Bernoulli **22**(1), 2016, 376–395
DOI: 10.3150/14-BEJ662

Integral representation of random variables with respect to Gaussian processes

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It was shown in Mishura *et al.* (*Stochastic Process. Appl.* **123** (2013) 2353–2369), that any random variable can be represented as improper pathwise integral with respect to fractional Brownian motion. In this paper, we extend this result to cover a wide class of Gaussian processes. In particular, we consider a wide class of processes that are Hölder continuous of order $\alpha > 1/2$ and show that only local properties of the covariance function play role for such results.

Keywords: Föllmer integral; Gaussian processes; generalised Lebesgue–Stieltjes integral; integral representation

1. Introduction

In stochastic analysis and its applications such as financial mathematics, it is an interesting question what kind of random variables one can replicate with stochastic integrals. In order to answer this question, first one needs to consider in which sense the stochastic integral exists. In particular, if the driving process X is not a semimartingale it is not clear how to define integrals with respect to X and what kind of integrands can be integrated with the given definition of the integral.

The motivation for our work originates back to Dudley [3] who showed that any functional ξ of a standard Brownian motion W can be replicated as an Itô integral $\int_0^1 \Psi(s)\,\mathrm{d}W_s$, where Ψ is an adapted process satisfying $\int_0^1 \Psi^2(s)\,\mathrm{d}s < \infty$ a.s. Moreover, under additional assumption $\int_0^1 \mathbb{E}[\Psi^2(s)]\,\mathrm{d}s < \infty$ one can cover only centered random variables with finite variance. On the other hand, in this case the process Ψ is unique.

Later on Mishura *et al.* [7] considered the same problem where standard Brownian motion W was replaced with fractional Brownian motion (fBm) B^H with Hurst index $H > \frac{1}{2}$. In this case the authors considered generalised Lebesgue–Stieltjes integrals with respect to fBm which can be defined, thanks to results of Azmoodeh *et al.* [1], for integrands of form $f(B_u^H)$ where f is a function of locally bounded variation. As an application of the results in [7], the authors considered financial implications of the results and gave a negative answer to the problem of zero integral; does $\int_0^1 \psi(s)\,\mathrm{d}B_s^H = 0$ imply that $\psi(s) = 0$. This problem was open for fBm for some time, and in addition the result was known only for Brownian motion.

It is interesting to note that while the stochastic integrals are defined in different ways, the results for standard Brownian motion and fBm are quite similar. On the other hand, the key idea to obtain representation for arbitrary processes with integrals with respect to some given process

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is to use idea of "tracking": first define a sequence which obviously converges and then track that sequence. The simplest way to do this is to define an integrand on a given time interval which diverges in the limit and then use stopping times. This idea was first used by Dudley [3] for Brownian motion and then by Mishura *et al.* [7] for fBm.

In this article, motivated by these two contributing works, we study the problem for more general class of Gaussian processes. In particular, we also consider generalised Lebesgue–Stieltjes integrals and show that the brilliant construction introduced in [7] for fBm applies, with small modifications, for more general Gaussian processes. We also note that the integrals exist also as forward integrals in the sense of Föllmer [4]. Our class of Gaussian processes consists of wide class of processes which has versions that are Hölder continuous of order $\alpha > \frac{1}{2}$. More precisely, our class of processes consist of Hölder continuous Gaussian processes X which also satisfy several mild extra conditions given for the corresponding covariance function R. In particular, the class includes many stationary and stationary increment processes that are Hölder continuous of sufficient order. In order to obtain such result for general class of Gaussian processes, we show that for the construction introduced in [7] the only required facts are local properties of the corresponding covariance function. Moreover, we show that the replication can be done in arbitrary small amount of time which has significant implications to the finance. As such, this article is a hybrid of discussing review paper and an original research article; We prove similar results as for fBm and use the same idea of tracking so the proofs are quite similar with only minor changes needed and no unnecessary complexity is added. On the other hand, the results are extended to much wider class of processes, the needed properties for such results are identified and it is also shown that the replication can be done in any time interval. We also discuss applications such as implications to finance and the problem of zero integral. In particular, the results of this paper indicate that with pathwise integrals the answer to the problem of zero integral is usually false.

The rest of the paper is organised as follows. We start Section 2 by recalling the findings obtained in [7] for fBm. Moreover, we introduce the key properties of fBm under which the authors in [7] obtained their results. We end the Section 2 by introducing our notation and assumptions. We also recall basic facts on generalised Lebesgue–Stieltjes integrals and Föllmer integrals. In Section 3, we introduce and prove the main results for our general class of processes. We end the paper by discussion in Section 4 where we shortly discuss financial applications, uniqueness of the representation and the problem of zero integral.

2. Auxiliary facts

Key properties for fractional Brownian motion

In [7], the authors proved the following:

- For any distribution function F there exists an adapted process Φ such that $\int_0^1 \Phi(s)\,\mathrm{d}B_s^H$ is well-defined (in the sense of generalised Lebesgue–Stieltjes integral) and has distribution F.
- Any measurable random variable ξ can be represented as an improper integral, that is, $\xi = \lim_{t\to 1-} \int_0^t \Psi(s)\,\mathrm{d}B_s^H$.
- A measurable random variable ξ which is an end value of some Hölder continuous process can be represented as a proper integral.

Our aim is to establish similar results for general class of Gaussian processes. By studying the paper [7], one can see that in a sense the following facts are the main ingredients for such results:

1. Itô's formula: for every locally bounded variation function f we have
$$F\big(B_T^H\big) = \int_0^T f\big(B_u^H\big)\,\mathrm{d}B_u^H,$$
where $F(x) = \int_0^x f(y)\,\mathrm{d}y$,
2. fBm has stationary increments,
3. a crossing bound at zero: there exists a constant C such that for every $0 < s < t \le T$ we have
$$\mathbb{P}\big(B_s^H < 0 < B_t^H\big) \le C(t-s)^H t^{-H},$$
4. small ball probability: there exists a constant C such that for every T and ε we have
$$\mathbb{P}\Big(\sup_{0\le t\le T}\big|B_t^H\big| \le \varepsilon\Big) \le \exp\big(-CT\varepsilon^{-1/H}\big)$$
provided that $\varepsilon \le T^H$.

For our purposes, we have results similar to conditions 1 and 3 for more general class of processes obtained by Sottinen and Viitasaari [11] (see subsection below). The conditions 2 and 4 we replace with weaker assumptions on the covariance structure of the Gaussian process X.

Definitions and auxiliary results

Throughout the paper, we are restricted on a bounded interval $[0, T]$ which is usually omitted in notation.

