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Stochastic differential equations driven by fractional Brownian motion and Poisson point process

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In this paper, we study a class of stochastic differential equations with additive noise that contains a fractional Brownian motion (fBM) and a Poisson point process of class (QL). The differential equation of this kind is motivated by the reserve processes in a general insurance model, in which the long term dependence between the claim payment and the past history of liability becomes the main focus. We establish some new fractional calculus on the fractional Wiener–Poisson space, from which we define the weak solution of the SDE and prove its existence and uniqueness. Using an extended form of Krylov-type estimate for the combined noise of fBM and compound Poisson, we prove the existence of the strong solution, along the lines of Gyöngy and Pardoux (*Probab. Theory Related Fields* **94** (1993) 413–425). Our result in particular extends the one by Mishura and Nualart (*Statist. Probab. Lett.* **70** (2004) 253–261).

Keywords: discontinuous fractional calculus; fractional Brownian motion; fractional Wiener–Poisson space; Krylov estimates; Poisson point process; stochastic differential equations

1. Introduction

In this paper, we are interested in the following stochastic differential equation (SDE):

$$X_t = x + \int_0^t b(s, X_s)\,\mathrm{d}s + \sigma B_t^H - L_t, \qquad t \in [0, T], \tag{1.1}$$

where $B^H = \{B_t^H : t \ge 0\}$ is a *fractional Brownian motion* with Hurst parameter $H \in (0, 1)$, defined on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$, with $\mathbb{F} = \{\mathcal{F}_t : t \ge 0\}$ being a filtration that satisfies the *usual hypotheses* (cf., e.g., [17]); and $L = \{L_t : t \ge 0\}$ is a Poisson point process of class (QL), independent of B^H. More precisely, we assume that L takes the form

$$L_t = \int_0^t \int_{\mathbb{R}} f(s, x) N_p(\mathrm{d}s, \mathrm{d}x), \qquad t \ge 0, \tag{1.2}$$

where f is a deterministic function, and p is a stationary Poisson point process whose counting measure N_p is a Poisson random measure with Lévy measure ν (see Section 2 for more details).

1350-7265

One of the motivations for our study is to consider a general reserve process of an insurance company, perturbed by an additive noise that has long term dependency. A commonly seen perturbed reserve (or surplus) model is of the following form:

$$U_t = x + c(1+\rho)t + \varepsilon W_t - L_t, \qquad t \in [0,T]. \tag{1.3}$$

Here $x \geq 0$ denotes the initial surplus, $c > 0$ is the premium rate, $\rho > 0$ is the "safety" (or expense) loading, $\varepsilon > 0$ is the perturbation parameter, $W = \{W_t : t \geq 0\}$ is a Brownian motion, which represents an additional uncertainty coming from either the aggregated claims or the premium income, L_t denotes cumulated claims up to time t, and finally, $T > 0$ is a fixed time horizon. We refer the reader to the well-referred book [19], Chapter 13, and the references therein for more explanations of such models.

In this paper, we are particularly interested in the case where the diffusion perturbation term possesses long-range dependence. Such a phenomenon has been noted in insurance models based on the observations that the claims often display long memories due to extreme weather, natural disasters, and also noted in casualty insurance such as automobile third-party liability (cf. e.g., [3,5–7,10,13,14] and references therein). A reasonable refinement that reflects the long memory but also retains the original features of the aggregated claims is to assume that the Brownian motion W in (1.3) is replaced by a fractional Brownian motion B^H, for a certain Hurst parameter $H \in (0,1)$. In fact, if we assume further that in addition to the premium income, the company also receives interest of its reserves at time with interest rate $r > 0$, and that the safety loading ρ also depends on the current reserve value, one can argue that the reserve process X should satisfy an SDE of the form of (1.1) with

$$b(t,x) = rx + c\big(1 + \rho(t,x)\big), \qquad (t,x) \in [0,T] \times \mathbb{R}.$$

The main purpose of this paper is to find the minimum conditions on the function b under which the SDE (1.1) is well posed, in both weak and strong sense. In the case when $L \equiv 0$, the SDE (1.1) becomes one driven by an (additive) fBM and the similar issues were investigated by Nualart and Ouknine [16] and Hu, Nualart and Song [9]. One of the main results is that, unlike the ordinary differential equation case, the well-posedness of the SDE can be established under only some integrability conditions, and in particular, no Lipschitz continuity is required for uniqueness. The main idea is to use a Krylov-type estimate to obtain a comparison theorem, whence the pathwise uniqueness. Such a scheme was utilized by Gyöngy and Pardoux [8] when studying the quasi-linear SPDEs, and has been a frequently used tool to treat the SDEs with non-Lipschitz coefficients, as an alternative to the well-known Yamada–Watanabe theorem. In fact, this method is even more crucial in the current case, as the usual Yamada–Watanabe theorem type of argument does not seem to work due to the lack of independent increment property of an fBM.

The main difficulty in the study of SDE (1.1), however, is the presence of the jumps. In the case when $H > 1/2$, Mishura and Nualart [15] studied the existence of weak solution of SDE (1.1) with $L \equiv 0$, and the coefficient b is allowed to have finitely many discontinuities in its spatial variable x. By a simple transformation (e.g., setting $\tilde{X} = X - L$), our result in a sense extends their result to a more general case in which b possesses countably many discontinuities in x. More importantly, we remove the extra assumption that $H < (1+\sqrt{5})/4$ in [15] when the number of jumps is finite. To our best knowledge, the fractional calculus applying to SDE driven by both fBM and Poisson point process is new.

The rest of the paper is organized as follows. In Section 2, we review briefly the basics on fBM and some fractional calculus that is needed in this paper. In Section 3, we prove a Girsanov theorem and in Section 4 we apply it to study the existence of the weak solution. In Section 5, we address the uniqueness issue, in both weak and strong forms, and in Section 6 we study the existence of the strong solution.

2. Preliminaries

In this section, we review some of the basic concepts in fractional calculus and introduce the notion of (canonical) fractional Wiener–Poisson spaces which will be the basis of our study. Throughout this paper, we denote $\mathbf{E}$ (also $\mathbf{E}_1, \ldots$) for a generic Euclidean space, whose inner products and norms will be denoted as the same ones $\langle\cdot,\cdot\rangle$ and $|\cdot|$, respectively; and denote $\|\cdot\|$ to be the norm of a generic Banach space. Let $\mathcal{U} \subset \mathbf{E}$ be a measurable subset. We shall denote by $L^p(\mathcal{U};\mathbf{E}_1)$, $0 \le p < \infty$, the space of all $\mathbf{E}_1$-valued measurable function $\phi(\cdot)$ defined on $\mathcal{U}$ such that $\int_{\mathcal{U}} |\phi(t)|^p \,\mathrm{d}t < \infty$ ($p = 0$ means merely measurable). For each $n \in \mathbb{N}$, $\mathbb{C}^n(\mathcal{U};\mathbf{E}_1)$ denotes all the $\mathbf{E}_1$-valued, nth continuously differentiable functions on $\mathcal{U}$, with the usual sup-norm.

2.1. Fractional calculus

We begin with a brief review of the deterministic fractional calculus. We refer to the book Samko, Kilbas and Marichev [20] for an exhaustive survey on the subject. We first recall some basic definitions.

Let $-\infty < a < b < \infty$, and $\varphi \in L^1([a,b])$. The integrals

$$\bigl(I^\alpha_{a+}\varphi\bigr)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}\,\mathrm{d}t, \qquad x > a, \tag{2.1}$$

$$\bigl(I^\alpha_{b-}\varphi\bigr)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{\varphi(t)}{(t-x)^{1-\alpha}}\,\mathrm{d}t, \qquad x < b, \tag{2.2}$$

are called *fractional integrals of order* α, where $\Gamma(\cdot)$ is the Gamma-function and $\alpha \in [0,\infty)$. Both I^α_{a+} and I^α_{b-} are the so-called Riemann–Liouville fractional integrals, and they are often called "left" and "right" fractional integrals, respectively. We shall denote the image of $L^p([a,b])$ under the fractional integration operator I^α_{a+} (resp. I^α_{b-}) by $I^\alpha_{a+}(L^p([a,b]))$ (resp. $I^\alpha_{b-}(L^p([a,b]))$). Moreover, in what follows we shall often use left-fractional integration, which has the following properties:

$$\begin{gathered}\bigl[I^\alpha_{a+}I^\beta_{a+}\varphi\bigr](\cdot) = \bigl[I^{\alpha+\beta}_{a+}\varphi\bigr](\cdot),\\ t^\alpha I^\beta_{0+}t^{-\alpha-\beta}I^\alpha_{0+}t^\beta\varphi(\cdot) = I^\alpha_{0+}I^\beta_{0+}\varphi(\cdot) = I^{\alpha+\beta}_{0+}\varphi(\cdot), \qquad \alpha > 0, \beta > 0.\end{gathered} \tag{2.3}$$

We note that (2.3) holds for a.e. $x \in [a,b]$. If $\varphi \in \mathbb{C}([a,b])$, then (2.3) holds for all $x \in [a,b]$.

The (Riemann–Liouville) fractional derivatives are defined, naturally, as the inverse operator of the fractional integration. To wit, for any function $f \in L^0([a,b])$, we define

$$\big(\mathcal{D}^{\alpha}_{a+} f\big)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x \frac{f(t)}{(x-t)^{\alpha}}\,\mathrm{d}t, \tag{2.4}$$

$$\big(\mathcal{D}^{\alpha}_{b-} f\big)(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_x^b \frac{f(t)}{(t-x)^{\alpha}}\,\mathrm{d}t, \tag{2.5}$$

whenever they exist. We call $\mathcal{D}^{\alpha}_{a+} f$ (resp. $\mathcal{D}^{\alpha}_{b-} f$) the *left (resp. right) fractional derivative of order* α, $0 < \alpha < 1$. We note that if $f(t) \in \mathbb{C}^1([a,b])$, then it is easy to verify that (see [20], page 224)

$$\mathcal{D}^{\alpha}_{a+} f = \frac{f(x)}{\Gamma(1-\alpha)(x-a)^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)} \int_a^x \frac{f(x)-f(t)}{(x-t)^{1+\alpha}}\,\mathrm{d}t \overset{\triangle}{=} D^{\alpha}_{a+} f. \tag{2.6}$$

The derivative $D^{\alpha}_{a+} f$ is called Marchaud fractional derivative. We should note that the right-hand side of (2.6) is not only well-defined for differentiable functions, but for example, for function $f(x)$ that is β-Hölder continuous, with $\beta > \alpha$. For more general functions, the fractional Marchaud derivative (2.6) should be understood as (cf. [20])

$$D^{\alpha}_{a+} f \overset{\triangle}{=} \lim_{\varepsilon\to 0} D^{\alpha}_{a+,\varepsilon} f, \tag{2.7}$$

where the limit is in the space L^p, and

$$\big[D^{\alpha}_{a+,\varepsilon} f\big](x) \overset{\triangle}{=} \frac{f(x)}{\Gamma(1-\alpha)(x-a)^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)} \int_a^{x-\varepsilon} \frac{f(x)-f(t)}{(x-t)^{1+\alpha}}\,\mathrm{d}t. \tag{2.8}$$

We collect some of the important properties of the fractional integral and derivative in the follow theorem. The proofs can be found in [20].

Theorem 2.1.

(i) *For any* $\varphi \in L^1([a,b])$ *and* $0 < \alpha < 1$, *it holds that*

$$D^{\alpha}_{a+} I^{\alpha}_{a+} \varphi = \lim_{\varepsilon\to 0} D^{\alpha}_{a+,\varepsilon} I^{\alpha}_{a+} \varphi = \mathcal{D}^{\alpha}_{a+} I^{\alpha}_{a+} \varphi = \varphi. \tag{2.9}$$

(ii) *For any* $f \in I^{\alpha}_{a+}(L^1([a,b]))$ *and* $\alpha > 0$, *it holds that*

$$I^{\alpha}_{a+} D^{\alpha}_{a+} f = I^{\alpha}_{a+} \mathcal{D}^{\alpha}_{a+} f = f. \tag{2.10}$$

(iii) *Let* $\psi \in L^p([0,b])$, $b > 0$, $1 < p < \infty$. *Then* ψ *has the representation* $\psi(x) = I^{\alpha}_{0+} x^{\mu} f(x)$, *a.e.* $x \in [0,b]$, *for some* $f \in L^p([0,b])$, $\alpha > 0$, *and* $p(1+\mu) > 1$ *if and only if* ψ *takes one of the following two forms*:

(a) $\psi(x) = x^{\mu}[I^{\alpha}_{0+} g](x)$, *a.e.* $x \in [0,b]$, $g \in L^p([0,b])$;

(b) $\psi(x) = x^{\mu-\varepsilon}[I^{\alpha}_{0+} x^{\varepsilon} g_1](x)$, *a.e.* $x \in [0,b]$, $g_1 \in L^p([0,b])$, $p(1+\varepsilon) > 1$.

2.2. Fractional Wiener–Poisson space

We recall that a stochastic process $B^H = \{B_t^H, t \in [0,T]\}$, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0})$, is called an $\mathbb{F}$-fractional Brownian motion (fBM) with Hurst parameter $H \in (0,1)$ if

(i) B^H is a Gaussian process with continuous paths and $B_0^H = 0$;
(ii) for each $t \geq 0$, B_t^H is $\mathcal{F}_t$-measurable and $\mathbb{E}B_t^H = 0$, for each $t \geq 0$;
(iii) for all $s, t \geq 0$, it holds that

$$\mathbb{E}\big(B_t^H B_s^H\big) = R_H(t,s) = \tfrac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big). \tag{2.11}$$

It follows from (2.11) that $\mathbb{E}|B_t^H - B_s^H|^2 = |t-s|^{2H}$, that is, B^H has stationary increments. Furthermore, by Kolmogorov's continuity criterion, B_t^H has α-Hölder continuous paths for all $\alpha < H$. In particular, if $H = 1/2$, then B^H becomes a standard Brownian motion; and if $H = 1$, then $\{B_t^1; t \geq 0\}$ has the same law as $\{\xi t; t \geq 0\}$, where ξ is an $N(0,1)$ random variable.

