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Approximating the first crossing-time density for a curved boundary

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This paper is concerned with the problem of approximating the density of the time at which a Brownian path first crosses a curved boundary in cases where the exact density is not known or is difficult to compute. Two methods are proposed involving the use of images, and the square root boundary provides an example for numerical comparison. Two-sided boundaries are also discussed.

Keywords: Brownian motion; curved boundary; first crossing-time density; sequential analysis

1. Introduction

A problem which is important in sequential analysis and constantly arises in other contexts is to evaluate the probability $P(t)$ that a Brownian path does not cross a curved boundary before time t. When the boundary is linear the solution is well known, but otherwise few exact solutions are available and even when a formula is known for a particular boundary it usually involves heavy computation. It is very desirable, therefore, to have a reasonably good approximation which is easy to calculate. The present paper attempts to address the question by looking for workable approximate solutions which can be applied without too much effort to a variety of situations. We have preferred to consider the first crossing-time density $g(t)$ rather than the distribution function $1 - P(t)$ since it gives a more vivid impression of the form of the distribution.

Suppose we have standard Brownian motion $X(t)$ and a one-sided absorbing boundary $\xi(t)$. A well-known result which is widely applicable is the *tangent approximation* (TA), where $\xi(t)$ is replaced at each t by its tangent at t. It was first introduced by Strassen (1967) as an asymptotic approximation for large $\xi(t)$ as $t \to 0$, and independently Daniels (1974), Cusick (1981), Jennen and Lerche (1981) established the result for general t. It was developed as such by Lerche (1986) and his colleagues who devised improved asymptotic versions. However, it seems to work quite well in many cases when $\xi(t)$ is not large and an asymptotic justification is not available. Our object is to explore ways of improving on the TA when $\xi(t)$ is not large.

Two different approaches are discussed, both exploiting the use of images (see Daniels 1982). In the first approach the TA, which can be derived by introducing a suitable negative image source, is modified by adding a second image source. This replaces the tangent to $\xi(t)$ at $t = T$ by another boundary $\bar{\xi}(t, T)$ which has higher-order contact with $\xi(t)$ at $t = T$ and has an easily computable first crossing-time density. In the second approach we look for a distribution of weighted images on the vertical axis which, together with the original source at the origin, will generate the given absorbing boundary $\xi(t)$. An approximate solution is found by discretizing the resulting integral equation, thus reducing the problem to the solution of a set of linear equations.

The results presented here must be regarded as a preliminary look at the problem, the suggested methods being tried out on a particular form of $\xi(t)$, namely the square-root boundary $\xi(t) = \ell\sqrt{t+1}$. The aim has been to balance accuracy of approximation against ease of calculation.

In the final section, two-sided boundaries are discussed. It is shown that the TA can be improved by considering both branches simultaneously.

2. The tangent approximation

Consider a standard Brownian motion $X(t)$ with $X(0) = 0$, constrained by an absorbing boundary $\xi(t) > 0$. We require the probability of not crossing $\xi(t)$ before t, i.e.

$$P(t) = P\{X(\tau) < \xi(\tau) | 0 \leq \tau < t\}. \tag{2.1}$$

Equivalently, we consider the first crossing-time density (fcd).

$$g(t) = -\frac{\partial P}{\partial t}. \tag{2.2}$$

For general $\xi(t)$ the problem is usually difficult to solve analytically, so we seek a good approximation which is easy to compute.

There is another formula for the fcd which is more convenient than (2.2) for the present purpose. Let $p(x, t)$ be the density at (x, t) given that $x(\tau) < \xi(\tau)$, $0 \leq \tau < t$. By definition $p(\xi(t), t) = 0$. Then

$$g(t) = -\frac{1}{2} \frac{\partial p(x, t)}{\partial x}\bigg|_{x \uparrow \xi(t)}. \tag{2.3}$$

The usual proof is as follows.