Definition 2.1. *Let X be a centered Gaussian process. We denote by $R_X(t,s)$, $W_X(t,s)$, and $V_X(t)$ its covariance, incremental variance and variance, that is,*
$$R_X(t,s) = \mathbb{E}[X_t X_s],$$
$$W_X(t,s) = \mathbb{E}\big[(X_t - X_s)^2\big],$$
$$V_X(t) = \mathbb{E}\big[X_t^2\big].$$
We denote by $w_X^*(t)$ the "worst case" incremental variance
$$w_X^*(t) = \sup_{0\le s\le T-t} W_X(s, s+t).$$

Let now $\alpha \in (\frac{1}{2}, 1)$. We consider the following class of processes.

Definition 2.2. *A centered continuous Gaussian process $X = (X_t)_{t\in[0,T]}$ with covariance R_X belongs to the* class $\mathcal{X}_T^\alpha$ *if there is a constant δ such that for every $u \in [T-\delta, T)$ the process $Y_t = X_{t+u} - X_u$ for $t \in [0, T-u]$ satisfies*:

1. $R_Y(s,t) > 0$ *for every* $s, t > 0$,
2. *the "worst case" incremental variance satisfies*

$$w_Y^*(t) = \sup_{0\le s\le T-t-u} W_Y(s, s+t) \le Ct^{2\alpha},$$

   where $C > 0$,
3. *there exist* $c, \hat{\delta} > 0$ *such that*

$$V_Y(s) \ge cs^2$$

   provided $s \le \hat{\delta}$,
4. *there exists a* $\hat{\delta} > 0$ *such that*

$$\sup_{0<t<2\hat{\delta}} \sup_{t/2\le s\le t} \frac{R_Y(s,s)}{R_Y(t,s)} < \infty.$$

The class depends also on parameter δ which will be omitted on the notation.

Note that the definition is quite technical. However, the conditions are needed in order to have Itô formula and crossing bound for incremental process Y close to time T. Moreover, the results for fBm relies on the fact that B^H has stationary increments. For our class we simply need certain structure for covariance close to T. The idea on the results is that before some point $t = T - \delta$ we simply wait and do nothing. Moreover, the following remarks and examples show that the assumptions are not very restrictive and are satisfied for many Gaussian processes. For further discussion and details, see [11] where the class was first introduced such that the covariance of X itself satisfy properties 1–4.

Remark 2.3.

1. Note that the first condition means that the increments of the process are positively correlated close to time T. More precisely, we need

$$R_X(t+u, s+u) + R_X(u,u) > R_X(t+u, u) + R_X(u, s+u).$$

   In other words, the covariance function should have positive increments on rectangles.
2. The second condition implies that Y has version which is Hölder continuous of any order $a < \alpha$. For the rest of the paper, we assume that this version is chosen.
3. A special subclass of $\mathcal{X}_T^\alpha$ are processes with stationary increments. In this case, we have

$$\begin{aligned} R_Y(t,s) &= R_X(t,s) = \tfrac{1}{2}\big[V(t) + V(s) - V(t-s)\big], \\ W_Y(t,s) &= W_X(t,s) = V_X(t-s), \\ w_Y^*(t) &= w_X^*(t) = V_X(t). \end{aligned}$$

Especially, stationary increment processes with $W_X(t,s) \sim |t-s|^{2\alpha}$ at zero with $\alpha > \frac{1}{2}$ belong to $\mathcal{X}_T^\alpha$ for every T. In particular, fBm with Hurst index $H > \frac{1}{2}$ belongs to $\mathcal{X}_T^\alpha$.

4. Another special subclass of $\mathcal{X}_T^\alpha$ are stationary processes. In this case, we have

$$\begin{aligned} R_X(t,s) &= r(t-s), \\ W_X(t,s) &= 2\big[r(0) - r(t-s)\big], \\ V_X(t) &= r(0), \\ w_X^*(t) &= 2\big[r(0) - r(t)\big] \end{aligned}$$

and

$$\begin{aligned} R_Y(t,s) &= r(t-s) + r(0) - r(t) - r(s), \\ W_Y(t,s) &= W_X(t,s), \\ V_Y(t) &= W_X(t+u,u) = w_X^*(t), \\ w_Y^*(t) &= w_X^*(t). \end{aligned}$$

Consequently, for a stationary process X with covariance function $r(t)$ we have $X \in \mathcal{X}_T^\alpha$ if $r(t)$ satisfies

$$\begin{aligned} r(t-s) + r(0) &> r(t) + r(s), \\ ct^2 \le r(0) - r(t) &\le Ct^{2\alpha} \end{aligned}$$

and

$$\sup_{0<t<2\hat{\delta}} \sup_{t/2 \le s \le t} \frac{r(0) - r(s)}{r(t-s) + r(0) - r(t) - r(s)} < \infty.$$

Especially, processes with strictly decreasing covariance at zero satisfy assumptions 1 and 4. In particular, stationary processes with strictly decreasing covariance and $W_X(t,s) \sim |t-s|^{2\alpha}$ at zero with $\alpha > \frac{1}{2}$ belongs to $\mathcal{X}_T^\alpha$ for every T. As an example, the process X with covariance function

$$r(t) = \exp\big(-|t|^{2\alpha}\big)$$

with $\frac{1}{2} < \alpha < 1$ belongs to $\mathcal{X}_T^\alpha$. We will use this process as a motivating example throughout the paper, and we will denote this process by $\tilde{X}$.

The following statement derived in Sottinen and Viitasaari [11] is one of the main ingredients for our study.

Theorem 2.4. *Let $X \in \mathcal{X}_T^\alpha$ with $\alpha > \frac{1}{2}$ and let f be a function of locally bounded variation. Set $F(x) = \int_0^x f(y)\,\mathrm{d}y$. Then*

$$F(X_T - X_u) = \int_u^T f(X_s - X_u)\,\mathrm{d}X_s \tag{2.1}$$

provided $u \in [T-\delta, T)$, where the integral can be understood as a generalised Lebesgue–Stieltes integral or as Föllmer integral.

Remark 2.5. In the original paper [11], the authors considered only convex functions. However, by examining the proof it is evident that the result holds also for functions of locally bounded variation.

Furthermore, we make the following assumption for small ball probabilities. The examples are discussed in the next subsection.

Assumption 2.6. *There exist constants $C, \delta > 0$ such that for every $s, t \in [T-\delta, T]$ with $t = s + \Delta$ it holds*

$$\mathbb{P}\Big(\sup_{s \le u \le t} |X_u - X_s| \le \varepsilon\Big) \le \exp\big(-C\Delta\varepsilon^{-1/\alpha}\big) \tag{2.2}$$

provided that $\varepsilon \le \Delta^\alpha$.

