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Approximations of fractional Brownian motion

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Approximations of fractional Brownian motion using Poisson processes whose parameter sets have the same dimensions as the approximated processes have been studied in the literature. In this paper, a special approximation to the one-parameter fractional Brownian motion is constructed using a two-parameter Poisson process. The proof involves the tightness and identification of finite-dimensional distributions.

Keywords: fractional Brownian motion; Poisson process; weak convergence

1. Introduction

The fractional Brownian motion with Hurst index $H \in (0, 1)$ is a centered Gaussian process $B^H = \{B^H(t), t \geq 0\}$ with covariance function

$$\mathbb{E}[B^H(t)B^H(s)] = \tfrac{1}{2}(|s|^{2H} + |t|^{2H} - |t-s|^{2H}). \tag{1.1}$$

It follows from (1.1) that B^H is self-similar with index H and has stationary increments. Unless $H = 1/2$ (i.e., B^H is Brownian motion), B^H is not Markovian. Moreover, it is known that B^H has long-range dependence if $H \in (1/2, 1)$ and short-range dependence if $H \in (0, 1/2)$ (see Samorodnitsky and Taqqu [14]). These properties have made B^H not only important theoretically, but also very popular as stochastic models in many areas including telecommunications, biology, hydrology and finance.

Weak convergence to fractional Brownian motion has been studied extensively since the works of Davydov [7] and Taqqu [16]. In recent years many new results on approximations of fractional Brownian motion have been established. For example, Enriquez [9] showed that fractional Brownian motion can be approximated in law by appropriately normalized correlated random walks. Meyer, Sellan and Taqqu [13] proved that the law of B^H can be approximated by those of a random wavelet series. By extending Stroock [15], Bardina *et al.* [4] and Delgado and Jolis [8] have established approximations in law to fractional Brownian motions by processes constructed using Poisson processes.

Let $\{N(t), t \geq 0\}$ be a standard Poisson process, and for all $\varepsilon > 0$, define the processes $X_\varepsilon = \{X_\varepsilon(t), t \geq 0\}$ by

$$X_\varepsilon(t) = \frac{1}{\varepsilon}\int_0^t (-1)^{N(r/\varepsilon^2)}\,\mathrm{d}r, \qquad t \geq 0.$$

Stroock [15] proved that as ε tends to zero, the laws of X_ε converge weakly in the Banach space $\mathcal{C}[0,1]$ (i.e., the space of continuous functions on $[0,1]$) to the law of Brownian motion. Delgado and Jolis [8] proved that every Gaussian process of the form

$$X_t = \int_0^1 K(t,s)\,\mathrm{d}B_s,$$

where B is a one-dimensional Brownian motion and K a sufficiently regular deterministic kernel, can be weakly approximated by the family of processes

$$Y^\varepsilon(t) = \frac{1}{\varepsilon}\int_0^1 K(t,r)(-1)^{N(r/\varepsilon^2)}\,\mathrm{d}r.$$

Their result can be applied to fractional Brownian motion. In addition, Bardina and Jolis [2] proved that as ε tends to 0, the family of two-parameter random fields Y_ε defined by

$$Y_\varepsilon(s,t) = \int_0^t\int_0^s \frac{1}{\varepsilon^2}\sqrt{xy}(-1)^{N(x/\varepsilon,y/\varepsilon)}\,\mathrm{d}x\,\mathrm{d}y \tag{1.2}$$

converges in law in the space of continuous functions on $[0,1]^2$ to the standard Brownian sheet. Bardina, Jolis and Tudor showed in [4] that as ε tends to 0, the family of two-parameter random fields

$$\hat{Y}_\varepsilon(s,t) = \int_0^1\int_0^1 \frac{1}{\varepsilon^2} f(s,t,x,y)\sqrt{xy}(-1)^{N(x/\varepsilon,y/\varepsilon)}\,\mathrm{d}x\,\mathrm{d}y \tag{1.3}$$

converges in law to the two-parameter Gaussian process

$$W(s,t) = \int_0^1\int_0^1 f(s,t,x,y)B(\mathrm{d}x,\mathrm{d}y),$$

where B is a standard Brownian sheet and the deterministic kernel

$$f(s,t,x,y) = f_1(s,x)f_2(t,y) \tag{1.4}$$

can be separated by the integration variables and satisfies certain conditions. As examples, the authors include the fractional Brownian sheet, among others. For more information, see Bardina, Jolis and Rovira [3], where an approximation to the d-parameter Wiener process by a d-parameter Poisson process was provided, and Bardina and Bascompte [1], where two independent Gaussian processes were constructed using a unique Poisson process.