$$\begin{aligned} \frac{\partial P}{\partial t} &= \frac{\partial}{\partial t} \int_{-\infty}^{\xi(t)} p(x, t) \mathrm{d}x = \int_{-\infty}^{\xi(t)} \frac{\partial p(x, t)}{\partial t} \mathrm{d}x \qquad (p(\xi(t), t) = 0) \\ &= \frac{1}{2} \int_{-\infty}^{\xi(t)} \frac{\partial^2 p}{\partial x^2} \mathrm{d}x = \frac{1}{2} \frac{\partial p(x, t)}{\partial x}\bigg|_{x \uparrow \xi(t)} \end{aligned}$$

using the diffusion equation. But this proof does not work for two-sided boundaries because the lower bound $\eta(t)$ replaces $-\infty$ and the final term is

$$\frac{1}{2} \frac{\partial p(x, t)}{\partial x}\bigg|_{x \uparrow \xi(t)} - \frac{1}{2} \frac{\partial p(x, t)}{\partial x}\bigg|_{x \downarrow \eta(t)}.$$

The contributions from each branch cannot therefore be separated, even though (2.3) is actually correct for the upper branch, and, with the appropriate changes, for the lower branch also. A probabilistic proof avoiding the difficulty is sketched in Daniels (1982).

The tangent approximation consists in replacing the boundary $\xi(t)$ near $t = T$ by its tangent $\bar{\xi}(t, T) = \alpha + \beta t$, where $\alpha = \xi(T) - T\xi'(T), \beta = \xi'(T)$, the idea being that when $\xi(t)$ is large the only Brownian paths contributing to the fcd at T are those crossing within a small neighbourhood of T. An alternative approach using a renewal-type integral equation is given in Daniels (1974). The density $\bar{p}(x, t)$ for the tangent boundary is found in the usual way by putting a negative image of weight $\kappa = \exp(-2\alpha\beta)$ at $x = 2\alpha, t = 0$, so that

$$\bar{p}(x, t) = \frac{e^{-x^2/2t}}{\sqrt{2\pi t}} - \frac{e^{-2\alpha\beta-(x-2\alpha)^2/2t}}{\sqrt{2\pi t}}$$

$$= \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}\{1 - e^{2\alpha(x-\alpha-\beta t)/t}\} \tag{2.4}$$

which vanishes on the absorbing boundary $\bar{\xi} = \alpha + \beta t$. The fcd is, from (2.3),

$$\bar{g}(t) = \frac{1}{2\sqrt{2\pi}t^{3/2}}\{\bar{\xi}e^{-\bar{\xi}^2/2t} - (\bar{\xi} - 2\alpha)\,e^{-2\alpha\beta-(\bar{\xi}-2\alpha)^2/2t}\} \tag{2.5}$$

which reduces to

$$\bar{g}(t) = \frac{\alpha}{\sqrt{2\pi}t^{3/2}}\,e^{-\bar{\xi}^2/2t} = \frac{\alpha}{\sqrt{2\pi}t^{3/2}}\,e^{-2\alpha\beta-(\bar{\xi}-2\alpha)^2/2t}, \tag{2.6}$$

the second form generalizing naturally to the more complicated case considered later.

An important condition for the TA to work is that the intercept α on the vertical axis is positive. This implies that $\xi(T)/T > \xi'(T)$.

It helps the later manipulation to apply the time inversion

$$u = 1/t, \qquad z = x/t, \qquad \zeta(u) = \xi(t)/t, \tag{2.7}$$

in terms of which the tangent boundary becomes

$$\bar{\zeta} = \alpha u + \beta, \qquad \alpha = \zeta'(U), \qquad \beta = \zeta(U) - U\zeta'(U), \tag{2.8}$$

where $U = 1/T$.

The TA is a surprisingly good approximation in many cases. It is asymptotically exact as $\xi(t) \to \infty$ for fixed t, but as has been mentioned it seems to work well for quite moderate $\xi(t)$. In the numerical example shown in Table 1 for the boundary $\xi(t) = \ell\sqrt{t+1}$ the agreement is quite good over the whole range even when $\ell = 0.5$.

3. Improvements on the TA

Various attempts have been made to improve the TA. Ferebee (1983) and Jennen (1985) get higher-order terms in an asymptotic expansion using the integral equation approach. Jennen also introduces an extra term to allow for possible crossings before T. In an important paper, Durbin (1992) developed a sequence of approximations starting with the TA which actually converge to the correct answer. Unfortunately his method involves heavy computation, but it appears not to rely on asymptotic considerations.

