all $\mu \in (0, \theta_1)$ and all sequences $\delta_n \to 0$, we have*

$$M_n(\delta_n, \mu) := \sup_{|\mathbf{u}-\mathbf{v}|\le\delta_n} \frac{|\alpha_n(\mathbf{u}) - \alpha_n(\mathbf{v})|}{|\mathbf{u}-\mathbf{v}|^{\mu} \vee n^{-\mu}} = o_P(1).$$

Condition 4.1 can, for instance, be verified in the i.i.d. case with $\theta_1 = 1/2$, exploiting a bound for the multivariate oscillation modulus derived in Proposition A.1 in [32] and relying on results in [15].

Condition 4.2. *The empirical process α_n converges weakly in $\ell^\infty([0,1]^d)$ to some limit process α_C which has continuous sample paths, almost surely.*

For i.i.d. samples, the latter condition is satisfies with α_C being a C-Brownian bridge, that is, a centered Gaussian process with continuous sample paths, a.s., and with $\mathrm{Cov}\{\alpha_C(\mathbf{u}), \alpha_C(\mathbf{v})\} = C(\mathbf{u} \wedge \mathbf{v}) - C(\mathbf{u})C(\mathbf{v})$.

Condition 4.3. *There exist $\theta_2 \in (0, 1/2]$ and $\theta_3 \in (1/2, 1]$ such that, for any $\omega \in (0, \theta_2)$, any $\lambda \in (0, \theta_3)$ and all $j = 1, \dots, d$, we have*

$$\sup_{u_j\in(0,1)} \left| \frac{\alpha_{nj}(u_j)}{u_j^{\omega}(1-u_j)^{\omega}} \right| = O_P(1), \qquad \sup_{u_j\in(1/n^{\lambda},1-1/n^{\lambda})} \left| \frac{\beta_{nj}(u_j)}{u_j^{\omega}(1-u_j)^{\omega}} \right| = O_P(1),$$

where $\alpha_{nj}(u_j) = \sqrt{n}\{G_{nj}(u_j) - u_j\}$ and $\beta_{nj}(u_j) = \sqrt{n}\{G_{nj}^{-}(u_j) - u_j\}$.

Here, $G_{nj}(u_j) = n^{-1}\sum_{i=1}^{n} \mathbb{1}(U_{ij} \le u_j)$ and, for a distribution function H on the reals, H^- denotes the (left-continuous) generalized inverse function of H defined as

$$H^-(u) := \inf\{x \in \mathbb{R} : H(x) \ge u\}, \qquad 0 < u \le 1,$$

and $H^-(0) = \sup\{x \in \mathbb{R} : H(x) = 0\}$. In the i.i.d. case, Condition 4.3 is a mere consequence of results in [10], with $\theta_2 = 1/2$, $\theta_3 = 1$.

The following proposition shows that the (probabilistic) Conditions 4.1, 4.2 and 4.3 are satisfied for sequences that are exponentially alpha-mixing.

Proposition 4.4. *Suppose that $\mathbf{X}_1, \mathbf{X}_2, \dots$ is a stationary, alpha-mixing sequence with $\alpha^{[\mathbf{X}]}(k) = O(a^k)$, as $k \to \infty$, for some $a \in (0, 1)$. Then, Conditions 4.1, 4.2 and 4.3 are satisfied with $\theta_1 = \theta_2 = 1/2$ and $\theta_3 = 1$.*

Here, Condition 4.3 is a mere consequence of results in [33] and [11], whereas Condition 4.2 has been shown in [29]. For the proof of Condition 4.1, we can rely on results from [26]. The precise arguments are given in Section B.1 in the supplementary material [3].

The following theorem can be regarded as a generalization of Theorem 2.2: weighted weak convergence of the empirical copula process takes place provided the abstract Conditions 4.1, 4.2 and 4.3 are met. The proof is given in Section 4.2 below.

Theorem 4.5 (Weighted weak convergence of empirical copula processes). *Suppose Conditions 2.1, 4.1 and 4.3 are met. Then, for any $c \in (0,1)$ and any $\omega \in (0, \frac{\theta_1}{2(1-\theta_1)} \wedge \frac{\theta_2}{2(1-\theta_2)} \wedge (\theta_3 - 1/2))$,*

$$\sup_{\mathbf{u}\in[c/n,1-c/n]^d} \left| \frac{\hat{\mathbb{C}}_n(\mathbf{u})}{g_\omega(\mathbf{u})} - \frac{\bar{\mathbb{C}}_n(\mathbf{u})}{g_\omega(\mathbf{u})} \right| = o_P(1).$$

If additionally Condition 4.2 is met, then $\bar{\mathbb{C}}_n/\tilde{g}_\omega \rightsquigarrow \mathbb{C}_C/\tilde{g}_\omega$ in $(\ell^\infty([0,1]^d), \|\cdot\|_\infty)$.

Proof of Theorem 2.2. The theorem is a mere consequence of Proposition 4.4 and Theorem 4.5. □

4.2. Proof of Theorem 4.5

Throughout the proof, we will use the following additional notations. Set

$$C_n(\mathbf{u}) = G_n\{\mathbf{G}_n^-(\mathbf{u})\}, \qquad \mathbf{G}_n^-(\mathbf{u}) = \bigl(G_{n1}^-(u_1), \dots, G_{nd}^-(u_d)\bigr)$$

and define a version of the empirical copula process based on C_n by

$$\mathbf{u} \mapsto \mathbb{C}_n(\mathbf{u}) = \sqrt{n}\{C_n(\mathbf{u}) - C(\mathbf{u})\}.$$

Moreover, for $0 < a < b < 1/2$, define

$$N(a,b) = \{\mathbf{u} \in [0,1]^d \mid a < g_1(\mathbf{u}) \le b\}.$$

Note that $[0,1]^d = \{\mathbf{u} : g_1(\mathbf{u}) = 0\} \cup N(0,a) \cup N(a,1/2)$. The set $N(a,1/2)$ consists of those vectors such that all of their coordinates are larger than a and such that at most $d-2$ coordinates are larger than or equal to $1-a$. In particular, for $d=2$, we have $N(a,1/2) = (a,1-a)^2$.

The proof of Theorem 4.5 will be based on the following sequence of lemmas. All convergences are with respect to $n \to \infty$.

Lemma 4.6. *Under the conditions of Theorem 4.5,*

$$\sup_{\mathbf{u}\in N(cn^{-1},1/2)} \left| \frac{\hat{\mathbb{C}}_n(\mathbf{u})}{g_\omega(\mathbf{u})} - \frac{\mathbb{C}_n(\mathbf{u})}{g_\omega(\mathbf{u})} \right| = o_P(1).$$

Lemma 4.7. *Under the conditions of Theorem 4.5,*

$$\sup_{\mathbf{u}\in N(n^{-1/2},1/2)} \left| \frac{\mathbb{C}_n(\mathbf{u})}{g_\omega(\mathbf{u})} - \frac{\bar{\mathbb{C}}_n(\mathbf{u})}{g_\omega(\mathbf{u})} \right| = o_P(1).$$

Lemma 4.8. *Under the conditions of Theorem 4.5, for any $\delta_n \downarrow 0$ such that $\delta_n \ge cn^{-1}$,*

$$\sup_{\mathbf{u}\in N(cn^{-1},\delta_n)} \left| \frac{\mathbb{C}_n(\mathbf{u})}{g_\omega(\mathbf{u})} \right| = o_P(1).$$

Lemma 4.9. *Under the conditions of Theorem 4.5, for any $\delta_n \downarrow 0$,*

$$\sup_{\mathbf{u}\in N(0,\delta_n)} \left| \frac{\bar{\mathbb{C}}_n(\mathbf{u})}{g_\omega(\mathbf{u})} \right| = o_P(1).$$

Lemma 4.10. *Under the conditions of Theorem 4.5, for any $\delta_n \downarrow 0$*

$$\sup_{\mathbf{u},\mathbf{u}'\in[c/n,1-c/n]^d:|\mathbf{u}-\mathbf{u}'|\le\delta_n} \left| \frac{\mathbb{C}_n(\mathbf{u})}{g_\omega(\mathbf{u})} - \frac{\mathbb{C}_n(\mathbf{u}')}{g_\omega(\mathbf{u}')} \right| = o_P(1) \tag{4.1}$$

and

$$\sup_{\mathbf{u},\mathbf{u}'\in[0,1]^d:|\mathbf{u}-\mathbf{u}'|\le\delta_n} \left| \frac{\bar{\mathbb{C}}_n(\mathbf{u})}{\tilde{g}_\omega(\mathbf{u})} - \frac{\bar{\mathbb{C}}_n(\mathbf{u}')}{\tilde{g}_\omega(\mathbf{u}')} \right| = o_P(1). \tag{4.2}$$

Proof of Theorem 4.5. Set $\delta_n = dn^{-1/2}$. Given $\mathbf{u} \in [\frac{c}{n}, 1 - \frac{c}{n}]^d$, choose $\mathbf{u}' \in [\frac{1}{\sqrt{n}}, 1 - \frac{1}{\sqrt{n}}]^d$ such that $|\mathbf{u} - \mathbf{u}'| \le \delta_n$. Since

$$\begin{aligned}\left| \frac{\hat{\mathbb{C}}_n(\mathbf{u})}{g_\omega(\mathbf{u})} - \frac{\bar{\mathbb{C}}_n(\mathbf{u})}{g_\omega(\mathbf{u})} \right| \le{}& \left| \frac{\hat{\mathbb{C}}_n(\mathbf{u})}{g_\omega(\mathbf{u})} - \frac{\mathbb{C}_n(\mathbf{u})}{g_\omega(\mathbf{u})} \right| + \left| \frac{\mathbb{C}_n(\mathbf{u})}{g_\omega(\mathbf{u})} - \frac{\mathbb{C}_n(\mathbf{u}')}{g_\omega(\mathbf{u}')} \right| \\ &+ \left| \frac{\mathbb{C}_n(\mathbf{u}')}{g_\omega(\mathbf{u}')} - \frac{\bar{\mathbb{C}}_n(\mathbf{u}')}{g_\omega(\mathbf{u}')} \right| + \left| \frac{\bar{\mathbb{C}}_n(\mathbf{u}')}{g_\omega(\mathbf{u}')} - \frac{\bar{\mathbb{C}}_n(\mathbf{u})}{g_\omega(\mathbf{u})} \right|\end{aligned}$$

the first assertion of the theorem follows from Lemma 4.6, 4.7 and 4.10.

