Asymptotics of empirical copula processes under non-restrictive smoothness assumptions

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Weak convergence of the empirical copula process is shown to hold under the assumption that the first-order partial derivatives of the copula exist and are continuous on certain subsets of the unit hypercube. The assumption is non-restrictive in the sense that it is needed anyway to ensure that the candidate limiting process exists and has continuous trajectories. In addition, resampling methods based on the multiplier central limit theorem, which require consistent estimation of the first-order derivatives, continue to be valid. Under certain growth conditions on the second-order partial derivatives that allow for explosive behavior near the boundaries, the almost sure rate in Stute's representation of the empirical copula process can be recovered. The conditions are verified, for instance, in the case of the Gaussian copula with full-rank correlation matrix, many Archimedean copulas, and many extreme-value copulas.

Keywords: Archimedean copula; Brownian bridge; empirical copula; empirical process; extreme-value copula; Gaussian copula; multiplier central limit theorem; tail dependence; weak convergence

1. Introduction

A flexible and versatile way to model dependence is via copulas. A fundamental tool for inference is the empirical copula, which basically is equal to the empirical distribution function of the sample of multivariate ranks, rescaled to the unit interval. The asymptotic behavior of the empirical copula process was studied in, amongst others, Stute [29], Gänssler and Stute [10], Chapter 5, van der Vaart and Wellner [32], page 389, Tsukahara [30,31], Fermanian *et al.* [9], Ghoudi and Rémillard [15], and van der Vaart and Wellner [33]. Weak convergence is shown typically for copulas that are continuously differentiable on the closed hypercube, and rates of convergence of certain remainder terms have been established for copulas that are twice continuously differentiable on the closed hypercube. Unfortunately, for many (even most) popular copula families, even the first-order partial derivatives of the copula fail to be continuous at some boundary points of the hypercube.

Example 1.1 (Tail dependence). Let C be a bivariate copula with first-order partial derivatives $\dot{C}_1$ and $\dot{C}_2$ and positive lower tail dependence coefficient $\lambda = \lim_{u \downarrow 0} C(u, u)/u > 0$. On the one hand, $\dot{C}_1(u, 0) = 0$ for all $u \in [0, 1]$ by the fact that $C(u, 0) = 0$ for all $u \in [0, 1]$. On the other hand, $\dot{C}_1(0, v) = \lim_{u \downarrow 0} C(u, v)/u \geq \lambda > 0$ for all $v \in (0, 1]$. It follows that $\dot{C}_1$ cannot be continuous at the point $(0, 0)$; similarly for $\dot{C}_2$. For copulas with a positive upper tail dependence coefficient, the first-order partial derivatives cannot be continuous at the point $(1, 1)$.

Likewise, for the Gaussian copula with non-zero correlation parameter ρ, the first-order partial derivatives fail to be continuous at the points $(0, 0)$ and $(1, 1)$ if $\rho > 0$ and at the points $(0, 1)$ and $(1, 0)$ if $\rho < 0$; see also Example 5.1 below. As a consequence, the cited results on the empirical copula process do not apply to such copulas. This problem has been largely ignored in the literature, and unjustified calls to the above results abound. A notable exception is the paper by Omelka, Gijbels, and Veraverbeke [22]. On page 3031 of that paper, it is claimed that weak convergence of the empirical copula process still holds if the first-order partial derivatives are continuous at $[0, 1]^2 \setminus \{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

It is the aim of this paper to remedy the situation by showing that the earlier cited results on the empirical copula process actually do hold under a much less restrictive assumption, including indeed many copula families that were hitherto excluded. The assumption is non-restrictive in the sense that it is needed anyway to ensure that the candidate limiting process exists and has continuous trajectories. The results are stated and proved in general dimensions. When specialized to the bivariate case, the condition is substantially weaker still than the above-mentioned condition in Omelka, Gijbels, and Veraverbeke [22].

Let F be a d-variate cumulative distribution function (c.d.f.) with continuous margins $F_1, \ldots, F_d$ and copula C, that is, $F(x) = C(F_1(x_1), \ldots, F_d(x_d))$ for $x \in \mathbb{R}^d$. Let $X_1, \ldots, X_n$ be independent random vectors with common distribution F, where $X_i = (X_{i1}, \ldots, X_{id})$. The empirical copula was defined in Deheuvels [5] as

$$C_n(u) = F_n(F_{n1}^{-1}(u_1), \ldots, F_{nd}^{-1}(u_d)), \qquad u \in [0, 1]^d, \tag{1.1}$$

where F_n and F_{nj} are the empirical joint and marginal cdfs of the sample and where F_{nj}^{-1} is the marginal quantile function of the jth coordinate sample; see Section 2 below for details. The empirical copula C_n is invariant under monotone increasing transformations on the data, so it depends on the data only through the ranks. Indeed, up to a difference of order $1/n$, the empirical copula can be seen as the empirical c.d.f. of the sample of normalized ranks, as, for instance, in Rüschendorf [25]. For convenience, the definition in equation (1.1) will be employed throughout the paper.

The empirical copula process is defined by

$$\mathbb{C}_n = \sqrt{n}(C_n - C), \tag{1.2}$$

to be seen as a random function on $[0, 1]^d$. We are essentially interested in the asymptotic distribution of $\mathbb{C}_n$ in the space $\ell^\infty([0, 1]^d)$ of bounded functions from $[0, 1]^d$ into $\mathbb{R}$ equipped with the topology of uniform convergence. Weak convergence is to be understood in the sense used in the monograph by van der Vaart and Wellner [32], in particular their Definition 1.3.3.

Although the empirical copula is itself a rather crude estimator of C, it plays a crucial rule in more sophisticated inference procedures on C, much in the same way as the empirical c.d.f. F_n is a fundamental object for creating and understanding inference procedures on F or parameters thereof. For instance, the empirical copula is a basic building block when estimating copula densities (Chen and Huang [3], Omelka, Gijbels and Veraverbeke [22]) or dependence measures and functions (Schmid *et al.* [27], Genest and Segers [14]), for testing for independence (Genest and Rémillard [12], Genest, Quessy and Rémillard [11], Kojadinovic and Holmes [17]), for testing for shape constraints (Denuit and Scaillet [6], Scaillet [26], Kojadinovic and Yan [18]), for resampling (Rémillard and Scaillet [24], Bücher and Dette [2]), and so forth.

After some preliminaries in Section 2, the principal result of the paper is given in Section 3, stating weak convergence of the empirical copula process under the condition that for every $j \in \{1, \ldots, d\}$, the jth first-order partial derivative $\dot{C}_j$ exists and is continuous on the set $\{u \in [0, 1]^d : 0 < u_j < 1\}$. The condition is non-restrictive in the sense that it is necessary for the candidate limiting process to exist and have continuous trajectories. Moreover, the resampling method based on the multiplier central limit theorem proposed in Rémillard and Scaillet [24] is shown to be valid under the same condition. Section 4 provides a refinement of the main result: under certain bounds on the second-order partial derivatives that allow for explosive behavior near the boundaries, the almost sure error bound on the remainder term in Stute [29] and Tsukahara [31] can be entirely recovered. The result hinges on an exponential inequality for a certain oscillation modulus of the multivariate empirical process detailed in the Appendix; the inequality is a generalization of a similar inequality in Einmahl [7] and was communicated by Hideatsu Tsukahara. Section 5 concludes the paper with a number of examples of copulas that do or do not verify certain sets of conditions.