Table 1. Square-root boundary: (a) $\ell = 0.5$; (b) $\ell = 1.0$

(a)

t	g	g_0	g_1	g_2	g_3
0.05	1.2871	1.2928	1.2927	1.2928	1.2917
0.10	1.5976	1.5967	1.5963	1.5964	1.5948
0.15	1.3181	1.3201	1.3192	1.3195	1.3209
0.20	1.0570	1.0578	1.0564	1.0569	1.0591
0.25	0.8592	0.8595	0.8575	0.8581	0.8598
0.30	0.7085	0.7123	0.7099	0.7107	0.7116
0.35	0.5978	0.6015	0.5987	0.5996	0.5998
0.40	0.5137	0.5163	0.5132	0.5140	0.5139
0.45	0.4461	0.4494	0.4459	0.4468	0.4465
0.50	0.3924	0.3958	0.3920	0.3930	0.3925
0.6	0.3122	0.3161	0.3119	0.3129	0.3124
0.7	0.2568	0.2603	0.2557	0.2569	0.2563
0.8	0.2152	0.2196	0.2147	0.2159	0.2153
0.9	0.1837	0.1888	0.1837	0.1848	0.1842
1.0	0.1598	0.1648	0.1595	0.1607	0.1600
1.2	0.1253	0.1302	0.1246	0.1259	0.1250
1.4	0.1012	0.1067	0.1010	0.1022	0.1011
1.6	0.0848	0.0898	0.0840	0.0859	0.0840
1.8	0.0720	0.0772	0.0714	0.0726	0.0711
2.0	0.0621	0.0675	0.0617	0.0629	0.0613
2.2	0.0545	0.0598	0.0541	0.0553	0.0535
2.4	0.0485	0.0536	0.0479	0.0491	0.0472

(b)

t	g	g_0	g_1	g_2	g_3
0.2	0.2234	0.2230	0.2229	0.2229	0.2224
0.3	0.2810	0.2805	0.2801	0.2803	0.2799
0.4	0.2772	0.2779	0.2771	0.2774	0.2773
0.5	0.2559	0.2570	0.2557	0.2562	0.2559
0.6	0.2311	0.2325	0.2309	0.2316	0.2311
0.7	0.2081	0.2094	0.2074	0.2082	0.2076
0.8	0.1871	0.1889	0.1865	0.1875	0.1868
0.9	0.1685	0.1710	0.1684	0.1694	0.1687
1.0	0.1529	0.1557	0.1527	0.1539	0.1531
1.2	0.1278	0.1309	0.1275	0.1289	0.1277
1.4	0.1089	0.1122	0.1084	0.1099	0.1083
1.6	0.0946	0.0976	0.0937	0.0952	0.0931
1.8	0.0827	0.0862	0.0820	0.0837	0.0810
2.0	0.0732	0.0769	0.0726	0.0743	0.0711
2.5	0.0564	0.0603	0.0557	0.0576	0.0534

Table 1. continued

t	g	g_0	g_1	g_2	g_3
3.0	0.0455	0.0493	0.0446	0.0465	0.0417
3.5	0.0376	0.0415	0.0369	0.0388	0.0336
4.0	0.0320	0.0358	0.0312	0.0331	0.0277
4.5	0.0277	0.0314	0.0269	0.0288	0.0233
5.0	0.0243	0.0280	0.0235	0.0254	0.0199
5.5	0.0216	0.0252	0.0208	0.0227	0.0173
6.0	0.0195	0.0229	0.0186	0.0205	0.0151

$\xi(t) = t\sqrt{t+1}$,
g is 'exact', g_0 is TA,
g_1 has $\alpha_1 = \xi(0)$, g_2 has $\alpha_1 = \xi(0)$, $\beta_1 = \xi'(0)$
g_3 by second approach (Section 6)

Roberts and Shortland (1993) work with the hazard rate $r(t)$ rather than the fcd $g(t)$:

$$r(t) = g(t)/G(t), \qquad G(t) = \int_t^\infty g(\tau)\mathrm{d}\tau. \tag{3.1}$$