Next, let us show that $\bar{\mathbb{C}}_n/\tilde{g}_\omega \rightsquigarrow \mathbb{C}_C/\tilde{g}_\omega$ in $(\ell^\infty([0,1]^d), \|\cdot\|_\infty)$. From Problem 2.1.5 in [37] and Lemma 4.10 we obtain that $\bar{\mathbb{C}}_n/\tilde{g}_\omega$ is asymptotically equicontinuous. Furthermore, Condition 4.2 yields that the finite dimensional distributions of $\bar{\mathbb{C}}_n/\tilde{g}_\omega$ converge weakly to the finite dimensional distributions of $\mathbb{C}_C/\tilde{g}_\omega$. Note that $\mathbb{C}_C/\tilde{g}_\omega(\mathbf{u}) = \bar{\mathbb{C}}_n/\tilde{g}_\omega(\mathbf{u}) = 0$ for any $\mathbf{u}$ with at least one entry equal to 0 or with $d-1$ entries equal to 1. □

Proof of Lemma 4.6. It suffices to show that, there exists $\mu \in (\omega, \theta_1)$ such that

$$\sup_{\mathbf{u}\in[0,1]^d} \big|\hat{C}_n(\mathbf{u}) - C_n(\mathbf{u})\big| = o_P\big(n^{-1/2-\mu}\big).$$

Note that $F_{nj}(X_{ij}) = G_{nj}(U_j)$, whence

$$\begin{aligned}\sup_{\mathbf{u}\in[0,1]^d} \big|\hat{C}_n(\mathbf{u}) - C_n(\mathbf{u})\big| \le{}& \sup_{\mathbf{u}\in[0,1]^d} \left| \frac{1}{n}\sum_{i=1}^n \mathbb{1}\left\{\mathbf{G}_n(\mathbf{U}_i) \le \frac{n+1}{n}\mathbf{u}\right\} - \mathbb{1}\big\{\mathbf{G}_n(\mathbf{U}_i) \le \mathbf{u}\big\} \right| \\ &+ \sup_{\mathbf{u}\in[0,1]^d} \left| \frac{1}{n}\sum_{i=1}^n \mathbb{1}\big\{\mathbf{G}_n(\mathbf{U}_i) \le \mathbf{u}\big\} - \mathbb{1}\big\{\mathbf{U}_i \le \mathbf{G}_n^-(\mathbf{u})\big\} \right|\end{aligned}$$

$$\le \sum_{j=1}^{d} \Bigg[\sup_{u\in[0,1]} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\Big\{ u < G_{nj}(U_{ij}) \le \frac{n+1}{n} u \Big\}$$

$$+ \sup_{u\in[0,1]} \frac{1}{n} \sum_{i=1}^{n} \big| \mathbb{1}\big\{ G_{nj}(U_{ij}) \le u \big\} - \mathbb{1}\big\{ U_{ij} \le G_{nj}^{-}(u) \big\} \big| \Bigg].$$

From the definition of the empirical distribution function and the generalized inverse function we have that, for any fixed u, both $\sum_{i=1}^{n} \mathbb{1}\{u < G_{nj}(U_{ij}) \le \frac{n+1}{n} u\}$ and $\sum_{i=1}^{n} |\mathbb{1}\{G_{nj}(U_{ij}) \le u\} - \mathbb{1}\{U_{ij} \le F_{nj}^{-}(u)\}|$ are bounded by the maximal number of U_{ij} which are equal. Note that this maximal number is equal to $n \times \sup_{u\in[0,1]} |G_{nj}(u) - G_{nj}(u-)|$. Provided there are no ties among $U_{1j}, \ldots, U_{nj}$, for any $j = 1, \ldots, d$ (which, e.g., occurs in the i.i.d. case), this expression is equal to 1 and the lemma is proven. In the general case, we have

$$\begin{aligned}
\sup_{u\in[0,1]} \big| G_{nj}(u) - G_{nj}(u-) \big| &\le \sup_{\substack{u,v\in[0,1] \\ |u-v|\le 1/n}} \big| G_{nj}(u) - G_{nj}(v) \big| \\
&\le \sup_{\substack{u,v\in[0,1] \\ |u-v|\le 1/n}} \big| G_{nj}(u) - G_{nj}(v) - (u-v) \big| + \frac{1}{n} \qquad (4.3) \\
&\le \frac{1}{\sqrt{n}} \sup_{\substack{\mathbf{u},\mathbf{v}\in[0,1]^d \\ |\mathbf{u}-\mathbf{v}|\le 1/n}} \big| \alpha_n(\mathbf{u}) - \alpha_n(\mathbf{v}) \big| + \frac{1}{n}.
\end{aligned}$$

Then, the assertion follows from Condition 4.1. □

Proof of Lemma 4.7. First of all, we write

$$\mathbb{C}_n(\mathbf{u}) - \bar{\mathbb{C}}_n(\mathbf{u}) = (B_{n1} + B_{n2} + B_{n3})(\mathbf{u}),$$

where

$$B_{n1}(\mathbf{u}) = \alpha_n\big\{\mathbf{G}_n^{-}(\mathbf{u})\big\} - \alpha_n(\mathbf{u}),$$

$$B_{n2}(\mathbf{u}) = \sqrt{n}\big[C\big\{\mathbf{G}_n^{-}(\mathbf{u})\big\} - C(\mathbf{u})\big] - \sum_{j=1}^{d} \dot{C}_j(\mathbf{u})\beta_{nj}(u_j),$$

$$B_{n3}(\mathbf{u}) = \sum_{j=1}^{d} \dot{C}_j(\mathbf{u})\big\{\beta_{nj}(u_j) + \alpha_{nj}(u_j)\big\}.$$

For $p = 1, 2, 3$, set $A_{np}(\mathbf{u}) = B_{np}(\mathbf{u})/g_\omega(\mathbf{u})$. The lemma is proved if we show uniform negligibility of each term individually.

Treatment of A_{n1}. Let Ω_n denote the event that $\sup_{\mathbf{u}\in[0,1]^d} |\mathbf{G}_n^{-}(\mathbf{u}) - \mathbf{u}| \le \delta_n = n^{-1/2+\kappa}$, with $\kappa > 0$ to be specified later on. Note that the probability of Ω_n converges to 1. Exploiting Condition 4.1 and the fact that $|g_\omega(\mathbf{u})|^{-1} \le n^{\omega/2}$ for $\mathbf{u} \in N(n^{-1/2}, 1/2)$ we obtain, for any $\mu \in (0, \theta_1)$,

$$\begin{aligned}\sup_{\mathbf{u}\in N(n^{-1/2},1/2)} \big|A_{n1}(\mathbf{u})\big| &\le n^{\omega/2} \sup_{\mathbf{u}\in[0,1]^d} \big|\alpha_n\{\mathbf{G}_n^-(\mathbf{u})\} - \alpha_n(\mathbf{u})\big| \\ &\le n^{\omega/2} M_n(\delta_n, \mu) \sup_{\mathbf{u}\in[0,1]^d} \big\{\big|\mathbf{G}_n^-(\mathbf{u}) - \mathbf{u}\big|^\mu \vee n^{-\mu}\big\} \mathbb{1}_{\Omega_n} + o_P(1) \\ &\le n^{\omega/2-\mu/2+\kappa\mu} o_P(1) + o_P(1).\end{aligned}$$

The right-hand side is $o_P(1)$ if we choose $\mu \in (\omega, \theta_1)$ sufficiently large and $\kappa > 0$ sufficiently small such that $\omega < \mu(1 - 2\kappa)$.

Treatment of A_{n2}. Fix $\mathbf{u} \in N(n^{-1/2}, 1/2)$. Let $S = S_{\mathbf{u}}$ denote the set of all $j \in \{1, \dots, d\}$ such that $u_j \in [n^{-1/2}, 1 - n^{-\gamma}]$, with $\gamma > 1/2$ to be specified later. Let $(\mathbf{G}_n^-(\mathbf{u}))_S$ denote the vector in $\mathbb{R}^d$ whose jth coordinate is equal to $G_{nj}^-(u_j)\mathbb{1}(j \in S) + u_j \mathbb{1}(j \notin S)$. Write $A_{n2}(\mathbf{u}) = D_{n1}(\mathbf{u}) + D_{n2}(\mathbf{u})$, where

$$D_{n1}(\mathbf{u}) = \left(\sqrt{n}\big[C\{\mathbf{G}_n^-(\mathbf{u})\} - C\{(\mathbf{G}_n^-(\mathbf{u}))_S\}\big] - \sum_{j\notin S} \dot{C}_j(\mathbf{u})\beta_{nj}(u_j)\right) g_\omega^{-1}(\mathbf{u}),$$

$$D_{n2}(\mathbf{u}) = \left(\sqrt{n}\big[C\{(\mathbf{G}_n^-(\mathbf{u}))_S\} - C(\mathbf{u})\big] - \sum_{j\in S} \dot{C}_j(\mathbf{u})\beta_{nj}(u_j)\right) g_\omega^{-1}(\mathbf{u}).$$

Since $\dot{C}_j \in [0, 1]$, we can bound

$$D_{n1}(\mathbf{u}) \le 2\sum_{j\notin S} \left|\frac{\beta_{nj}(u_j)}{g_\omega(\mathbf{u})}\right| \le 2\sum_{j=1}^d \sup_{u_j\in[1-n^{-\gamma},1]} \left|\frac{\beta_{nj}(u_j)}{n^{-\omega/2}}\right|.$$