Which processes satisfy the Assumption 2.6?

In this subsection, we briefly review what kind of processes $X \in \mathcal{X}_T^\alpha$ satisfy the Assumption 2.6. In general, the small ball probabilities are an interesting subject of study and a survey on small ball probabilities is given by Li and Shiao [6] where also the following theorem can be found.

Theorem 2.7. *Let $\{X_t, t \in [0,1]\}$ be a centered Gaussian process with $X_0 = 0$. Assume that there is a function $\sigma^2(h)$ such that*

$$\forall 0 \le s, t \le 1, \qquad \mathbb{E}(X_s - X_t)^2 \le \sigma^2\big(|t-s|\big),$$

and that there are $0 < c_1 \le c_2 < 1$ such that $c_1\sigma(2h \wedge 1) \le \sigma(h) \le c_2\sigma(2h \wedge 1)$ for $0 < h < 1$. Then there exists $K > 0$ depending only on c_1 and c_2 such that

$$\mathbb{P}\Big(\sup_{0 \le t \le 1} |X_t| \le \sigma(\varepsilon)\Big) \ge \exp\Big(-\frac{K}{\varepsilon}\Big).$$

Example 2.8. It is straightforward that fBm satisfies the assumptions for any $H \in (0,1)$.

As a direct consequence, we obtain the following statement.

Corollary 2.9. *Let $X \in \mathcal{X}_T^\alpha$. Then for every $t \in [0, T]$ there exist $\Delta > 0$ and $K > 0$ such that*

$$\mathbb{P}\Big(\sup_{s \le u \le t} |X_u - X_s| \le \varepsilon\Big) \ge \exp\big(-K\Delta\varepsilon^{-1/\alpha}\big),$$

provided that $|t-s| \le \Delta$.

According to this corollary the bound given in Assumption 2.6 is the best possible in terms of Δ and ε. The upper bound is more difficult to obtain. Moreover, it is pointed out in [6] that the incremental variance is not an appropriate tool for the upper bound. However, in many cases of interest we can have the required upper bound. In particular, many cases of interest have stationary increments or are stationary processes. For processes with stationary increments, the following theorem can be used to study the upper bound. For the proof, we refer to [5] where a slightly more general setup was considered.

Theorem 2.10. *Assume that the centered process X has stationary increments and the incremental variance $W(t,s) = W(0,t-s)$ satisfies*:

1. *There exists $\theta \in (0,4)$ such that for every $x \in [0,\frac{1}{2}]$ we have*
$$W(0,2x) \le \theta W(0,x).$$
2. *For every $0 < x < 1$ and $2 \le j \le \frac{1}{x} - 2$, we have*
$$\begin{aligned} &6W(0,jx) + W\big(0,(j+2)x\big) + W\big(0,(j-2)x\big) \\ &\quad \ge 4W\big(0,(j+1)x\big) + 4W\big(0,(j-1)x\big). \end{aligned} \tag{2.3}$$

Then there exists a constant $K > 0$ such that for every $\varepsilon \in (0,1)$ we have
$$\mathbb{P}\Big(\sup_{0\le t\le 1} |X_t - X_0| \le \sqrt{W(0,\varepsilon)}\Big) \le \exp\bigg(-\frac{K}{\varepsilon}\bigg).$$

Remark 2.11. In the original theorem, it was stated that instead of (2.3) it is also sufficient that the incremental variance $W(t,s)$ is concave. Note that in our case usually $W(0,t) \sim t^{2\alpha}$ with $\alpha > \frac{1}{2}$. Hence, $W(t,s)$ cannot be concave.

Remark 2.12. We remark that the result holds also for stationary Gaussian processes.

Corollary 2.13. *Assume that $X \in \mathcal{X}_T^\alpha$ has stationary increments or is stationary such that $W(0,t) \sim t^{2\alpha}$. Then Assumption 2.6 is satisfied.*

Proof. It is straightforward to see that a function $W(0,x) = x^{2\alpha}$ satisfies (2.3) provided $\alpha > \frac{1}{2}$. It remains to note that with δ small enough, we have $W(0,t-s) \sim C|t-s|^{2\alpha}$ provided $|t-s| \le \Delta$. □

Example 2.14. As special examples we note that fBm B^H with $H > \frac{1}{2}$ and the process $\tilde{X}$ satisfy the Assumption 2.6.

For general processes, $X \in \mathcal{X}_T^\alpha$ it is not clear when Assumption 2.6 is satisfied. In principle, one can derive similar result as Theorem 2.10 under similar conditions. However, in this case the incremental variance function $W(t+s,s)$ depends also on the starting point s. Consequently, one needs to check the condition when s is close to T. Hence in this case, the structure of the covariance function is more important.

Pathwise integrals

In this section, we briefly introduce two kinds of pathwise integrals.

Generalized Lebesgue–Stieltjes Integral

The generalized Lebesgue–Stieltjes integral is based on fractional integration and fractional Besov spaces. For details on these topics, we refer to [9] and [8].

Recall first the definitions for fractional Besov norms and Lebesgue–Liouville fractional integrals and derivatives.

Definition 2.15. *Fix* $0 < \beta < 1$.

1. *The* fractional Besov space $W_1^\beta = W_1^\beta([0,T])$ *is the space of real-valued measurable functions* $f : [0,T] \to \mathbb{R}$ *such that*

$$\|f\|_{1,\beta} = \sup_{0 \le s < t \le T} \left(\frac{|f(t) - f(s)|}{(t-s)^\beta} + \int_s^t \frac{|f(u) - f(s)|}{(u-s)^{1+\beta}} \, \mathrm{d}u \right) < \infty.$$

2. *The* fractional Besov space $W_2^\beta = W_2^\beta([0,T])$ *is the space of real-valued measurable functions* $f : [0,T] \to \mathbb{R}$ *such that*

$$\|f\|_{2,\beta} = \int_0^T \frac{|f(s)|}{s^\beta} \, \mathrm{d}s + \int_0^T \int_0^s \frac{|f(u) - f(s)|}{(u-s)^{1+\beta}} \, \mathrm{d}u \, \mathrm{d}s < \infty.$$

In this paper, we study the norm $\|f\|_{2,\beta}$ on different intervals $[0,t]$. Hence we use short notation $\|f\|_{t,\beta}$.