They approximate this by

$$\bar{r}(t) = \bar{g}(t)/\{\Phi(\bar{\xi}/\sqrt{t}) - \mathrm{e}^{-2\alpha\beta}\Phi([\bar{\xi} - 2\alpha]/\sqrt{t})\}, \tag{3.2}$$

where $\bar{\xi} = \alpha + \beta t$ and $\bar{g}(t)$ is the fcd for the TA. Apparently they get improved accuracy for a variety of boundaries and can state bounds on the error. Also the tail probability and fcd are approximated by

$$\bar{G}(t) = 1 - \exp\left\{-\int_0^t \bar{r}(\tau)\mathrm{d}\tau\right\} \tag{3.3}$$

$$\bar{g}(t) = \bar{r}(t)\exp\left\{-\int_0^t \bar{r}(\tau)\mathrm{d}\tau\right\}. \tag{3.4}$$

We now describe another way of improving the accuracy of the TA by replacing the tangent with a boundary which fits $\xi(t)$ more closely.

4. A more flexible approximate boundary

Daniels (1969; 1982) showed that by introducing two negative images, at $2\alpha_1, 2\alpha_2$, $(\alpha_1 < \alpha_2)$ with weights $\kappa_1 = \exp(-2\alpha_1\beta_1)$, $\kappa_2 = \exp(-2\alpha_2\beta_2)$, one could obtain curved boundaries with easily computable fcds. These can be used for $\bar{\xi}(t, T)$ instead of the previous straight-line approximation.

The density in $x(t) < \bar{\xi}(t, T), 0 \le t < T$, is

$$\bar{p}(x,t) = \frac{1}{\sqrt{2\pi t}}\{e^{-x^2/2t} - e^{-2\alpha_1\beta_1-(x-2\alpha_1)^2/2t} - e^{-2\alpha_2\beta_2-(x-2\alpha_2)^2/2t}\}$$

$$= \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}\{1 - e^{2\alpha_1(x-\alpha_1-\beta_1 t)/t} - e^{2\alpha_2(x-\alpha_2-\beta_2 t)/t}\}. \tag{4.1}$$

The absorbing boundary $\bar{\xi}(t, T)$ satisfying $p(\bar{\xi}, t) = 0$ is then given implicitly by

$$e^{2\alpha_1(\bar{\xi}-\alpha_1-\beta_1 t)/t} + e^{2\alpha_2(\bar{\xi}-\alpha_2-\beta_2 t)/t} = 1 \tag{4.2}$$

and its fcd is

$$\bar{g}(t) = \frac{1}{\sqrt{2\pi}\, t^{3/2}}\{\alpha_1 e^{-2\alpha_1\beta_1-(\bar{\xi}-2\alpha_1)^2/2t} + \alpha_2 e^{-2\alpha_2\beta_2-(\bar{\xi}-2\alpha_2)^2/2t}\}. \tag{4.3}$$

When t is small the image at $2\alpha_1$ dominates and $\bar{\xi} \sim \alpha_1 + \beta_1 t$. It follows that in general the intercept is α_1, which must necessarily be positive.

In the special case where $\alpha_1 = \alpha, \alpha_2 = 2\alpha$ there is an explicit formula

$$\bar{\xi} = \alpha - \frac{t}{2\alpha}\log\left\{\frac{\kappa_1}{2} + \left(\frac{1}{4}\kappa_1^2 + \kappa_2 e^{-4\alpha^2/t}\right)^{\frac{1}{2}}\right\} \tag{4.4}$$

which is often used as a boundary with known fcd for trying out approximations.

Since there are now four constants $\alpha_1, \beta_1, \alpha_2, \beta_2$, it is possible to introduce two extra conditions to improve the fit of $\bar{\xi}(t, T)$ to $\xi(t)$, either by matching higher derivatives of $\xi(t)$ at T or by introducing further constraints on $\bar{\xi}(t, T)$, or by both. The fact that the TA turns out to be too large may be because it does not allow for crossings near $t = 0$ where $\xi(t)$ is small. We shall attempt to compensate for this by equating the intercepts of $\bar{\xi}(t, T)$ and $\xi(t)$, i.e. by choosing $\alpha_1 = \xi(0)$, leaving enough freedom to match $\bar{\xi}$ and ξ at T up to the second derivative.