In what follows, we shall consider the *canonical space* with respect to an fBM or the fractional Wiener space. Let $\Omega^1 = \mathbb{C}_0([0,T])$, the space of all continuous functions, null at zero, and endowed with the usual sup-norm. Let $\mathcal{F}_t^1 \triangleq \sigma\{\omega(\cdot \wedge t) | \omega \in \Omega^1\}$, $t \geq 0$, $\mathcal{F}^1 \triangleq \mathcal{F}_T^1$, $\mathbb{F}^1 = \{\mathcal{F}_t^1, t \in [0,T]\}$ and $\mathbb{P}^{B^H}$ is the probability measure on $(\Omega^1, \mathcal{F}^1)$ under which the *canonical process*

$$B_t^H(\omega) \triangleq \omega(t), \qquad (t,\omega) \in [0,T] \times \Omega^1$$

is an fBM of Hurst parameter H.

For any $H \in (0,1)$, we define

$$R_H(t,s) = \int_0^{t \wedge s} K_H(t,r) K_H(s,r)\, \mathrm{d}r, \tag{2.12}$$

where K_H is the square integrable kernel given by

$$K_H(t,s) \triangleq \Gamma\left(H + \frac{1}{2}\right)^{-1} (t-s)^{H-1/2} F\left(H - \frac{1}{2}, \frac{1}{2} - H, H + \frac{1}{2}, 1 - \frac{t}{s}\right), \tag{2.13}$$

and $F(a,b,c,z)$ is the Gaussian hypergeometric function:

$$F(a,b,c,z) = \sum_{k=0}^{\infty} \frac{a^{(k)} b^{(k)}}{c^{(k)} k!} z^k, \qquad a, b \in \mathbb{R}, |z| < 1, c \neq 0, -1, \ldots,$$

where $a^{(k)}$, $b^{(k)}$, $c^{(k)}$ are the Pochhammer symbol for the *rising factorial*: $x^{(0)} = 1$, $x^{(k)} = \frac{\Gamma(x+k)}{\Gamma(x)}$.

Now, let $\mathscr{E}$ be the set of all step functions on $[0, T]$, and let $\mathscr{H}$ be the so-called *Reproducing Kernel Hilbert space*, defined as the closure of $\mathscr{E}$ with respect to the scalar product

$$\langle I_{[0,t]}, I_{[0,s]}\rangle_{\mathscr{H}} = R_H(t,s), \qquad s,t \in [0,T]. \tag{2.14}$$

For any $H \in (0,1)$, we define a linear operator $K_H : L^2([0,T]) \to L^2([0,T])$ by

$$[K_H f](t) \stackrel{\triangle}{=} \int_0^t K_H(t,s) f(s)\,\mathrm{d}s, \qquad f \in L^2\bigl([0,T]\bigr),\ t \in [0,T]. \tag{2.15}$$

Also, for any $f \in L^0([0,T])$ and $\beta > 0$, we shall denote

$$[\![f]\!]^\beta(t) \stackrel{\triangle}{=} t^\beta f(t), \qquad t \in [0,T], \tag{2.16}$$

and $I_{0+}^{\alpha,\beta}(L^p([0,T])) = \{f \in L^0([0,T]) : [\![f]\!]^\beta \in I_{0+}^\alpha(L^p([0,T]))\}$. Then we have the following result (cf., e.g., [2], Theorem 2.1, or [20], Theorem 10.4).

Theorem 2.2. *For each $H \in (0,1)$, the operator K_H is an isomorphism between $L^2([0,T])$ and $I_{0+}^{H+1/2}(L^2([0,T]))$. Furthermore, it holds that*

$$K_H f = \begin{cases} I_{0+}^{2H}[\![I_{0+}^{1/2-H}[\![f]\!]^{H-1/2}]\!]^{1/2-H}, & H < 1/2, \\ I_{0+}^{1}[\![I_{0+}^{H-1/2}[\![f]\!]^{1/2-H}]\!]^{H-1/2}, & H > 1/2. \end{cases} \tag{2.17}$$

From (2.17) it is easy to check that the inverse operator K_H^{-1} on an absolutely continuous function h satisfies

$$K_H^{-1} h = \begin{cases} [\![I_{0+}^{1/2-H}[\![h']\!]^{1/2-H}]\!]^{H-1/2}, \\ \quad \text{if } h' \in L^1\bigl([0,T]\bigr), \text{ and } H < 1/2, \\ [\![D_{0+}^{H-1/2}[\![h']\!]^{1/2-H}]\!]^{H-1/2}, \\ \quad \text{if } h' \in I_{0+}^{H-1/2,1/2-H}\bigl(L^1\bigl([0,T]\bigr)\bigr) \cap L^1\bigl([0,T]\bigr), \text{ and } H > 1/2, \end{cases} \tag{2.18}$$

where h' is the derivative of h (cf., e.g., [20], Theorem 10.6, and [16]).

Next, let K_H^* be the adjoint of K_H on $L^2([0,T])$, that is, for any $f \in \mathscr{E}$, $g \in L^2([0,T])$,

$$\int_0^T \bigl[K_H^* f\bigr](t) g(t)\,\mathrm{d}t = \int_0^T f(t)[K_H g](t)\,\mathrm{d}t.$$

Then, it can be shown by Fubini and integration by parts that for any $f \in \mathscr{E}$,

$$\bigl[K_H^* f\bigr](t) = K_H(T,t)\varphi(t) + \int_t^T \bigl(f(s) - f(t)\bigr)\frac{\partial K_H}{\partial s}(s,t)\,\mathrm{d}s, \qquad t \in [0,T].$$

In particular, for $\varphi, \psi \in \mathscr{E}$, we have (see, e.g., [1])

$$\bigl\langle K_H^*\varphi, K_H^*\psi\bigr\rangle_{L^2((0,T))} = \langle\varphi,\psi\rangle_{\mathcal{H}}.$$

Consequently, the operator K_H^* is an isometry between the Hilbert spaces $\mathscr{H}$ and $L^2([0,T])$. Furthermore, it can be shown that the process $W = \{W_t, t \in [0,T]\}$ defined by

$$W_t = B^H\big(\big(K_H^*\big)^{-1}(I_{[0,t]})\big) \tag{2.19}$$

is a Wiener process, and the process B^H has an integral representation of the form

$$B_t^H = \int_0^t K_H(t,s)\,\mathrm{d}W_s, \qquad t \in [0,T]. \tag{2.20}$$

We now turn our attention to the Poisson part. We first consider a Poisson random measure $N(\cdot,\cdot)$ on $[0,T]\times\mathbb{R}$, defined on a given probability space $(\Omega,\mathcal{F},\mathbb{P})$, with mean measure $\hat{N}(\mathrm{d}t,\mathrm{d}x) = \mathrm{d}t\,\nu(\mathrm{d}x)$, where ν is the Lévy measure, a σ-finite measure on $\mathbb{R}^* \stackrel{\triangle}{=} \mathbb{R}\setminus\{0\}$ satisfying the standard integrability condition:

$$\int_{\mathbb{R}^*}\big(1\wedge|x|^2\big)\nu(\mathrm{d}x) < +\infty.$$

In this paper, we shall be interested in a *Poisson point process* of class (QL), namely a point process whose counting measure, defined by $N_L((0,t]\times A) = \#\{s\in(0,t]: \Delta L_s \in A\} = \sum_{0<s\le t}\mathbf{1}_{\{\Delta L_s\in A\}}$, $t\ge 0$, $A\in\mathscr{B}(\mathbb{R}^*)$, has a deterministic and continuous compensator (cf. [11]). In light of the representation theorem [11], Theorem II-7.4, we shall assume without loss of generality that the process L takes the following form:

$$L_t = \int_0^t\int_{\mathbb{R}^*} f(s,x)N(\mathrm{d}s,\mathrm{d}x), \qquad t\ge 0, \tag{2.21}$$

where $f\in L^1(\mathrm{d}t\times\mathrm{d}\nu)$ is a deterministic function. Then, the counting measure $N_L(\mathrm{d}t,\mathrm{d}x)$ can be written as

$$N_L\big((0,t]\times A\big) = \int_0^t\int_{\mathbb{R}^*}\mathbf{1}_A\big(f(s,x)\big)N(\mathrm{d}s,\mathrm{d}x), \tag{2.22}$$

and its compensator is therefore $\widehat{N}_L(\mathrm{d}t,\mathrm{d}x) = \mathbb{E}N_L(\mathrm{d}t,\mathrm{d}x) = f(t,x)\,\mathrm{d}t\,\nu(\mathrm{d}x)$. Clearly, if $f(s,x)\equiv g(x)$, then L is a stationary Poisson point process. In particular, if we assume that $g(x)\equiv x$ and $\nu(\mathrm{d}x) = \lambda F(\mathrm{d}x)$, where $F(\cdot)$ is a probability measure on $\mathbb{R}$, then L is a *compound Poisson process* with jump intensity λ and jump size distribution F.

Throughout this paper, we shall assume that

$$\mathbb{E}\left\{\int_0^T |L|_t^2\,\mathrm{d}t + \mathrm{e}^{\beta|\tilde{L}|_T}\right\} < \infty, \qquad \forall\beta>0, \tag{2.23}$$

where $|L|_t \stackrel{\triangle}{=} \sum_{0\le s\le t}|\Delta L_s|$ and $|\tilde{L}|_t \stackrel{\triangle}{=} \sum_{0\le s\le t}(|\Delta L_s|\vee 1)$, $t\in[0,T]$.

Remark 2.1. We note that (2.23) contains in particular the compound Poisson case. Indeed, if $L_t = \sum_{i=1}^{N_t} U_i$, where N is a standard Poisson process with intensity $\lambda>0$, and $\{U_i\}$ are i.i.d.

random variables with finite moment generating function $M_{|U_1|}(t) \stackrel{\triangle}{=} \mathbb{E}\{e^{t|U_1|}\} < \infty$, $\forall t \geq 0$. Then we can easily calculate that

$$\begin{aligned}
&\mathbb{E}\left\{\int_0^T |L|_t^2 \,dt + e^{\beta|\tilde{L}|_T}\right\} \\
&\quad = \frac{(\lambda\mathbb{E}|U_1|)^2T^3}{3} + \frac{\lambda\mathbb{E}\{|U_1|^2\}T^2}{2} + \sum_{k=0}^{\infty}\mathbb{E}\left\{e^{\beta\sum_{i=1}^k(|U_i|\vee 1)}|N_T = k\right\}\frac{(\lambda T)^k}{k!}e^{-\lambda T} \qquad (2.24)\\
&\quad = \frac{(\lambda\mathbb{E}|U_1|)^2T^3}{3} + \frac{\lambda\mathbb{E}\{|U_1|^2\}T^2}{2} + e^{\lambda T(\mathbb{E}[e^{\beta(|U_1|\vee 1)}]-1)} < \infty.
\end{aligned}$$

We can also consider the canonical space for a given Poisson point process of class (QL). Let $\Omega^2 = \mathbb{D}([0,T])$, the space of all real-valued, càdlàg (right-continuous with left limit) functions, endowed with the Skorohod topology, and let $\mathcal{F}_t^2 \stackrel{\triangle}{=} \sigma\{\omega(\cdot \wedge t)|\omega \in \Omega^2\}$, $t \geq 0$, $\mathcal{F}^2 \stackrel{\triangle}{=} \mathcal{F}_T^2$, $\mathbb{F}^2 = \{\mathcal{F}_t^2, t \in [0,T]\}$. Let $\mathbb{P}^L$ be the law of the process L on $\mathbb{D}([0,T])$. Then, the coordinate process, by a slight abuse of notations,

$$L_t(\omega) = \omega(t), \qquad (t,\omega) \in [0,T] \times \Omega^2,$$

is a Poisson point process, defined on $(\Omega^2, \mathcal{F}^2, \mathbb{P}^L)$, whose compensated counting measure is $\widehat{N}_L(dt, dz) = \mathbb{E}[N_L(dt, dz)] = f(t,z)\nu(dz)\,dt$, where ν is a Lévy measure and (2.23) holds.

Combining the discussions above, we now consider two canonical spaces $(\Omega^1, \mathcal{F}^1, \mathbb{P}^{B^H}; \mathbb{F}^1)$ and $(\Omega^2, \mathcal{F}^2, \mathbb{P}^L; \mathbb{F}^2)$, where $\Omega^1 = \mathbb{C}([0,T])$ and $\Omega^2 = \mathbb{D}([0,T])$. We define the *fractional Wiener–Poisson space* to simply be the product space:

$$\begin{aligned}
&\Omega \stackrel{\triangle}{=} \Omega^1 \times \Omega^2; \qquad \mathcal{F} \stackrel{\triangle}{=} \mathcal{F}^1 \otimes \mathcal{F}^2;\\
&\mathbb{P} \stackrel{\triangle}{=} \mathbb{P}^{B^H} \otimes \mathbb{P}^L; \qquad \mathcal{F}_t \stackrel{\triangle}{=} \mathcal{F}_t^1 \otimes \mathcal{F}_t^2, \qquad t \in [0,T].
\end{aligned} \tag{2.25}$$

We write the element of Ω as $\omega = (\omega^1, \omega^2) \in \Omega$. Then, the two marginal coordinate processes defined by

$$B_t^H(\omega) \stackrel{\triangle}{=} \omega^1(t), \qquad L_t(\omega) \stackrel{\triangle}{=} \omega^2(t), \qquad (t,\omega) \times [0,T] \times \Omega, \tag{2.26}$$

will be the fractional Brownian motion and Poisson point process, respectively, with the given laws. Note that under our assumptions B^H and L are always independent (cf., e.g., [11], Theorem II-6.3). Also, we can assume without loss of generality that the filtration $\mathbb{F}$ is right continuous, and is augmented by all the $\mathbb{P}$-null sets so that it satisfies the *usual hypotheses*.

To end this section, we recall that if $\mathcal{X}$ is a metric space, X is a $\mathcal{X}$-valued Gaussian random variable, and $g(\cdot)$ is a seminorm on $\mathcal{X}$, such that and $\mathbb{P}(g(X) < \infty) > 0$. Then it follows from the Fernique Theorem (cf. [4]) that there exists $\varepsilon > 0$ such that $\mathbb{E}[\exp(\lambda g^2(X))] < \infty$, for all $0 < \lambda < \varepsilon$. It is then easy to see that for all $0 < \rho < 2$, one has

$$\mathbb{E}\big[\exp\big(\lambda g^\rho(X)\big)\big] < \infty, \qquad \forall\lambda > 0. \tag{2.27}$$

This fact is useful in our analysis, similar to, for example, [16].