We note that in the serial works [1–4,8], the dimension of the parameter set is always the same for the approximating and the approximated processes. Naturally, we will be interested in the problem of whether we can approximate the d-parameter fractional Brownian motions by r-parameter Poisson processes if $d \neq r$. The purpose of this paper is to study this problem in the case of $d=1$ and $r=2$. We find that for a special deterministic kernel function which cannot be separated with respect to the integration variables, the answer is affirmative. Below, we introduce the deterministic kernel function.

In order to study a non-Gaussian and non-stable process arising as the limit of sums of rescaled renewal processes under the condition of intermediate growth, Gaigalas [11], page 451, introduced the function $h(t, x, y)$, defined as follows. For $x, t \geq 0$ and $y \in \mathbb{R}$,

$$\begin{aligned} h(t,x,y) &= \big((t+y)\wedge 0 + x\big)_+ - (y\wedge 0 + x)_+ \\ &= \int_{-t}^{0} \mathbf{1}_{[0,x]}(u-y)\,\mathrm{d}u = \begin{cases} t, & -y < x,\ y < -t, \\ x+t+y, & -t-y < x \leq -y,\ y < -t, \\ -y, & -y < x,\ -t \leq y < 0, \\ x, & 0 < x \leq -y,\ -t \leq y < 0, \\ 0, & \text{otherwise}. \end{cases} \end{aligned} \tag{1.5}$$

Note that Kaj and Taqqu [12], page 388, interpreted the function

$$K_t(y,x) := (t-y)_+ \wedge x - (-y)_+ \wedge x = h(t, x, -x-y)$$

in the context of the infinite source Poisson model as a function of the starting time y and the duration x of a session that measures the length of the time interval contained in $[0, t]$ during which the session is active. Therefore, $h(t, x, y) = K_t(-x-y, x)$ measures the length of the time interval contained in $[0, t]$ during which the session with duration x and finishing time $-y$ is active. Obviously, by this definition, $h(t, x, y) \neq 0$ if and only if $y < 0$ and $-y-x < t$, that is, $x + y > -t$.

In this paper, we define

$$\begin{aligned} g_s(t,x,y) &:= h(t+s,x,y) - h(s,x,y) \\ &= \int_{-t-s}^{0} \mathbf{1}_{[0,x]}(u-y)\,\mathrm{d}u - \int_{-s}^{0} \mathbf{1}_{[0,x]}(u-y)\,\mathrm{d}u \\ &= \int_{-t-s}^{-s} \mathbf{1}_{[0,x]}(u-y)\,\mathrm{d}u = \int_{-t}^{0} \mathbf{1}_{[0,x]}(u-y-s)\,\mathrm{d}u = h(t,x,y+s) \end{aligned} \tag{1.6}$$

for all $t \geq 0$, $x \geq 0$, $y \in \mathbb{R}$ and any given $s > 0$. Then, according to the definition of an integral of random measure (see [14], Chapter 3), we can directly verify that for $H \in (\frac{1}{2}, 1)$,

$$\begin{aligned} &\sqrt{C_H}\int_0^\infty\int_{\mathbb{R}_-} g_s(t,x,y)x^{H-2}W(\mathrm{d}x,\mathrm{d}y) \\ &\quad = \sqrt{C_H}\int_0^\infty\int_{\mathbb{R}_-} h(t,x,y+s)x^{H-2}W(\mathrm{d}x,\mathrm{d}y) \\ &\quad \overset{d}{=} \sqrt{C_H}\int_0^\infty\int_{\mathbb{R}_-} h(t,x,y)x^{H-2}W(\mathrm{d}x,\mathrm{d}y) \overset{d}{=} B^H(t), \end{aligned} \tag{1.7}$$

where $W(\mathrm{d}x, \mathrm{d}y)$ is a Gaussian random measure on $\mathbb{R}_+ \times \mathbb{R}$ with control measure $\mathrm{d}x\,\mathrm{d}y$ (see Section 2 for its definition), $C_H = H(2H-1)(1-H)(3-2H)$ and the notation $\overset{d}{=}$ denotes identification of finite-dimensional distributions. In fact, the last equality is taken from Gaigalas [11], page 454, although the constant C_H is omitted in Gaigalas' representation. Therefore,

$$C_H \int_0^\infty \int_{\mathbb{R}_-} \frac{g_s^2(t,x,y)}{x^{4-2H}} \,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2}\mathbb{E}[(B^H(t))^2] = \frac{t^{2H}}{2}. \tag{1.8}$$

From the representations (1.2) and (1.7), inspired by (1.3), it seems reasonable that the law of the process B^H can be approximated by that of some process similar to $\hat{Y}_\varepsilon$ with kernel $g_s(t,x,y)$.