It is convenient to work with $u = 1/t, \zeta(u) = \xi(t)/t$. The boundary $\bar{\zeta}(u, U)$ is then given by

$$e^{2\alpha_1(\bar{\zeta}-\alpha_1 u-\beta_1)} + e^{2\alpha_2(\bar{\zeta}-\alpha_2 u-\beta_2)} = 1. \tag{4.5}$$

Differentiating (4.5) twice gives

$$\alpha_1(\bar{\zeta}' - \alpha_1)e^{2\alpha_1(\bar{\zeta}-\alpha_1 u-\beta_1)} + \alpha_2(\bar{\zeta}' - \alpha_2)e^{2\alpha_2(\bar{\zeta}-\alpha_2 u-\beta_2)} = 0 \tag{4.6}$$

$$\alpha_1\{\bar{\zeta}'' + 2\alpha_1(\bar{\zeta}' - \alpha_1)^2\}e^{2\alpha_1(\bar{\zeta}-\alpha_1 u-\beta_1)} + \alpha_2\{\bar{\zeta}'' + 2\alpha_2(\bar{\zeta}' - \alpha_2)^2\}e^{2\alpha_2(\bar{\zeta}-\alpha_2 u-\beta_2)} = 0 \tag{4.7}$$

Then

$$\bar{\zeta}'' + 2(\bar{\zeta}' - \alpha_1)(\bar{\zeta}' - \alpha_2)(\bar{\zeta}' - \alpha_1 - \alpha_2) = 0 \tag{4.8}$$

$$\beta_1 = \bar{\zeta} - \alpha_1 u - \frac{1}{2\alpha_1}\log\left\{\frac{\alpha_2(\alpha_2 - \bar{\zeta}')}{(\alpha_2 - \alpha_1)(\alpha_1 + \alpha_2 - \bar{\zeta}')}\right\} \tag{4.9}$$

$$\beta_2 = \bar{\zeta} - \alpha_2 u - \frac{1}{2\alpha_2}\log\left\{\frac{\alpha_1(\bar{\zeta}' - \alpha_1)}{(\alpha_2 - \alpha_1)(\alpha_1 + \alpha_2 - \bar{\zeta}')}\right\}. \tag{4.10}$$

We have chosen

$$\alpha_1 = \xi(0) = \zeta'(\infty) = \zeta_0', \quad \text{say}, \tag{4.11}$$

so that from (4.8), at $u = U$,

$$\alpha_2 = \zeta' - \tfrac{1}{2}\zeta_0' + \{\tfrac{1}{4}(\zeta_0')^2 - \tfrac{1}{2}\zeta''/(\zeta' - \zeta_0')\}^{\frac{1}{2}}, \tag{4.12}$$

and β_1, β_2 follow from (4.9), (4.10) with $u = U$.

Alternatively, $\bar{\xi}(t, T)$ may be constrained to fit $\xi(t)$ more closely near $t = 0$ by choosing $\alpha_1 = \xi(0), \beta_1 = \xi'(0)$, but then we can only fit $\bar{\xi} = \xi, \bar{\xi}' = \xi'$ at $t = T$ using α_2, β_2. From (4.5), (4.6) it is found that

$$\alpha_2(\zeta' - \alpha_2) = \frac{\alpha_1(\alpha_1 - \zeta')e^{2\alpha_1(\zeta - \alpha_1 U - \beta_1)}}{1 - e^{2\alpha_1(\zeta - \alpha_1 U - \beta_1)}} \tag{4.13}$$

$$\beta_2 = \zeta - \alpha_2 U - \frac{1}{2\alpha_2}\log\{1 - e^{2\alpha_1(\zeta - \alpha_1 U - \beta_1)}\}, \tag{4.14}$$

where $U = 1/T$, $\zeta(U) = \xi(T)/T$, $\zeta'(U) = \xi(T) - T\xi'(T)$.

A third possibility is to match derivatives of $\bar{\xi}$ and ξ up to the third at $t = T$, ignoring any constraint near $t = 0$. The formulae are more elaborate, involving the iterative solution of a cubic equation, and the results are no better than those obtained by the previous two methods, so we have not reproduced them.