3. The problem

In this paper, we are interested in the following stochastic differential equation with additive noise:

$$X_t = x + \int_0^t b(s, X_s)\,\mathrm{d}s + B_t^H - L_t, \qquad t \in [0, T], \tag{3.1}$$

where b is a Borel function on $[0, T] \times \mathbb{R}$, B^H is an fBM with Hurst parameter $H \in (0, 1)$ and L is a Poisson point process of class (QL), both defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$. We assume that B^H and L are both $\mathbb{F}$-adapted, and they are independent. We often consider the filtration generated by (B^H, L), denoted by $\mathbb{F}^{(B^H,L)} = \{\mathcal{F}_t^{(B^H,L)} : t \geq 0\}$ where

$$\mathcal{F}_t^{(B^H,L)} \triangleq \sigma\{(B_s^H, L_s) : 0 \leq s \leq t\}, \qquad t \geq 0, \tag{3.2}$$

and we assume that $\mathbb{F}^{(B^H,L)}$ is augmented by all the $\mathbb{P}$-null sets so that it satisfies the *usual hypotheses*. As usual, we have the following definitions of solutions to the SDE (3.1).

Definition 3.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which are defined an fBM B^H, $H \in (0, 1)$, and a Poisson point process L, independent of B^H and of class (QL). A process X defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a strong solution to (3.1) if*

(i) *X is $\mathbb{F}^{(B^H,L)}$-adapted;*
(ii) *X satisfies (3.1), $\mathbb{P}$-almost surely.*

Definition 3.2. *A seven-tuple $(\Omega, \mathcal{F}, P, \mathbb{F}, X, B^H, L)$ is called a weak solution to (3.1) if*

(i) *$(\Omega, \mathcal{F}, P; \mathbb{F})$ is a filtered probability space;*
(ii) *B^H is an $\mathbb{F}$-fBM, and L is an $\mathbb{F}$-Poisson point process of class (QL);*
(iii) *(X, B^H, L) satisfies (3.1), $\mathbb{P}$-almost surely.*

For simplicity, we often say that (X, B^H, L) (or simply X) is a weak solution to (3.1) without specifying the associated probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$ when the context is clear. It is readily seen from (3.1) that if (X, B^H, L) is a weak solution, then $\mathbb{F}^{(B^H,L)} \subseteq \mathbb{F}^X$. The well-known example of Tanaka indicates that the converse is not necessarily true, even in the case when $H = 1/2$ and $L \equiv 0$.

Throughout this paper, we shall make use of the following *standing assumptions*:

Assumption 3.1. *The function $b : [0, T] \times \mathbb{R} \mapsto \mathbb{R}$ satisfies the following assumptions for $H \in (0, 1/2)$ and $H \in (1/2, 1)$, respectively:*

(i) *If* $H < 1/2$, *then for some* $0 < \rho \le 1$ *and* $K > 0$, *it holds that*

$$\big|b(t,x)\big| \le K\big(1+|x|^{\rho}\big), \qquad \forall (t,x) \in [0,T]\times\mathbb{R}. \tag{3.3}$$

(ii) *If* $H > 1/2$, *then* b *is Hölder-*γ *continuous in* t *and Hölder-*α *in* x, *where* $\gamma > H - 1/2$, *and* $1 - \frac{1}{2H} < \alpha < 1$. *That is, for some* $K > 0$,

$$\big|b(t,x) - b(s,y)\big| \le K\big(|x-y|^{\alpha} + |t-s|^{\gamma}\big), \qquad \forall (t,x),(s,y) \in [0,T]\times\mathbb{R}. \tag{3.4}$$

Remark 3.1. (1) We note that in the case when $H < 1/2$ we do not require any regularity on the coefficient b. To discuss the well-posedness under such a weak condition on the coefficient, is only possible due to the presence of the "noises" B^H and L (see also [16] for the case when $L \equiv 0$), and it is quite different from the theory of ordinary differential equations, for example.

(2) Compared to [16], we require that b grows only sub-linearly in the case $H < 1/2$. This is due to the possible infinite jumps of L. In fact, Remark 4.1 below shows that the problem could be ill-posed if $\rho > 1/2$. Such a constraint can be removed when L has only finitely many jumps.

We end this section by making the following observation. Denote $\tilde{X} = X + L$, and

$$\tilde{b}(t,x,\omega) \stackrel{\triangle}{=} b\big(t, x - L_t(\omega)\big), \qquad (t,x,\omega) \in [0,T]\times\mathbb{R}\times\Omega.$$

Then the SDE (3.1) becomes

$$\tilde{X}_t = x + \int_0^t \tilde{b}(s,\tilde{X}_s)\,\mathrm{d}s + B_t^H, \qquad t \in [0,T]. \tag{3.5}$$

Thus the problem is reduced to the case studied by [16], except that the coefficient $\tilde{b}$ is now random. However, if we consider the problem on the canonical Wiener–Poisson space in which $(B_t^H(\omega), L_t(\omega)) = (\omega^1(t), \omega^2(t))$, $t \in [0,T]$, then we can formally consider the SDE (3.5) as one on $(\Omega^1, \mathcal{F}^1, \mathbb{P}^{B^H})$:

$$\tilde{X}_t = x + \int_0^t b^{\omega^2}(s,\tilde{X}_s)\,\mathrm{d}s + B_t^H, \qquad t \in [0,T], \tag{3.6}$$

where $b^{\omega^2}(t,x) \stackrel{\triangle}{=} b(t, x-\omega^2(t)) = \tilde{b}(t,x,\omega^2)$, for each fixed $\omega^2 \in \Omega^2$. In other words, we can apply the result of [16] to obtain the well-posedness for each $\omega^2 \in \Omega^2$, provided that the coefficient b^{ω^2} satisfies the assumptions in [16]. We should note, however, that such a seemingly simple argument is actually rather difficult to implement, especially for the weak solution case, due to some subtle measurability issues caused by the lack of regularity of b in the case $H < 1/2$, and the discontinuity of the paths of L (whence $\tilde{b}$ in the temporal variable t), in the case $H > 1/2$.

4. Existence of a weak solution ($H < 1/2$)

In this section, we shall validate the argument presented at the end of the last section, in the case $H < 1/2$. Namely, we shall prove that the SDE (3.5) possesses a weak solution, along the lines of the arguments of [16].

Recall from Assumption 3.1 that in the case $H < 1/2$ the function b satisfies (3.3). Consider the canonical Wiener–Poisson space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P} = \mathbb{P}^{B^H} \otimes \mathbb{P}^L$, with a given Hurst parameter $H \in (0, 1/2)$, a Lévy measure $\nu(\mathrm{d}z)$, and a deterministic function $f : [0, T] \times \mathbb{R} \mapsto \mathbb{R}$ so that $\widehat{N}_L(\mathrm{d}t, \mathrm{d}z) = \mathbb{E}[N_L(\mathrm{d}t, \mathrm{d}z)] = f(t, z)\nu(\mathrm{d}z)\,\mathrm{d}t$ satisfies (2.23). Let (B^H, L) be the canonical process. Define $u_t \stackrel{\triangle}{=} -b(t, B_t^H - L_t + x)$ and

$$v_t \stackrel{\triangle}{=} -K_H^{-1}\left(\int_0^{\cdot} b\big(r, B_r^H - L_r + x\big)\,\mathrm{d}r\right)(t) = K_H^{-1}\left(\int_0^{\cdot} u_r\,\mathrm{d}r\right)(t), \qquad t \in [0, T], \tag{4.1}$$

where K_H^{-1} is defined by (2.18). We have the following lemma.

Lemma 4.1. *Assume $H < 1/2$ and (3.3) is in force with $0 < \rho < 1/2$. Then the process v defined by (4.1) enjoys the following properties*:

(1) $\mathbb{P}\{v \in L^2([0, T])\} = 1$;
(2) *v satisfies the Novikov condition*:

$$\mathbb{E}\left\{\exp\left(\frac{1}{2}\int_0^T |v_t|^2\,\mathrm{d}t\right)\right\} < \infty. \tag{4.2}$$

Furthermore, if L has only finitely many jumps, then the results hold under (3.3) for any $\rho \in (0, 1]$.

Proof. (1) In what follows, we denote $C > 0$ to be a generic constant depending only on the coefficient b, the constants in Assumption 3.1, and the Hurst parameter H; and is allowed to vary from line to line. Since $H < 1/2$, and (3.3) holds, some simple computation, together with assumption (2.23), shows that

$$\begin{aligned}
\mathbb{E}\int_0^T |u_t|^2\,\mathrm{d}t &= \mathbb{E}\int_0^T \big|b\big(t, B_t^H - L_t + x\big)\big|^2\,\mathrm{d}s \le C\mathbb{E}\int_0^T \big(1 + \big|B_t^H - L_t + x\big|\big)^2\,\mathrm{d}t \\
&\le C\left[\big(1 + |x|\big)^2 T + \mathbb{E}\int_0^T \big|B_t^H\big|^2\,\mathrm{d}t + \mathbb{E}\int_0^T |L|_t^2\,\mathrm{d}t\right] \\
&= C\left[\big(1 + |x|\big)^2 T + \frac{T^{2H+1}}{2H+1} + \mathbb{E}\int_0^T |L|_t^2\,\mathrm{d}t\right] < \infty.
\end{aligned}$$

Therefore, $\int_0^T |u_t|^2\,\mathrm{d}s < \infty$, $\mathbb{P}$-a.s. Since $H < 1/2$, $[\![u]\!]^{1/2-H}$ belongs to $L^2([0, T])$, $\mathbb{P}$-a.s. as well. Thus, applying [20], Theorem 5.3, $I_{0+}^{1/2-H}[\![u]\!]^{1/2-H} \in L^q([0, T])$, $\mathbb{P}$-a.s., for some $q = \frac{2}{1-2(1/2-H)} = \frac{1}{H} > 2$. In particular, $I_{0+}^{1/2-H}[\![u]\!]^{1/2-H} \in L^2([0, T])$, $\mathbb{P}$-a.s. Let $N \subset \Omega$ be the exceptional $\mathbb{P}$-null set. Then for any $\omega \notin N$, we can apply Theorem 2.1(iii)(a) to find $h^\omega \in L^2([0, T])$ such that

$$\big[I_{0+}^{1/2-H}[\![u]\!]^{1/2-H}(\omega)\big](t) = t^{1/2-H}\big[I_{0+}^{1/2-H} h^\omega\big](t), \qquad \omega \notin N.$$

Now recall from (2.18) we see that this implies that for each $\omega \notin N$, it holds that

$$K_H^{-1}\left(\int_0^\cdot u_r(\omega)\,\mathrm{d}r\right) = I_{0+}^{1/2-H} h^\omega.$$

Thus, applying [20], Theorem 5.3, again we have $K_H^{-1}(\int_0^\cdot u_r(\cdot)\,\mathrm{d}r) \in L^q([0,T])$, $\mathbb{P}$-a.s., for some $q = \frac{2}{1-2(1/2-H)} = \frac{1}{H} > 2$. In particular, (1) holds.

(2) Using the Assumption 3.3 again we have, $\mathbb{P}$-almost surely,

$$\begin{aligned} |v_s| &= \left| s^{H-1/2} I_{0+}^{1/2-H} [\![u]\!]^{1/2-H}(s) \right| \\ &= C s^{H-1/2} \left| \int_0^s (s-r)^{-1/2-H} r^{1/2-H} b\big(r, B_r^H - L_r + x\big)\,\mathrm{d}r \right| \\ &\le C T^{1/2-H}\big(1 + |x|^\rho + \|B^H\|_\infty^\rho + |L|_T^\rho\big), \end{aligned}$$

where $\|B^H\|_\infty \stackrel{\triangle}{=} \sup_{0\le s\le T} |B_s^H|$. Note that L and B^H are independent we have

$$\begin{aligned} &\mathbb{E}\left\{\exp\left(\frac{1}{2}\int_0^T |v_t|^2\,\mathrm{d}t\right)\right\} \\ &\quad \le \mathrm{e}^{CT^{2-2H}(1+|x|^{2\rho})} \mathbb{E}\big\{\exp\big(CT^{2-2H}\|B^H\|_\infty^{2\rho}\big)\big\} \mathbb{E}\big\{\mathrm{e}^{CT^{2-2H}|L|_T^{2\rho}}\big\}. \end{aligned} \tag{4.3}$$

Note that $2\rho < 1$ by (3.3) in Assumption 3.1, we have

$$\mathbb{E}\big\{\mathrm{e}^{CT^{2-2H}|L|_T^{2\rho}}\big\} \le \mathbb{E}\big\{\mathrm{e}^{CT^{2-2H}(|L|_T+1)}\big\} < \infty, \tag{4.4}$$

thanks to (2.23). Note that $\rho < 1/2$ also guarantees that $\mathbb{E}\{\exp(CT^{2-2H}\|B^H\|_\infty^{2\rho})\} < \infty$ for all $T > 0$ with $\mathcal{X} = \mathbb{C}([0,T])$, $X = B^H$, and $g(\cdot) = \|\cdot\|_\infty$ in (2.27). This, together with (4.3) and (4.4), proves (4.2).

Finally, note that if L has only finitely many jumps, then $\Delta L_t = 0$ for all but finitely many $t \in [0,T]$. Thus (4.4) holds for all $\rho \in (0,1]$. This proof is now complete. □

Remark 4.1. We note that unlike the finite jump case (see also [16] for the continuous case) where we only assume $0 < \rho \le 1$, in general it is necessary to assume $\rho < 1/2$ to guarantee the finiteness of $\mathbb{E}\{\mathrm{e}^{|L|_T^{2\rho}}\}$. In fact, if $\rho > 1/2$, then even in the simplest standard Poisson case $L_t \equiv N_t$ we have

$$\mathbb{E}\mathrm{e}^{(N_T)^{2\rho}} = \sum_{n=0}^\infty \mathrm{e}^{n^{2\rho}} \frac{\lambda^n}{n!} \mathrm{e}^{-\lambda}.$$

If we denote $a_n = \mathrm{e}^{n^{2\rho}} \frac{\lambda^n}{n!}$, then $\ln a_n = n^{2\rho} + n\ln\lambda - \ln n!$. Since $\ln n! < n\ln n$, and

$$\lim_{n\to\infty} \frac{n\ln n}{n^{2\rho} + n\ln\lambda} = 0,$$

a simple calculation then shows that

$$\begin{aligned}\lim_{n\to\infty}\ln a_n &= \lim_{n\to\infty}\{n^{2\rho}+n\ln\lambda-\ln n!\}\\ &= \lim_{n\to\infty}\{n^{2\rho}+n\ln\lambda\}\left\{1-\frac{\ln n!}{n^{2\rho}+n\ln\lambda}\right\}=+\infty.\end{aligned}$$

That is, $a_n\to+\infty$, and consequently $\mathbb{E}e^{(N_T)^{2\rho}}=\infty$.