The right-hand side is $o_P(1)$ by Lemma 5.3. Regarding D_{n2}, by Taylor's theorem, $|D_{n2}(\mathbf{u})| = \frac{1}{2}\sum_{j_1,j_2\in S} D_{n2}^{j_1 j_2}(\mathbf{u})$, where

$$D_{n2}^{j_1 j_2}(\mathbf{u}) = n^{-1/2} \ddot{C}_{j_1 j_2}(\boldsymbol{\xi}_n)\beta_{nj_1}(u_{j_1})\beta_{nj_2}(u_{j_2}) g_\omega(\mathbf{u})^{-1},$$

and where $\boldsymbol{\xi}_n = (\xi_{n1}, \dots, \xi_{nd})'$ is an intermediate point between $(\mathbf{G}_n^-(\mathbf{u}))_S$ and $\mathbf{u}$. By Condition 2.1, we have

$$\big|\ddot{C}_{j_1 j_2}(\boldsymbol{\xi}_n)\big| \le K\{\xi_{nj_1}(1 - \xi_{nj_1})\}^{-1/2}\{\xi_{nj_2}(1 - \xi_{nj_2})\}^{-1/2}.$$

Therefore, since $g_\omega(\mathbf{u})^{-1} \le n^{\omega/2}$,

$$\begin{aligned}\big|D_{n2}^{j_1 j_2}(\mathbf{u})\big| \le K n^{-1/2+\omega/2} \sup_{\mathbf{u}\in[n^{-1/2},1-n^{-1/2}]^d} \Bigg| &\left\{\frac{u_{j_1}(1 - u_{j_1})}{\xi_{nj_1}(1 - \xi_{nj_1})}\right\}^{1/2} \times \left\{\frac{u_{j_2}(1 - u_{j_2})}{\xi_{nj_2}(1 - \xi_{nj_2})}\right\}^{1/2} \\ &\times \frac{|\beta_{nj_1}(u_{j_1})|}{\{u_{j_1}(1 - u_{j_1})\}^\omega} \times \frac{|\beta_{nj_2}(u_{j_2})|}{\{u_{j_2}(1 - u_{j_2})\}^\omega} \times \big\{u_{j_1}(1 - u_{j_1})u_{j_2}(1 - u_{j_2})\big\}^{\omega-1/2}\Bigg|.\end{aligned}$$

By an application of Lemma 5.2 and by Condition 4.3, the right-hand side is of order $O_P(n^{-1/2+\omega/2+\gamma(1-2\omega)}) = o_P(1)$, provided we choose $\gamma \in (1/2, \{1/2 + \omega/(2-4\omega)\} \wedge \{1/(2(1-\theta_2))\} \wedge \theta_3)$. Since $\mathbf{u} \in N(n^{-1/2}, 1/2)$ was arbitrary, we can conclude that $\sup_{\mathbf{u}\in N(n^{-1/2},1/2)} |A_{n2}(\mathbf{u})| = o_P(1)$.

Treatment of A_{n3}. Since $|\dot{C}_j(\mathbf{u})| \le 1$ for any $\mathbf{u} \in [0,1]^d$, we have

$$\sup_{\mathbf{u}\in N(n^{-1/2},1/2)} \bigl|A_{n3}(\mathbf{u})\bigr| \le n^{\omega/2} \sum_{j=1}^{d} \sup_{u_j\in[0,1]} \bigl|\beta_{nj}(u_j) + \alpha_{nj}\bigl\{G_{nj}^{-}(u_j)\bigr\}\bigr| + n^{\omega/2} \sum_{j=1}^{d} \sup_{u_j\in[0,1]} \bigl|\alpha_{nj}\bigl\{G_{nj}^{-}(u_j)\bigr\} - \alpha_{nj}(u_j)\bigr|.$$

The second sum on the right-hand side is of order $o_P(1)$ as shown in the preceding treatment of the term A_{n1}. Negligibility of the first sum follows from Lemma 5.1, observing that $\alpha_{nj}\{G_{nj}^{-}(u_j)\} = \sqrt{n}[G_{nj}\{G_{nj}^{-}(u_j)\} - G_{nj}^{-}(u_j)]$ from the definition of α_n. $\square$

Proof of Lemma 4.8. Note that, by a monotonicity argument, it suffices to treat sequences δ_n such that $\delta_n \gg n^{-1/2}$, that is, $\delta_n\sqrt{n} \to \infty$. First of all, choose γ such that $1/2 + \omega < \gamma < 1/\{2(1-\theta_2)\} \wedge \theta_3$. Set $M_{n\gamma} = N(n^{-\gamma}, \delta_n) \cap (n^{-\gamma}, 1-n^{-\gamma})^d$ and $M_{n\gamma}^c = N(n^{-\gamma}, \delta_n) \setminus (n^{-\gamma}, 1-n^{-\gamma})^d$, and note that $N(cn^{-1}, \delta_n) = N(cn^{-1}, n^{-\gamma}) \cup M_{n\gamma} \cup M_{n\gamma}^c$. Therefore,

$$\sup_{\mathbf{u}\in N(cn^{-1},\delta_n)} \left|\frac{\mathbb{C}_n(\mathbf{u})}{g_\omega(\mathbf{u})}\right| = R_n\bigl\{N\bigl(cn^{-1}, n^{-\gamma}\bigr)\bigr\} \vee R_n(M_{n\gamma}) \vee R_n\bigl(M_{n\gamma}^c\bigr), \tag{4.4}$$

where, for $A \subset [0,1]^d$, $R_n(A) = \sup_{\mathbf{u}\in A} |\mathbb{C}_n(\mathbf{u})/g_\omega(\mathbf{u})|$. It suffices to show negligibility of each term on the right-hand side of (4.4).

Treatment of $R_n\{N(cn^{-1}, n^{-\gamma})\}$. We will distinguish the cases that either $g_\omega(\mathbf{u}) = u_1^\omega$ or $g_\omega(\mathbf{u}) = (1-u_1)^\omega$. The cases $g_\omega(\mathbf{u}) = u_j^\omega$ or $g_\omega(\mathbf{u}) = (1-u_j)^\omega$ for some $j > 1$ can be treated similarly.

Let us first consider $\mathbf{u}$ such that $g_\omega(\mathbf{u}) = u_1^\omega$. Obviously,

$$\bigl|C_n(\mathbf{u}) - C(\mathbf{u})\bigr| \le \bigl|C_n(\mathbf{u}) - C_n(0, u_2, \dots, u_d)\bigr| + \bigl|C(0, u_2, \dots, u_d) - C(\mathbf{u})\bigr|.$$

By Lipschitz-continuity of the copula function C, the second term on the right-hand side can be bounded by $u_1 = g_1(\mathbf{u})$. For the first term, note that

$$\begin{aligned} \bigl|C_n(\mathbf{u}) - C_n(0, u_2, \dots, u_d)\bigr| &= \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\bigl\{\mathbf{U}_i \le \mathbf{G}_n^{-}(\mathbf{u})\bigr\} \\ &\le \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\bigl\{U_{i1} \le G_{n1}^{-}(u_1)\bigr\} = G_{n1}\bigl\{G_{n1}^{-}(u_1)\bigr\}. \end{aligned} \tag{4.5}$$

By Lemma 5.1 the last expression is equal to $u_1 + o_P(n^{-1/2-\mu}) = g_1(\mathbf{u}) + o_P(n^{-1/2-\mu})$ for any $\mu \in (\omega, \theta_1)$, where the residual term is uniformly in $u_1 \in [0,1]$. Combined, this yields $|\mathbb{C}_n(\mathbf{u})| \le \sqrt{n}2g_1(\mathbf{u}) + o_P(n^{-\mu})$, and hence

$$\sup_{\mathbf{u}\in N(cn^{-1},n^{-\gamma}),g_\omega(\mathbf{u})=u_1^\omega} \left|\frac{\mathbb{C}_n(\mathbf{u})}{g_\omega(\mathbf{u})}\right| \le 2n^{1/2+\omega-\gamma} + o_P\bigl(n^{-\mu+\omega}\bigr) = o_P(1).$$

Now, consider the case $g_\omega(\mathbf{u}) = (1-u_1)^\omega$, that is, $1-u_1 = 1-\min_{j\neq k} u_j$ for some $k \in \{2,\dots,d\}$ and without loss of generality we may assume that $k=2$. Then, in particular, $1-u_1 \le 1-u_2$ and $1-u_1 \ge 1-u_j$ for all $j \ge 3$. Now, decompose

$$\bigl|C_n(\mathbf{u}) - C(\mathbf{u})\bigr| \le \bigl|C_n(\mathbf{u}) - C_n\bigl(\mathbf{u}^{(2)}\bigr)\bigr| + \bigl|C_n\bigl(\mathbf{u}^{(2)}\bigr) - C\bigl(\mathbf{u}^{(2)}\bigr)\bigr| + \bigl|C\bigl(\mathbf{u}^{(2)}\bigr) - C(\mathbf{u})\bigr|.$$

Again by Lipschitz-continuity of the copula function, we have

$$\bigl|C\bigl(\mathbf{u}^{(2)}\bigr) - C(\mathbf{u})\bigr| \le \sum_{j\neq 2} |1-u_j| \le (d-1)|1-u_1| = (d-1)g_1(\mathbf{u}).$$

Furthermore, we have

$$\begin{aligned}\bigl|C_n(\mathbf{u}) - C_n\bigl(\mathbf{u}^{(2)}\bigr)\bigr| \le{}& \bigl|C_n(\mathbf{u}) - C_n\{1,u_2,\dots,u_d\}\bigr| \\ &+ \bigl|C_n\{1,u_2,\dots,u_d\} - C_n\{1,u_2,1,u_4,\dots,u_d\}\bigr| \\ &+ \dots + \bigl|C_n\{1,u_2,1,1,\dots,1,u_d\} - C_n\bigl(\mathbf{u}^{(2)}\bigr)\bigr|\end{aligned} \tag{4.6}$$

and thus, by similar arguments as in (4.5), $|C_n(\mathbf{u}) - C_n(\mathbf{u}^{(2)})| \le (d-1)g_1(\mathbf{u}) + o_P(n^{-1/2-\mu})$, uniformly in $\mathbf{u}$. Finally, from Lemma 5.1

$$\bigl|C_n\bigl(\mathbf{u}^{(2)}\bigr) - C\bigl(\mathbf{u}^{(2)}\bigr)\bigr| = \bigl|G_{n2}\bigl\{G_{n2}^-(u_2)\bigr\} - u_2\bigr| = o_P\bigl(n^{-1/2-\mu}\bigr).$$