2. Preliminaries

Let $X_i = (X_{i1}, \ldots, X_{id})$, $i \in \{1, 2, \ldots\}$, be independent random vectors with common c.d.f. F whose margins $F_1, \ldots, F_d$ are continuous and whose copula is denoted by C. Define $U_{ij} = F_j(X_{ij})$ for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, d\}$. The random vectors $U_i = (U_{i1}, \ldots, U_{id})$ constitute an i.i.d. sample from C. Consider the following empirical distribution functions: for $x \in \mathbb{R}^d$ and for $u \in [0, 1]^d$,

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{(-\infty, x]}(X_i), \qquad F_{nj}(x_j) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{(-\infty, x_j]}(X_{ij}),$$

$$G_n(u) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{[0, u]}(U_i), \qquad G_{nj}(u_j) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{[0, u_j]}(U_{ij}).$$

Here, order relations on vectors are to be interpreted componentwise, and $\mathbb{1}_A(x)$ is equal to 1 or 0 according to whether x is an element of A or not. Let $X_{1:n, j} < \cdots < X_{n:n, j}$ and $U_{1:n, j} < \cdots < U_{n:n, j}$ be the vectors of ascending order statistics of the jth coordinate samples $X_{1j}, \ldots, X_{nj}$ and $U_{1j}, \ldots, U_{nj}$, respectively. The marginal quantile functions associated to F_{nj} and G_{nj} are

$$F_{nj}^{-1}(u_j) = \inf\{x \in \mathbb{R} : F_{nj}(x) \geq u_j\}$$

$$= \begin{cases} X_{k:n, j}, & \text{if } (k-1)/n < u_j \leq k/n, \\ -\infty, & \text{if } u_j = 0; \end{cases}$$

$$G_{nj}^{-1}(u_j) = \inf\{u \in [0, 1] : G_{nj}(u) \geq u_j\}$$

$$= \begin{cases} U_{k:n, j}, & \text{if } (k-1)/n < u_j \leq k/n, \\ 0, & \text{if } u_j = 0. \end{cases}$$

Some thought shows that $X_{ij} \leq F_{nj}^{-1}(u_j)$ if and only if $U_{ij} \leq G_{nj}^{-1}(u_j)$, for all $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, d\}$ and $u_j \in [0, 1]$. It follows that the empirical copula in equation (1.1) is given by

$$C_n(u) = G_n(G_{n1}^{-1}(u_1), \ldots, G_{nd}^{-1}(u_d)).$$

In particular, without loss of generality we can work directly with the sample $U_1, \ldots, U_n$ from C.

The empirical processes associated to the empirical distribution functions G_n and G_{nj} are given by

$$\alpha_n(u) = \sqrt{n}\big(G_n(u) - C(u)\big), \qquad \alpha_{nj}(u_j) = \sqrt{n}\big(G_{nj}(u_j) - u_j\big), \tag{2.1}$$

for $u \in [0, 1]^d$ and $u_j \in [0, 1]$. Note that $\alpha_{nj}(0) = \alpha_{nj}(1) = 0$ almost surely. We have

$$\alpha_n \rightsquigarrow \alpha \qquad (n \to \infty)$$

in $\ell^\infty([0, 1]^d)$, the arrow '$\rightsquigarrow$' denoting weak convergence as in Definition 1.3.3 in van der Vaart and Wellner [32]. The limit process α is a C-Brownian bridge, that is, a tight Gaussian process, centered and with covariance function

$$\mathrm{cov}(\alpha(u), \alpha(v)) = C(u \wedge v) - C(u)C(v),$$

for $u, v \in [0, 1]^d$; here $u \wedge v = (\min(u_1, v_1), \ldots, \min(u_d, v_d))$. Tightness of the process α and continuity of its mean and covariance functions implies the existence of a version of α with continuous trajectories. Without loss of generality, we assume henceforth that α is such a version.

For $j \in \{1, \ldots, d\}$, let e_j be the jth coordinate vector in $\mathbb{R}^d$. For $u \in [0, 1]^d$ such that $0 < u_j < 1$, let

$$\dot{C}_j(u) = \lim_{h \to 0} \frac{C(u + he_j) - C(u)}{h},$$

be the jth first-order partial derivative of C, provided it exists.

Condition 2.1. *For each $j \in \{1, \ldots, d\}$, the jth first-order partial derivative $\dot{C}_j$ exists and is continuous on the set $V_{d,j} := \{u \in [0, 1]^d : 0 < u_j < 1\}$.*

Henceforth, assume Condition 2.1 holds. To facilitate notation, we will extend the domain of $\dot{C}_j$ to the whole of $[0, 1]^d$ by setting

$$\dot{C}_j(u) = \begin{cases} \limsup\limits_{h \downarrow 0} \dfrac{C(u + he_j)}{h}, & \text{if } u \in [0, 1]^d, u_j = 0, \\[2ex] \limsup\limits_{h \downarrow 0} \dfrac{C(u) - C(u - he_j)}{h}, & \text{if } u \in [0, 1]^d, u_j = 1. \end{cases} \tag{2.2}$$

In this way, $\dot{C}_j$ is defined everywhere on $[0, 1]^d$, takes values in $[0, 1]$ (because $|C(u) - C(v)| \leq \sum_{j=1}^d |u_j - v_j|$), and is continuous on the set $V_{d,j}$, by virtue of Condition 2.1. Also note that $\dot{C}_j(u) = 0$ as soon as $u_i = 0$ for some $i \neq j$.

3. Weak convergence

In Proposition 3.1, Condition 2.1 is shown to be sufficient for the weak convergence of the empirical copula process $\mathbb{C}_n$. In contrast to earlier results, Condition 2.1 does not require existence or continuity of the partial derivatives on certain boundaries. Although the improvement is seemingly small, it dramatically enlarges the set of copulas to which it applies; see Section 5. Similarly, the unconditional multiplier central limit theorem for the empirical copula process based on estimated first-order partial derivatives continues to hold (Proposition 3.2). This result is useful as a justification of certain resampling procedures that serve to compute critical values for test statistics based on the empirical copula in case of a composite null hypothesis, for instance, in the context of goodness-of-fit testing as in Kojadinovic and Yan [18].