Remark 2.16. Let $C^\alpha = C^\alpha([0,T])$ denote the space of Hölder continuous functions of order α on $[0,T]$ and let $0 < \varepsilon < \beta \wedge (1-\beta)$. Then

$$C^{\beta+\varepsilon} \subset W_1^\beta \subset C^{\beta-\varepsilon} \quad \text{and} \quad C^{\beta+\varepsilon} \subset W_2^\beta.$$

Definition 2.17. *Let* $t \in [0,T]$. *The* Riemann–Liouville fractional integrals I_{0+}^β *and* I_{t-}^β *of order* $\beta > 0$ *on* $[0,T]$ *are*

$$\left(I_{0+}^\beta f\right)(s) = \frac{1}{\Gamma(\beta)} \int_0^s f(u)(s-u)^{\beta-1} \, \mathrm{d}u,$$
$$\left(I_{t-}^\beta f\right)(s) = \frac{\mathrm{e}^{\mathrm{i}\pi\beta}}{\Gamma(\beta)} \int_s^t f(u)(u-s)^{\beta-1} \, \mathrm{d}u,$$

where Γ *is the Gamma-function. The* Riemann–Liouville fractional derivatives D_{0+}^β *and* D_{t-}^β *are the left-inverses of the corresponding integrals* I_{0+}^β *and* I_{t-}^β. *They can be also define via the* Weyl representation *as*

$$\left(D_{0+}^{\beta}f\right)(s)=\frac{1}{\Gamma(1-\beta)}\left(\frac{f(s)}{s^{\beta}}+\beta\int_0^s\frac{f(s)-f(u)}{(s-u)^{\beta+1}}\,\mathrm{d}u\right),$$

$$\left(D_{t-}^{\beta}f\right)(s)=\frac{\mathrm{e}^{\mathrm{i}\pi\beta}}{\Gamma(1-\beta)}\left(\frac{f(s)}{(t-s)^{\beta}}+\beta\int_s^t\frac{f(s)-f(u)}{(u-s)^{\beta+1}}\,\mathrm{d}u\right)$$

if $f\in I_{0+}^{\beta}(L^1)$ *or* $f\in I_{t-}^{\beta}(L^1)$, *respectively.*

Denote $g_{t-}(s)=g(s)-g(t-)$.

The generalized Lebesgue–Stieltjes integral is defined in terms of fractional derivative operators according to the next proposition.

Proposition 2.18 ([8]). *Let* $0<\beta<1$ *and let* $f\in W_2^{\beta}$ *and* $g\in W_1^{1-\beta}$. *Then for any* $t\in(0,T]$ *the* generalized Lebesgue–Stieltjes integral *exists as the following Lebesgue integral*

$$\int_0^t f(s)\,\mathrm{d}g(s)=\int_0^t\left(D_{0+}^{\beta}f_{0+}\right)(s)\left(D_{t-}^{1-\beta}g_{t-}\right)(s)\,\mathrm{d}s$$

and is independent of β.

We will use the following estimate to prove the existence of Föllmer integrals.

Theorem 2.19 ([8]). *Let* $f\in W_2^{\beta}$ *and* $g\in W_1^{1-\beta}$. *Then we have the bound*

$$\left|\int_0^t f(s)\,\mathrm{d}g(s)\right|\le\sup_{0\le s<t\le T}\left|D_{t-}^{1-\beta}g_{t-}(s)\right|\|f\|_{2,\beta}.$$

Föllmer integral

We also recall the definition of a forward-type Riemann–Stieltjes integral due to Föllmer [4] (for English translation, see [10]).

Definition 2.20. *Let* $(\pi_n)_{n=1}^{\infty}$ *be a sequence of partitions* $\pi_n=\{0=t_0^n<\cdots<t_{k(n)}^n=T\}$ *such that* $|\pi_n|=\max_{j=1,\dots,k(n)}|t_j^n-t_{j-1}^n|\to 0$ *as* $n\to\infty$. *Let* X *be a continuous process. The* Föllmer integral along the sequence $(\pi_n)_{n=1}^{\infty}$ *of* Y *with respect to* X *is defined as*

$$\int_0^t Y_u\,\mathrm{d}X_u=\lim_{n\to\infty}\sum_{t_j^n\in\pi_n\cap(0,t]}Y_{t_{j-1}^n}\left(X_{t_j^n}-X_{t_{j-1}^n}\right),$$

if the limit exists a.s.

The Föllmer integral is a natural choice for applications such as finance. However, usually it is difficult to prove the existence of the Föllmer integral. For instance, for finite quadratic variation processes the existence of the integral is a consequence of the Itô's formula. On the other hand, generalised Lebesgue–Stieltjes integrals provides a tool to obtain the existence of Föllmer integral. For instance, in [11] the authors proved first the existence of a generalised Lebesgue–Stieltjes integral and then obtained the existence of Föllmer integral by applying Theorem 2.19.

3. Main results

We begin with the following technical lemma which gives the diverging integrand. In our case, it can be defined similarly as for fBm. Hence, we simply present the key points of the proof.

Lemma 3.1. *Let $X \in \mathcal{X}_T^\alpha$ such that Assumption 2.6 is satisfied. Then one can construct $\mathbb{F}$-adapted process ϕ_T on $[0,T]$ such that the integral*

$$\int_0^s \phi_T(s)\,\mathrm{d}X_s$$

exists for every $s < T$ and

$$\lim_{s\to T-}\int_0^s \phi_T(s)\,\mathrm{d}X_s = \infty \tag{3.1}$$

a.s.

Proof. Fix numbers $\gamma \in (1, \frac{1}{\alpha})$ and $\eta \in (0, \frac{1}{\gamma\alpha} - 1)$. Furthermore, set $t_0 = 0$ and $t_n = \sum_{k=1}^n \Delta_k$, $n \geq 1$ where $\Delta_n = \frac{Tn^{-\gamma}}{\sum_{k=1}^\infty k^{-\gamma}}$, and define a function $f_\eta(x) = (1+\eta)|x|^\eta \operatorname{sign}(x)$. Note that we can assume without loss of generality that conditions of Definition 2.2 hold in the whole interval. Otherwise set $t_1 = T - \delta$ and start after t_1. Finally, we set

$$\tau_n = \min\{t \geq t_{n-1}\colon |X_t - X_{t_{n-1}}| \geq n^{-1/(1+\eta)}\} \wedge t_n$$

and

$$\phi_T(s) = \sum_{n=1}^\infty f_\eta(X_s - X_{t_{n-1}})\mathbf{1}_{[t_{n-1},\tau_n)}(s).$$

In order to complete the proof, we have to show that $\|\phi_T\|_{s,\beta} < \infty$ a.s. for every $s < T$ and that (3.1) holds. The fact that $\|\phi_T\|_{s,\beta} < \infty$ can be proved similarly as for fBm case in [7] together with Theorem 2.4. Hence, it remains to show that (3.1) holds.