Altogether, we obtain

$$\sup_{\mathbf{u}\in N(cn^{-1},n^{-\gamma}),g_\omega(\mathbf{u})=(1-u_1)^\omega} \left|\frac{\mathbb{C}_n(\mathbf{u})}{g_\omega(\mathbf{u})}\right| \le 2(d-1)n^{1/2+\omega-\gamma} + o_P\bigl(n^{-\mu+\omega}\bigr) = o_P(1).$$

Treatment of $R_n(M_{n\gamma})$. Again, let us first treat the case where $g_\omega(\mathbf{u}) = u_1^\omega$. We can write $\mathbb{C}_n(\mathbf{u})/g_\omega(\mathbf{u}) = S_{1n}(\mathbf{u}) + S_{2n}(\mathbf{u}) + S_{3n}(\mathbf{u})$, where

$$\begin{aligned} S_{1n}(\mathbf{u}) &= \sqrt{n}\bigl[G_n\bigl\{\mathbf{G}_n^-(\mathbf{u})\bigr\} - C\bigl\{\mathbf{G}_n^-(\mathbf{u})\bigr\}\bigr]/g_\omega(\mathbf{u}), \\ S_{2n}(\mathbf{u}) &= \sqrt{n}\bigl[C\bigl\{\mathbf{G}_n^-(\mathbf{u})\bigr\} - C\bigl\{G_{n1}^-(u_1),u_2,\dots,u_d\bigr\}\bigr]/g_\omega(\mathbf{u}), \\ S_{3n}(\mathbf{u}) &= \sqrt{n}\bigl[C\bigl\{G_{n1}^-(u_1),u_2,\dots,u_d\bigr\} - C(\mathbf{u})\bigr]/g_\omega(\mathbf{u}). \end{aligned}$$

Lipschitz continuity of the copula C together with Condition 4.3 implies that

$$\sup_{\mathbf{u}\in M_{n\gamma},g_\omega(\mathbf{u})=u_1^\omega} \bigl|S_{3n}(\mathbf{u})\bigr| = o_P(1).$$

Regarding S_{1n}, let Ω_n denote the event that $\sup_{u_1\in[0,\delta_n]} G_{n1}^-(u_1) \le 2\delta_n$. On Ω_n^c, we have $\sqrt{n}\delta_n < \sup_{u_1\in[0,\delta_n]} \sqrt{n}|G_{n1}^-(u_1) - u_1| = O_P(1)$, whence, by the assumption that $\sqrt{n}\delta_n \to \infty$, we get $\Pr(\Omega_n^c) \to 0$. Therefore, by Condition 4.1, for any $\mu \in (0, \theta_1)$, we have

$$\begin{aligned}
\bigl|S_{1n}(\mathbf{u})\bigr| &= \left|\frac{\alpha_n\{\mathbf{G}_n^-(\mathbf{u})\} - \alpha_n\{0, G_{n2}^-(u_2), \dots, G_{nd}^-(u_d)\}}{u_1^\omega}\right| \\
&\le M_n(2\delta_n, \mu)\left|\frac{\{G_{n1}^-(u_1)\}^\mu \vee n^{-\mu}}{u_1^\omega}\right| \mathbb{1}_{\Omega_n} + o_P(1) \\
&\le o_P(1)\left\{\frac{|G_{n1}^-(u_1) - u_1|^\mu}{u_1^\omega} + u_1^{\mu-\omega}\right\} \vee n^{-\mu+\gamma\omega} + o_P(1),
\end{aligned}$$

where we used subadditivity of the function $x \mapsto x^\mu$, $x \ge 0$. By Condition 4.3, we have

$$\begin{aligned}
\sup_{u_1\in[n^{-\gamma},\delta_n]} \frac{|G_{n1}^-(u_1) - u_1|^\mu}{u_1^\omega} &\le n^{-\mu/2} \sup_{u_1\in[n^{-\gamma},\delta_n]} \bigl|u_1^{\omega(\mu-1)}\bigr| O_P(1) \\
&= O_P\bigl(n^{-\mu/2-\gamma\omega(\mu-1)}\bigr).
\end{aligned}$$

Exploit that $\gamma < 1$ and choose $\mu \in (\omega/(\omega + 1/2), \theta_1)$ to obtain that, as $n \to \infty$,

$$\sup_{\mathbf{u}\in M_{n\gamma}, g_\omega(\mathbf{u})=u_1^\omega} \bigl|S_{1n}(\mathbf{u})\bigr| = o_P(1).$$

Finally, we turn to S_{2n}. The mean value theorem allows to write

$$S_{2n}(\mathbf{u}) = \sum_{j=2}^d \frac{\dot{C}_j\{G_{n1}^-(u_1), \zeta_2, \dots, \zeta_d\}\sqrt{n}\{G_{nj}^-(u_j) - u_j\}}{g_\omega(\mathbf{u})} =: \sum_{j=2}^d S_{2nj}(\mathbf{u})$$

for some intermediate values ζ_j between u_j and $G_{nj}^-(u_j)$, for $j = 2, \dots, d$. We may consider each summand individually; let us fix $j \in \{2, \dots, d\}$ and distinguish two cases. First, suppose that $1 - u_j < u_1 = g_1(\mathbf{u})$. Then, with $\omega' \in (\omega, \theta_1)$,

$$\bigl|S_{2nj}(\mathbf{u})\bigr| \le \frac{\sqrt{n}|G_{nj}^-(u_j) - u_j|}{(1 - u_j)^{\omega'}}(1 - u_j)^{\omega'-\omega} = o_P(1),$$

by Condition 4.3 and the fact that $n^{-\gamma} < (1 - u_j) \le \delta_n$. Now, suppose that $1 - u_j \ge u_1 = g_1(\mathbf{u}) > n^{-\gamma}$. Since $\dot{C}_j(0, u_2, \dots, u_d) = 0$ for any $j = 2, \dots, d$, another application of the mean value theorem allows to write

$$S_{2nj}(\mathbf{u}) = \frac{\ddot{C}_{j1}(\boldsymbol{\xi}_j)G_{n1}^-(u_1)\sqrt{n}\{G_{nj}^-(u_j) - u_j\}}{u_1^\omega},$$

where $\boldsymbol{\xi}_j = (\xi_{j1}, \zeta_2, \dots, \zeta_d)$ satisfies $\xi_{j1} \in (0, G_{n1}^-(u_1))$. Now, fix $\omega' \in (0, \theta_2)$ such that $\omega' > (1 - \frac{1}{2\gamma}) \vee \omega$. By Condition 2.1 and Lemma 5.2, we have

$$
\begin{aligned}
|S_{2nj}(\mathbf{u})| &\le \frac{G_{n1}^-(u_1)}{u_1^\omega} \left| \frac{\sqrt{n}\{G_{nj}^-(u_j) - u_j\}}{\{u_j(1-u_j)\}^{\omega'}} \right| \times K \frac{\{u_j(1-u_j)\}^{\omega'}}{\xi_{jj}(1-\xi_{jj})} \\
&\le \left\{ n^{-1/2} \frac{\sqrt{n}|G_{n1}^-(u_1) - u_1|}{u_1^\omega} + u_1^{1-\omega} \right\} \{u_j(1-u_j)\}^{\omega'-1} O_P(1).
\end{aligned}
\tag{4.7}
$$

Observing that $u_j \ge u_1$ as a consequence of $g_\omega(\mathbf{u}) = u_1^\omega$ and that $1 - u_j \ge u_1$ by assumption, we obtain

$$
\{u_j(1-u_j)\}^{\omega'-1} \le [\{u_j \wedge (1-u_j)\}/2]^{\omega'-1} \le 2^{1-\omega'} u_1^{\omega'-1} \le 2u_1^{\omega'-1},
$$

where we used the fact that $u(1-u) \ge \{u \wedge (1-u)\}/2$ for all $u \in [0,1]$. Therefore, we can bound the right-hand side of (4.7) by

$$
\{n^{-1/2} u_1^{\omega'-1} O_P(1) + u_1^{\omega'-\omega}\} \times O_P(1),
$$

where all O_P-terms are uniform in $\{\mathbf{u} \in M_{n\gamma} : g_\omega(\mathbf{u}) = u_1^\omega\}$. Thus, by the choice of γ and ω', $\sup_{\mathbf{u} \in M_{n\gamma}, g_\omega(\mathbf{u}) = u_1^\omega} |S_{2n}(\mathbf{u})| = o_P(1)$.