5. The square-root boundary

The explicit formula (4.4) is often used as a convenient boundary for comparison purposes. Unfortunately, it cannot be used to test our approximations because it belongs to the same family of boundaries from which they are constructed, so that our approximations will reproduce it exactly! Not many other exact solutions are known or tabulated and the results of simulation are in general too coarse.

Keilson and Ross (1975) tabulated the probability for a standard Ornstein–Uhlenbeck process to cross a horizontal boundary at height ℓ before time τ. Interpolation at $t = \frac{1}{2}\log(1 + \tau)$ gives the probability of Brownian motion crossing the square-root boundary $\xi(t) = \ell\sqrt{1 + 1}$ before time t. Numerical differentiation then gives the fcd $g(t)$. The original table has four-figure accuracy so the last figure for $P(t)$ and $g(t)$ is not reliable. Also the values for very small t are not accurate. Tables 1a, 1b, compare the 'exact' fcd $g(t)$ for $\ell = 0.5, 1.0$, with the approximations $g_0(t)$ for the TA and the refinements $g_1(t), g_2(t)$ obtained by the two methods described. The TA results are in all cases too high. Of the other two, g_1 seems on the whole better than g_2. (The last column, g_3, refers to the results of Section 6.)

6. Another approach using images

The required fcd can also be approximated in the following way. Assume there is a distribution of images on the vertical axis at points θ in the interval $2\xi(0) \leq \theta < \infty$, with weights $-dK(\theta)$, which will reproduce the given absorbing boundary $\xi(t)$. Such a distribution must satisfy the integral equation

$$\frac{e^{-\xi^2(t)/2t}}{\sqrt{2\pi t}} = \int_{2\xi(0)}^{\infty} \frac{e^{-\{\xi(t) - \theta\}^2/2t}}{\sqrt{2\pi t}} dK(\theta), \tag{6.1}$$

which simplifies to

$$1 = \int_{2\xi(0)}^{\infty} e^{\theta\xi(t)/t-\theta^2/2t} dK(\theta). \tag{6.2}$$

Lerche (1986) studied this integral equation on the assumption that $dK(\theta)$ is non-negative. He showed that for such weights the resulting absorbing boundary $\xi(t)$ is unique and concave. But with a general $\xi(t)$, even if concave, it need not be true that $dK(\theta)$ is non-negative or that $\xi(t)$ is unique. In fact uniqueness is not a necessary requirement. What is essential is that the density $p(x,t)$ for the constrained Brownian motion is positive for all $x(t) < \xi(t)$, otherwise a new absorbing boundary will appear below $\xi(t)$ which competes with $\xi(t)$.

In general (6.2) is difficult to solve analytically, but an approximation can be sought by discretizing the problem. Weights $-\kappa_r$ are attached to images at θ_r where $2\xi(0) = \theta_1 < \theta_2 < \ldots < \theta_N$, and (6.2) is replaced by the set of linear equations

$$1 = \sum_{r=1}^{N} \kappa_r e^{\theta_r\bar{\xi}(t_s)/t_s - \theta_r^2/2t_s}, \qquad s = 1, 2, \ldots, N, \tag{6.3}$$

where $t_1 < t_2 < \ldots < t_N$ is a set of times chosen to cover the time-range of interest. The weights found by solving (6.3) will then lead to an absorbing boundary $\bar{\xi}(t)$ which coincides with $\xi(t)$ at the specified times t_s, and the fcd is approximated by

$$g(t) = \frac{1}{2\sqrt{2\pi}\,t^{3/2}} \sum_{r=1}^{N} \kappa_r\theta_r e^{-\{\bar{\xi}(t)-\theta_r\}^2/2t}. \tag{6.4}$$

The condition that $p(x,t) > 0$ for $x(t) < \xi(t)$ can be checked in any particular case.

It is not obvious how best to choose the values of θ_r for a given value of N and a specified range of t_s. As an exploratory example we have considered the square-root boundary with $\ell = 0.5$ and $\ell = 1.0$. Taking $N = 5$ and equally spaced values of t_s from 0.05 to 1.00 when $\ell = 0.5$ and from 0.2 to 1.0 when $\mathscr{L} = 1.0$, the weights κ_r found for the indicated values of θ_r are shown in Table 2. The corresponding values of the approximate fcd $g_3(t)$ are shown in the final columns of Tables 1a and 1b. The fit is good over the chosen ranges of t_s but begins to deteriorate as t increases beyond this range. Incidentally the observed values of κ_r suggest that $dK(\theta)$ in (6.2) may oscillate about zero for this $\xi(t)$.