Assume first that the first-order partial derivatives $\dot{C}_j$ exist and are continuous throughout the closed hypercube $[0, 1]^d$. For $u \in [0, 1]^d$, define

$$\mathbb{C}(u) = \alpha(u) - \sum_{j=1}^{d} \dot{C}_j(u) \alpha_j(u_j), \tag{3.1}$$

where $\alpha_j(u_j) = \alpha(1, \ldots, 1, u_j, 1, \ldots, 1)$, the variable u_j appearing at the jth entry. By continuity of $\dot{C}_j$ throughout $[0, 1]^d$, the trajectories of $\mathbb{C}$ are continuous. From Fermanian *et al.* [9] and Tsukahara [31] we learn that $\mathbb{C}_n \leadsto \mathbb{C}$ as $n \to \infty$ in the space $\ell^{\infty}([0, 1]^d)$.

The structure of the limit process $\mathbb{C}$ in equation (3.1) can be understood as follows. The first term, $\alpha(u)$, would be there even if the true margins F_j were used rather than their empirical counterparts F_{nj}. The terms $-\dot{C}_j(u) \alpha_j(u_j)$ encode the impact of not knowing the true quantiles $F_j^{-1}(u_j)$ and having to replace them by the empirical quantiles $F_{nj}^{-1}(u_j)$. The minus sign comes from the Bahadur–Kiefer result stating that $\sqrt{n}(G_{nj}^{-1}(u_j) - u_j)$ is asymptotically undistinguishable from $-\sqrt{n}(G_{nj}(u_j) - u_j)$; see, for instance, Shorack and Wellner [28], Chapter 15. The partial derivative $\dot{C}_j(u)$ quantifies the sensitivity of C with respect to small deviations in the jth margin.

Now consider the same process $\mathbb{C}$ as in equation (3.1) but under Condition 2.1 and with the domain of the partial derivatives extended to $[0, 1]^d$ as in equation (2.2). Since the trajectories of α are continuous and since $\alpha_j(0) = \alpha_j(1) = 0$ for each $j \in \{1, \ldots, d\}$, the trajectories of $\mathbb{C}$ are continuous, even though $\dot{C}_j$ may fail to be continuous at points $u \in [0, 1]^d$, such that $u_j \in \{0, 1\}$. The process $\mathbb{C}$ is the weak limit in $\ell^{\infty}([0, 1]^d)$ of the sequence of processes

$$\tilde{\mathbb{C}}_n(u) = \alpha_n(u) - \sum_{j=1}^{d} \dot{C}_j(u) \alpha_{nj}(u_j), \qquad u \in [0, 1]^d. \tag{3.2}$$

The reason is that the map from $\ell^{\infty}([0, 1]^d)$ into itself that sends a function f to $f - \sum_{j=1}^{d} \dot{C}_j \pi_j(f)$, where $(\pi_j(f))(u) = f(1, \ldots, 1, u_j, 1, \ldots, 1)$, is linear and bounded.

Proposition 3.1. *If Condition* 2.1 *holds, then, with* $\tilde{\mathbb{C}}_n$ *as in equation* (3.2),

$$\sup_{u \in [0,1]^d} |\mathbb{C}_n(u) - \tilde{\mathbb{C}}_n(u)| \xrightarrow{p} 0 \qquad (n \to \infty).$$

As a consequence, in $\ell^\infty([0, 1]^d)$,

$$\mathbb{C}_n \rightsquigarrow \mathbb{C} \qquad (n \to \infty).$$

Proof. It suffices to show the first statement of the proposition. For $u \in [0, 1]^d$, put

$$R_n(u) = |\mathbb{C}_n(u) - \tilde{\mathbb{C}}_n(u)|, \qquad u \in [0, 1]^d.$$

If $u_j = 0$ for some $j \in \{1, \ldots, d\}$, then obviously $\mathbb{C}_n(u) = \tilde{\mathbb{C}}_n(u) = 0$, so $R_n(u) = 0$ as well. The vector of marginal empirical quantiles is denoted by

$$v_n(u) = (G_{n1}^{-1}(u_1), \ldots, G_{nd}^{-1}(u_d)), \qquad u \in [0, 1]^d. \tag{3.3}$$

We have

$$\begin{aligned}
\mathbb{C}_n(u) &= \sqrt{n}\big(C_n(u) - C(u)\big) \\
&= \sqrt{n}\{G_n(v_n(u)) - C(v_n(u))\} + \sqrt{n}\{C(v_n(u)) - C(u)\} \tag{3.4} \\
&= \alpha_n(v_n(u)) + \sqrt{n}\{C(v_n(u)) - C(u)\}.
\end{aligned}$$

Since α_n converges weakly in $\ell^\infty([0, 1]^d)$ to a C-Brownian bridge α, whose trajectories are continuous, the sequence $(\alpha_n)_n$ is asymptotically uniformly equicontinuous; see Theorem 1.5.7 and Addendum 1.5.8 in van der Vaart and Wellner [32]. As $\sup_{u_j \in [0,1]} |G_{nj}^{-1}(u_j) - u_j| \to 0$ almost surely, it follows that

$$\sup_{u \in [0,1]^d} |\alpha_n(v_n(u)) - \alpha_n(u)| \xrightarrow{P} 0 \qquad (n \to \infty).$$

Fix $u \in [0, 1]^d$. Put $w(t) = u + t\{v_n(u) - u\}$ and $f(t) = C(w(t))$ for $t \in [0, 1]$. If $u \in (0, 1]^d$, then $v_n(u) \in (0, 1)^d$, and therefore $w(t) \in (0, 1)^d$ for all $t \in (0, 1]$, as well. By Condition 2.1, the function f is continuous on $[0, 1]$ and continuously differentiable on $(0, 1)$. By the mean value theorem, there exists $t^* = t_n(u) \in (0, 1)$ such that $f(1) - f(0) = f'(t^*)$, yielding

$$\sqrt{n}\{C(v_n(u)) - C(u)\} = \sum_{j=1}^d \dot{C}_j(w(t^*))\sqrt{n}\big(G_{nj}^{-1}(u_j) - u_j\big). \tag{3.5}$$

If one or more of the components of u are zero, then the above display remains true as well, no matter how $t^* \in (0, 1)$ is defined, because both sides of the equation are equal to zero. In particular, if $u_k = 0$ for some $k \in \{1, \ldots, d\}$, then the kth term on the right-hand side vanishes because $G_{nk}^{-1}(0) = 0$ whereas the terms with index $j \neq k$ vanish because the kth component of the vector $w(t^*)$ is zero, and thus the first-order partial derivatives $\dot{C}_j$ vanish at this point.