In this paper we define a sequence of processes $\{Y_n(t), t \in [0,1]\}_{n\geq 1}$ as follows:

$$Y_n(t) = n\sqrt{2C_H} \int_0^n \int_{-n}^0 g_s(t,x,y) x^{H-2} \sqrt{x|y|} (-1)^{N_n(x,y)} \,\mathrm{d}x\,\mathrm{d}y \tag{1.9}$$

for $H \in (1/2, 1)$. Here, $\{N_n(x,y), (x,y) \in \mathbb{R}_+ \times \mathbb{R}_-\}$ is a Poisson process with intensity n (see Definition 2.1).

The main purpose of this paper is to show that the law of Y_n converges to the law of B^H for $H \in (1/2, 1)$. We note that the kernel $g_s(t,x,y)$ in (1.9) cannot be separated by the arguments (x,y), unlike the kernel function in (1.3). This difference is not trivial. As we will see in Remark 3.1, it causes many real technical difficulties.

The rest of the paper is organized as follows. Section 2 is devoted to introducing the necessary definitions, notation and the main result. In Section 3 we prove the family of processes $\{Y_n(t)\}_{n\geq 1}$ given by (1.9) is tight in $\mathcal{C}[0,1]$. In Section 4 we prove that as n tends to infinity, the finite-dimensional distributions of $\{Y_n(t)\}$ converge weakly to those of the fractional Brownian motion B^H with $H \in (1/2, 1)$ and hence $\{Y_n(t)\}$ converges weakly in $\mathcal{C}[0,1]$ to the fractional Brownian motion B^H.

2. Preliminaries

We now give the definitions of the Brownian sheet and Poisson processes on $\mathbb{R} \times \mathbb{R}$. Let $\mathcal{F}$ be the Borel algebra on $\mathbb{R} \times \mathbb{R}$. ν and μ denote a σ-finite measure and the Lebesgue measure on $\mathbb{R} \times \mathbb{R}$, respectively.

Definition 2.1. *Given a positive constant $\beta > 0$, a random set function $N(\cdot)$ on the measure space $(\mathbb{R} \times \mathbb{R}, \mathcal{F}, \nu)$ is called the Poisson random measure with density measure $\beta\nu$ if it satisfies the following conditions*:

- (i) *for every $A \in \mathcal{F}$ with $\mu(A) < \infty$, $N(A)$ is a Poisson random variable with parameter $\beta\nu(A)$ defined on the same probability space*;
- (ii) *if $A_1, \dots, A_n \in \mathcal{F}$ are disjoint and all have finite measure, then the random variables $N(A_1), \dots, N(A_n)$ are independent*;
- (iii) *if $A_1, A_2, \dots \in \mathcal{F}$ are disjoint and $\nu(\bigcup_{i=1}^\infty A_i) < \infty$, then $N(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty N(A_i)$ a.s.*

If $N(\cdot)$ is a Poisson random measure with density measure $\beta\mu$, then we define

$$N(s,t)=\begin{cases} N([0,s]\times[0,t]), & s\geq 0,\ t\geq 0,\\ N([0,s]\times[t,0]), & s\geq 0,\ t\leq 0,\\ N([s,0]\times[0,t]), & s\leq 0,\ t\geq 0,\\ N([s,0]\times[t,0]), & s\leq 0,\ t\leq 0\end{cases}$$

and call $N=\{N(s,t),(s,t)\in\mathbb{R}\times\mathbb{R}\}$ the two-parameter Poisson process with intensity β in $\mathbb{R}\times\mathbb{R}$.

It is not hard to see that $N=\{N(s,t),(s,t)\in\mathbb{R}_+\times\mathbb{R}_-\}$ is independent with $N_1=\{N(s,t),(s,t)\in\mathbb{R}_+\times\mathbb{R}_+\}$, which is the ordinary two-parameter process in $\mathbb{R}_+\times\mathbb{R}_+$, and for any $(s,t)\in\mathbb{R}_+\times\mathbb{R}_-$, $\{N(s,t)\}$ has the same finite-dimensional distributions as those of $\{N(s,|t|)\}$.