One would expect to get a more accurate fit by choosing N large and arranging the values of t_s to cover the whole effective time-range as estimated from the TA. Unfortunately large

Table 2. Square-root boundary: discretized images

$\ell = 0.5$			$\ell = 1.0$		
t_s	θ_r	κ_r	t_s	θ_r	κ_r
0.0500	1.00	0.7830	0.2	2.00	0.3919
0.2875	1.25	−0.0406	0.4	2.25	−0.0777
0.5250	1.50	0.1916	0.6	2.50	0.3162
0.7625	1.75	−0.2652	0.8	2.75	−0.3957
1.0000	2.00	0.1794	1.0	3.00	0.2791

N leads to numerical instability. A way of reducing the value of N while retaining comparable accuracy might be to replace summation in (6.3) by a more accurate numerical integration formula, thus making (6.3) more like (6.2).

7. Two-sided boundaries

We now consider two-sided boundaries with branches $\xi_+(t) > 0, \xi_-(t) < 0$. Asymptotically the one-sided TA for each branch is valid, but from the present point of view the accuracy can be improved by considering both branches together. The discussion is confined to symmetric boundaries where $\xi_+(t) = \xi(t)$, $\xi_-(t) = -\xi(t)$, but the argument can be extended to the more general case.

The two-sided tangent approximation replaces $\xi_+(t)$ and $\xi_-(t)$ at $t = T$ by the tangents

$$\bar{\xi}_+(t, T) = \alpha + \beta t, \qquad \bar{\xi}_-(t, T) = -(\alpha + \beta t), \tag{7.1}$$

where $\alpha = \xi(T) - T\xi'(T), \beta = \xi'(T)$. For the density $p(x, t)$ to vanish on both $\bar{\xi}_+$ and $\bar{\xi}_-$ an infinite series of images is required. When $\beta = 0$ we have the classical Wald test with parallel boundaries at $\pm\alpha$. In that case the density is

$$\bar{p}(x, t) = \frac{1}{\sqrt{2\pi t}} \sum_{r=-\infty}^{\infty} (-1)^r e^{-(x-2\alpha r)^2/2t}. \tag{7.2}$$

This is derived in the usual way by starting with the terms for $r = 0, 1$ corresponding to the single boundary at α produced by an image at 2α. To incorporate the boundary at $-\alpha$ successive images are then placed at $-4\alpha, 6\alpha \ldots$, each successive image being added to correct the disturbance produced by the previous image.

When $\beta \neq 0$ all that is required is to modify (7.2) by multiplying the rth term by $\exp(-2\alpha\beta r^2)$, giving the density

$$\bar{p}(x, t) = \frac{1}{\sqrt{2\pi t}} \sum_{r=-\infty}^{\infty} (-1)^r e^{-2\alpha\beta r^2 - (x-2\alpha r)^2/2t} \tag{7.3}$$

with the corresponding fcd, conditional on crossing the upper boundary,

$$\bar{g}(t) = \frac{\alpha}{\sqrt{2\pi}\, t^{3/2}} \sum_{r=1}^{\infty} (-1)^{r-1} r e^{-2\alpha\beta r^2} \{e^{-(\bar{\xi}-2\alpha r)^2/2t} - e^{-(\bar{\xi}+2\alpha r)^2/2t}\}. \tag{7.4}$$

Daniels (1982) established result (7.3) in a discussion of T.W. Anderson's sequential test with converging linear boundaries. It can, however, be verified directly in the following way. Rearrange the terms of (7.3) by pairing $r = -2s$ and $r = 2s + 1$:

$$\begin{aligned}\bar{p}(x, t) &= \frac{1}{\sqrt{2\pi t}} \sum_{s=-\infty}^{\infty} \{e^{-2\alpha\beta(2s)^2 - \{x+2\alpha(2s)\}^2/2t} - e^{-2\alpha\beta(2s+1)^2 - \{x-2\alpha(2s+1)\}^2/2t}\} \\ &= \frac{1}{\sqrt{2\pi t}} \sum_{s=-\infty}^{\infty} e^{-2\alpha\beta(2s)^2 - \{x+2\alpha(2s)\}^2/2t} \{1 - e^{2\alpha(4s+1)(x-\alpha-\beta t)/t}\}. \\ &= 0 \quad \text{on} \quad \bar{\xi}_+ = \alpha + \beta t.\end{aligned}$$