It is known since Kiefer [16] that

$$\sup_{u_j \in [0,1]} \big|\sqrt{n}\big(G_{nj}^{-1}(u_j) - u_j\big) + \alpha_{nj}(u_j)\big| \xrightarrow{P} 0 \qquad (n \to \infty).$$

Since $0 \leq \dot{C}_j \leq 1$, we find

$$\sup_{u \in [0,1]^d} \left| \sqrt{n} \{ C(v_n(u)) - C(u) \} + \sum_{j=1}^d \dot{C}_j \left(u + t^* \{ v_n(u) - u \} \right) \alpha_{nj}(u_j) \right| \overset{p}{\to} 0$$

as $n \to \infty$. It remains to be shown that

$$\sup_{u \in [0,1]^d} D_{nj}(u) \overset{p}{\to} 0 \qquad (n \to \infty)$$

for all $j \in \{1, \ldots, d\}$, where

$$D_{nj}(u) = \left| \dot{C}_j \left(u + t^* \{ v_n(u) - u \} \right) - \dot{C}_j(u) \right| |\alpha_{nj}(u_j)|. \tag{3.6}$$

Fix $\varepsilon > 0$ and $\delta \in (0, 1/2)$. Split the supremum over $u \in [0,1]^d$ according to the cases $u_j \in [\delta, 1-\delta]$ on the one hand and $u_j \in [0, \delta) \cup (1-\delta, 1]$ on the other hand. We have

$$\Pr\left(\sup_{u \in [0,1]^d} D_{nj}(u) > \varepsilon \right) \leq \Pr\left(\sup_{u \in [0,1]^d, u_j \in [\delta, 1-\delta]} D_{nj}(u) > \varepsilon \right)$$

$$+ \Pr\left(\sup_{u \in [0,1]^d, u_j \notin [\delta, 1-\delta]} D_{nj}(u) > \varepsilon \right).$$

Since $\sup_{u \in [0,1]^d} |v_n(u) - u| \to 0$ almost surely, since $\dot{C}_j$ is uniformly continuous on $\{ u \in [0,1]^d : \delta/2 \leq u_j \leq 1 - \delta/2 \}$, and since the sequence $\sup_{u_{nj} \in [0,1]} |\alpha_{nj}(u_j)|$ is bounded in probability, the first probability on the right-hand side of the previous display converges to zero. As $|x - y| \leq 1$ whenever $x, y \in [0, 1]$ and since $0 \leq \dot{C}_j(w) \leq 1$ for all $w \in [0,1]^d$, the second probability on the right-hand side of the previous display is bounded by

$$\Pr\left(\sup_{u_j \in [0, \delta) \cup (1-\delta, 1]} |\alpha_{nj}(u_j)| > \varepsilon \right).$$

By the portmanteau lemma, the lim sup of this probability as $n \to \infty$ is bounded by

$$\Pr\left(\sup_{u_j \in [0, \delta) \cup (1-\delta, 1]} |\alpha_j(u_j)| \geq \varepsilon \right).$$

The process α_j being a standard Brownian bridge, the above probability can be made smaller than an arbitrarily chosen $\eta > 0$ by choosing δ sufficiently small. We find

$$\limsup_{n \to \infty} \Pr\left(\sup_{u \in [0,1]^d} D_{nj}(u) > \varepsilon \right) \leq \eta.$$

As η was arbitrary, the claim is proven. $\square$

An alternative to the direct proof above is to invoke the functional delta method as in Fermanian *et al.* [9]. Required then is a generalization of Lemma 2 in the cited paper asserting

Hadamard differentiability of a certain functional under Condition 2.1. This program is carried out for the bivariate case in Bücher [1], Lemma 2.6.

For purposes of hypothesis testing or confidence interval construction, resampling procedures are often required; see the references in the introduction. In Fermanian *et al.* [9], a bootstrap procedure for the empirical copula process is proposed, whereas in Rémillard and Scaillet [24], a method based on the multiplier central limit theorem is employed. Yet another method is proposed in Bücher and Dette [2]. In the latter paper, the finite-sample properties of all these methods are compared in a simulation study, and the multiplier approach by Rémillard and Scaillet [24] is found to be best overall. Although the latter approach requires estimation of the first-order partial derivatives, it remains valid under Condition 2.1, allowing for discontinuities on the boundaries.

Let $\xi_1, \xi_2, \ldots$ be an i.i.d. sequence of random variables, independent of the random vectors $X_1, X_2, \ldots$, and with zero mean, unit variance, and such that $\int_0^\infty \sqrt{\Pr(|\xi_1| > x)}\, dx < \infty$. Define

$$\alpha_n'(u) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i \big(\mathbb{1}\{X_{i1} \le F_{n1}^{-1}(u_1), \ldots, X_{id} \le F_{nd}^{-1}(u_d)\} - C_n(u)\big). \tag{3.7}$$

In $(\ell^\infty([0,1]^d))^2$, we have by Lemma A.1 in Rémillard and Scaillet [24],

$$(\alpha_n, \alpha_n') \rightsquigarrow (\alpha, \alpha') \qquad (n \to \infty), \tag{3.8}$$

where α' is an independent copy of α. Further, let $\hat{\dot{C}}_{nj}(u)$ be an estimator of $\dot{C}_j(u)$; for instance, apply finite differencing to the empirical copula at a spacing proportional to $n^{-1/2}$ as in Rémillard and Scaillet [24]. Define

$$\mathbb{C}_n'(u) = \alpha_n'(u) - \sum_{j=1}^d \hat{\dot{C}}_{nj}(u)\alpha_{nj}'(u_j), \tag{3.9}$$

where $\alpha_{nj}'(u_j) = \alpha_n'(1, \ldots, 1, u_j, 1, \ldots, 1)$, the variable u_j appearing at the jth coordinate.

Proposition 3.2. *Assume Condition 2.1. If there exists a constant K such that $|\hat{\dot{C}}_{nj}(u)| \le K$ for all n, j, u, and if*

$$\sup_{u \in [0,1]^d : u_j \in [\delta, 1-\delta]} |\hat{\dot{C}}_{nj}(u) - \dot{C}_j(u)| \overset{p}{\to} 0 \qquad (n \to \infty) \tag{3.10}$$

for all $\delta \in (0, 1/2)$ and all $j \in \{1, \ldots, d\}$, then in $(\ell^\infty([0,1]^d))^2$, we have

$$(\mathbb{C}_n, \mathbb{C}_n') \rightsquigarrow (\mathbb{C}, \mathbb{C}') \qquad (n \to \infty),$$

where $\mathbb{C}'$ is an independent copy of $\mathbb{C}$.

Proof. Recall the process α_n' in equation (3.7), and define

$$\tilde{\mathbb{C}}_n'(u) = \alpha_n'(u) - \sum_{j=1}^d \dot{C}_j(u)\alpha_{nj}'(u_j), \qquad u \in [0,1]^d.$$

The difference with the process $\mathbb{C}'_n$ in equation (3.9) is that the true partial derivatives of C are used rather than the estimated ones. By Proposition 3.1 and equation (3.8), we have

$$(\mathbb{C}_n, \tilde{\mathbb{C}}'_n) \rightsquigarrow (\mathbb{C}, \mathbb{C}') \qquad (n \to \infty)$$

in $(\ell^\infty([0, 1]^d))^2$. Moreover,

$$|\mathbb{C}'_n(u) - \tilde{\mathbb{C}}'_n(u)| \leq \sum_{j=1}^{d} |\hat{\dot{C}}_{nj}(u) - \dot{C}_j(u)||\alpha'_{nj}(u_j)|.$$