For the treatment of $R_n(M_{n\gamma})$, it remains to consider the case $g_\omega(\mathbf{u}) = (1-u_1)^\omega$, that is, $1 - u_1 = 1 - \min_{j \ne k} u_j$ for some $k \in \{2, \dots, d\}$. Again, without loss of generality, we may assume that $k = 2$, which implies that $1 - u_1 \le 1 - u_2$ and $1 - u_1 \ge 1 - u_j$ for all $j \ge 3$. Note that, additionally, $1 - u_j > n^{-\gamma}$ for all $j = 1, \dots, d$ since $\mathbf{u} \in M_{n\gamma}$. Now,

$$
\frac{\mathbb{C}_n(\mathbf{u})}{g_\omega(\mathbf{u})} = \frac{\alpha_n\{\mathbf{G}_n^-(\mathbf{u})\} + \sqrt{n}[C\{\mathbf{G}_n^-(\mathbf{u})\} - C(\mathbf{u})]}{g_\omega(\mathbf{u})} = \sum_{p=1}^4 T_{pn}(\mathbf{u})
$$

with

$$
\begin{aligned}
T_{1n}(\mathbf{u}) &= \frac{\alpha_n\{\mathbf{G}_n^-(\mathbf{u})\} - \alpha_n\{1, G_{n2}^-(u_2), 1, \dots, 1\}}{g_\omega(\mathbf{u})}, \\
T_{2n}(\mathbf{u}) &= \frac{\alpha_n\{1, G_{n2}^-(u_2), 1, \dots, 1\} + \sqrt{n}\{G_{n2}^-(u_2) - u_2\}}{g_\omega(\mathbf{u})}, \\
T_{3n}(\mathbf{u}) &= \frac{\sqrt{n}[C\{\mathbf{G}_n^-(\mathbf{u})\} - C\{G_{n1}^-(u_1), u_2, G_{n3}^-(u_3), \dots, G_{nd}^-(u_d)\}]}{g_\omega(\mathbf{u})} - \frac{\sqrt{n}\{G_{n2}^-(u_2) - u_2\}}{g_\omega(\mathbf{u})}, \\
T_{4n}(\mathbf{u}) &= \frac{\sqrt{n}[C\{G_{n1}^-(u_1), u_2, G_{n3}^-(u_3), \dots, G_{nd}^-(u_d)\} - C(\mathbf{u})]}{g_\omega(\mathbf{u})}.
\end{aligned}
$$

Concerning T_{1n}, we can proceed similar as for S_{1n} above. Define the event Ω_n by $|\mathbf{G}_n^-(\mathbf{u}) - (1, G_{n2}^-(u_2), 1, \dots, 1)'| \le 2d\delta_n$ and note that $\mathbb{P}(\Omega_n^c) \to 0$. Then, by Condition 4.1 applied with

$\mu \in (\omega/(\omega + \frac{1}{2}), \theta_1)$,

$$\left|T_{1n}(\mathbf{u})\right| \le M_n(2d\delta_n, \mu) \frac{|\mathbf{G}_n^-(\mathbf{u}) - (1, G_{n2}^-(u_2), 1, \dots, 1)'|^\mu \vee n^{-\mu}}{(1-u_1)^\omega} \mathbb{1}_{\Omega_n} + o_P(1).$$

Use the fact that $\gamma < 1$ and $1 - u_1 \ge 1 - u_j \ge n^{-\gamma}$ for $j \ge 3$ and subadditivity of $x \mapsto x^\mu$ to bound the right-hand side by

$$\begin{aligned} &O_P(1) \sum_{j \ne 2} \frac{|G_{nj}^-(u_j) - u_j|^\mu + |1-u_j|^\mu}{(1-u_j)^\omega} + o_P(1) \\ &\quad \le O_P(1) \left\{ \sum_{j \ne 2} n^{-\mu/2 + \omega - \omega\mu} \left\{ \frac{\sqrt{n}|G_{nj}^-(u_j) - u_j|}{(1-u_j)^\omega} \right\}^\mu + \delta_n^{\mu-\omega} \right\} + o_P(1). \end{aligned}$$

Therefore, by Condition 4.3 and by the choice of μ, $|T_{1n}(\mathbf{u})| = o_P(1)$ uniformly in $\{\mathbf{u} \in M_{n\gamma} : g_\omega(\mathbf{u}) = (1-u_1)^\omega\}$.

Regarding T_{2n}, by the definition of α_n and since $g_1(\mathbf{u}) = 1 - u_1 \ge n^{-1}$,

$$\sup_{\mathbf{u} \in M_{n\gamma}, g_1(\mathbf{u}) = 1-u_1} \left|T_{2n}(\mathbf{u})\right| \le n^\omega \sup_{u_2 \in [0,1]} \sqrt{n} \left|G_{n2}\left\{G_{n2}^-(u_2)\right\} - u_2\right|.$$

An application of Lemma 5.1 with $\mu \in (\omega, \theta_1)$ yields that the right-hand side is of order $o_P(n^{-\mu+\omega}) = o_P(1)$.

Regarding T_{3n}, choose $\omega' \in (\omega \vee (1 - \frac{1}{2\gamma}), \theta_2)$. By the mean-value theorem, we can write

$$T_{3n}(\mathbf{u}) = \frac{\sqrt{n}[\dot{C}_2\{G_{n1}^-(u_1), \zeta_2, G_{n3}^-(u_3), \dots, G_{nd}^-(u_d)\} - 1]\{G_{n2}^-(u_2) - u_2\}}{g_\omega(\mathbf{u})}$$

for some intermediate value ζ_2 between $G_{n2}^-(u_2)$ and u_2. Due to the fact that $\dot{C}_2\{1, \zeta_2, 1, \dots, 1\} = 1$, a second application of the mean value theorem allows to write the right-hand side of the last display as

$$T_{3n}(\mathbf{u}) = \sum_{j \ne 2} \frac{\sqrt{n}\ddot{C}_{2j}(\boldsymbol{\xi})\{G_{n2}^-(u_2) - u_2\}\{G_{nj}^-(u_j) - 1\}}{g_\omega(\mathbf{u})}$$

for some $\boldsymbol{\xi}$ lying between $\mathbf{G}_n^-(\mathbf{u})$ and $\mathbf{u}$. Hence, by Conditions 2.1, 4.3 and Lemma 5.2, we can bound T_{3n} as follows:

$$\begin{aligned} \left|T_{3n}(\mathbf{u})\right| &\le \frac{\sqrt{n}|G_{n2}^-(u_2) - u_2|}{\{u_2(1-u_2)\}^{\omega'}} \frac{\{u_2(1-u_2)\}^{\omega'}}{(1-u_1)^\omega} \frac{O_P(1)}{u_2(1-u_2)} \sum_{j \ne 2} \left|G_{nj}^-(u_j) - 1\right| \\ &= O_P(1) \left\{u_2(1-u_2)\right\}^{\omega'-1} \sum_{j \ne 2} \left\{ \frac{|G_{nj}^-(u_j) - u_j|}{(1-u_j)^\omega} + (1-u_j)^{1-\omega} \right\}. \end{aligned}$$

Since $1-u_2 \ge 1-u_1$ and $u_2 \ge 1-u_1$, the right-hand side is of order $O_P\{n^{-1/2}(1-u_1)^{\omega'-1} + (1-u_1)^{\omega'-\omega}\} = o_P(1)$ uniformly in $\mathbf{u} \in M_{n\gamma}$ such that $g_\omega(\mathbf{u}) = (1-u_1)^\omega$, by the choice of ω'.

Finally, regarding T_{4n}, Lipschitz-continuity of the copula function and Condition 4.3 immediately imply that for any $\omega' \in (\omega, \theta_2)$

$$\left|T_{4n}(\mathbf{u})\right| \le \sum_{j\ne 2}(1-u_1)^{-\omega}(1-u_j)^{\omega'} \frac{\sqrt{n}|G_{nj}^{-}(u_j)-u_j|}{(1-u_j)^{\omega'}} = O_P\bigl((1-u_1)^{\omega'-\omega}\bigr),$$

which is of order $o_P(1)$ uniformly in $\{\mathbf{u} \in M_{n\gamma} : g_\omega(\mathbf{u}) = (1-u_1)^\omega\}$.

Treatment of $R_n(M_{n\gamma}^c)$. First note that, from the definition of $N(n^{-\gamma}, \delta_n)$, for every $\mathbf{u} \in M_{n\gamma}^c$ there are at most $d-2$ components larger than or equal to $1-n^{-\gamma}$. For that reason, we can write

$$M_{n\gamma}^c = \bigcup_{\boldsymbol{\ell}=(\ell_1,\dots,\ell_d)\in\{0,1\}^d;|\boldsymbol{\ell}|\ge 2} S_{\ell_1}\times\cdots\times S_{\ell_d},$$

where $|\boldsymbol{\ell}| = \sum_{j=1}^d \ell_j$, $S_0 = [1-n^{-\gamma}, 1]$ and $S_1 = (n^{-\gamma}, 1-n^{-\gamma})$. In order to show negligibility of $R_n(M_{n\gamma}^c)$, it suffices to fix a vector ℓ with $|\ell| \ge 2$ and to show uniform negligibility of $\mathbb{C}_n/g_\omega$ over $\mathbf{u} \in S_{\boldsymbol{\ell}} := S_{\ell_1} \times \cdots \times S_{\ell_d}$.

For $\mathbf{u} \in [0,1]^d$, let $\mathbf{u}^{(\boldsymbol{\ell})}$ denote the vector whose jth component (with $j = 1,\dots,d$) is equal to $\mathbb{1}(\ell_j = 0) + u_j\mathbb{1}(\ell_j = 1)$. Then,

$$\begin{aligned}\sup_{\mathbf{u}\in S_{\boldsymbol{\ell}}}\left|\frac{\mathbb{C}_n(\mathbf{u})}{g_\omega(\mathbf{u})}\right| &\le \sup_{\mathbf{u}\in S_{\boldsymbol{\ell}}} \frac{\sqrt{n}|G_n\{\mathbf{G}_n^-(\mathbf{u})\} - G_n\{\mathbf{G}_n^-(\mathbf{u})^{(\boldsymbol{\ell})}\}|}{n^{-\omega\gamma}} \\ &\quad + \sup_{\mathbf{u}\in S_{\boldsymbol{\ell}}} \frac{\sqrt{n}|G_n\{\mathbf{G}_n^-(\mathbf{u})^{(\boldsymbol{\ell})}\} - C(\mathbf{u}^{(\boldsymbol{\ell})})|}{g_\omega(\mathbf{u})} \\ &\quad + \sup_{\mathbf{u}\in S_{\boldsymbol{\ell}}} \frac{\sqrt{n}|C(\mathbf{u}^{(\boldsymbol{\ell})}) - C(\mathbf{u})|}{n^{-\omega\gamma}} \\ &=: I_{n1} + I_{n2} + I_{n3}.\end{aligned}$$

For I_{n3}, by Lipschitz-continuity of C and by the choice of γ,

$$I_{n3} \le n^{1/2+\omega\gamma}\sqrt{d}\bigl|\mathbf{u}-\mathbf{u}^{(\boldsymbol{\ell})}\bigr| = O\bigl(n^{1/2+\omega\gamma-\gamma}\bigr) = o(1).$$

For the treatment of I_{n1}, we can proceed similar as in (4.6) to obtain that $|G_n\{\mathbf{G}_n^-(\mathbf{u})\} - G_n\{\mathbf{G}_n^-(\mathbf{u})^{(\boldsymbol{\ell})}\}| \le (d-2)n^{-\gamma} + o_P(n^{-1/2-\mu})$ for any $\mu \in (\omega, \theta_1)$. This yields $I_{n1} = o_P(n^{1/2+\omega\gamma-\gamma} + n^{\omega\gamma-\mu}) = o_P(1)$.