First by Theorem 2.4, we get that for every $s \in [t_{n-1}, t_n)$

$$\int_0^s \phi_T(u)\,\mathrm{d}X_u = \sum_{k=1}^{n-1}|X_{\tau_k} - X_{t_{k-1}}|^{1+\eta} + |X_{s\wedge\tau_n} - X_{t_{n-1}}|^{1+\eta}.$$

Now, as in the case of fBm, it is enough to show that only finite numbers of events A_n happen where A_n is defined by

$$A_n = \Big\{ \sup_{t_{n-1} \le t \le t_n} |X_t - X_{t_{n-1}}| < n^{-1/(\eta+1)} \Big\}.$$

But now, by Assumption 2.6, we have

$$\mathbb{P}(A_n) \le \mathrm{e}^{-Cn^{-\gamma+1/(\alpha(\eta+1))}}$$

for n large enough. Noting our choices of γ and η we obtain $\sum_{n\ge1} \mathbb{P}(A_n) < \infty$, and thus the result follows from Borel–Cantelli lemma. □

Remark 3.2. Same result can be obtained for integrals over any interval $[s, t] \subset [T - \delta, T]$.

Remark 3.3. It was remarked in paper by Mishura *et al.* [7] that for fBm it is easy to see that $\|\phi_T\|_{t,\beta} < \infty$ even for random times $t < T$. This is indeed natural, since the Itô's formula (2.1) holds also for any bounded random time τ (see [11] for details).

Remark 3.4. It was shown in [1] that for fBm one can approximate the integral of Itô's formula (2.1) with Riemann–Stieltjes sums along uniform partition, i.e. the integral exists also as Föllmer integral. Moreover, it was pointed out in [11] that this is true for more general processes $X \in \mathcal{X}_T^\alpha$ and any partition. Hence for any n, the integral

$$\int_{t_{n-1}}^{t_n} f_\eta(X_s - X_{t_{n-1}})\, \mathrm{d}X_s$$

exists also as Föllmer integral. Now by noting that $\phi_T(s)$ is defined as a linear combination of functions of this form it is evident that the integral

$$\int_0^t \phi_T(s)\, \mathrm{d}X_s$$

exists also as Föllmer integral for every $t < T$. The same conclusion holds true also for other results presented in this paper.

As a direct corollaries, we obtain that integral with respect to X_t can have any distribution and that any measurable random variable can be represented as an improper integral; same results as for fBm. For the sake of completeness, we present the results.

Corollary 3.5. *For any cdf F one can construct adapted process $\psi_T(s)$ such that $\int_0^T \psi_T(s)\, \mathrm{d}X_s$ has distribution F.*

Proof. The proof follows same arguments as for fBm in [7] except that since we do not know how the process X behaves before some time close to T, we have to choose some point $v < T$ such that X_v has non-vanishing variance. The rest follows with same arguments with obvious changes. □

Remark 3.6. Note that the result remains true if replace the process X with $Y = h(X)$, where h is strictly monotone C^1 function. In this case the integrals of form

$$\int_0^T \psi_T(s)\,\mathrm{d}Y_s$$

are well defined by results in [11]. We remark that the result is still valid even if the function h is uniformly bounded.

Theorem 3.7. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with left-continuous filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ and let $X \in \mathcal{X}_T^\alpha$ such that Assumption 2.6 is satisfied. Then for any $\mathcal{F}_T$ measurable random variable ξ one can construct $\mathbb{F}$-adapted process Ψ_T on $[0, T]$ such that the integral*

$$\int_0^s \Psi_T(s)\,\mathrm{d}X_s$$

exists for every $s < T$ and

$$\lim_{s\to T-} \int_0^s \Psi_T(s)\,\mathrm{d}X_s = \xi$$

a.s.

Proof. As in the proof of Lemma 3.1, we can assume that assumptions of Definition 2.2 are satisfied for the whole interval. Put first $Y_t = \tan \mathbb{E}[\arctan \xi|\mathcal{F}_t]$. Now Y_t is adapted, and we have $Y_t \to \xi$ as $t \to T-$ a.s. by martingale convergence theorem and left continuity of $\mathbb{F}$. Next for a sequence t_n increasing to T, set $\delta_n = Y_{t_n} - Y_{t_{n-1}}$ and $\tau_n = \inf\{t \geq t_n\colon\ Z_t^n = |\delta_n|\}$, where $Z_t^n = \int_{t_n}^t \phi_{t_{n+1}}(s)\,\mathrm{d}X_s$, and $\phi_{t_{n+1}}(s)$ is the process constructed in Lemma 3.1 such that $Z_t^n \to \infty$ as $t \to t_{n+1}$. By setting

$$\Psi_T(s) = \sum_{n\geq 1} \phi_{t_{n+1}}(s)\mathbf{1}_{[t_n,\tau_n]}(s)\,\mathrm{sign}(\delta_n)$$

we can repeat the arguments in [7] to conclude that

$$V_t := \lim_{t\to T-} \int_0^t \Psi_T(s)\,\mathrm{d}X_s = \xi. \qquad \square$$

Remark 3.8. Consider an arbitrary $\mathbb{F}$ measurable process Y_t. If for every $t \in (0, T]$ we have $X \in \mathcal{X}_t^\alpha$, then by Theorem 3.7 we have that for every t there is a process $\Psi_t(u)$ such that the process

$$V_t := \lim_{s\to t-} \int_0^s \Psi_t(u)\,\mathrm{d}X_u$$

is a version of Y_t.

For the proof of our main theorem we also need a bound for the probability that a Gaussian process X crosses a zero level. The bound is a consequence of the following more general result proved in [11].

Lemma 3.9. *Let X be a centered Gaussian process with strictly positive and bounded covariance function R, $0 < s < t \le T$ and $a \in \mathbb{R}$. Then there exists a universal constant $C = C(T)$ such that*

$$\begin{aligned}&\mathbb{P}(X_s < a < X_t)\\ &\quad\le C\frac{\sqrt{W(t,s)}}{\sqrt{V(s)}}\left[1 + \frac{R(s,s)}{R(t,s)} + \frac{|a|\mathrm{e}^{-a^2/(2V^*)}}{\sqrt{V(s)}}\max\left(1, \frac{R(s,s)}{R(t,s)}\right)\right],\end{aligned}$$

where

$$V^* = \sup_{s\le T} V(s).$$

Corollary 3.10. *Let X be a centered Gaussian process with positive and bounded covariance function $R(s,t)$, and let $0 < s \le t \le T$ be fixed. Then there exists a constant $C = C(T)$ such that*

$$\mathbb{P}(X_s < 0 < X_t) \le C\sqrt{\frac{W(t,s)}{V(s)}}\left[1 + \frac{R(s,s)}{R(t,s)}\right].$$

In [7] the authors also studied when a random variable ξ can be viewed as a proper integral, that is,

$$\xi = \int_0^1 \Psi(s)\,\mathrm{d}B_s^H$$

for some process $\Psi(s)$. As a result it was shown in [7] that this is true if ξ can be viewed as an endpoint of some stochastic process which is Hölder continuous of some order $a > 0$. Moreover, under assumption that Ψ is continuous the authors also proved that the conditions are necessary. As the proof is based on similar arguments as the proofs of previous theorems, it is not a surprise that we can derive similar results for our general class of processes. However, in our general case we have to modify the proof accordingly by choosing parameters differently. Consequently, we can only cover processes ξ which are Hölder continuous of order $a > 1 - \alpha$. For extensions, see Remark 3.12 below.