It suffices to show that each of the d terms on the right-hand side converges to 0 in probability, uniformly in $u \in [0, 1]^d$. The argument is similar to the one at the end of the proof of Proposition 3.1. Pick $\delta \in (0, 1/2)$, and split the supremum according to the cases $u_j \in [\delta, 1 - \delta]$ and $u_j \in [0, \delta) \cup (1 - \delta, 1]$. For the first case, use equation (3.10) together with tightness of α'_{nj}. For the second case, use the assumed uniform boundedness of the partial derivative estimators and the fact that the limit process $\hat{\alpha}_j$ is a standard Brownian bridge, having continuous trajectories and vanishing at 0 and 1. □

4. Almost sure rate

Recall the empirical copula process $\mathbb{C}_n$ in equation (1.2) together with its approximation $\tilde{\mathbb{C}}_n$ in equation (3.2). If the second-order partial derivatives of C exist and are continuous on $[0, 1]^d$, then the original result by Stute [29], proved in detail in Tsukahara [30], reinforces the first claim of Proposition 3.1 to

$$\sup_{u \in [0,1]^d} |\mathbb{C}_n(u) - \tilde{\mathbb{C}}_n(u)|$$
$$= O(n^{-1/4}(\log n)^{1/2}(\log\log n)^{1/4}) \qquad (n \to \infty) \text{ almost surely.} \tag{4.1}$$

For many copulas, however, the second-order partial derivatives explode near certain parts of the boundaries. The question then is how this affects the above rate. Recall $V_{d,j} = \{u \in [0, 1]^d : 0 < u_j < 1\}$ for $j \in \{1, \ldots, d\}$.

Condition 4.1. *For every $i, j \in \{1, \ldots, d\}$, the second-order partial derivative $\ddot{C}_{ij}$ is defined and continuous on the set $V_{d,i} \cap V_{d,j}$, and there exists a constant $K > 0$ such that*

$$|\ddot{C}_{ij}(u)| \leq K \min\left(\frac{1}{u_i(1 - u_i)}, \frac{1}{u_j(1 - u_j)}\right), \qquad u \in V_{d,i} \cap V_{d,j}.$$

Condition 4.1 holds, for instance, for absolutely continuous bivariate Gaussian copulas and for bivariate extreme-value copulas whose Pickands dependence functions are twice continuously differentiable and satisfy a certain bound; see Section 5.

Under Condition 4.1, the rate in equation (4.1) can be entirely recovered. The following proposition has benefited from a suggestion of John H.J. Einmahl leading to an improvement of a result

in an earlier version of the paper claiming a slightly slower rate. Furthermore, part of the proof is an adaptation due to Hideatsu Tsukahara of the end of the proof of Theorem 4.1 in Tsukahara [30], upon which the present result is based.

Proposition 4.2. *If Conditions* 2.1 *and* 4.1 *are verified, then equation* (4.1) *holds.*

Proof. Combining equations (3.4) and (3.5) in the proof of Proposition 3.1 yields

$$\mathbb{C}_n(u) = \alpha_n(v_n(u)) + \sum_{j=1}^{d} \dot{C}_j(w(t^*))\sqrt{n}\big(G_{nj}^{-1}(u_j) - u_j\big), \qquad u \in [0,1]^d,$$

with α_n the ordinary multivariate empirical process in equation (2.1), $v_n(u)$ the vector of marginal empirical quantiles in equation (3.3), and $w(t^*) = u + t^*\{v_n(u) - u\}$ a certain point on the line segment between u and $v_n(u)$ with local coordinate $t^* = t_n(u) \in (0, 1)$. In view of the definition of $\tilde{\mathbb{C}}_n(u)$ in equation (3.2), it follows that

$$\sup_{u \in [0,1]^d} |\mathbb{C}_n(u) - \tilde{\mathbb{C}}_n(u)| \leq \mathrm{I}_n + \mathrm{II}_n + \mathrm{III}_n,$$

where

$$\mathrm{I}_n = \sup_{u \in [0,1]^d} |\alpha_n(v_n(u)) - \alpha_n(u)|,$$

$$\mathrm{II}_n = \sum_{j=1}^{d} \sup_{u \in [0,1]^d} \big|\sqrt{n}\big(G_{nj}^{-1}(u_j) - u_j\big) + \alpha_{nj}(u_j)\big|,$$

$$\mathrm{III}_n = \sum_{j=1}^{d} \sup_{u \in [0,1]^d} D_{nj}(u),$$

with $D_{nj}(u)$ as defined in equation (3.6). By Kiefer [16], the term II_n is $O(n^{-1/4}(\log n)^{1/2} \times (\log\log n)^{1/4})$ as $n \to \infty$, almost surely. It suffices to show that the same almost sure rate is valid for I_n and III_n, too.

The term I_n. The argument is adapted from the final part of the proof of Theorem 4.1 in Tsukahara [30], and its essence was kindly provided by Hideatsu Tsukahara. We have

$$\mathrm{I}_n \leq M_n(A_n), \qquad A_n = \max_{j \in \{1,\dots,d\}} \sup_{u_j \in [0,1]} |G_{nj}^{-1}(u_j) - u_j|,$$

and $M_n(a)$ is the oscillation modulus of the multivariate empirical process α_n defined in equation (A.1). We will employ the exponential inequality for $\Pr\{M_n(a) \geq \lambda\}$ stated in Proposition A.1, which generalizes Inequality 3.5 in Einmahl [7]. Set $a_n = n^{-1/2}(\log\log n)^{1/2}$. By the Chung–Smirnov law of the iterated logarithm for empirical distribution functions (see, e.g.,

Shorack and Wellner [28], page 504),

$$\limsup_{n\to\infty} \frac{1}{a_n} \sup_{u_j\in[0,1]} |G_{nj}^{-1}(u_j) - u_j| = \limsup_{n\to\infty} \frac{1}{a_n} \sup_{v_j\in[0,1]} |v_j - G_{nj}(v_j)|$$

$$= 1/\sqrt{2} \qquad \text{almost surely.} \tag{4.2}$$

Choose $\lambda_n = 2K_2^{-1/2} n^{-1/4} (\log n)^{1/2} (\log\log n)^{1/4}$ for K_2 as in Proposition A.1. Since $\lambda_n/(n^{1/2} a_n) \to 0$ as $n \to \infty$, and since the function ψ in equation (A.2) below is decreasing with $\psi(0) = 1$, it follows that $\psi(\lambda_n/(n^{1/2} a_n)) \geq 1/2$ for sufficiently large n. Furthermore, we have

$$\sum_{n\geq 2} \frac{1}{a_n} \exp\left(-\frac{K_2\lambda_n^2}{2a_n}\right) = \sum_{n\geq 2} \frac{1}{n^{3/2}(\log\log n)^{1/2}} < \infty.$$

By the Borel–Cantelli lemma and Proposition (A.1), as $n \to \infty$,

$$\text{I}_n \leq M_n(A_n) \leq M_n(a_n) = \text{O}(n^{-1/4}(\log n)^{1/2}(\log\log n)^{1/4}) \qquad \text{almost surely.}$$

The term III_n. Let

$$\delta_n = n^{-1/2}(\log n)(\log\log n)^{-1/2}.$$

Fix $j \in \{1, \ldots, d\}$. We split the supremum of $D_{nj}(u)$ over $u \in [0, 1]^d$ according to the cases $u_j \in [0, \delta_n) \cup (1 - \delta_n, 1]$ and $u_j \in [\delta_n, 1 - \delta_n]$.