Finally, regarding I_{n2}, note that $g_\omega(\mathbf{u}) = g_\omega(\mathbf{u}^{(\boldsymbol{\ell})})$. Therefore,

$$I_{n2} = \sup_{\mathbf{u}\in S_\ell} \frac{|\mathbb{C}_n(\mathbf{u}^{(\boldsymbol{\ell})})|}{g_\omega(\mathbf{u}^{(\boldsymbol{\ell})})}.$$

All coordinates of vectors in S_ℓ which are not equal to 1 lie in $(n^{-\gamma}, 1-n^{-\gamma})$. Therefore, I_{n2} can be treated similar as $R_n(M_{n\gamma})$. $\square$

Proof of Lemma 4.9. The proof is similar to the proof of Lemma 4.8 and is therefore postponed to Section B.2 in the supplementary material [3]. $\square$

Proof of Lemma 4.10. Let us first show (4.1). As in the proof of Lemma 4.8, by a monotonicity argument, it suffices to treat sequences δ_n such that $\delta_n \ge n^{-1/2}$. We split the proof into two cases and begin by considering $\mathbf{u} \in N(cn^{-1}, 2\delta_n^{1/2})$. Obviously, $|\mathbf{u}-\mathbf{u}'| \le \delta_n$ implies $\mathbf{u}' \in N(cn^{-1}, 2\delta_n^{1/2}+\delta_n) \subset N(cn^{-1}, 3\delta_n^{1/2})$. Thus, by Lemma 4.8, we obtain

$$\sup_{\mathbf{u},\mathbf{u}'\in[c/n,1-c/n]^d, |\mathbf{u}-\mathbf{u}'|\le\delta_n, \mathbf{u}\in N(cn^{-1},2\delta_n^{1/2})} \left|\frac{\mathbb{C}_n(\mathbf{u})}{g_\omega(\mathbf{u})} - \frac{\mathbb{C}_n(\mathbf{u}')}{g_\omega(\mathbf{u}')}\right| = o_P(1).$$

Now, consider the case $\mathbf{u} \in N(2\delta_n^{1/2}, 1/2)$. Then, $|\mathbf{u}-\mathbf{u}'| \le \delta_n$ implies that $\mathbf{u}' \in N(2\delta_n^{1/2} - \delta_n, 1/2) \subset N(\delta_n^{1/2}, 1/2)$. Hence, Lemma 4.7 implies that

$$\begin{aligned}
&\sup_{\mathbf{u},\mathbf{u}'\in[c/n,1-c/n]^d, |\mathbf{u}-\mathbf{u}'|\le\delta_n, \mathbf{u}\in N(2\delta_n^{1/2},1/2)} \left|\frac{\mathbb{C}_n(\mathbf{u})}{g_\omega(\mathbf{u})} - \frac{\mathbb{C}_n(\mathbf{u}')}{g_\omega(\mathbf{u}')}\right| \\
&\quad\le \sup_{\mathbf{u},\mathbf{u}'\in[c/n,1-c/n]^d\cap N(\delta_n^{1/2},1/2), |\mathbf{u}-\mathbf{u}'|\le\delta_n} \left|\frac{\bar{\mathbb{C}}_n(\mathbf{u})}{g_\omega(\mathbf{u})} - \frac{\bar{\mathbb{C}}_n(\mathbf{u}')}{g_\omega(\mathbf{u}')}\right| + o_P(1).
\end{aligned}$$

Therefore, in order to prove (4.1), it suffices to show that

$$\sup_{\mathbf{u},\mathbf{u}'\in N(\delta_n^{1/2},1/2), |\mathbf{u}-\mathbf{u}'|\le\delta_n} \left|\frac{\bar{\mathbb{C}}_n(\mathbf{u}) - \bar{\mathbb{C}}_n(\mathbf{u}')}{g_\omega(\mathbf{u})}\right| = o_P(1), \tag{4.8}$$

$$\sup_{\mathbf{u},\mathbf{u}'\in N(\delta_n^{1/2},1/2), |\mathbf{u}-\mathbf{u}'|\le\delta_n} \left|\bar{\mathbb{C}}_n(\mathbf{u}')\left(\frac{1}{g_\omega(\mathbf{u})} - \frac{1}{g_\omega(\mathbf{u}')}\right)\right| = o_P(1). \tag{4.9}$$

The respective proofs will be given below at the end of this proof.

For the proof of (4.2), note that $\bar{\mathbb{C}}_n(\mathbf{u})/\tilde{g}_\omega(\mathbf{u}) = 0$ for $g_\omega(\mathbf{u}) = 0$. Therefore, we can bound

$$\sup_{|\mathbf{u}-\mathbf{u}'|\le\delta_n, g_1(\mathbf{u})=0, g_1(\mathbf{u}')>0} \left|\frac{\bar{\mathbb{C}}_n(\mathbf{u})}{\tilde{g}_\omega(\mathbf{u})} - \frac{\bar{\mathbb{C}}_n(\mathbf{u}')}{\tilde{g}_\omega(\mathbf{u}')}\right| \le \sup_{\mathbf{u}'\in N(0,\delta_n)} \left|\frac{\bar{\mathbb{C}}_n(\mathbf{u}')}{g_\omega(\mathbf{u}')}\right| = o_P(1)$$

by Lemma 4.9. The suprema over $\{\mathbf{u} : g_1(\mathbf{u}) > 0, g_1(\mathbf{u}') = 0\}$ or $\{\mathbf{u} : g_1(\mathbf{u}) = g_1(\mathbf{u}') = 0\}$ can be treated analogously, whereas the suprema over $\{\mathbf{u} : g_1(\mathbf{u}) > 0, g_1(\mathbf{u}') > 0\}$ can be handled by (4.8), (4.9) and Lemma 4.9. This proves (4.2).

It remains to be shown that (4.8) and (4.9) are valid.

Proof of (4.8). By Conditions 2.1 and 4.1 and the fact that $\dot{C}_j \in [0,1]$ we have, for $\mathbf{u}, \mathbf{u}' \in N(\delta_n^{1/2}, 1/2)$, $|\mathbf{u}-\mathbf{u}'| \le \delta_n$ and any $\mu \in (0, \theta_1)$,

$$\left|\frac{\bar{\mathbb{C}}_n(\mathbf{u}) - \bar{\mathbb{C}}_n(\mathbf{u}')}{g_\omega(\mathbf{u})}\right| \le \left|\frac{\alpha_n(\mathbf{u}) - \alpha_n(\mathbf{u}')}{g_\omega(\mathbf{u})}\right| + \sum_{j=1}^d \left|\frac{\dot{C}_j(\mathbf{u})\{\alpha_n(\mathbf{u}^{(j)}) - \alpha_n(\mathbf{u}'^{(j)})\}}{g_\omega(\mathbf{u})}\right| + \sum_{j=1}^d \left|\frac{\{\dot{C}_j(\mathbf{u}) - \dot{C}_j(\mathbf{u}')\}\alpha_n(\mathbf{u}'^{(j)})}{g_\omega(\mathbf{u})}\right|.$$

The right-hand side can be further bounded by

$$(d+1)\frac{|\mathbf{u}-\mathbf{u}'|^\mu \vee n^{-\mu}}{g_\omega(\mathbf{u})} M_n(\delta_n, \mu) + \sum_{j=1}^d \left|\frac{\{\dot{C}_j(\mathbf{u}) - \dot{C}_j(\mathbf{u}')\}\alpha_n(\mathbf{u}'^{(j)})}{g_\omega(\mathbf{u})}\right|.$$

Since $g_\omega(\mathbf{u}) \ge \delta_n^{\omega/2}$ for $\mathbf{u} \in N(\delta_n^{1/2}, 1/2)$, the first summand on the right of the last display is of order $O_P(\delta_n^{\mu-\omega/2})$, which is $o_P(1)$ if we choose $\mu > \omega/2$. For the second term, we fix j and will consider two cases for each summand separately. First, suppose $1-u'_j < \delta_n^{1/2}$. In this case, Condition 4.3 yields, for arbitrary $\omega' \in (0, \theta_2)$,

$$\left|\frac{\{\dot{C}_j(\mathbf{u}) - \dot{C}_j(\mathbf{u}')\}\alpha_n(\mathbf{u}'^{(j)})}{g_\omega(\mathbf{u})}\right| \le 2\delta_n^{-\omega/2}\{u'_j(1-u'_j)\}^{\omega'} \frac{|\alpha_n(\mathbf{u}'^{(j)})|}{\{u'_j(1-u'_j)\}^{\omega'}} = O_P(\delta_n^{-\omega/2+\omega'/2}).$$

Since we can choose $\omega' \in (\omega, \theta_2)$ the latter is $o_P(1)$.