Theorem 3.11. *Let $X \in \mathcal{X}_T^\alpha$ such that Assumption 2.6 is satisfied, and let ξ be $\mathcal{F}_T$ measurable random variable. If there exists a Hölder continuous process Z_s of order $a > 1 - \alpha$ such that $Z_T = \xi$, then one can construct $\mathbb{F}$-adapted process Ψ_T on $[0, T]$ such that the integral*

$$\int_0^T \Psi_T(s)\,\mathrm{d}X_s$$

exists and

$$\int_0^T \Psi_T(s)\,\mathrm{d}X_s = \xi$$

a.s.

As in the proof of Lemma 3.1 and without loss of generality, we assume that conditions of Definition 2.2 are satisfied for the whole interval. Otherwise we simply choose t_1 large enough such that we are close to T.

Proof of Theorem 3.11. Without loss of generality, we can assume $a < \alpha$. Let $\beta \in (1-\alpha, a \wedge \frac{1}{2})$ and fix $\gamma > \frac{1}{a-\beta} \vee 1$. We put $\Delta_n = \frac{Tn^{-\gamma}}{\sum_{k=1}^{\infty} k^{-\gamma}}$ and set $t_0 = 0$, $t_n = \sum_{k=1}^{n-1} \Delta_k$, $n \geq 2$. Note that with our choice of γ and β we have $\gamma(\alpha - \beta) - 1 > \gamma(\alpha - a)$. Hence, we can choose some $\kappa \in (\gamma(\alpha - a), \gamma(\alpha - \beta) - 1)$. Next, we proceed as for fBm case and divide the proof into three steps:

1. Set $\Psi_T(t) = 0$ on interval $[t_0, t_1]$. To proceed the construction is done recursively on intervals $(t_n, t_{n+1}]$ and the construction is divided into two steps depending on whether we have $Y_{t_{n-1}} = Z_{t_{n-2}}$ (Case A) or $Y_{t_{n-1}} \neq Z_{t_{n-2}}$ (Case B). For the sake of completeness and clearness, we present the steps.

   Put $Y_t = \int_0^t \Psi_T(s)\,\mathrm{d}X_s$ and assume that $\Psi_T(s)$ is constructed on $[0, t_{n-1}]$ for some $n \geq 2$. If we have Case A, then we set

   $$\tau_n = \inf\bigl\{t \geq t_{n-1} \colon n^{\kappa}|X_t - X_{t_{n-1}}| = |Z_{t_{n-1}} - Z_{t_{n-2}}|\bigr\} \wedge t_n$$

   and for $s \in [t_{n-1}, t_n)$,

   $$\Psi_T(s) = n^{\kappa} \operatorname{sign}(X_s - X_{t_{n-1}}) \operatorname{sign}(Z_{t_{n-1}} - Z_{t_{n-2}}) \mathbf{1}_{[0,\tau_n]}(s).$$

   Now if $\tau_n < t_n$, we obtain by Itô's formula (2.1) that

   $$Y_{t_n} = Z_{t_{n-1}}.$$

   Assume next that we have Case B. Then we proceed as in Theorem 3.7 and set

   $$Y_t^n = \int_{t_{n-1}}^t \phi_{t_n}(s)\,\mathrm{d}X_s,$$

   where $\phi_{t_n}(s)$ is the process constructed in Lemma 3.1 such that $Y_t^n \to \infty$ as $t \to t_n$,

   $$\tau_n = \inf\bigl\{t \geq t_{n-1} \colon Y_t^n = |Z_{t_{n-1}} - Y_{t_{n-1}}|\bigr\},$$

   and for $s \in [t_{n-1}, t_n)$,

   $$\Psi_T(s) = \phi_{t_n}(s) \operatorname{sign}(Z_{t_{n-1}} - Y_{t_{n-1}}) \mathbf{1}_{[0,\tau_n]}(s).$$

   Then $Y_{t_n} = Z_{t_{n-1}}$.

2. Next, note that for a fixed n, the only possibility that $Y_{t_n} \neq Z_{t_{n-1}}$ is that we have case A and $\tau_n \geq t_n$. Hence, it suffices to show that the event

$$C_n = \left\{ \sup_{t_{n-1} \leq t \leq t_n} n^{\kappa} |X_t - X_{t_{n-1}}| \leq |Z_{t_{n-1}} - Z_{t_{n-2}}| \right\}$$

happens only finite number of times. For this we take $b \in (\alpha - \frac{\kappa}{\gamma}, a)$, and the arguments in [7] implies that it is sufficient to show that only finite number of events

$$D_n = \left\{ \sup_{t_{n-1} \leq t \leq t_n} n^{\kappa} |X_t - X_{t_{n-1}}| \leq \Delta_n^b \right\}$$

happen. Recall that now we have $b > \alpha - \frac{\kappa}{\gamma}$ which can be written as $\gamma b + \kappa > \gamma \alpha$. Hence we can apply the small ball estimate (2.2) of Assumption 2.6 together with Borel–Cantelli lemma to obtain the result.