Since $0 \leq \dot{C}_j \leq 1$, the supremum over $u \in [0, 1]^d$ such that $u_j \in [0, \delta_n) \cup (1 - \delta_n, 1]$ is bounded by

$$\sup_{u\in[0,1]^d : u_j\in[0,\delta_n)\cup(1-\delta_n,1]} D_{nj}(u) \leq \sup_{u_j\in[0,\delta_n)\cup(1-\delta_n,1]} |\alpha_{nj}(u_j)|.$$

By Theorem 2.(iii) in Einmahl and Mason [8] applied to $(d, v, k_n) = (1, 1/2, n\delta_n)$, the previous supremum is of the order

$$\sup_{u_j\in[0,\delta_n)\cup(1-\delta_n]} |\alpha_{nj}(u_j)| = \text{O}(\delta_n^{1/2}(\log\log n)^{1/2})$$

$$= \text{O}(n^{-1/4}(\log n)^{1/2}(\log\log n)^{1/4}) \qquad (n \to \infty) \text{ almost surely.} \tag{4.3}$$

Next let $u \in [0, 1]^d$ be such that $\delta_n \leq u_j \leq 1 - \delta_n$. By Lemma 4.3 below and by convexity of the function $(0, 1) \ni s \mapsto 1/\{s(1 - s)\}$,

$$D_{nj}(u) = \left|\dot{C}_j\big(u + \lambda_n(u)\{v_n(u) - u\}\big) - \dot{C}_j(u)\right| |\alpha_{nj}(u_j)|$$

$$\leq K\max\left(\frac{1}{u_j(1 - u_j)}, \frac{1}{G_{nj}^{-1}(u_j)(1 - G_{nj}^{-1}(u_j))}\right) \|v_n(u) - u\|_1 |\alpha_{nj}(u_j)|,$$

with $\|x\|_1 = \sum_{j=1}^d |x_j|$. Let $b_n = (\log n)^{1/2} \log \log n$; clearly $\sum_{n=2}^\infty n^{-1} b_n^{-2} < \infty$. By Csáki [4] or Mason [20],

$$\Pr\left(\sup_{0 < s < 1} \frac{|\alpha_{nj}(s)|}{(s(1-s))^{1/2}} > b_n \text{ infinitely often} \right) = 0.$$

It follows that, with probability one, for all sufficiently large n,

$$|\alpha_{nj}(u_j)| \leq \left(u_j(1-u_j) \right)^{1/2} b_n, \qquad u_j \in [0, 1].$$

Let I denote the identity function on $[0, 1]$, and let $\|\cdot\|_\infty$ denote the supremum norm. For $u_j \in [\delta_n, 1 - \delta_n]$,

$$G_{nj}^{-1}(u_j) = u_j \left(1 + \frac{G_{nj}^{-1}(u_j) - u_j}{u_j} \right) \geq u_j \left(1 - \frac{\|G_{nj}^{-1} - I\|_\infty}{\delta_n} \right),$$

$$1 - G_{nj}^{-1}(u_j) \geq (1 - u_j) \left(1 - \frac{\|G_{nj}^{-1} - I\|_\infty}{\delta_n} \right).$$

By the law of the iterated logarithm (see (4.2))

$$\|G_{nj}^{-1} - I\| = o(\delta_n) \qquad (n \to \infty) \text{ almost surely.}$$

We find that with probability one, for all sufficiently large n and for all $u \in [0, 1]^d$ such that $u_j \in [\delta_n, 1 - \delta_n]$,

$$D_{nj}(u) \leq 2K \left(u_j(1 - u_j) \right)^{-1/2} \|v_n(u) - u\|_1 b_n.$$

We use again the law of the iterated logarithm in (4.2) to bound $\|v_n(u) - u\|_1$. As a consequence, with probability one,

$$\sup_{u \in [0,1]^d : u_j \in [\delta_n, 1-\delta_n]} D_{nj}(u) = O(\delta_n^{-1/2} (\log \log n)^{1/2} n^{-1/2} b_n)$$

$$= O(n^{-1/4} (\log \log n)^{7/4}) \qquad (n \to \infty) \text{ almost surely.} \tag{4.4}$$

The bound in (4.4) is dominated by the one in (4.3). The latter therefore yields the total rate. $\square$

Lemma 4.3. *If Conditions* 2.1 *and* 4.1 *hold, then*

$$|\dot{C}_j(v) - \dot{C}_j(u)| \leq K \max\left(\frac{1}{u_j(1-u_j)}, \frac{1}{v_j(1-v_j)} \right) \|v - u\|_1, \tag{4.5}$$

for every $j \in \{1, \ldots, d\}$ *and for every* $u, v \in [0, 1]^d$ *such that* $0 < u_j < 1$ *and* $0 < v_j < 1$; *here* $\|x\|_1 = \sum_{i=1}^d |x_i|$ *denotes the* L_1-*norm.*

Proof. Fix $j \in \{1, \ldots, d\}$ and $u, v \in [0, 1]^d$ such that $u_j, v_j \in (0, 1)$. Consider the line segment $w(t) = u + t(v - u)$ for $t \in [0, 1]$, connecting $w(0) = u$ with $w(1) = v$; put $w_i(t) = u_i + t(v_i - u_i)$ for $i \in \{1, \ldots, d\}$. Clearly $0 < w_j(t) < 1$ for all $t \in [0, 1]$. Next, consider the function $f(t) = \dot{C}_j(w(t))$ for $t \in [0, 1]$. The function f is continuous on $[0, 1]$ and continuously differentiable on $(0, 1)$. Indeed, if $u_i \neq v_i$ for some $i \in \{1, \ldots, d\}$, then $0 < w_i(t) < 1$ for all $t \in (0, 1)$; if $u_i = v_i$, then $w_i(t) = u_i = v_i$ does not depend on t at all. Hence, the derivative of f in $t \in (0, 1)$ is given by

$$f'(t) = \sum_{i \in I} (v_i - u_i) \ddot{C}_{ij}(w(t)),$$

where $\mathcal{I} = \{i \in \{1, \ldots, d\}: u_i \neq v_i\}$. By the mean-value theorem, we obtain that for some $t^* \in (0, 1)$,

$$\dot{C}_j(v) - \dot{C}_j(u) = f(1) - f(0) = f'(t^*) = \sum_{i \in \mathcal{I}} (v_i - u_i) \ddot{C}_{ij}(w(t^*)).$$

As a consequence,

$$|\dot{C}_j(u) - \dot{C}_j(v)| \leq \|v - u\|_1 \max_{i \in \mathcal{I}} \sup_{0 < t < 1} |\ddot{C}_{ij}(w(t))|.$$