We can now construct a weak solution to (3.1), in the case $H<1/2$, as follows. Define

$$\tilde{B}_t^H \triangleq B_t^H-\int_0^t b\big(s, B_s^H-L_s+x\big)\,\mathrm{d}s = B_t^H+\int_0^t u_s\,\mathrm{d}s, \qquad t\in[0,T]. \tag{4.5}$$

Using the representation (2.20), we can write

$$\tilde{B}_t^H = B_t^H+\int_0^t u_s\,\mathrm{d}s=\int_0^t K_H(t,s)\,\mathrm{d}W_s+\int_0^t u_s\,\mathrm{d}s=\int_0^t K_H(t,s)\,\mathrm{d}\tilde{W}_s,$$

where

$$\tilde{W}_t = W_t+\int_0^t\left(K_H^{-1}\left(\int_0^{\cdot} u_s\,\mathrm{d}s\right)(r)\right)\mathrm{d}r=W_t+\int_0^t v_r\,\mathrm{d}r. \tag{4.6}$$

By Lemma 4.1, the process v satisfies the Novikov condition (4.2). Thus, if we define a new probability measure $\tilde{P}$ on the canonical fractional Wiener–Poisson space $(\Omega,\mathcal{F})$ by

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}P}\triangleq\exp\left\{-\int_0^T v_s\,\mathrm{d}W_s-\frac{1}{2}\int_0^T v_s^2\,\mathrm{d}s\right\}, \tag{4.7}$$

then, under $\tilde{P}$, $\tilde{W}$ is an $\mathbb{F}$-Brownian motion, and $\tilde{B}^H$ is an $\mathbb{F}$-fractional Brownian motion with Hurst parameter H (cf. Decreusefond and Üstunel [2]).

Furthermore, since B^H and L are independent, we can easily check, by following the arguments of Brownian case (cf., e.g., [21], Theorem 124, [11], Theorem II-6.3) that L_t is still a Poisson point process of class (QL) with same parameters, and is independent of $\tilde{B}^H$. We now define $X_t=x+B_t^H-L_t$, $t\in[0,T]$. Then, it follows from (4.5) that

$$\tilde{B}_t^H=(X_t-x+L_t)-\int_0^t b(t,X_s)\,\mathrm{d}s, \qquad t\in[0,T]. \tag{4.8}$$

In other words, $(\Omega,\mathcal{F},\tilde{\mathbb{P}},\mathbb{F},X,\tilde{B}^H,L)$ is a weak solution of (3.1). That is, we have proved the following theorem.

Theorem 4.1. *Assume $H<1/2$ and that the assumptions of Lemma 4.1 are in force. Then for any $T>0$, the SDE (3.1) has at least one weak solution on $[0,T]$.*

5. Existence of a weak solution ($H > 1/2$)

In this section, we study the existence of the weak solution in the case when $H > 1/2$. We note that even though the coefficient b is Hölder continuous in both variables by Assumption 3.1(ii) (3.4), the coefficient $\tilde{b}$ of the reduced SDE (3.5) will have discontinuity on the variable t, thus the Assumption 3.1(ii) is no longer valid for $\tilde{b}$, and therefore the results of [16] cannot be applied directly. We shall, however, using the same scheme as in the last section to prove the existence of the weak solution, although the arguments is much more involved.

We begin with some preparations. Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be the canonical fractional Wiener–Poisson space, and let (B^H, L) be the canonical process. For fixed $x \in \mathbb{R}$, consider again the process

$$u_t(\omega) = -b\big(t, B_t^H(\omega) - L_t(\omega) + x\big) = -b\big(t, \omega^1(t) - \omega^2(t) + x\big), \qquad (t, \omega) \in [0, T] \times \Omega,$$

and define $v_t(\omega) = K_H^{-1}(\int_0^\cdot u_r(\omega)\,\mathrm{d}r)(t)$, $(t, \omega) \in [0, T] \times \Omega$, where K_H^{-1} is given by (2.18) in the case $H > 1/2$. As in the previous section, we shall again argue that Lemma 4.1 holds. The main difference between our case and [16], however, is that the paths of u are discontinuous despite the Assumption 3.1(ii), thus the fractional calculus will need to be modified.

We first note that, by the Fubini theorem,

$$\mathbb{P}\big\{v \in L^2\big([0, T]\big)\big\} = \int_{\Omega^2} \mathbb{P}^{B^H}\left\{\int_0^T \big|v_s\big(\omega^1, \omega^2\big)\big|^2\,\mathrm{d}s < \infty\right\}\mathbb{P}^L\big(\mathrm{d}\omega^2\big).$$

Thus to show $\mathbb{P}\{v \in L^2([0, T])\} = 1$, it suffices to show that, for $\mathbb{P}^L$-a.e., $\omega^2 \in \Omega^2$, it holds that

$$\mathbb{P}^{B^H}\left\{\int_0^T \big|v_s^{\omega^2}\big(\omega^1\big)\big|^2\,\mathrm{d}s < \infty\right\} = 1,$$

where $v_s^{\omega^2}(\omega^1) \stackrel{\triangle}{=} v_s(\omega^1, \omega^2)$ is the "ω^2-section" of v_t. But in light of (2.18), we need first show that, for $\mathbb{P}^L$-a.e. $\omega^2 \in \Omega^2$, $u^{\omega^2} \in I_{0+}^{H-1/2,1/2-H}(L^1([0, T])) \cap L^1([0, T])$, $\mathbb{P}^{B^H}$-a.s., where

$$u_t^{\omega^2}(\omega_1) \stackrel{\triangle}{=} u_t\big(\omega_1, \omega^2\big) = -b^{\omega^2,x}\big(t, B_t^H(\omega_1)\big), \qquad \big(t, \omega^1\big) \in [0, T] \times \Omega^1 \tag{5.1}$$

and

$$b^{\omega^2,x}(t, y) \stackrel{\triangle}{=} b\big(t, y - \omega^2(t) + x\big), \qquad (t, y) \in [0, T] \times \mathbb{R}. \tag{5.2}$$

Since we are considering only the canonical process $L(\omega) = L(\omega^2) = \omega^2$, which is a Poisson process under $\mathbb{P}^L$ and thus does not have fixed time jumps (i.e., $\mathbb{P}^L\{\Delta L_t \neq 0\} = 0$, $\forall t \geq 0$). We can, modulo a $\mathbb{P}^L$-null set, assume without of generality that ω^2 is piecewise constant, and jumps at $0 < \sigma_1(\omega^2) < \cdots < \sigma_{N_T(\omega^2)}(\omega^2) < T$, where $N_t(\omega^2)$ denotes the number of jumps of $L(\omega^2)$ up to time $t > 0$. For notational convenience in what follows, we shall also denote $\sigma_0(\omega^2) = 0$, $\sigma_{N_T(\omega^2)+1}(\omega^2) = T$, although they do not represent jump times. Then by Assumption 3.1(ii) we see that $t \mapsto b^{\omega^2,x}(t, B_t^H)$ is μ-Hölder continuous on every interval (σ_i, σ_{i+1}), $i = 0, 1, \ldots, N_T(\omega^2)$, with $\mu = H - \frac{1}{2} + \varepsilon$ for some $\varepsilon > 0$. Thus, by virtue of Theorem 6.5

in [20], $u^{\omega^2} \in I^{H-1/2}_{\sigma_i+}(L^2(\sigma_i,\sigma_{i+1}))$, $\mathbb{P}^{B^H}$-a.s., for all $i=0,\dots,N_T(\omega^2)$. It then follows from Theorem 13.11 of [20] that $u^{\omega^2} \in I^{H-1/2}_{0+}(L^2([0,T]))$, $\mathbb{P}^{B^H}$-a.s. Therefore, there exists a $\mathbb{P}^{B^H}$-null set $N \subset \Omega^1$, so that for any $\omega^1 \notin N$, we can apply Theorem 2.1(iii)(a) or Lemma 3.2 in [20] to find a function $h^{\omega^1,\omega^2} \in L^2([0,T])$, such that:

$$[\![u^{\omega^2}]\!]^{1/2-H}(t,\omega^1) = t^{1/2-H}u^{\omega^2}_t(\omega^1) = I^{H-1/2}_{0+}t^{1/2-H}h^{\omega^1,\omega^2}(t), \qquad t \in [0,T].$$

That is, $u^{\omega^2} \in I^{H-1/2,1/2-H}_{0+}(L^1([0,T]))$, $\mathbb{P}^{B^H}$-a.s. On the other hand, since $u^{\omega^2} \in I^{H-1/2}_{0+}(L^2[0,T])$ implies $u^{\omega^2} \in L^2([0,T])$, thanks to Theorem 5.3 of [20], we conclude that (2.18) holds with $h(\cdot) = \int_0^\cdot u_r\,\mathrm{d}r$, $\mathbb{P}^{B^H}$-a.s. That is, $v_t = K_H^{-1}(\int_0^\cdot u_r\,\mathrm{d}r)(t)$, $t \in [0,T]$, belongs to $L^2([0,T])$, $\mathbb{P}^{B^H}$-a.s. Note that the argument is valid for $\mathbb{P}^L$-a.e. $\omega^2 \in \Omega^2$, we obtain that $\mathbb{P}\{v \in L^2([0,T])\} = 1$. We now prove an analogue of Lemma 4.1 for the case $H > 1/2$.

Lemma 5.1. *Assume that $H > 1/2$, and that Assumption 3.1*(ii) *holds with $1 - \frac{1}{2H} < \alpha < 1 - H$. Then the conclusion of Lemma 4.1 remains valid.*

Furthermore, if L has only finitely many jumps, then the constraint $\alpha < 1 - H$ can be removed.

Proof. We have already argued that the process $v_t = K_H^{-1}(\int_0^\cdot u_r\,\mathrm{d}r)(t)$, $t \in [0,T]$, satisfies $\mathbb{P}\{v \in L^2([0,T])\} = 1$ in the beginning of this section. We shall show that the process v also satisfies the Novikov condition (4.2), whence part (2) of Lemma 4.1.

To this end, first note that on the canonical space $\Omega^2 = \mathbb{D}([0,T])$, and under the probability $\mathbb{P}^L$, the canonical process $L(\omega) = \omega^2$ is a Poisson point process of class (QL). Now, for fixed $T > 0$, denote $\Omega^2_n \stackrel{\triangle}{=} \{\omega^2 : N_T(\omega^2) = n\}$ for $n = 0,1,\dots$; and for $\omega^2 \in \Omega^2_n$, again denote $0 < \sigma_1(\omega^2) < \cdots < \sigma_n(\omega^2) < T$ be the jump times of $L(\omega^2)$, and $\sigma_0(\omega^2) = 0$, $\sigma_{n+1}(\omega^2) = T$. Finally, denote $S_k(\omega^2) \stackrel{\triangle}{=} \sum_{i=1}^k \Delta L_{\sigma_i}(\omega^2)$, $k = 1,2,\dots$, and $S_0(\omega^2) = 0$. In what follows, we often suppress the variable ω^2 when the context is clear.

Now recall from (2.18) that, for $H > 1/2$,

$$v^{\omega^2}_t = K_H^{-1}\left(\int_0^\cdot u^{\omega^2}_r\,\mathrm{d}r\right)(t) = t^{H-1/2}D^{H-1/2}_{0+}[\![u^{\omega^2}]\!]^{1/2-H}(t), \qquad t \in [0,T]. \tag{5.3}$$

We shall calculate $D^{H-1/2}_{0+}[\![u^{\omega^2}]\!]^{1/2-H}$ for $\omega^2 \in \Omega^2_n$, for each $n = 0,1,2,\dots$. To see this, fix $n \in \mathbb{N}$, and let $\omega^2 \in \Omega^2_n$. For notational simplicity, in what follows we denote

$$u^{\omega^2,k}_t(\omega^1) = -b\bigl(t, B^H_t(\omega^1) - S_{k-1}(\omega^2) + x\bigr), \qquad \bigl(t,\omega^1\bigr) \in [0,T] \times \Omega^1, k \geq 1, \tag{5.4}$$

so that $u^{\omega^2}_t = \sum_{k=1}^{n+1} u^{\omega^2,k}_t \mathbf{1}_{[\sigma_{k-1}(\omega^2),\sigma_k(\omega^2))}(t)$, $t \in [0,T]$, $\mathbb{P}^1$-a.s. Then, for $t \in [0,\sigma_1(\omega^2))$, by definition (2.7) and (2.8) with $p = 2$ we have

$$D^{H-1/2}_{0+}[\![u^{\omega^2}]\!]^{1/2-H}(t)$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(3/2-H)} \frac{[\![u^{\omega^2,1}]\!]^{1/2-H}(t)}{t^{H-1/2}} \\
&\quad + \frac{H-1/2}{\Gamma(3/2-H)} \int_0^t \frac{[\![u^{\omega^2,1}]\!]^{1/2-H}(t) - [\![u^{\omega^2,1}]\!]^{1/2-H}(r)}{(t-r)^{H+1/2}} \, \mathrm{d}r \\
&\triangleq \Phi_1(t).
\end{aligned} \tag{5.5}
$$