3. To complete the proof, we have to show that $\|\Psi_T\|_{T,\beta} < \infty$ a.s. For this, we go through the main steps which are different from the case of fBm. We write

$$A_n = \left\{ \text{We have Case A on } (t_{n-1}, t_n] \right\}, \qquad B_n = A_n^C,$$

and

$$\begin{aligned} \Psi_T(s) &= \sum_{n\geq 2} \Psi_T(s) \mathbf{1}_{(t_{n-1},t_n]}(s) \mathbf{1}_{A_n} \\ &\quad + \sum_{n\geq 2} \Psi_T(s) \mathbf{1}_{(t_{n-1},t_n]}(s) \mathbf{1}_{B_n} \\ &=: \Psi_T^A(s) + \Psi_T^B(s). \end{aligned}$$

As for fBm case, it is evident that $\|\Psi_T^B(s)\|_{T,\beta} < \infty$ since only finite numbers of events B_n happen. Furthermore, we can write

$$\begin{aligned} \mathbb{E}\left[\left\| \Psi_T^A(s) \right\|_{T,\beta} \right] &= \int_0^T \frac{\mathbb{E}|\Psi_T^A(s)|}{s^\beta} \\ &\quad + \sum_{n=2}^{\infty} \int_{t_{n-1}}^{t_n} \int_0^{t_{n-1}} \frac{\mathbb{E}|\Psi_T^A(t) - \Psi_T^A(s)|}{(t-s)^{\beta+1}} \mathrm{d}s\, \mathrm{d}t \\ &\quad + \sum_{n=2}^{\infty} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t} \frac{\mathbb{E}|\Psi_T^A(t) - \Psi_T^A(s)|}{(t-s)^{\beta+1}} \mathrm{d}s\, \mathrm{d}t \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

The finiteness of I_1 and I_2 are easy to show and we omit the details. For I_3 we set $\lambda_n(t) = \operatorname{sign}(X_t - X_{t_{n-1}})$ and obtain

$$
\begin{aligned}
I_3 &= \sum_{n=2}^{\infty} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t} \frac{\mathbb{E}|\Psi_T^A(t) - \Psi_T^A(s)|}{(t-s)^{\beta+1}} \,\mathrm{d}s\,\mathrm{d}t \\
&= \sum_{n=2}^{\infty} n^{\kappa} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t} \frac{\mathbb{E}|\lambda_n(t)\mathbf{1}_{s\le\tau_n} - \lambda_n(s)\mathbf{1}_{s\le\tau_n}|\mathbf{1}_{A_n}}{(t-s)^{\beta+1}} \,\mathrm{d}s\,\mathrm{d}t \\
&\le \sum_{n=2}^{\infty} n^{\kappa} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t} \frac{\mathbb{E}[|\lambda_n(t) - \lambda_n(s)| + \mathbf{1}_{s\le\tau_n<t}]}{(t-s)^{\beta+1}} \,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}
$$

Now note that

$$
\left|\lambda_n(t) - \lambda_n(s)\right| = \mathbf{1}_{\{X_s - X_{t_{n-1}} \le 0 \le X_t - X_{t_{n-1}}\}} + \mathbf{1}_{\{X_s - X_{t_{n-1}} \ge 0 \ge X_t - X_{t_{n-1}}\}},
$$

and by taking expectation together with symmetry it is sufficient to consider probability

$$
\mathbb{P}(X_s - X_{t_{n-1}} \le 0 \le X_t - X_{t_{n-1}}).
$$

Let us study the integral

$$
\int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t} \frac{\mathbb{P}(X_s - X_{t_{n-1}} \le 0 \le X_t - X_{t_{n-1}})}{(t-s)^{\beta+1}} \,\mathrm{d}s\,\mathrm{d}t.
$$

By change of variable, we obtain that it is sufficient to study

$$
\begin{aligned}
&\int_0^{t_n - t_{n-1}} \int_0^{t} \frac{\mathbb{P}(X_{s+t_{n-1}} - X_{t_{n-1}} \le 0 \le X_{t+t_{n-1}} - X_{t_{n-1}})}{(t-s)^{\beta+1}} \,\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^{t_n - t_{n-1}} \int_0^{t/2} \frac{\mathbb{P}(X_{s+t_{n-1}} - X_{t_{n-1}} \le 0 \le X_{t+t_{n-1}} - X_{t_{n-1}})}{(t-s)^{\beta+1}} \,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \int_0^{t_n - t_{n-1}} \int_{t/2}^{t} \frac{\mathbb{P}(X_{s+t_{n-1}} - X_{t_{n-1}} \le 0 \le X_{t+t_{n-1}} - X_{t_{n-1}})}{(t-s)^{\beta+1}} \,\mathrm{d}s\,\mathrm{d}t \\
&=: J_1 + J_2.
\end{aligned}
$$

For J_1 we can bound the probability with one and get

$$
J_1 \le C\Delta_n^{1-\beta}.
$$

Consider next the term J_2. By assumption 1 of Definition 2.2 the covariance of Gaussian processes $X_{s+t_{n-1}} - X_{t_{n-1}}$ and $X_{t+t_{n-1}} - X_{t_{n-1}}$ is positive for every n and every $s, t \in [0, t_n - t_{n-1}]$. Thus we can apply Corollary 3.10 and assumption 4 to obtain

$$
\mathbb{P}(X_s - X_{t_{n-1}} \le 0 \le X_t - X_{t_{n-1}}) \le C \frac{\sqrt{W_n(t,s)}}{\sqrt{\mathbb{E}(X_s - X_{t_{n-1}})^2}},
$$

where

$$W_n(t,s) = \mathbb{E}\big(X_{t+t_{n-1}} - X_{t_{n-1}} - (X_{s+t_{n-1}} - X_{t_{n-1}})\big)^2 \\ \le C(t-s)^{2\alpha},$$

and

$$\mathbb{E}(X_{s+t_{n-1}} - X_{t_{n-1}})^2 \ge Cs^2$$

by assumptions. Hence, by symmetry of probabilities $P(X_s - X_{t_{n-1}} \le 0 \le X_t - X_{t_{n-1}})$ and $P(X_s - X_{t_{n-1}} \ge 0 \ge X_t - X_{t_{n-1}})$, we obtain

$$\begin{aligned} J_2 &\le C \int_0^{t_n - t_{n-1}} \int_{t/2}^t \frac{(t-s)^{\alpha-\beta-1}}{s} \, \mathrm{d}s \, \mathrm{d}t \\ &\le C \int_0^{t_n - t_{n-1}} t^{\alpha-\beta-1} \, \mathrm{d}t \\ &\le C\Delta_n^{\alpha-\beta}. \end{aligned}$$

To conclude, we note that

$$\int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^t \frac{\mathbf{1}_{s \le \tau_n < t}}{(t-s)^{\beta+1}} \, \mathrm{d}s \, \mathrm{d}t \le C\Delta_n^{1-\beta},$$

and hence

$$I_3 \le C \sum_{n=2}^{\infty} n^{\kappa - \gamma(\alpha-\beta)} < \infty$$

by our choice of κ, γ, and β. □

Remark 3.12. With our general assumptions, we can only cover Hölder continuous variables of order $a > 1 - \alpha$. However, under additional assumption that for s close to T and small enough Δ the incremental variance satisfies

$$\mathbb{E}[X_{s+\Delta} - X_s]^2 \ge C\Delta^{2\theta}$$

with some constant C and some parameter $\theta \in (\alpha, 1)$, we can cover more. More precisely, we can cover Hölder continuous processes of order $a > \theta - \alpha$. Especially this is the case if the process X is stationary or has stationary increments with $W_X(0,t) \sim t^{2\alpha}$. In particular case of fBm one can cover Hölder continuous processes of any order $a > 0$. Similarly, with a process $\tilde{X}$ one can cover Hölder continuous processes of any order $a > 0$.