By Condition 4.1,

$$|\dot{C}_j(u) - \dot{C}_j(v)| \leq \|v - u\|_1 K \sup_{0 < t < 1} \frac{1}{w_j(t)\{1 - w_j(t)\}}.$$

Finally, since the function $s \mapsto 1/\{s(1 - s)\}$ is convex on $(0, 1)$ and since $w_j(t)$ is a convex combination of u_j and v_j, the supremum of $1/[w_j(t)\{1 - w_j(t)\}]$ over $t \in [0, 1]$ must be attained at one of the endpoints u_j or v_j. Equation (4.5) follows. $\qquad \square$

5. Examples

Example 5.1 (Gaussian copula). Let C be the d-variate Gaussian copula with correlation matrix $R \in \mathbb{R}^{d \times d}$, that is,

$$C(u) = \Pr\left(\bigcap_{j=1}^d \{\Phi(X_j) \leq u_j\}\right), \qquad u \in [0, 1]^d,$$

where $X = (X_1, \ldots, X_d)$ follows a d-variate Gaussian distribution with zero means, unit variances, and correlation matrix R; here Φ is the standard normal c.d.f. It can be checked readily that if the correlation matrix R is of full rank, then Condition 2.1 is verified, and Propositions 3.1 and 3.2 apply.

Still, if $0 < \rho_{1j} = \mathrm{corr}(X_1, X_j) < 1$ for all $j \in \{2, \ldots, d\}$, then on the one hand we have $\lim_{u_1 \downarrow 0} \dot{C}_1(u_1, u_{-1}) = 1$ for all $u_{-1} \in (0, 1]^{d-1}$, whereas on the other hand we have $\dot{C}_1(u) = 0$ as soon as $u_j = 0$ for some $j \in \{2, \ldots, d\}$. As a consequence, $\dot{C}_1$ cannot be extended continuously to the set $\{0\} \times ([0, 1]^{d-1} \setminus (0, 1]^{d-1})$.

In the bivariate case, Condition 4.1 can be verified by direct calculation, provided the correlation parameter ρ satisfies $|\rho| < 1$.

Example 5.2 (Archimedean copulas). Let C be a d-variate Archimedean copula; that is,

$$C(u) = \phi^{-1}\big(\phi(u_1) + \cdots + \phi(u_d)\big), \qquad u \in [0,1]^d,$$

where the generator $\phi : [0,1] \to [0,\infty]$ is convex, decreasing, finite on $(0,1]$, and vanishes at 1, whereas $\phi^{-1} : [0,\infty) \to [0,1]$ is its generalized inverse, $\phi^{-1}(x) = \inf\{u \in [0,1] : \phi(u) \le x\}$; in fact, if $d \ge 3$, more conditions on ϕ are required for C to be a copula; see McNeil and Nešlehová [21].

Suppose ϕ is continuously differentiable on $(0,1]$ and $\phi'(0+) = -\infty$. Then the first-order partial derivatives of C are given by

$$\dot{C}_j(u) = \frac{\phi'(u_j)}{\phi'(C(u))}, \qquad u \in [0,1]^d, 0 < u_j < 1.$$

If $u_i = 0$ for some $i \ne j$, then $C(u) = 0$ and $\phi'(C(u)) = -\infty$, so indeed $\dot{C}_j(u) = 0$. We find that Condition 2.1 is verified, so Propositions 3.1 and 3.2 apply.

In contrast, $\dot{C}_j$ may easily fail to be continuous at some boundary points. For instance, if $\phi'(1) = 0$, then $\dot{C}_j$ cannot be extended continuously at $(1, \ldots, 1)$. Or if ϕ^{-1} is long-tailed, that is, if $\lim_{x \to \infty} \phi^{-1}(x+y)/\phi^{-1}(x) = 1$ for all $y \in \mathbb{R}$, then $\lim_{u_1 \downarrow 0} C(u_1, u_{-1})/u_1 = 1$ for all $u_{-1} \in (0,1]^{d-1}$, whereas $\dot{C}_1(u) = 0$ as soon as $u_j = 0$ for some $j \in \{2, \ldots, d\}$; it follows that $\dot{C}_1$ cannot be extended continuously to the set $\{0\} \times ([0,1]^{d-1} \setminus (0,1]^{d-1})$.

Example 5.3 (Extreme-value copulas). Let C be a d-variate extreme-value copula; that is,

$$C(u) = \exp(-\ell(-\log u_1, \ldots, -\log u_d)), \qquad u \in (0,1]^d,$$

where the tail dependence function $\ell : [0,\infty)^d \to [0,\infty)$ verifies

$$\ell(x) = \int_{\Delta_{d-1}} \max_{j \in \{1,\ldots,d\}} (w_j x_j) H(dw), \qquad x \in [0,\infty)^d,$$

with H a non-negative Borel measure (called spectral measure) on the unit simplex $\Delta_{d-1} = \{w \in [0,1]^d : w_1 + \cdots + w_d = 1\}$ satisfying the d constraints $\int w_j H(dw) = 1$ for all $j \in \{1, \ldots, d\}$; see, for instance, Leadbetter and Rootzén [19] or Pickands [23]. It can be verified that ℓ is convex, is homogeneous of order 1, and that $\max(x_1, \ldots, x_d) \le \ell(x) \le x_1 + \cdots + x_d$ for all $x \in [0,\infty)^d$.

Suppose that the following holds:

For every $j \in \{1, \ldots, d\}$, the first-order partial derivative $\dot{\ell}_j$ of ℓ with respect to x_j exists and is continuous on the set $\{x \in [0,\infty)^d : x_j > 0\}$. $\qquad$ (5.1)

Then the first-order partial derivative of C in u with respect to u_j exists and is continuous on the set $\{u \in [0, 1]^d : 0 < u_j < 1\}$. Indeed, for $u \in [0, 1]^d$ such that $0 < u_j < 1$, we have

$$
\dot{C}_j(u) = \begin{cases} \dfrac{C(u)}{u_j} \dot{\ell}_j(-\log u_1, \ldots, -\log u_d), & \text{if } u_i > 0 \text{ for all } i, \\ 0, & \text{if } u_i = 0 \text{ for some } i \neq j. \end{cases}
$$

The properties of ℓ imply that $0 \leq \dot{\ell}_j \leq 1$ for all $j \in \{1, \ldots, d\}$. Therefore, if $u_i \downarrow 0$ for some $i \neq j$, then $\dot{C}_j(u) \to 0$, as required. Hence if (5.1) is verified, Condition 2.1 is verified as well and Propositions 3.1 and 3.2 apply.