Similarly, for $\sigma_{k-1}(\omega^2) \le t < \sigma_k(\omega^2)$ with $1 < k \le n+1$, we have

$$
\begin{aligned}
&D_{0+}^{H-1/2} [\![u^{\omega^2}]\!]^{1/2-H}(t) \\
&\quad = \frac{1}{\Gamma(3/2-H)} \frac{[\![u^{\omega^2}]\!]^{1/2-H}(t)}{t^{H-1/2}} \\
&\qquad + \frac{H-1/2}{\Gamma(3/2-H)} \int_0^t \frac{[\![u^{\omega^2}]\!]^{1/2-H}(t) - [\![u^{\omega^2}]\!]^{1/2-H}(r)}{(t-r)^{H+1/2}} \, \mathrm{d}r \\
&\quad = \frac{1}{\Gamma(3/2-H)} \frac{[\![u^{\omega^2,k}]\!]^{1/2-H}(t)}{t^{H-1/2}} \\
&\qquad + \frac{H-1/2}{\Gamma(3/2-H)} \sum_{i=1}^{k-1} \int_{\sigma_{i-1}}^{\sigma_i} \frac{[\![u^{\omega^2,k}]\!]^{1/2-H}(t) - [\![u^{\omega^2,i}]\!]^{1/2-H}(r)}{(t-r)^{H+1/2}} \, \mathrm{d}r \\
&\qquad + \frac{H-1/2}{\Gamma(3/2-H)} \int_{\sigma_{k-1}}^t \frac{[\![u^{\omega^2,k}]\!]^{1/2-H}(t) - [\![u^{\omega^2,k}]\!]^{1/2-H}(r)}{(t-r)^{H+1/2}} \, \mathrm{d}r \triangleq \Phi_k(t).
\end{aligned} \tag{5.6}
$$

Consequently, we obtain the following formula:

$$
D_{0+}^{H-1/2} [\![u^{\omega^2}]\!]^{1/2-H}(t) = \sum_{k=1}^{n+1} \Phi_k(t) \mathbf{1}_{[\sigma_{k-1}(\omega^2), \sigma_k(\omega^2))}(t), \qquad t \in [0,T), \mathbb{P}^1\text{-a.s.} \tag{5.7}
$$

That is,

$$
v_t^{\omega^2} = t^{H-1/2} D_{0+}^{H-1/2} [\![u^{\omega^2}]\!]^{1/2-H}(t) = t^{H-1/2} \sum_{i=1}^{n+1} \Phi_k(t) \mathbf{1}_{[\sigma_{k-1}(\omega^2) < t \le \sigma_k(\omega^2))}(t), \tag{5.8}
$$

where Φ_k's are defined by (5.5) and (5.6). We now estimate each term in (5.8). Note that for $t \in [\sigma_{k-1}, \sigma_k)$ we have

$$
\begin{aligned}
&\frac{H-1/2}{\Gamma(3/2-H)} \sum_{i=1}^{k-1} \int_{\sigma_{i-1}}^{\sigma_i} \frac{[\![u^{\omega^2,k}]\!]^{1/2-H}(t)}{(t-r)^{H+1/2}} \, \mathrm{d}r \\
&\quad = \frac{1}{\Gamma(3/2-H)} \left\{ \frac{1}{(t-\sigma_{k-1})^{H-1/2}} - \frac{1}{t^{H-1/2}} \right\} [\![u^{\omega^2,k}]\!]^{1/2-H}(t).
\end{aligned}
$$

It then follows from (5.6) that, for $t \in [\sigma_{k-1}, \sigma_k)$,

$$\begin{aligned}
t^{H-1/2}\Phi_k(t) &= t^{H-1/2}\bigg\{\frac{1}{\Gamma(3/2-H)}\frac{[\![u^{\omega^2,k}]\!]^{1/2-H}(t)}{t^{H-1/2}} \\
&\quad + \frac{H-1/2}{\Gamma(3/2-H)}\sum_{i=1}^{k-1}\int_{\sigma_{i-1}}^{\sigma_i}\frac{[\![u^{\omega^2,k}]\!]^{1/2-H}(t)-[\![u^{\omega^2,i}]\!]^{1/2-H}(r)}{(t-r)^{H+1/2}}\,\mathrm{d}r \\
&\quad + \frac{H-1/2}{\Gamma(3/2-H)}\int_{\sigma_{k-1}}^{t}\frac{[\![u^{\omega^2,k}]\!]^{1/2-H}(t)-[\![u^{\omega^2,k}]\!]^{1/2-H}(r)}{(t-r)^{H+1/2}}\,\mathrm{d}r\bigg\} \\
&= C_1^H\frac{t^{H-1/2}[\![u^{\omega^2,k}]\!]^{1/2-H}(t)}{(t-\sigma_{k-1})^{H-1/2}} - C_2^H t^{H-1/2}\sum_{i=1}^{k-1}\int_{\sigma_{i-1}}^{\sigma_i}\frac{[\![u^{\omega^2,i}]\!]^{1/2-H}(t)}{(t-r)^{H+1/2}}\,\mathrm{d}r \\
&\quad + C_2^H t^{H-1/2}\sum_{i=1}^{k-1}\int_{\sigma_{i-1}}^{\sigma_i}\frac{[\![u^{\omega^2,i}]\!]^{1/2-H}(t)-[\![u^{\omega^2,i}]\!]^{1/2-H}(r)}{(t-r)^{H+1/2}}\,\mathrm{d}r \\
&\quad + C_2^H t^{H-1/2}\int_{\sigma_{k-1}}^{t}\frac{[\![u^{\omega^2,k}]\!]^{1/2-H}(t)-[\![u^{\omega^2,k}]\!]^{1/2-H}(r)}{(t-r)^{H+1/2}}\,\mathrm{d}r \triangleq A^k(t)+B^k(t),
\end{aligned}$$

where $C_1^H \triangleq \frac{1}{\Gamma(3/2-H)}$, $C_2^H \triangleq \frac{H-1/2}{\Gamma(3/2-H)} = (H-1/2)C_1^H$, and

$$A^k(t) \triangleq C_1^H\frac{t^{H-1/2}[\![u^{\omega^2,k}]\!]^{1/2-H}(t)}{(t-\sigma_{k-1})^{H-1/2}} - C_2^H t^{H-1/2}\sum_{i=1}^{k-1}\int_{\sigma_{i-1}}^{\sigma_i}\frac{[\![u^{\omega^2,i}]\!]^{1/2-H}(t)}{(t-r)^{H+1/2}}\,\mathrm{d}r, \tag{5.9}$$

$$\begin{aligned}
B^k(t) &\triangleq C_2^H t^{H-1/2}\sum_{i=1}^{k-1}\int_{\sigma_{i-1}}^{\sigma_i}\frac{[\![u^{\omega^2,i}]\!]^{1/2-H}(t)-[\![u^{\omega^2,i}]\!]^{1/2-H}(r)}{(t-r)^{H+1/2}}\,\mathrm{d}r \\
&\quad + C_2^H t^{H-1/2}\int_{\sigma_{k-1}}^{t}\frac{[\![u^{\omega^2,k}]\!]^{1/2-H}(t)-[\![u^{\omega^2,k}]\!]^{1/2-H}(r)}{(t-r)^{H+1/2}}\,\mathrm{d}r.
\end{aligned} \tag{5.10}$$

It is readily seen that (suppressing $\omega = (\omega^1, \omega^2)$'s)

$$\begin{aligned}
|A^k(t)| &= \bigg|C_1^H\sum_{i=1}^{k-1} b\big(t, B_t^H - S_{i-1}+x\big)\bigg[\frac{1}{(t-\sigma_{i-1})^{H-1/2}} - \frac{1}{(t-\sigma_i)^{H-1/2}}\bigg] \\
&\qquad + C_1^H\frac{b(t, B_t^H - S_{k-1}+x)}{(t-\sigma_{k-1})^{H-1/2}}\bigg| \\
&\le \bigg|C_1^H\sum_{i=1}^{k-1}\big[b\big(t, B_t^H - S_{i-1}+x\big) - b\big(t, B_t^H + x\big)\big]\bigg[\frac{1}{(t-\sigma_{i-1})^{H-1/2}} - \frac{1}{(t-\sigma_i)^{H-1/2}}\bigg]
\end{aligned}$$

$$
\begin{aligned}
&\qquad + C_1^H \frac{(b(t, B_t^H - S_{k-1} + x) - b(t, B_t^H + x)}{(t-\sigma_{k-1})^{H-1/2}} \Bigg| \\
&\qquad + \Bigg| C_1^H \sum_{i=1}^{k-1} b\big(t, B_t^H + x\big) \bigg[\frac{1}{(t-\sigma_{i-1})^{H-1/2}} - \frac{1}{(t-\sigma_i)^{H-1/2}} \bigg] \\
&\qquad\quad + C_1^H \frac{b(t, B_t^H + x)}{(t-\sigma_{k-1})^{H-1/2}} \Bigg| \\
&\le C_1^H \max_{1 \le i \le k} \big| b\big(t, B_t^H - S_{i-1} + x\big) - b\big(t, B_t^H + x\big) \big| \\
&\qquad \times \Bigg| \sum_{i=1}^{k-1} \bigg[\frac{1}{(t-\sigma_i)^{H-1/2}} - \frac{1}{(t-\sigma_{i-1})^{H-1/2}} \bigg] + \frac{1}{(t-\sigma_{k-1})^{H-1/2}} \Bigg| \\
&\qquad + C_1^H t^{1/2-H} \big| b\big(t, B_t^H + x\big) \big| \\
&\le C(t-\sigma_{k-1})^{1/2-H} |L|_T^\alpha + C t^{1/2-H} \big(\big| b(0,x) \big| + |t|^\gamma + \| B^H \|_\infty^\alpha \big),
\end{aligned}
\tag{5.11}
$$

where C is a generic constant depending on H, α, and K, thanks to Assumption 3.1. On the other hand, we write $B^k(t) = -C(B_1^k(t) + B_2^k(t))$, where

$$
\begin{aligned}
B_1^k(t) = t^{H-1/2} \sum_{i=1}^{k-1} \bigg[& b\big(t, B_t^H - S_{i-1} + x\big) \int_{\sigma_{i-1}}^{\sigma_i} \frac{t^{1/2-H} - r^{1/2-H}}{(t-r)^{1/2+H}} \, \mathrm{d}r \\
& + \int_{\sigma_{i-1}}^{\sigma_i} \frac{b(t, B_t^H - S_{i-1} + x) - b(r, B_t^H - S_{i-1} + x)}{(t-r)^{1/2+H}} r^{1/2-H} \, \mathrm{d}r \bigg] \\
& + t^{H-1/2} b\big(t, B_t^H - S_{k-1} + x\big) \int_{\sigma_{k-1}}^{t} \frac{t^{1/2-H} - r^{1/2-H}}{(t-r)^{1/2+H}} \, \mathrm{d}r \\
& + t^{H-1/2} \int_{\sigma_{k-1}}^{t} \frac{b(t, B_t^H - S_{k-1} + x) - b(r, B_t^H - S_{k-1} + x)}{(t-r)^{1/2+H}} r^{1/2-H} \, \mathrm{d}r,
\end{aligned}
$$

and

$$
\begin{aligned}
B_2^k(t) = t^{H-1/2} \sum_{i=1}^{k-1} & \int_{\sigma_{i-1}}^{\sigma_i} \frac{b(r, B_t^H - S_{i-1} + x) - b(r, B_r^H - S_{i-1} + x)}{(t-r)^{1/2+H}} r^{1/2-H} \, \mathrm{d}r \\
& + t^{H-1/2} \int_{\sigma_{k-1}}^{t} \frac{b(r, B_t^H - S_{k-1} + x) - b(r, B_r^H - S_{k-1} + x)}{(t-r)^{1/2+H}} r^{1/2-H} \, \mathrm{d}r.
\end{aligned}
$$

Then, it is easy to see that, for each fixed $0<\varepsilon<H-\frac{H-1/2}{\alpha}$ (recall Assumption 3.1(ii)), and denoting $G \triangleq \sup_{0\le t<r\le T}\frac{|B^H_t-B^H_r|}{|t-r|^{H-\varepsilon}}$, we have

$$
\begin{aligned}
\left|B_2^k(t)\right| &\le t^{H-1/2}\sum_{i=1}^{k-1}\int_{\sigma_{i-1}}^{\sigma_i}\frac{|B^H_t-B^H_r|^\alpha}{(t-r)^{1/2+H}}r^{1/2-H}\,\mathrm{d}r \\
&\quad + t^{H-1/2}\int_{\sigma_{k-1}}^{t}\frac{|B^H_t-B^H_r|^\alpha}{(t-r)^{1/2+H}}r^{1/2-H}\,\mathrm{d}r \\
&= t^{H-1/2}\int_0^t\frac{|B^H_t-B^H_r|^\alpha}{(t-r)^{1/2+H}}r^{1/2-H}\,\mathrm{d}r \le Ct^{1/2-H+\alpha(H-\varepsilon)}G^\alpha.
\end{aligned}
\tag{5.12}
$$

Furthermore, by the same argument as in (5.11) we also have

$$
\begin{aligned}
&\left|B_1^k(t)\right| = t^{H-1/2}\max_{1\le i\le k}\left|b\left(t,B^H_t-S_{i-1}+x\right)\right| \\
&\qquad\times\left[\sum_{i=1}^{k-1}\int_{\sigma_{i-1}}^{\sigma_i}\frac{r^{1/2-H}-t^{1/2-H}}{(t-r)^{1/2+H}}\,\mathrm{d}r+\int_{\sigma_{k-1}}^{t}\frac{r^{1/2-H}-t^{1/2-H}}{(t-r)^{1/2+H}}\,\mathrm{d}r\right] \\
&\qquad + Kt^{H-1/2}\left[\sum_{i=1}^{k-1}\int_{\sigma_{i-1}}^{\sigma_i}\frac{|t-r|^\gamma}{(t-r)^{1/2+H}}r^{1/2-H}\,\mathrm{d}r+\int_{\sigma_{k-1}}^{t}\frac{|t-r|^\gamma}{(t-r)^{1/2+H}}r^{1/2-H}\,\mathrm{d}r\right] \\
&\quad\le\left[\left|b(0,x)\right|+K\left(|t|^\gamma+\left|B^H_t\right|^\alpha+|L_T|^\alpha\right)\right]t^{H-1/2}\int_0^t\frac{r^{1/2-H}-t^{1/2-H}}{(t-r)^{1/2+H}}\,\mathrm{d}r \\
&\qquad + Kt^{H-1/2}\int_0^t\frac{|t-r|^\gamma}{(t-r)^{1/2+H}}r^{1/2-H}\,\mathrm{d}r \\
&\quad\le C\left\{\left[\left|b(0,x)\right|+|t|^\gamma+\left|B^H_t\right|^\alpha+|L_T|^\alpha\right]t^{1/2-H}+t^{\gamma+1/2-H}\right\} \\
&\quad\le Ct^{1/2-H}\left[\left|b(0,x)\right|+|t|^\gamma+\left\|B^H\right\|_\infty^\alpha+|L|_T^\alpha\right].
\end{aligned}
\tag{5.13}
$$