Now, suppose $1-u'_j \ge \delta_n^{1/2}$. Then, the mean value theorem allows to write

$$\left|\frac{\{\dot{C}_j(\mathbf{u}) - \dot{C}_j(\mathbf{u}')\}\alpha_n(\mathbf{u}'^{(j)})}{g_\omega(\mathbf{u})}\right| \le \sum_{\ell=1}^d \left|\frac{\ddot{C}_{j\ell}(\boldsymbol{\xi}_j)\alpha_n(\mathbf{u}'^{(j)})(u_\ell - u'_\ell)}{g_\omega(\mathbf{u})}\right|,$$

where $\boldsymbol{\xi}_j$ denotes an intermediate point between $\mathbf{u}$ and $\mathbf{u}'$. In particular, the components of $\boldsymbol{\xi}_j = (\xi_{j1}, \ldots, \xi_{jd})$ satisfy $\xi_{j\ell} \ge \sqrt{\delta_n}$ and $1-\xi_{jj} \ge \sqrt{\delta_n} - \delta_n \ge \sqrt{\delta_n}/2$, for sufficiently large n. Then, by Condition 2.1, the sum on the right-hand side of the last display can be bounded by

$$d\frac{K}{\xi_{jj}(1-\xi_{jj})}|\alpha_n(\mathbf{u}'^{(j)})|\delta_n^{1-\omega/2} = O_P(\delta_n^{1/2-\omega/2}) = o_P(1).$$

Proof of (4.9). Note that it is sufficient to bound $|g_\omega(\mathbf{u})^{-1} - g_\omega(\mathbf{u}')^{-1}|$, because $\sup_{\mathbf{u}\in[0,1]^d} |\bar{\mathbb{C}}_n(\mathbf{u})| = O_P(1)$. To this end, we first observe that, for $\mathbf{u}, \mathbf{u}' \in N(\delta_n^{1/2}, 1/2)$ and $|\mathbf{u}-\mathbf{u}'| \le \delta_n$, we have

$$|g_\omega(\mathbf{u}) - g_\omega(\mathbf{u}')| \le \omega\delta_n^{(\omega-1)/2}|g_1(\mathbf{u}) - g_1(\mathbf{u}')| = O(\delta_n^{(\omega+1)/2}),$$

where we used the mean value theorem and the fact that g_1 is Lipschitz-continuous on $N(\delta_n^{1/2}, 1/2)$. Therefore,

$$\left|\frac{1}{g_\omega(\mathbf{u})} - \frac{1}{g_\omega(\mathbf{u}')}\right| = \left|\frac{g_\omega(\mathbf{u}') - g_\omega(\mathbf{u})}{g_\omega(\mathbf{u})g_\omega(\mathbf{u}')}\right| = O\big(\delta_n^{(\omega+1)/2-\omega}\big) = o(1),$$

which implies (4.9). □

4.3. Proof of Theorem 3.3

Let $n \geq 2$. Decompose $\sqrt{n}\{R_n - \mathbb{E}[J(\mathbf{U})]\} = A_n - r_{n1}$, where

$$A_n = \sqrt{n} \int_{(1/2n,1-1/2n]^2} J(\mathbf{u})\,\mathrm{d}(\hat{C}_n - C)(\mathbf{u}),$$

$$r_{n1} = \sqrt{n} \int_{\{(1/2n,1-1/2n]^2\}^c} J(\mathbf{u})\,\mathrm{d}C(\mathbf{u}),$$

where A^c denotes the complement of a set A in $(0,1)^2$. From integration by parts for Lebesgue–Stieltjes integrals (see Theorem A.6 in the supplementary material [3]) we have that $A_n = B_n + r_{n2} + r_{n3}$, where

$$B_n = \int_{(1/2n,1-1/2n]^2} \hat{\mathbb{C}}_n(\mathbf{u})\,\mathrm{d}J(\mathbf{u}),$$

where

$$\begin{aligned} r_{n2} = {} & \Delta\left(\hat{\mathbb{C}}_n J, \frac{1}{2n}, \frac{1}{2n}, 1-\frac{1}{2n}, 1-\frac{1}{2n}\right) - \int_{(1/2n,1-1/2n]} \hat{\mathbb{C}}_n\left(u, 1-\frac{1}{2n}\right) J\left(\mathrm{d}u, 1-\frac{1}{2n}\right) \\ & + \int_{(s1/2n,1-1/2n]} \hat{\mathbb{C}}_n\left(u, \frac{1}{2n}\right) J\left(\mathrm{d}u, \frac{1}{2n}\right) \\ & - \int_{(1/2n,1-1/2n]} \hat{\mathbb{C}}_n\left(1-\frac{1}{2n}, v\right) J\left(1-\frac{1}{2n}, \mathrm{d}v\right) \\ & + \int_{(1/2n,1-1/2n]} \hat{\mathbb{C}}_n\left(\frac{1}{2n}, v\right) J\left(\frac{1}{2n}, \mathrm{d}v\right), \end{aligned}$$

with $\Delta(f, a_1, a_2, b_1, b_2) = f(b_1, b_2) - f(a_1, b_2) - f(b_1, a_2) + f(a_1, a_2)$ for $f : (0,1)^2 \to \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in (0,1)^2$ and where

$$\begin{aligned} r_{n3} = \int_{(1/2n,1-1/2n]^2} & \nu_n\left(\{u\} \times \left(v, 1-\frac{1}{2n}\right]\right) + \nu_n\left(\left(u, 1-\frac{1}{2n}\right] \times \{v\}\right) \\ & + \nu_n\big(\big\{(u,v)\big\}\big)\,\mathrm{d}J(u,v) \end{aligned}$$

$$+\int_{(1/2n,1-1/2n]} \nu_n\left(\{u\}\times\left(\frac{1}{2n},1-\frac{1}{2n}\right]\right) J\left(\mathrm{d}u,\frac{1}{2n}\right)$$
$$+\int_{(1/2n,1-1/2n]} \nu_n\left(\left(\frac{1}{2n},1-\frac{1}{2n}\right]\times\{v\}\right) J\left(\frac{1}{2n},\mathrm{d}v\right),$$

with ν_n denoting the unique signed measure on $[\frac{1}{2n}, 1-\frac{1}{2n}]$ associated with $\hat{\mathbb{C}}_n$ (see Theorem A.4 in the supplementary material [3]).

For the arguments that follow, we remark that by Proposition 4.4 the conditions of Theorem 2.2 imply those of Theorem 4.5. Thus, all results from the proof of Theorem 4.5 are applicable here.

Regarding weak convergence of B_n, observe that by Theorem 2.2, Lemma 4.9 and the integrability condition in (3.2)

$$\begin{aligned} B_n &= \int_{(0,1)^2} \mathbb{1}\left\{\mathbf{u}\in\left(\frac{1}{2n},1-\frac{1}{2n}\right]^2\right\}\frac{\bar{\mathbb{C}}_n(\mathbf{u})}{g_\omega(\mathbf{u})} g_\omega(\mathbf{u})\,\mathrm{d}J(\mathbf{u})+o_P(1) \\ &= \int_{(0,1)^2} \frac{\bar{\mathbb{C}}_n(\mathbf{u})}{g_\omega(\mathbf{u})} g_\omega(\mathbf{u})\,\mathrm{d}J(\mathbf{u})+o_P(1). \end{aligned}$$

Now, the integrability condition in (3.2) implies that the functional $f\mapsto\int_{(0,1)^2} f\tilde{g}_\omega\,\mathrm{d}J$ is continuous when viewed as a map from $(\ell^\infty((0,1)^2),\|\cdot\|_\infty)$ to $\mathbb{R}$, and thus B_n converges weakly to $\int_{(0,1)^2}\mathbb{C}_C(\mathbf{u})\,\mathrm{d}J(\mathbf{u})$ by Theorem 2.2 and the continuous mapping theorem. Hence, it remains to be shown that r_{n1}, r_{n2} and r_{n3} are $o_P(1)$.

Regarding r_{n1}, since $|J(u,v)|\le \text{const}\times g_\omega(u,v)^{-1}$, we can bound

$$|r_{n1}|\le\sqrt{n}\int_{([1/2n,1-1/2n]^2)^c} g_\omega(u,v)^{-1}\,\mathrm{d}C(u,v).$$

The set $\{(\frac{1}{2n},1-\frac{1}{2n}]^2\}^c$ consists of vectors where either both components or only one component is close to the boundary of $[0,1]^2$. In order to bound the integral on the right-hand side of the last display, we distinguish these cases and exemplarily consider the integral over $(0,\frac{1}{2n}]^2$ and the one over $(0,\frac{1}{2n}]\times(\frac{1}{2n},1-\frac{1}{2n}]$. Integrals over the remaining subsets can be treated in the same way. First, since $g_\omega(u,v)^{-1}\le u^{-\omega}+v^{-\omega}$ for $u,v\in(0,\frac{1}{2n}]$, we have

$$\sqrt{n}\int_{(0,1/2n]^2} g_\omega(u,v)^{-1}\,\mathrm{d}C(u,v)\le\sqrt{n}\int_{(0,1/2n]^2} u^{-\omega}+v^{-\omega}\,\mathrm{d}C(u,v).$$

Let us only consider the integral over $u^{-\omega}$ on the right-hand side, the one over $v^{-\omega}$ can be treated analogously. We have

$$\begin{aligned} \sqrt{n}\int_{(0,1/2n]^2} u^{-\omega}\,\mathrm{d}C(u,v) &\le \sqrt{n}\int_{(0,1/2n]\times[0,1]} u^{-\omega}\,\mathrm{d}C(u,v) \\ &= \sqrt{n}\int_{(0,1/2n]} u^{-\omega}\,\mathrm{d}u = O\left(n^{-1/2+\omega}\right)=o(1). \end{aligned}$$

Second, on $(0, \frac{1}{2n}] \times (\frac{1}{2n}, 1-\frac{1}{2n}]$, we have $g_\omega(u,v)^{-1} = u^{-\omega}$, whence, by a similar reasoning,

$$\sqrt{n}\int_{(0,1/2n]\times(1/2n,1-1/2n]} g_\omega(u,v)^{-1}\,\mathrm{d}C(u,v) \le \sqrt{n}\int_{(0,1/2n]} u^{-\omega}\,\mathrm{d}u = O\bigl(n^{-1/2+\omega}\bigr).$$

Regarding r_{n2}, use Theorem 2.2 and (3.3) and (3.4) to replace $\hat{\mathbb{C}}_n/g_\omega$ by $\bar{\mathbb{C}}_n/g_\omega$ at the cost of a negligible remainder (note that $g_\omega(u,\delta) = \delta^\omega$ for $u \in (\delta, 1-\delta]$). Then, the four integrals in the definition of r_{n2} are $o_P(1)$ by (3.3), (3.4), Lemma 4.9 and Proposition 4.4, while $\Delta(\bar{\mathbb{C}}_n J, \frac{1}{2n}, \frac{1}{2n}, 1-\frac{1}{2n}, 1-\frac{1}{2n})$ converges to 0 by Lemma 4.9, Proposition 4.4 and the fact that $|J(\mathbf{u})| \le \text{const} \times g_\omega(\mathbf{u})^{-1}$ for $\mathbf{u} \in (0,1)^2$.