Table 3. Two-sided square-root boundary: (a) $\ell = 0.5$; (b) $\ell = 1.0$

(a) 'exact' TA for ξ_+			(b) 'exact' TA for ξ_+		
t	g	g_0	t	g	g_0
0.05	1.2874	1.2928	0.2	0.2221	0.2230
0.10	1.6002	1.5965	0.3	0.2812	0.2805
0.15	1.3169	1.3170	0.4	0.2761	0.2779
0.20	1.0427	1.0449	0.5	0.2575	0.2569
0.25	0.8304	0.8308	0.6	0.2303	0.2324
0.30	0.6624	0.6661	0.7	0.2071	0.2091
0.35	0.5352	0.5388	0.8	0.1864	0.1884
0.40	0.4362	0.4395	0.9	0.1678	0.1702
0.45	0.3546	0.3614	1.0	0.1518	0.1545
0.50	0.2959	0.2984			
			1.2	0.1256	0.1290
0.6	0.2026	0.2098	1.4	0.1056	0.1095
0.7	0.1440	0.1508	1.6	0.0900	0.0944
0.8	0.1047	0.1108	1.8	0.0776	0.0824
0.9	0.0808	0.0831	2.0	0.0676	0.0727
1.0	0.0587	0.0635			
			2.5	0.0497	0.0553
1.2	0.0325	0.0388	3.0	0.0380	0.0440
1.4	0.0202	0.0251	3.5	0.0301	0.0362
1.6	0.0128	0.0170	4.0	0.0244	0.0305
1.8	0.0086	0.0119	4.5	0.0202	0.0262
2.0	0.0057	0.0087	5.0	0.0169	0.0229
2.5	0.0039	0.0044	5.5	0.0144	0.0203
3.0	0.0021	0.0025	6.0	0.0124	0.0182

$\xi(t) = \pm\ell\sqrt{t+1}$

Alternatively the terms of (7.3) can be rearranged by pairing $r = 2s - 1$ and $r = -2s$:

$$\bar{p}(x,t) = \frac{1}{\sqrt{2\pi t}} \sum_{s=-\infty}^{\infty} \{e^{-2\alpha\beta(2s)^2 - \{x+2\alpha(2s)\}^2/2t} - e^{-2\alpha\beta(2s-1)^2 - \{x-2\alpha(2s-1)\}^2/2t}\}$$

$$= \frac{1}{\sqrt{2\pi t}} \sum_{s=-\infty}^{\infty} e^{-2\alpha\beta(2s)^2 - \{x+2\alpha(2s)\}^2/2t}\{1 - e^{-2\alpha(4s-1)(x+\alpha+\beta t)/t}\}$$

$$= 0 \quad \text{on} \quad \bar{\xi}_- = -(\alpha + \beta t).$$

The particular pairing of the terms is suggested by the order in which the images were originally introduced.

Tables 3a and 3b compare the two-sided TA with the 'exact' values for $\ell = 0.5$ and 1.0. They may be compared with the one-sided values in Table 1. The two-sided TA is consistently greater than 'exact'. We have not yet extended our improved approximations to two-sided boundaries.

8. General comments

We have considered one particular boundary which is convex and can be small for small t. Other boundaries may need different approximations – for example, using the first approach, if $\xi(t)$ is not too small near $t = 0$ it may be worth fitting $\bar{\xi}(t, T)$ up to the third derivative. Clearly, improved approximations could be arrived at by including more images but the computations become much more elaborate. The second approach provides an alternative way of introducing more images without complicating the formulae, relying instead on a direct numerical solution.

The techniques presented here need a firmer mathematical basis. In particular, some way has to be found for providing bounds on the fit when the exact $g(t)$ is not known. Comparisons with particular cases can be misleading, as the example of (4.4) shows, though they can also be suggestive.

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