Combining (5.12) and (5.13), we have for any $t\in[0,T]$,

$$
\left|B^k(t)\right|\le Ct^{1/2-H}\left[\left|b(0,x)\right|+|t|^\gamma+\left\|B^H\right\|_\infty^\alpha+|L|_T^\alpha+t^{\alpha(H-\varepsilon)}G^\alpha\right].
\tag{5.14}
$$

Now, combining (5.11) and (5.14), and denoting $\mathbb{E}_n[\cdot]=\mathbb{E}[\cdot|N_T=n]$, we have

$$
\begin{aligned}
&\mathbb{E}\left\{\exp\left\{\frac12\int_0^T v^2(t)\,\mathrm{d}t\right\}\right\} \\
&\quad=\sum_{n=0}^\infty\mathbb{E}_n\left\{\exp\left\{\frac12\sum_{k=1}^n\int_{\sigma_{k-1}}^{\sigma_k}t^{2H-1}\Phi_k^2(t)\,\mathrm{d}t\right.\right.
\end{aligned}
\tag{5.15}
$$

$$+\frac{1}{2}\int_{\sigma_n}^{T} t^{2H-1}\Phi_{n+1}^2(t)\,\mathrm{d}t\Big\}\Big\}P(N_T=n)$$

$$=\sum_{n=0}^{\infty}\mathbb{E}_n\left\{\exp\left\{C\sum_{k=1}^{n+1}\int_{\sigma_{k-1}}^{\sigma_k}\big(|A^k(t)|+|B^k(t)|\big)^2\,\mathrm{d}t\right\}\right\}P(N_T=n).$$

By (5.11) and (5.14) and using the fact

$$\frac{\sum_{i=1}^{n+1}x_i^{2-2H}}{n+1}\le\left(\frac{\sum_{i=1}^{n+1}x_i}{n+1}\right)^{2-2H},\qquad x_i>0,$$

we have

$$\begin{aligned}
&\sum_{k=1}^{n+1}\int_{\sigma_{k-1}}^{\sigma_k}\big(\big|A^k(t)\big|+\big|B^k(t)\big|\big)^2\,\mathrm{d}t\\
&\quad\le C\sum_{k=1}^{n+1}\int_{\sigma_{k-1}}^{\sigma_k}(t-\sigma_{k-1})^{1-2H}|L|_T^{2\alpha}\,\mathrm{d}t\\
&\qquad+C\int_0^T t^{1-2H}\big(\big|b^2(0,x)\big|+|t|^{2\gamma}+\big\|B^H\big\|_\infty^{2\alpha}+t^{2\alpha(H-\varepsilon)}G^{2\alpha}\big)\,\mathrm{d}t\\
&\quad\le C\sum_{k=1}^{n+1}(\sigma_k-\sigma_{k-1})^{2-2H}|L|_T^{2\alpha}\\
&\qquad+C\int_0^T t^{1-2H}\big(\big|b^2(0,x)\big|+|t|^{2\gamma}+\big\|B^H\big\|_\infty^{2\alpha}+t^{2\alpha(H-\varepsilon)}G^{2\alpha}\big)\,\mathrm{d}t\\
&\quad\le C(n+1)^{2H-1}|L|_T^{2\alpha}+C\big[1+\big\|B^H\big\|_\infty^{2\alpha}+G^{2\alpha}\big].
\end{aligned}\tag{5.16}$$

Putting (5.16) into (5.15), we obtain

$$\begin{aligned}
&\mathbb{E}\big\{\mathrm{e}^{1/2\int_0^T v^2(t)\,\mathrm{d}t}\big\}\\
&\quad\le\mathbb{E}\big\{\exp\big\{C\big[1+\big\|B^H\big\|_\infty^{2\alpha}+G^{2\alpha}\big]\big\}\big\}\mathbb{E}\big\{\exp\big\{C(N_T+1)^{2H-1}|L|_T^{2\alpha}\big\}\big\}.
\end{aligned}\tag{5.17}$$

By the same argument as Lemma 4.1, it is easy to prove that $\mathbb{E}\{\mathrm{e}^{C\|B^H\|_\infty^{2\alpha}+G^{2\alpha}}\}<\infty$.

We need to show that $\mathbb{E}\{\mathrm{e}^{C(N_T+1)^{2H-1}|L|_T^{2\alpha}}\}<\infty$. Note that $\alpha<1-H$ in Assumption 3.1(ii) implies that $2H-1+2\alpha<1$, and recall $\tilde{L}$ from (2.23), we have

$$\begin{aligned}
\mathbb{E}\exp\big\{C(N_T+1)^{2H-1}|L|_T^{2\alpha}\big\}&\le\mathbb{E}\exp\left\{C\left(\sum_{i=1}^{N_T}\big(|\Delta L_{\sigma_i}|\vee 1\big)+1\right)^{2H-1+2\alpha}\right\}\\
&\le\mathbb{E}\exp\left\{C\left(\sum_{i=1}^{N_T}|\Delta\tilde{L}_{\sigma_i}|+1\right)\right\}<\infty.
\end{aligned}\tag{5.18}$$

Therefore, we can show that $\mathbb{E}\{e^{1/2\int_0^T v^2(t)\,dt}\} < \infty$.

Finally, note that if L has only finitely many jumps, then $|\Delta L_{\sigma_i}| = 0$ for all but finitely many i's. Thus, (5.18) always holds for any $\alpha > 0$. The proof is complete. □

Remark 5.1. We observe that $1 - \frac{1}{2H} < \alpha < 1 - H$ implies $H < \frac{\sqrt{2}}{2}$. This is again due to the presence of possible infinite number of jumps. We note that a similar constraint $H < \frac{1+\sqrt{5}}{4}$ was also placed in [15], where only finitely many jumps were considered. But in that case we need only $1 - \frac{1}{2H} < \alpha < 1$, thus our result is still much stronger than that of [15].

We have the following analogues of Theorem 4.1.

Theorem 5.1. *Assume $H > 1/2$ and that the assumptions in Lemma 5.1 are in force. Then the SDE (3.1) has at least one weak solution on* $[0, T]$.

6. Uniqueness in law and pathwise uniqueness

In this section, we study the uniqueness of the weak solution. We shall first show that the weak solutions to (3.1) are unique in law. The argument is very similar to that of [16], we describe it briefly.

Let (X, B^H, L) be a weak solutions of (3.1), defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$, with the existence interval $[0, T]$. Let W be the $\mathbb{F}$-Brownian motion such that

$$B_t^H = \int_0^t K_H(t,s)\,dW_s, \qquad t \in [0,T]. \tag{6.1}$$

Define

$$v_t = K_H^{-1}\left(\int_0^{\cdot} b(r, X_r)\,dr\right)(t), \qquad t \in [0,T], \tag{6.2}$$

and let us assume that v satisfies the assumption (1) and (2) in Lemma 4.1. Then applying the Girsanov theorem we see that the process $\tilde{W}_t = W_t + \int_0^t v_s\,ds$, $t \in [0,T]$, is an $\mathbb{F}$-Brownian motion under the new probability measure $\tilde{\mathbb{P}}$, defined by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \xi_T(X) \triangleq \exp\left\{-\int_0^T v_t\,dW_t - \frac{1}{2}\int_0^T |v_t|^2\,dt\right\}. \tag{6.3}$$

Thus $\tilde{B}_t^H \triangleq \int_0^t K_H(t,s)\,d\tilde{W}_s$, $t \in [0,T]$, is an fBM under $\tilde{\mathbb{P}}$, and it holds that

$$X_t + L_t - x = \int_0^t b(s, X_s)\,ds + B_t^H = \int_0^t K_H(t,s)\,d\tilde{W}_s = \tilde{B}_t^H, \qquad t \in [0,T].$$

Since under the Girsanov transformation the process L remains a Poisson point process with the same parameters, and is automatically independent of the Brownian motion $\tilde{W}$ under $\tilde{\mathbb{P}}$ (cf. [11], Theorem II-6.3), we can then write X as the independent sum of $\tilde{B}^H$ and $-L$:

$$X_t = x + \tilde{B}_t^H - L_t, \qquad t \in [0, T].$$

Since the argument above can be applied to any weak solution, we have essentially proved the following weak uniqueness result.

Theorem 6.1. *Suppose that the assumptions of Lemma* 4.1 (*resp. Lemma* 5.1) *for* $H < 1/2$ (*resp.* $H > 1/2$) *are in force. Then two weak solutions of SDE* (3.1) *must have the same law, over their common existence interval* $[0, T]$.

Proof. We need only to show that the adapted process v defined by (6.2) satisfies (1) and (2) in Lemma 4.1. In what follows we let $C > 0$ denote a generic constant depending only on the constants H, K, α, γ in Assumption 3.1 and $T > 0$, and is allowed to vary from line to line. In the case $H < \frac{1}{2}$, denoting $u = b(\cdot, X_\cdot)$, for any $t \in [0, T]$ we have

$$\begin{aligned}
\mathbb{E}\int_0^t |u_r|^2\,\mathrm{d}r &= \mathbb{E}\int_0^t \big|b(r, X_r)\big|^2\,\mathrm{d}r \le C\mathbb{E}\int_0^t \big(1 + |X_r|^2\big)\,\mathrm{d}r \\
&\le C\mathbb{E}\int_0^t \left[1 + |x|^2 + \left|\int_0^r b(s, X_s)\,\mathrm{d}s\right|^2 + \big|B_r^H\big|^2 + |L_r|^2\right]\mathrm{d}r \\
&\le C\left\{\mathbb{E}\int_0^t r\int_0^r |u_s|^2\,\mathrm{d}s\,\mathrm{d}r + \big(1 + |x|^2\big)t + \frac{t^{2H+1}}{2H+1} + \mathbb{E}\int_0^T |L|_T^2\,\mathrm{d}r\right\} \\
&\le C_L\left\{\big(1 + |x|^2\big) + \int_0^t \mathbb{E}\int_0^r |u_s|^2\,\mathrm{d}s\,\mathrm{d}r\right\},
\end{aligned}$$

where $C_L > 0$ depends on C and L, thanks to (2.23). Thus by Growall's inequality, we obtain

$$\mathbb{E}\int_0^T |u_s|^2\,\mathrm{d}s = \mathbb{E}\int_0^T \big|b(s, X_s)\big|^2\,\mathrm{d}s \le C_L\big(1 + |x|^2\big)\mathrm{e}^{C_L T} < \infty.$$

Then, by the same argument as Lemma 4.1, we can check that $v = K_H^{-1}(\int_0^\cdot u_r\,\mathrm{d}r)$ satisfies (1) of Lemma 4.1. Furthermore, similarly to the proof Lemma 4.1 we can obtain that

$$|v_s| \le C_L T^{1/2-H}\big(1 + \|X\|_\infty^\rho\big),$$

where $\|X\|_\infty \stackrel{\triangle}{=} \sup_{0\le s\le T} |X_s|$. Applying Grownall's inequality again it is easy to show that

$$\|X\|_\infty \le \big(|x| + \big\|B^H\big\|_\infty + C_L T + |L|_T\big)\mathrm{e}^{C_L T}, \tag{6.4}$$

which then leads to (2) of Lemma 4.1.

We now assume $H > \frac{1}{2}$. Following the same argument of Lemma 5.1, it suffices to show that between two jump times of L, the process $u = b(\cdot, X_\cdot) \in I_{\sigma_{k-1}+}^{H-1/2}(L^2([\sigma_{k-1}, \sigma_k]))$, $\mathbb{P}$-almost surely. But note that between two jumps we have, by Assumption 3.1,

$$
\begin{aligned}
&\big|b(t,X_t)-b(s,X_s)\big| \\
&\quad\le C\big\{|t-s|^\gamma+|X_t-X_s|^\alpha\big\} \\
&\quad\le C\left\{|t-s|^\gamma+\left|\int_s^t b(u,X_u)\,\mathrm{d}u\right|^\alpha+\big|B_t^H-B_s^H\big|^\alpha\right\} \\
&\quad\le C\left\{|t-s|^\gamma+\left|\int_s^t \big(\big|b(0,x)\big|+|u|^\gamma+|X_u-x|^\alpha\big)\,\mathrm{d}u\right|^\alpha+\big|B_t^H-B_s^H\big|^\alpha\right\} \\
&\quad\le C\big\{|t-s|^\gamma+\big(\big|b(0,x)\big|+|T|^\gamma+\|X\|_\infty^\alpha+|x|^\alpha\big)|t-s|^\alpha+\big|B_t^H-B_s^H\big|^\alpha\big\}.
\end{aligned}
$$

Since $\gamma > H-\frac{1}{2}$ and $\alpha > 1-\frac{1}{2H} > H-\frac{1}{2}$, we see that between jumps the paths $t \mapsto b(t,X_t)$ are Hölder continuous of order $H-\frac{1}{2}+\varepsilon$ for some $\varepsilon>0$. By the same argument as in Section 4, it can be checked that $\mathbb{P}\{v \in L^2([0,T])\}=1$. Using the estimates

$$
\big|b(t,X_t)\big| \le C\big(\big|b(0,x)\big|+t^\gamma+|X_t-x|^\alpha\big)
$$

and $\|X\|_\infty \le C(1+|x|+\|B^H\|_\infty+|L|_T)$, we deduce that, for any $0\le r<t\le T$,

$$
\left|\int_r^t |u_s|\,\mathrm{d}s\right|^\alpha \le C\big(\big|b(0,x)\big|+t^\gamma+|x|^\alpha+\|X\|_\infty^\alpha\big)^\alpha (t-r)^\alpha. \tag{6.5}
$$