Regarding r_{n3}, since $\hat{C}_n$ and C are completely monotone, the (unique) measures in the Jordan decomposition of ν_n are given by $\nu_n^+ = \sqrt{n}\nu_{\hat{C}_n}$ and $\nu_n^- = \sqrt{n}\nu_C$, where $\nu_{\hat{C}_n}$ and ν_C denote the measures corresponding to $\hat{C}_n$ and C, respectively. Thus, continuity of the copula C yields

$$\nu_n\left(\{u\} \times \left(v, 1-\frac{1}{2n}\right]\right) = \sqrt{n}\nu_{\hat{C}_n}\left(\{u\} \times \left(v, 1-\frac{1}{2n}\right]\right) \le \sqrt{n}\bigl\{\hat{C}_n(u,1) - \hat{C}_n(u-,1)\bigr\}.$$

Since the last display is bounded by $n^{-1/2}$ times the maximum number of $\hat{U}_{i1}$ that are equal, a reasoning which is similar to the one used to obtain (4.3) yields that, for any $\mu \in (\omega, 1/2)$,

$$\nu_n\left(\{u\} \times \left(v, 1-\frac{1}{2n}\right]\right) = O_P\bigl(n^{-\mu}\bigr)$$

uniformly in $u, v \in (0,1)^2$. Similar estimations for the remaining terms in r_{n3} imply that $|r_{n3}|$ is of the order

$$O_P\bigl(n^{-\mu}\bigr)\left\{\int_{(1/2n,1-1/2n]^2} |\,\mathrm{d}J| + \int_{(1/2n,1-1/2n]}\left|J\left(\mathrm{d}u, \frac{1}{2n}\right)\right| + \int_{(1/2n,1-1/2n]}\left|J\left(\frac{1}{2n}, \mathrm{d}v\right)\right|\right\}.$$

By Conditions (3.2)–(3.4), these integrals are of order $O(n^\omega)$ which leads to $|r_{n3}| = O_P(n^{\omega-\mu}) = o_P(1)$.

5. Auxiliary results

Lemma 5.1. *Suppose Condition 4.1 is met. Then, for $j = 1, \dots, d$ and any $\mu \in [0, \theta_1)$, we have*

$$\sup_{u\in[0,1]}\bigl|G_{nj}\bigl\{G_{nj}^-(u)\bigr\} - u\bigr| = o_P\bigl(n^{-1/2-\mu}\bigr).$$

Proof. From the definition of the (left-continuous) generalized inverse, we have that $\sup_{u\in[0,1]}|H\{H^-(u)\} - u|$ is bounded by the maximum jump height of the function H, that is,

$$\sup_{u\in[0,1]}\bigl|G_{nj}\bigl\{G_{nj}^-(u)\bigr\} - u\bigr| \le \sup_{u\in[0,1]}\bigl|G_{nj}(u) - G_{nj}(u-)\bigr|.$$

Therefore, the assertion follows from (4.3) and Condition 4.1. □

Lemma 5.2. *Suppose Condition 4.3 is met. Then, for $j = 1, \dots, d$ and any $\gamma \in (0, \{1/[2(1 - \theta_2)]\} \wedge \theta_3)$, we have*

$$K_{nj}(\gamma) = \sup \left| \frac{u_j(1-u_j)}{\xi_j(1-\xi_j)} \right| = O_P(1),$$

where the supremum is taken over all $u_j \in [n^{-\gamma}, 1 - n^{-\gamma}]$ and all ξ_j between $G_{nj}^{-}(u_j)$ and u_j.

Proof. Since

$$K_{nj}(\gamma) \le K_{nj}^{(1)}(\gamma) \times K_{nj}^{(2)}(\gamma) := \sup \left| \frac{u_j}{\xi_j} \right| \times \sup \left| \frac{1-u_j}{1-\xi_j} \right|,$$

it suffices to treat both suprema on the right-hand side separately. In the following, we only consider the first one; the second one can be treated along similar lines. Obviously,

$$K_{nj}^{(1)}(\gamma) \le 1 \vee \sup_{u_j \in [n^{-\gamma}, 1-n^{-\gamma}]} \frac{u_j}{G_{nj}^{-}(u_j)}.$$

Let Ω_n denote the event that $\sup_{u_j \in [n^{-\gamma}, 1-n^{-\gamma}]} |\{G_{nj}^{-}(u_j) - u_j\}/u_j| \le 1/2$. Choose $\omega' \in (0 \vee (1 - \frac{1}{2\gamma}), \theta_2)$ and use Condition 4.3 to conclude that

$$\begin{aligned}
\sup_{u_j \in [n^{-\gamma}, 1-n^{-\gamma}]} \left| \frac{G_{nj}^{-}(u_j) - u_j}{u_j} \right| &\le \sup_{u_j \in [n^{-\gamma}, 1-n^{-\gamma}]} \left\{ \sqrt{n} \left| \frac{G_{nj}^{-}(u_j) - u_j}{u_j^{\omega'}} \right| \times \frac{u_j^{\omega'-1}}{\sqrt{n}} \right\} \\
&= O_P\big(n^{-1/2-\gamma(\omega'-1)}\big) = o_P(1).
\end{aligned}$$

Thus, $\mathbb{P}(\Omega_n^c) = o(1)$, which implies

$$\begin{aligned}
\sup_{u_j \in [n^{-\gamma}, 1-n^{-\gamma}]} \frac{u_j}{G_{nj}^{-}(u_j)} &= \sup_{u_j \in [n^{-\gamma}, 1-n^{-\gamma}]} \left(1 + \frac{G_{nj}^{-}(u_j) - u_j}{u_j} \right)^{-1} \mathbb{1}_{\Omega_n} + o_P(1) \\
&\le 2 + o_P(1) = O_P(1),
\end{aligned}$$

where we used that $1/(1+x) \le 1/(1-|x|)$ for $x \in [-1/2, 1/2]$. This yields the assertion. □

Lemma 5.3. *Under Conditions 4.1, 4.2 and 4.3 we have for any $\omega \in (0, \theta_1 \wedge \theta_2)$ and any $\gamma > 1/2$*

$$\sup_{u_j \in [1-n^{-\gamma}, 1]} \big|\beta_{nj}(u_j)\big| = o_P\big(n^{-\omega/2}\big).$$

Proof. Since the result is one-dimensional, we drop the index j in the following. Note that all the arguments that follow lead to bounds which are valid uniformly in $u \in [1 - n^{-\gamma}, 1]$. Now, fix

$u \in [1-n^{-\gamma}, 1]$ and choose $i \in \{0, \dots, n-1\}$ such that $u \in (\frac{i}{n}, \frac{i+1}{n}]$. Then, $G_n^-(u) = U_{i+1:n}$, where $U_{1:n} \le \cdots \le U_{n:n}$ denote the order statistics of $U_1, \dots, U_n$. Hence,

$$\begin{aligned} n^{\omega/2}\big|\beta_n(u)\big| &\le n^{\omega/2+1/2}\big\{|U_{i+1:n} - i/n| \vee \big|U_{i+1:n} - (i+1)/n\big|\big\} \\ &\le n^{\omega/2+1/2}|U_{i+1:n} - i/n| + n^{-1/2+\omega/2}. \end{aligned}$$

Now, as a consequence of Lemma 5.1, we have $G_n(U_{i+1:n}) = G_n\{G_n^-(u)\} = i/n + \kappa_{i,n}$, where $\max_{i=0}^{n-1} \kappa_{i,n} = o_P(n^{-\mu-1/2})$ with $\mu \in (\omega/2, \theta_1)$. Therefore,

$$n^{\omega/2+1/2}|U_{i+1:n} - i/n| \le n^{\omega/2+1/2}\big|G_n(U_{i+1:n}) - U_{i+1:n}\big| + n^{\omega/2+1/2}\kappa_{i,n}.$$

The second term on the right-hand side is $o_P(n^{-\mu+\omega/2}) = o_P(1)$. For the first term, we have

$$\begin{aligned} n^{\omega/2+1/2}\big|G_n(U_{i+1:n}) - U_{i+1:n}\big| &= \frac{\alpha_n(U_{i+1:n})}{(1-U_{i+1:n})^\omega} n^{\omega/2}(1-U_{i+1:n})^\omega \\ &\le \sup_{u\in(0,1)} \frac{|\alpha_n(u)|}{(1-u)^\omega} \times n^{\omega/2}(1-U_{i+1:n})^\omega \\ &= O_P(1) \times \big\{\sqrt{n}(1-U_{i+1:n})\big\}^\omega. \end{aligned}$$

For the factor on the right, since $u \ge 1 - n^\gamma$, we have, for any $w \in (\frac{i}{n}, \frac{i+1}{n}]$,

$$\begin{aligned} \sqrt{n}(1-U_{i+1:n}) &= \sqrt{n}\big\{w - G_n^-(w) + 1 - w\big\} \\ &\le \sup_{v\in[1-n^{-\gamma},1]} \big|\beta_n(v)\big| + n^{1/2-\gamma} \\ &\le \sup_{v\in[1-n^{-\gamma},1]} \big|\beta_n(v) - \beta_n\big(1-n^{-1/2}\big)\big| + \big|\beta_n\big(1-n^{-1/2}\big)\big| + n^{1/2-\gamma}. \end{aligned}$$

The first term in the expression above is $o_P(1)$ by asymptotic equicontinuity of β_n (which follows from weak convergence of β_n to a Gaussian process, this is a consequence of Condition 4.2 and the functional delta method), the second term is $o_P(1)$ by Condition 4.3, and the third term vanishes since $\gamma > 1/2$. $\square$

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Supplementary Material

Supplement to "Weak convergence of the empirical copula process with respect to weighted metrics" (DOI: 10.3150/15-BEJ751SUPP; .pdf). A detailed exposition on bounded variation and Lebesgue–Stieltjes integration for two-variate functions and the proofs of Proposition 4.4 and of Lemma 4.9 can be found in [3].

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