Let us consider the bivariate case in somewhat more detail. The function $A : [0, 1] \to [1/2, 1] : t \mapsto A(t) = \ell(1 - t, t)$ is called the Pickands dependence function of C. It is convex and satisfies $\max(t, 1 - t) \leq A(t) \leq 1$ for all $t \in [0, 1]$. By homogeneity of the function ℓ, we have $\ell(x, y) = (x + y)A(\frac{y}{x+y})$ for $(x, y) \in [0, \infty)^2 \setminus \{(0, 0)\}$. If A is continuously differentiable on $(0, 1)$, then (5.1) holds, and Condition 2.1 is verified. Nevertheless, if $A(1/2) < 1$, which is always true except in case of independence ($A \equiv 1$), the upper tail dependence coefficient $2\{1 - A(1/2)\}$ is positive so that the first-order partial derivatives fail to be continuous at the point $(1, 1)$; see Example 1.1. One can also see that $\dot{C}_1$ will not admit a continuous extension in the neighborhood of the point $(0, 0)$ in case $A'(0) = -1$.

We will now verify Condition 4.1 under the following additional assumption:

$$
\begin{aligned}
&\text{The function } A \text{ is twice continuously differentiable on } (0, 1) \\
&\text{and } M = \sup_{0 < t < 1}\{t(1 - t)A''(t)\} < \infty.
\end{aligned} \tag{5.2}
$$

In combination with Proposition 4.2, this will justify the use of the Stute–Tsukahara almost sure rate (4.1) in the proof of Theorem 3.2 in Genest and Segers [13]; in particular, see their equation (B.3). Note that the weight function $t(1-t)$ in the supremum in (5.2) is not unimportant: for the Gumbel extreme-value copula having dependence function $A(t) = \{t^{1/\theta} + (1 - t)^{1/\theta}\}^\theta$ with parameter $\theta \in (0, 1]$, it holds that $A''(t) \to \infty$ as $t \to 0$ or $t \to 1$ provided $1/2 < \theta < 1$, whereas condition (5.2) is verified for all $\theta \in (0, 1]$.

The copula density at the point $(u, v) \in (0, 1)^2$ is given by

$$
\ddot{C}_{12}(u, v) = \frac{C(u, v)}{uv}\left(\mu(t)v(t) - \frac{t(1 - t)A''(t)}{\log(uv)}\right),
$$

where

$$
t = \frac{\log(v)}{\log(uv)} \in (0, 1), \qquad \mu(t) = A(t) - tA'(t), \, v(t) = A(t) + (1 - t)A'(t).
$$

Note that if $A''(1/2) > 0$, then $\ddot{C}_{12}(w, w) \to \infty$ as $w \uparrow 1$. The properties of A imply $0 \leq \mu(t) \leq 1$ and $0 \leq v(t) \leq 1$. From $-\log(x) \geq 1 - x$, it follows that $-1/\log(uv) \leq \min\{1/(1 - u), 1/(1 - v)\}$ for $(u, v) \in (0, 1)^2$. Since $C(u, v) \leq \min(u, v)$ and since $\min(a, b)\min(c, d) \leq$

min$\{(ac), (bd)\}$ for positive numbers a, b, c, d, we find

$$0 \leq \ddot{C}_{12}(u, v) \leq \frac{\min(u, v)}{uv} \left\{ 1 + M \min\left(\frac{1}{1-u}, \frac{1}{1-v} \right) \right\}$$

$$\leq (1 + M) \min\left(\frac{1}{u(1-u)}, \frac{1}{v(1-v)} \right).$$

Similarly, for $(u, v) \in (0, 1) \times [0, 1]$,

$$\ddot{C}_{11}(u, v) = \begin{cases} \dfrac{C(u, v)}{u^2} \left(-\mu(t)\big(1 - \mu(t)\big) + \dfrac{t^2(1-t)A''(t)}{\log(u)} \right), & \text{if } 0 < v < 1, \\ 0, & \text{if } v \in \{0, 1\}. \end{cases}$$

Continuity at the boundary $v = 0$ follows from the fact that $C(u, v) \to 0$ as $v \to 0$; continuity at the boundary $v = 1$ follows from the fact that $t \to 0$ and $\mu(t) \to 0$ as $v \to 1$. Since $-\log(u) \leq (1-u)/u$, we find, as required,

$$0 \leq -\ddot{C}_{11}(u, v) \leq \frac{(1+M)}{u(1-u)}, \qquad (u, v) \in (0, 1) \times [0, 1].$$

Example 5.4 (If everything fails...). Sometimes, even Condition 2.1 does not hold: think, for instance, of the Fréchet lower and upper bounds, $C(u, v) = \max(u + v - 1, 0)$ and $C(u, v) = \min(u, v)$, and of the checkerboard copula with Lebesgue density $c = 2\mathbb{1}_{[0, 1/2]^2 \cup [1/2, 1]^2}$. In these cases, the candidate limiting process $\mathbb{C}$ has discontinuous trajectories, and the empirical copula process does not converge weakly in the topology of uniform convergence.

One may then wonder if weak convergence of the empirical copula process still holds in, for instance, a Skorohod-type topology on the space of càdlàg functions on $[0, 1]^2$. Such a result would be useful to derive, for instance, the asymptotic distribution of certain functionals of the empirical copula process, for example, suprema or integrals such as appearing in certain test statistics.

Appendix: Multivariate oscillation modulus

Let C be any d-variate copula and let $U_1, U_2, \ldots$ be an i.i.d. sequence of random vectors with common cumulative distribution function C. Let α_n be the multivariate empirical process in equation (2.1). Consider the oscillation modulus defined by

$$M_n(a) = \sup\{|\alpha_n(u) - \alpha_n(v)|: u, v \in [0, 1]^d, |u_j - v_j| \leq a \text{ for all } j\} \qquad \text{(A.1)}$$

for $a \in [0, \infty)$. Define the function $\psi : [-1, \infty) \to (0, \infty)$ by

$$\psi(x) = 2x^{-2}\{(1 + x)\log(1 + x) - x\}, \qquad x \in (-1, 0) \cup (0, \infty), \qquad \text{(A.2)}$$

together with $\psi(-1) = 2$ and $\psi(0) = 1$. Note that ψ is decreasing and continuous.

Proposition A.1 (John H. J. Einmahl, Hideatsu Tsukahara). *Let C be any d-variate copula. There exist constants K_1 and K_2, depending only on d, such that*

$$\Pr\{M_n(a) \geq \lambda\} \leq \frac{K_1}{a} \exp\left\{ -\frac{K_2 \lambda^2}{a} \psi\left(\frac{\lambda}{\sqrt{na}} \right) \right\}$$

for all $a \in (0, 1/2]$ and all $\lambda \in [0, \infty)$.

Proof. In Einmahl [7], Inequality 5.3, page 73, the same bound is proved in case C is the independence copula and for $a > 0$ such that $1/a$ is integer. As noted by Tsukahara, in a private communication, the only property of the joint distribution that is used in the proof is that the margins be uniform on the interval $(0, 1)$: Inequality 2.5 in Einmahl [7], page 12, holds for any distribution on the unit hypercube and equation (5.19) on page 72 only involves the margins. As a consequence, Inequality 5.3 in Einmahl [7] continues to hold for any copula C. Moreover, the assumption that $1/a$ be integer is easy to get rid of. □

Acknowledgments

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