In particular, we have

$$
\begin{aligned}
\left|\int_0^t |u_s|\,\mathrm{d}s\right|^\alpha &\le C\big(\big|b(0,x)\big|+t^\gamma+|x|^\alpha+\|X\|_\infty^\alpha\big)^\alpha t^\alpha \\
&\le C\big(1+\big|b(0,x)\big|+|t|^\gamma+|x|^\alpha+\big(|x|+\big\|B^H\big\|_\infty+|L|_T\big)\big)^\alpha T^\alpha \qquad (6.6)\\
&\le C\big[1+\big|b(0,x)\big|^\alpha+t^{\alpha\gamma}+|x|^\alpha+\big\|B^H\big\|_\infty^\alpha+|L|_T^\alpha\big].
\end{aligned}
$$

Furthermore, one can also check that, by applying (6.5) and (6.6), respectively,

$$
\begin{aligned}
\big|A^k(t)\big| &\le C_1^H \max_{1\le i\le k}\left|b\left(t,B_t^H+\int_0^t u_s\,\mathrm{d}s-S_{i-1}+x\right)-b\left(t,B_t^H+\int_0^t u_s\,\mathrm{d}s+x\right)\right| \\
&\qquad\times\left|\sum_{i=1}^{k-1}\left[\frac{1}{(t-\sigma_i)^{H-1/2}}-\frac{1}{(t-\sigma_{i-1})^{H-1/2}}\right]+\frac{1}{(t-\sigma_{k-1})^{H-1/2}}\right| \\
&\qquad+C_1^H t^{1/2-H}\left|b\left(t,B_t^H+\int_0^t u_s\,\mathrm{d}s+x\right)\right| \\
&\le C\Bigg\{(t-\sigma_{k-1})^{1/2-H}|L|_T^\alpha \qquad (6.7)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + t^{1/2-H}\left(\left|b(0,x)\right| + |t|^\gamma + \|B^H\|_\infty^\alpha + \left|\int_0^t |u_s|\,\mathrm{d}s\right|^\alpha\right)\Bigg\} \\
&\le C\big\{(t-\sigma_{k-1})^{1/2-H}|L|_T^\alpha \\
&\quad + t^{1/2-H}\|B^H\|_\infty^\alpha + t^{1/2-H}\big[1+|x|+\left|b(0,x)\right| + |t|^\gamma + |L|_T^\alpha\big]\big\} \\
&\le C\big\{(t-\sigma_{k-1})^{1/2-H}|L|_T^\alpha + t^{1/2-H}\|B^H\|_\infty^\alpha + t^{1/2-H}\big(1+|x|+\left|b(0,x)\right| + |t|^\gamma\big)\big\}
\end{aligned}
$$

and

$$
\begin{aligned}
\left|B_1^k(t)\right| &\le \max_{1\le i\le k}\left|b\left(t, B_t^H + \int_0^t u_s\,\mathrm{d}s - S_{i-1} + x\right)\right| t^{1/2-H} + K t^{\gamma+1/2-H} \\
&\le C t^{1/2-H}\left\{\left|b(0,x)\right| + |t|^\gamma + \|B^H\|_\infty^\alpha + |L_T|^\alpha + \left|\int_0^t u_s\,\mathrm{d}s\right|^\alpha\right\} \\
&\le C t^{1/2-H}\big\{1+|x|+\left|b(0,x)\right| + |L|_T^\alpha + \|B^H\|_\infty^\alpha + |t|^\gamma\big\},
\end{aligned} \tag{6.8}
$$

$$
\begin{aligned}
\left|B_2^k(t)\right| &\le t^{H-1/2}\int_0^t \frac{|\int_r^t u_s\,\mathrm{d}s + |B_t^H - B_r^H||^\alpha}{(t-r)^{1/2+H}} r^{1/2-H}\,\mathrm{d}r \\
&\le t^{H-1/2}\int_0^t \frac{|\int_r^t u_s\,\mathrm{d}s|^\alpha}{(t-r)^{1/2+H}} r^{1/2-H}\,\mathrm{d}r + t^{H-1/2}\int_0^t \frac{|B_t^H - B_r^H|^\alpha}{(t-r)^{1/2+H}} r^{1/2-H}\,\mathrm{d}r \\
&\le t^{H-1/2} C\big(1+\left|b(0,x)\right| + |x|^\alpha + \|X\|_\infty^\alpha\big)^\alpha \\
&\quad \times \int_0^t \frac{(t-r)^\alpha r^{1/2-H}}{(t-r)^{1/2+H}}\,\mathrm{d}r + C t^{1/2-H+\alpha(H-\varepsilon)} G^\alpha \\
&\le t^{\alpha+H-1/2} C\big(1+\left|b(0,x)\right| + |x|^\alpha + \|B^H\|_\infty^\alpha + |L|_T^\alpha\big) + C t^{1/2-H+\alpha(H-\varepsilon)} G^\alpha \\
&\le t^{\alpha+1/2-H} C\big\{1+|x|+\left|b(0,x)\right| + \|B^H\|_\infty^\alpha + |L|_T^\alpha\big\} + C t^{1/2-H+\alpha(H-\varepsilon)} G^\alpha.
\end{aligned} \tag{6.9}
$$

We can follow the same arguments of Lemma 5.1 to show that v also satisfies the Novikov condition (4.2), proving the theorem. □

Next, we show that the pathwise uniqueness holds for solutions to (3.1). The proof is more or less standard, see [18] or [21], we provide a sketch for completeness.

Theorem 6.2. *Suppose that Assumption 3.1 holds. Then two weak solutions of SDE (3.1) defined on the same filtered probability space with the same driving fBM B^H and Poisson point process L must coincide almost surely on their common existence interval.*

Proof. Let X^1 and X^2 be two weak solutions defined on the same filtered probability space with the same driving B^H and L. Define $Y^+ \triangleq X^1 \vee X^2$, and $Y^- \triangleq X^1 \wedge X^2$. One shows that both Y^+ and Y^- both satisfy (3.1). In fact, note that $X^1 - X^2$ involves only Lebesgue integral, the occupation density formula yields that the local time of $X^1 - X^2$ at 0 is identically zero. Thus, by Tanaka's formula,

$$(X_t^1 - X_t^2)^+ = \int_0^t \big(b(s, X_s^1) - b(s, X_s^2)\big) I_{\{X_s^1 - X_s^2 > 0\}}\, \mathrm{d}s.$$

Then, note that $Y^+ = X^2 + (X^1 - X^2)^+$, we have

$$\begin{aligned}
Y_t^+ &= x + \int_0^t b(s, X_s^2)\,\mathrm{d}s + B_t^H - L_t + \int_0^t \big(b(s, X_s^1) - b(s, X_s^2)\big) I_{\{X_s^1 - X_s^2 > 0\}}\,\mathrm{d}s \\
&= x + \int_0^t b(s, X_s^1) I_{\{X_s^1 - X_s^2 > 0\}}\,\mathrm{d}s + \int_0^t b(s, X_s^2) I_{\{X_s^1 - X_s^2 \le 0\}}\,\mathrm{d}s + B_t^H - L_t \\
&= x + \int_0^t b(s, Y_s^+)\,\mathrm{d}s + B_t^H - L_t.
\end{aligned}$$

Similarly one shows that Y_t^- satisfies SDE (3.1) as well. We claim that

$$\mathbb{P}\Big\{\sup_{0 \le t \le T} \big(Y_t^+ - Y_t^-\big) = 0\Big\} = 1. \tag{6.10}$$

Indeed, if $\mathbb{P}\{\sup_{0 \le t \le T}(Y_t^+ - Y_t^-) > 0\} > 0$, then there exists a rational number r and $t > 0$ such that $\mathbb{P}(Y_t^+ > r > Y_t^-) > 0$. Since $\{Y_t^+ > r\} = \{Y_t^- > r\} \cup \{Y_t^+ > r \ge Y_t^-\}$, we have

$$\mathbb{P}\big(Y_t^+ > r\big) = \mathbb{P}\big(Y_t^- > r\big) + \mathbb{P}\big(Y_t^+ > r \ge Y_t^-\big) > \mathbb{P}\big(Y_t^- > r\big).$$

This contradicts with the fact that Y_t^+ and Y_t^- have the same law, thanks to Theorem 6.1. Thus, (6.10) holds, and consequently, $X^1 \equiv X^2$, $\mathbb{P}$-a.s., proving the theorem. □

7. Existence of strong solutions

Having proved the existence of the weak solution and pathwise uniqueness, it is rather tempting to invoke the well-known Yamada–Watanabe Theorem to conclude the existence of the strong solution. However, there seem to be some fundamental difficulties in the proof of such a result, mainly because of the lack of the independent increment property for an fBM, which is crucial in the proof. It is also well known that, unlike an ODE, in the case of stochastic differential equations, the existence of the strong solution could be argued with assumptions on the coefficients being much weaker than Lipschitz, due to the presence of the "noise". We note that the argument in this section is quite similar to [8] and [16], with some necessary adjustments for the presence of the jumps.

We begin by observing that the SDE (3.1) can be solved pathwisely, as an ODE, when the coefficient b is regular enough (e.g., continuous in (t, x), and uniformly Lipschitz in x). Second, we claim that, under Assumption 3.1 it suffices to prove the existence of the strong solution when the coefficient b is uniformly bounded. Indeed, if we consider the following family of SDEs:

$$X_t = x + \int_0^t b_R(s, X_s)\,\mathrm{d}s + B_t^H - L_t, \qquad t \in [0,T],\ R > 0, \tag{7.1}$$

where b_R is the truncated version of b: $b_R(t,x) = b(t, (x \wedge R) \vee (-R))$, $(t,x) \in [0,T] \times \mathbb{R}$, then for each R, b_R is bounded, hence (7.1) has a strong solution, denoted by X^R, defined on $[0,T]$, and we can now assume that they all live on a common probability space. Now note that for $R_1 < R_2$, one has $b_{R_1} \equiv b_{R_2}$ whenever $|x| \le R_1$, thus by the pathwise uniqueness, it is easy to see that $X_t^{R_1} \equiv X_t^{R_2}$, for $t \in [0, \tau_{R_1}]$, $\mathbb{P}$-a.s., where $\tau_R \triangleq \inf\{t > 0 : |X_t^R| \ge R\} \wedge T$. Therefore, we can almost surely extend the solution to $[0, \tau)$, where $\tau \triangleq \lim_{R\to\infty} \tau_R$. Furthermore, it was shown (see, e.g., (6.4)) that X will never explode on $[0,\tau)$. Consequently, we must have $\tau = T$, $\mathbb{P}$-a.s.

We now give our main result of this section.

Theorem 7.1. *Assume that $b(t,x)$ satisfies Assumption 3.1. Then there exists a unique strong solution SDE (3.1).*

The proof of Theorem 7.1 follows an argument by Gyöngy and Pardoux [8], using the so-called Krylov estimate (cf. [12]). We note that by the argument preceding the theorem we need only consider the case when the coefficient b is bounded. The following lemma is thus crucial.

Lemma 7.1. *Suppose that the coefficient b satisfies Assumption 3.1 and is uniformly bounded by a constant $C > 0$. Suppose also that X is a strong solution to SDE (3.1). Then, there exist $\beta > 1$ and $\zeta > 1 + H$ such that for any measurable nonnegative function $g : [0,T] \times \mathbb{R} \mapsto \mathbb{R}_+$, it holds that*

$$\mathbb{E}\int_0^T g(t, X_t)\,\mathrm{d}t \le M\left(\int_0^T \int_{\mathbb{R}} g^{\beta\zeta}(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^{1/\beta\zeta}, \tag{7.2}$$

where M is a constant defined by

$$M \triangleq J^{1/\zeta'\beta} F^{1/\alpha}, \tag{7.3}$$

in which

$$F \triangleq \left\{\tilde{\mathbb{E}} \exp\left\{2\alpha^2 \int_0^T v_t^2\,\mathrm{d}t\right\}\right\}^{1/2}, \qquad J \triangleq \frac{(2\pi)^{1/2-\zeta'/2} T^{1+(1-\zeta')H}}{\sqrt{\zeta'}(1 + (1-\zeta')H)} \tag{7.4}$$

and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, $\frac{1}{\zeta} + \frac{1}{\zeta'} = 1$.

Proof. Let $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$ be a filtered probability space on which are defined a fBM B^H, a Poisson point process L of class (QL) and independent of B^H, and X is the strong solution to the corresponding SDE (3.1). Let W be an $\mathbb{F}$-Brownian motion such that $B^H = \int_0^{\cdot} K_H(t,s)\,\mathrm{d}W_s$. Recall from (6.2) the process $v = K_H^{-1}(\int_0^{\cdot} b(r, X_r)\,\mathrm{d}r)$, and define a new measure $\tilde{\mathbb{P}}$ by

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} \triangleq \exp\left\{-\int_0^T v_t\,\mathrm{d}W_t - \frac{1}{2}\int_0^T v_t^2\,\mathrm{d}t\right\} \triangleq Z_T^{-1}. \tag{7.5}$$

Then, in light of Lemmas 4.1 and 5.1, we know that $\tilde{\mathbb{P}}$ is a probability measure under which $\tilde{W}_t = W_t + \int_0^t v_r\,\mathrm{d}r$ is a Brownian motion, $\tilde{B}_t^H = \int_0^t K_H(t,s)\,\mathrm{d}\tilde{W}_s$ is a fBM, and L remains a Poisson point process with same parameters and is independent of $\tilde{B}^H$. Hence, under $\tilde{\mathbb{P}}$, $X_t = x + \tilde{B}_t^H - L_t$ has the density function:

$$p_t(y) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi} t^H} \mathrm{e}^{-(y+z-x)^2/2t^{2H}} f_L(t,z)\,\mathrm{d}z, \tag{7.6}$$

where $f_L(t,\cdot)$ is the density function of L_t.