Definition 2.2. *Consider a random set function $W(\cdot)$ on the measure space $(\mathbb{R}\times\mathbb{R},\mathcal{F},\nu)$ such that*:

(i) *for every $A\in\mathcal{F}$ with $\nu(A)<\infty$, $W(A)$ is a centered Gaussian random variable defined on the same probability space with variance $2\nu(A)$;*
(ii) *if $A_1,\dots,A_n\in\mathcal{F}$ are disjoint and have finite measure, then the random variables $W(A_1),\dots,W(A_n)$ are independent*;
(iii) *if $A_1,A_2,\dots\in\mathcal{F}$ are disjoint and $\nu(\bigcup_{i=1}^{\infty}A_i)<\infty$, then $W(\bigcup_{i=1}^{\infty}A_i)=\sum_{i=1}^{\infty}W(A_i)$ a.s.*

We then call $W(\cdot)$ a Gaussian random measure on $\mathbb{R}\times\mathbb{R}$ with control measure ν. In particular, if $W(\cdot)$ is a Gaussian random measure on $\mathbb{R}\times\mathbb{R}$ with control measure $\frac{1}{2}\mu$, then we define

$$B(s,t)=\begin{cases} W([0,s]\times[0,t]), & s\geq 0,\ t\geq 0,\\ W([0,s]\times[t,0]), & s\geq 0,\ t\leq 0,\\ W([s,0]\times[0,t]), & s\leq 0,\ t\geq 0,\\ W([s,0]\times[t,0]), & s\leq 0,\ t\leq 0\end{cases}$$

and call $B=\{B(s,t),(s,t)\in\mathbb{R}\times\mathbb{R}\}$ the two-parameter Brownian sheet in $\mathbb{R}\times\mathbb{R}$.

Similarly, we have that $B=\{B(s,t),(s,t)\in\mathbb{R}_+\times\mathbb{R}_-\}$ is independent of $B_1=\{B(s,t),(s,t)\in\mathbb{R}_+\times\mathbb{R}_+\}$, which is the ordinary Brownian sheet in $\mathbb{R}_+\times\mathbb{R}_+$, and for any $(s,t)\in\mathbb{R}\times\mathbb{R}$, we have $B(s,t)\stackrel{d}{=}B(|s|,|t|)$. Hence, from (1.7) it is easy to check that

$$B^H(t)\stackrel{d}{=}\sqrt{2C_H}\int_0^{\infty}\int_{\mathbb{R}_-}g_s(t,x,y)x^{H-2}B(\mathrm{d}x,\mathrm{d}y). \tag{2.1}$$

Let $\operatorname{sgn}(x)=1$ if $x\geq 0$ and $\operatorname{sgn}(x)=-1$ if $x<0$. We have the following conclusion, which essentially parallels Bardina and Jolis [2], Theorem 1.1. The proof is omitted.

Lemma 2.1. *Suppose that* $N = \{N(s,t), (s,t) \in \mathbb{R} \times \mathbb{R}\}$ *is a two-parameter Poisson process with intensity* 1 *in* $\mathbb{R} \times \mathbb{R}$. *For any* $S > 0$ *and* $T > 0$, *let*

$$B_n(u,v) = \operatorname{sgn}(uv) n \int_0^u \int_0^v \sqrt{|xy|}(-1)^{N(x\sqrt{n},y\sqrt{n})}\,\mathrm{d}x\,\mathrm{d}y \tag{2.2}$$

for any $|u| \le S$, $|v| \le T$. *The finite-dimensional distributions of* B_n *then converge weakly to those of a two-parameter Brownian sheet* $B = \{B(u,v), |u| \le S, |v| \le T\}$.

Naturally, (2.1) and (2.2) suggest that we consider the following approximation of B^H for $H \in (1/2, 1)$:

$$Y_n(t) = n\sqrt{2C_H} \int_0^n \int_{-n}^0 g_s(t,x,y) x^{H-2} \sqrt{x|y|}(-1)^{N_n(x,y)}\,\mathrm{d}x\,\mathrm{d}y \tag{2.3}$$

for $n \in \mathbb{N} = \{1, 2, \ldots\}$ and $t \in [0,1]$, where $N_n = \{N_n(x,y)\} = \{N(x\sqrt{n}, y\sqrt{n})\}$ is the two-parameter Poisson process with intensity n.

The main result of this paper is as follows.

Theorem 2.1. *For any fixed* $s > 0$, *the law of the process* $\{Y_n(t), t \in [0,1]\}$ *given by* (2.3) *converges weakly to the law of* $\{B^H(t), t \in [0,1]\}$ *for* $H \in (\frac{1}{2}, 1)$ *in* $\mathcal{C}[0,1]$.

Remark 2.1. If we define $\bar{h}(t,x,y) := h(t,x,-y)$, then $\bar{h}(t,x,y)$ has a more natural physical interpretation. It measures the length of the time interval contained in $[0,t]$ during which the session with duration x and finishing time y is active. Define

$$\bar{Y}_n(t) := n\sqrt{2C_H} \int_0^n \int_0^n \bar{g}_s(t,x,y) x^{H-2} \sqrt{xy}(-1)^{N_n(x,y)}\,\mathrm{d}x\,\mathrm{d}y,$$

where $\bar{g}_s(t,x,y) = \bar{h}(t+s,x,y) - \bar{h}(s,x,y) = \bar{h}(t,x,y-s)$. It is then easy to see that Theorem 2.1 holds for $\bar{Y}_n$.