Remark 3.13. In [7], the authors proved also that under additional assumption that Ψ is continuous, the assumption of the Theorem 3.11 is also necessary. The proof is based only to the Hölder continuity of fBm and well-known properties of Young integrals. Consequently, same conclusion remains for our general class of processes.

Corollary 3.14. *Let Z_t be a.s. Hölder continuous process of order $a > 1 - \alpha$ and for every $t \in (0, T]$ we have $X \in \mathcal{X}_t^\alpha$. Then for every t there exists $\mathbb{F}$-adapted process Ψ_t such that it holds, a.s.,*

$$\int_0^t \Psi_t(s)\,\mathrm{d}X_s = Z_t,$$

i.e. the integral $\int_0^t \Psi_t(s)\,\mathrm{d}X_s$ is a version of Z_t.

4. Applications and discussions

In the paper [7], the authors considered financial implications of their results to a model where the stock is driven by geometric fBm. In particular, the results indicate one more reason why geometric fBm is not a proper model in finance. Evidently, we could state similar results in our general setting and as a consequence, we can argue that processes $X \in \mathcal{X}_T^\alpha$ do not fit well as the driving process of stock prices. This is also discussed with details in [11] where the authors proved the pathwise Itô–Tanaka formula for processes in our class. For further details, we refer to [7] and [11], and the repetition of the arguments presented in [7] for more general processes $X \in \mathcal{X}_T^\alpha$ are left to the reader. However, we wish to give one remark on financial implications of our results. In [7], the authors proved that if the stock is driven by geometric fractional Brownian motion, then one can replicate essentially all interesting derivatives. On the other hand, we can never know whether the process driving the stock is geometric fBm or not. The benefit of our results is that in addition to the fact that the replication can be done with much more general class of processes, the replication can be done also in arbitrary small amount of time. This means that one can wait and observe the process up to some time arbitrary close to the maturity, and start the replication procedure after that point. Especially this is useful if there is no information on the stock dynamics. Assuming that the driving process is Gaussian, one can save time to estimate the covariance structure of the process and use this information for the replication.

On the uniqueness of representation

In the case of standard Brownian motion, every centered random variable ξ with finite variance can be represented as

$$\xi = \int_0^1 \Psi(s)\,\mathrm{d}W_s,$$

where $\int_0^1 \mathbb{E}[\Psi(s)]^2\,\mathrm{d}s < \infty$. Moreover, a direct consequence of the Itô isometry implies that in this case the process Ψ is unique. However, for generalised Lebesgue–Stieltjes integrals the representation is not unique. As an example, consider fractional Ornstein–Uhlenbeck process given by

$$U_t^\theta = \int_0^t \mathrm{e}^{-\theta(t-s)}\,\mathrm{d}B_s^H.$$

On the other hand, by Theorem 3.11 we know that

$$U_t^\theta = \int_0^t \Psi_t(s)\,\mathrm{d}B_s^H,$$

where $\Psi_t(s)$ is defined equally zero on interval $[0, t_1]$, and t_1 can be chosen arbitrary close to 1. Hence, the representation is clearly not unique in general with pathwise integrals. On the other hand, for Skorokhod integrals with respect to fBm the representation is unique (see [2]).

The problem of zero integral

Another application which was considered in [7] for fBm was the problem of zero integral, and we wish to end the paper by giving some remarks on zero integral problem for our general class of processes.

Recall that the zero integral problem refers to the question whether we have implication

$$\int_0^1 u_s\,\mathrm{d}X_s = 0, \qquad \text{a.s.} \quad \Rightarrow \quad u_s = 0, \qquad \mathbb{P}\otimes \mathrm{Leb}\big([0,T]\big) \qquad \text{a.e.} \tag{4.1}$$

For standard Brownian motion this is true under assumption $\int_0^T \mathbb{E}[u_s^2]\,\mathrm{d}s < \infty$, and the result is a direct consequence of the Itô isometry. On the other hand, if we only have that $\int_0^1 u_s^2\,\mathrm{d}s < \infty$ a.s., then the conclusion is false. In particular, one can construct an adapted process such that $\int_0^{1/2} u_s\,\mathrm{d}W_s = 1$ and $\int_{1/2}^1 u_s\,\mathrm{d}W_s = -1$.

Similarly for fBm, the authors in [7] explained that one can construct an adapted process such that $\int_0^{1/2} u_s\,\mathrm{d}B_s^H = 1$ and $\int_{1/2}^1 u_s\,\mathrm{d}B_s^H = -1$. Now the results presented in this paper indicate that the same conclusion remains true if we replace fBm B^H with more general Gaussian process X. This suggests that the problem of zero integral is not interesting in the first place since the conclusion is false in most of the interesting case unless one poses some extra assumptions. We also note that a negative answer to the question of zero integral is a direct consequence of the fact that the representation is not unique. As another example of this, consider a random variable $(X_1 - K)^+$. Clearly this random variable is an end value of Hölder continuous process, and thus Theorem 3.11 implies that there is a process $\Psi_1(s)$ such that

$$(X_1 - K)^+ = \int_0^1 \Psi_1(s)\,\mathrm{d}X_s.$$

Moreover, by construction of the process $\Psi_1(s)$ we have $\Psi_1(s) = 0$ on the interval $s \in [0, t_1]$. On the other hand, by Theorem 3.11 (assuming that the covariance R_X of the process X itself satisfies 1–4) we have

$$(X_1 - K)^+ = (X_0 - K)^+ + \int_0^1 \mathbf{1}_{X_s > K}\,\mathrm{d}X_s.$$

If now $X_0 \le K$ a.s., subtracting first equation from the second one, we obtain that

$$0 = \int_0^1 \Psi_1(s) - \mathbf{1}_{X_s > K} \, \mathrm{d}X_s.$$

Now $\Psi_1(s) = 0$ a.s. on $[0, t_1]$, and clearly the same is not true for process $\mathbf{1}_{X_s > K}$. This is another argument to show that the $\int_0^1 u_s \, \mathrm{d}X_s = 0$ does not imply $u_s = 0$ a.s. in general.

Acknowledgement

Lauri Viitasaari thanks The Finnish Graduate School in Stochastic and Statistics for financial support.

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Received March 2014 and revised June 2014