Now, applying Hölder's inequality we have

$$\mathbb{E}\int_0^T g(t,X_t)\,\mathrm{d}t = \tilde{\mathbb{E}}\left\{Z_T\int_0^T g(t,X_t)\,\mathrm{d}t\right\} \le \left\{\tilde{\mathbb{E}}\left[Z_T^\alpha\right]\right\}^{1/\alpha}\left\{\tilde{\mathbb{E}}\int_0^T g^\beta(t,X_t)\,\mathrm{d}t\right\}^{1/\beta}, \tag{7.7}$$

where $1/\alpha + 1/\beta = 1$. Rewriting v_t as $v_t = K_H^{-1}(\int_0^{\cdot} b(r, \tilde{B}_r^H - L_r + x)\,\mathrm{d}r)(t)$, we can follow the same argument as the proof of Lemmas 4.1 and 5.1 to get, $\tilde{\mathbb{E}}\mathrm{e}^{2\alpha^2\int_0^T v_t^2\,\mathrm{d}t} < \infty$. Therefore, $\exp\{2\alpha\int_0^t v_s\,\mathrm{d}\tilde{W}_s - 2\alpha^2\int_0^t v_s^2\,\mathrm{d}s\}$ is a $\tilde{\mathbb{P}}$-martingale, and consequently, applying Hölder's inequality we obtain

$$\begin{aligned}
\tilde{\mathbb{E}}\left[Z_T^\alpha\right] &= \tilde{\mathbb{E}}\exp\left\{\alpha\int_0^T v_t\,\mathrm{d}W_t + \frac{\alpha}{2}\int_0^T v_t^2\,\mathrm{d}t\right\} \\
&= \tilde{\mathbb{E}}\exp\left\{\alpha\int_0^T v_t\,\mathrm{d}\tilde{W}_t - \frac{\alpha}{2}\int_0^T v_t^2\,\mathrm{d}t\right\} \\
&= \tilde{\mathbb{E}}\exp\left\{\alpha\int_0^T v_t\,\mathrm{d}\tilde{W}_t - \alpha^2\int_0^T v_t^2\,\mathrm{d}t + \left(\alpha^2 - \frac{\alpha}{2}\right)\int_0^T v_t^2\,\mathrm{d}t\right\} \\
&\le \left(\tilde{\mathbb{E}}\exp\left\{2\alpha\int_0^T v_t\,\mathrm{d}\tilde{W}_t - 2\alpha^2\int_0^T v_t^2\,\mathrm{d}t\right\}\right)^{1/2}\left(\tilde{\mathbb{E}}\exp\left\{\left(2\alpha^2 - \alpha\right)\int_0^T v_t^2\,\mathrm{d}t\right\}\right)^{1/2} \\
&\le \left(\tilde{\mathbb{E}}\exp\left\{2\alpha^2\int_0^T v_t^2\,\mathrm{d}t\right\}\right)^{1/2} < \infty.
\end{aligned} \tag{7.8}$$

On the other hand, applying Hölder's inequality with $1/\zeta + 1/\zeta' = 1$, $\zeta > H+1$ yields

$$\begin{aligned}
\tilde{\mathbb{E}}\int_0^T g^\beta(t,X_t)\,\mathrm{d}t &= \int_0^T\int_{\mathbb{R}} g^\beta(t,y)p_t(y)\,\mathrm{d}y\,\mathrm{d}t \\
&\le \left\|g^\beta\right\|_{L^\zeta([0,T]\times\mathbb{R})}\left\|p_\cdot(\cdot)\right\|_{L^{\zeta'}([0,T]\times\mathbb{R})}.
\end{aligned} \tag{7.9}$$

Now, by the generalized Minkowski inequality (cf., e.g., [20], (1.33)), we have

$$\begin{aligned}\int_{\mathbb{R}}\big[p_t(y)\big]^{\gamma'}\,\mathrm{d}y &= \int_{\mathbb{R}}\bigg\{\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi}t^H}\mathrm{e}^{-(y+z-x)^2/2t^{2H}}f_L(t,z)\,\mathrm{d}z\bigg\}^{\zeta'}\mathrm{d}y\\ &\le \bigg\{\int_{\mathbb{R}}\bigg[\int_{\mathbb{R}}\bigg(\frac{1}{\sqrt{2\pi}t^H}\mathrm{e}^{-(y+z-x)^2/2t^{2H}}f_L(t,z)\bigg)^{\zeta'}\mathrm{d}y\bigg]^{1/\zeta'}\mathrm{d}z\bigg\}^{\zeta'} \qquad (7.10)\\ &= \bigg\{\int_{\mathbb{R}}f_L(t,z)\bigg[\int_{\mathbb{R}}\bigg(\frac{1}{\sqrt{2\pi}t^H}\mathrm{e}^{-(y+z-x)^2/2t^{2H}}\bigg)^{\zeta'}\mathrm{d}y\bigg]^{1/\zeta'}\mathrm{d}z\bigg\}^{\zeta'}.\end{aligned}$$

The direct calculation gives

$$\int_{\mathbb{R}}\bigg(\frac{1}{\sqrt{2\pi}t^H}\mathrm{e}^{-(y+z-x)^2/2t^{2H}}\bigg)^{\zeta'}\mathrm{d}y = (2\pi)^{1/2-\zeta'/2}\big(\zeta'\big)^{-1/2}t^{(1-\zeta')H}.$$

Plugging this into (7.10), we obtain

$$\begin{aligned}\int_{\mathbb{R}}\big[p_t(y)\big]^{\zeta'}\,\mathrm{d}y &= (2\pi)^{1/2-\zeta'/2}\big(\zeta'\big)^{-1/2}t^{(1-\zeta')H}\bigg(\int_{\mathbb{R}}f_L(t,z)\,\mathrm{d}z\bigg)^{\zeta'}\\ &= (2\pi)^{1/2-\zeta'/2}\big(\zeta'\big)^{-1/2}t^{(1-\sigma')H}.\end{aligned}$$

Since $\zeta > H+1$, this leads to that

$$\big\|p_\cdot(\cdot)\big\|_{L^{\zeta'}([0,T]\times\mathbb{R})} \le J^{1/\zeta'}, \qquad (7.11)$$

where J is defined by (7.4). Finally, noting that $\|g^\beta\|^{1/\beta}_{L^\zeta([0,T]\times\mathbb{R})} = \|g\|_{L^{\beta\zeta}([0,T]\times\mathbb{R})}$, the estimate (7.2) then follows from (7.7), (7.8), (7.9), and (7.11). □

Proof of Theorem 7.1. Since the proof is more or less standard, we only give a sketch for the completeness. We refer to [12], [8] and/or [16] for more details.

We need only prove the existence. We assume that the coefficient b is bounded (by $C>0$) and satisfies Assumption 3.1. Let $\{b_n(\cdot,\cdot)\}_{n=1}^\infty$ be a sequence of the mollifiers of b, so that all b_n's are smooth, have the same bound C, and satisfy Assumption 3.1 with the same parameters.

Next, for $n\le k$ we define $\tilde{b}_{n,k} \triangleq \bigwedge_{j=n}^k b_j$ and $\tilde{b}_n \triangleq \bigwedge_{j=n}^\infty b_j$. Then clearly, each $\tilde{b}_{n,k}$ is continuous, and uniformly Lipschitz in x, uniformly with respect to t. Furthermore, it holds that

$$\tilde{b}_{n,k}\downarrow\tilde{b}_n, \qquad \text{as } k\to\infty, \qquad \tilde{b}_n\uparrow b, \qquad \text{as } n\to\infty,$$

for almost all x. Now for fixed n, k, consider SDE

$$X_t = x+\int_0^t\tilde{b}_{n,k}(s,X_s)\,\mathrm{d}s + B_t^H - L_t, \qquad t\ge 0. \qquad (7.12)$$

As a pathwise ODE, (7.12) has a unique strong solution $\tilde{X}^{n,k}$, and comparison theorem holds, that is, $\{\tilde{X}^{n,k}\}$ decrease with k. Furthermore, since $\tilde{b}_{n,k}$'s are uniformly bounded by C, the solutions $\tilde{X}^{n,k}$ are pathwisely uniformly bounded, uniformly in n and k. Thus $X_t^n \overset{\triangle}{=} \lim_{k\to\infty} \tilde{X}_t^{n,k}$ exists, for all $t \in [0,T]$, $\mathbb{P}$-a.s. Since b_n's are still Lipschitz, the standard stability result of ODE then implies that $\tilde{X}^n$ solves

$$X_t = x + \int_0^t \tilde{b}_n(s, X_s)\,\mathrm{d}s + B_t^H - L_t, \qquad t \in [0,T].$$

Furthermore, the Dominated Convergence theorem leads to that the estimate (7.2) holds for all X^n's, for any bounded measurable function g.

Next, since $\tilde{X}^{n,k} \le \tilde{X}^{m,k}$, for $n \le m \le k$, we see that $\tilde{X}_n$ increases as n increases, thus $\tilde{X}^n$ converges, $\mathbb{P}$-almost surely, to some process X. The main task remaining is to show that X solves SDE (3.1), as b is no longer Lipschitz. In other words, we shall prove that

$$\lim_{n\to\infty} \mathbb{E}\int_0^T \big|\tilde{b}_n\big(t, X_t^n\big) - b(t, X_t)\big|\,\mathrm{d}t = 0. \tag{7.13}$$

To see this, we first note that

$$\mathbb{E}\int_0^T \big|\tilde{b}_n\big(t, X_t^n\big) - b(t, X_t)\big|\,\mathrm{d}s \le I_1^n + I_2^n, \tag{7.14}$$

where

$$\begin{aligned} I_1^n &\overset{\triangle}{=} \sup_k \mathbb{E}\int_0^T \big|\tilde{b}_k\big(t, X_t^n\big) - \tilde{b}_k(t, X_t)\big|\,\mathrm{d}t, \\ I_2^n &\overset{\triangle}{=} \mathbb{E}\int_0^T \big|\tilde{b}_n(t, X_t) - b(t, X_t)\big|\,\mathrm{d}t. \end{aligned} \tag{7.15}$$

Let $\kappa : \mathbb{R} \to \mathbb{R}$ be a smooth truncation function satisfying $0 \le \kappa(z) \le 1$ for every z, $\kappa(z) = 0$ for $|z| \ge 1$ and $\kappa(0) = 1$. Then by Bounded Convergence theorem one has

$$\lim_{R\to\infty} \mathbb{E}\int_0^T \big(1 - \kappa(X_t/R)\big)\,\mathrm{d}t = 0. \tag{7.16}$$

Now for any $R > 0$, we apply Lemma 7.1 with $\beta\zeta = 2$ and note that both $\tilde{b}_n$ and b are bounded by C to get

$$\begin{aligned} I_2^n &= \mathbb{E}\int_0^T \kappa(X_t/R)\big|\tilde{b}_n(t, X_t) - b(t, X_t)\big|\,\mathrm{d}t \\ &\quad + \mathbb{E}\int_0^T \big(1 - \kappa(X_t/R)\big)\big|\tilde{b}_n(t, X_t) - b(t, X_t)\big|\,\mathrm{d}t \\ &\le M\bigg(\int_0^T \int_{-R}^R \big|\tilde{b}_n(t, x) - b(t, x)\big|^2\,\mathrm{d}x\,\mathrm{d}t\bigg)^{1/2} + 2C\mathbb{E}\int_0^T \big(1 - \kappa(X_t/R)\big)\,\mathrm{d}t. \end{aligned} \tag{7.17}$$

First letting $n \to \infty$ and then letting $R \to \infty$, we get $\lim_{n\to\infty} I_2^n = 0$.

To show that $\lim_{n\to\infty} I_1^n = 0$, we first note that by (7.16), for any $\varepsilon > 0$, there exists $R_0 > 0$ such that

$$\mathbb{E}\int_0^T \left|1-\kappa(X_t/R_0)\right| \mathrm{d}t < \varepsilon. \tag{7.18}$$

Second, since $\{b_n\}$ converge to b almost everywhere, the Bounded Convergence theorem then shows that $\tilde{b}_n$ converges to b in $L^2_{T,R_0} \stackrel{\triangle}{=} L^2([0,T]\times[-R_0,R_0])$, hence $\{b_n, b\}_{n\ge 1}$ is a compact set in L^2_{T,R_0}. Thus, we can find finitely many bounded smooth function $H_1,\dots,H_N$ such that for each k, there is a H_{i_k} so that

$$\left(\int_0^T\int_{-R_0}^{R_0} \left|\tilde{b}_k(t,x) - H_{i_k}(t,x)\right|^2 \mathrm{d}r\,\mathrm{d}t\right)^{1/2} < \varepsilon. \tag{7.19}$$

Now, we write

$$I_1^n = \sup_k \mathbb{E}\int_0^T \left|\tilde{b}_k\big(t,X_t^n\big) - \tilde{b}_k(t,X_t)\right| \mathrm{d}t \le \sup_k I_1(n,k) + I_2(n) + \sup_k I_3(k),$$

where

$$\begin{cases} I_1(n,k) = \mathbb{E}\displaystyle\int_0^T \left|\tilde{b}_k\big(t,X_t^n\big) - H_{i_k}\big(t,X_t^n\big)\right| \mathrm{d}t; \\ I_2(n) = \displaystyle\sum_{j=1}^N \mathbb{E}\int_0^T \left|H_j\big(t,X_t^n\big) - H_j(t,X_t)\right| \mathrm{d}t; \\ I_3(k) = \mathbb{E}\displaystyle\int_0^T \left|\tilde{b}_k(t,X_t) - H_{i_k}(t,X_t)\right| \mathrm{d}t. \end{cases}$$

It is obvious that $\lim_{n\to\infty} I_2(n) = 0$. Furthermore, since the estimate (7.2) holds with $\beta\zeta = 2$ for all X^n's, similar to (7.17) we have

$$I_1(n,k) \le M\left(\int_0^T\int_{-R_0}^{R_0} \left|\tilde{b}_k(t,x) - H_{i_k}(t,x)\right|^2 \mathrm{d}x\,\mathrm{d}t\right)^{1/2} + C_1\mathbb{E}\int_0^T \left(1-\kappa\big(X_t^{(n)}/R_0\big)\right)\mathrm{d}t,$$

where C_1 is a constant depending on C and $\max_{1\le i\le N}\|H_i\|_\infty$. Hence, by (7.18), (7.19), and the Dominated Convergence theorem again we have

$$\lim_{n\to\infty}\sup_k I_1(n,k) \le M\varepsilon + C_1\mathbb{E}\int_0^T \big(1-\kappa(X_t/R_0)\big)\,\mathrm{d}t \le (M+C_1)\varepsilon.$$

Similarly, we have $\sup_k I_3(k) \le (M+C_1)\varepsilon$. Letting $\varepsilon \to 0$ we obtain $\lim_{n\to\infty} I_1^n = 0$. The proof is now complete. $\square$

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