Note that s is a given positive number. In the sequel we will treat it as a constant. In addition, since the parameter C_H cannot affect our discussion, we will take it to be 1 in order to simplify matters.

We now introduce some auxiliary notation. In the sequel, ∞ and $-\infty$ will denote positive infinity and negative infinity, respectively. For all $x' \ge x \ge 0$, $y' \ge y \ge 0$, by Cairoli and Walsh [6], we define

$$\Delta_{(x,y)} N(x',y') = N\big((x,x'] \times (y,y']\big). \tag{2.4}$$

For any $t \in [0,1]$, $x > 0$ and $y \le 0$, we let

$$\phi_t(x,y) := g_s(t,x,y)/x^{2-H} \ge 0.$$

Define a function $F_{n,f}(x, y)$ as follows:

$$F_{n,f}(x, y) = \sqrt{x|y|}n^2 \int_x^{\infty}\int_y^0 f(x_2, y_2)\sqrt{x_2|y_2|}\mathrm{e}^{-2n[x(y_2-y)-(x_2-x)y_2]}\,\mathrm{d}x_2\,\mathrm{d}y_2, \tag{2.5}$$

where $n > 0$, $x \geq 0$, $y \leq 0$ and f is a measurable function such that the integral is meaningful. Obviously, if $0 \leq f \leq g$, then $0 \leq F_{n,f}(x, y) \leq F_{n,g}(x, y)$.

3. Tightness of Y_n in $\mathcal{C}[0, 1]$

Let $Y_n = \{Y_n(t), t \in [0, 1]\}$ be the process defined by (2.3). The purpose of this section is to prove the tightness of the processes $\{Y_n\}_{n\geq 1}$.

Proposition 3.1. *The family of laws of the processes* $\{Y_n\}_{n\geq 1}$ *is tight in* $\mathcal{C}[0, 1]$.

To prove the proposition, we need the following lemmas.

Lemma 3.1. *Let* $\Omega_1 = \{0 < x_1 \leq x_2, y_2 \leq y_1 \leq 0\}$ *and* $\Omega_2 = \{0 < x_1 \leq x_2, y_1 < y_2 \leq 0\}$. *For a non-negative function* $f(x, y)$, *if* $\int_0^{\infty}\int_{-\infty}^0 f^2(x, y)\,\mathrm{d}x\,\mathrm{d}y < \infty$, *then for any* $u, v > 0$,

$$\begin{aligned} &\mathbb{E}\Big[\Big(n\int_0^u\int_{-v}^0 f(x, y)\sqrt{x|y|}(-1)^{N_n(x,y)}\,\mathrm{d}x\,\mathrm{d}y\Big)^2\Big] \\ &\quad\leq 2\big(I_1(n, f) + I_2(n, f)\big), \end{aligned} \tag{3.1}$$

where

$$I_1(n, f) = n^2\int_{\Omega_1}\prod_{i=1}^{2}\big(f(x_i, y_i)\sqrt{x_i|y_i|}\big)\mathrm{e}^{-2n(x_1y_1-x_2y_2)}\,\mathrm{d}x_1\,\mathrm{d}y_1\,\mathrm{d}x_2\,\mathrm{d}y_2, \tag{3.2}$$

$$I_2(n, f) = n^2\int_{\Omega_2}\prod_{i=1}^{2}\big(f(x_i, y_i)\sqrt{x_i|y_i|}\big)\mathrm{e}^{-2n[x_1(y_2-y_1)-(x_2-x_1)y_2]}\,\mathrm{d}x_1\,\mathrm{d}y_1\,\mathrm{d}x_2\,\mathrm{d}y_2. \tag{3.3}$$

Proof. Let $I(n, x_{1,2}, y_{1,2}) = \mathbb{E}[(-1)^{N_n(x_1,y_1)+N(x_2,y_2)}]$. By Fubini's theorem, the left-hand side of (3.1) is equal to

$$n^2\int_0^u\int_{-v}^0\int_0^u\int_{-v}^0 f(x_1, y_1)f(x_2, y_2)\sqrt{x_1|y_1|}\sqrt{x_2|y_2|}I(n, x_{1,2}, y_{1,2})\,\mathrm{d}x_1\,\mathrm{d}y_1\,\mathrm{d}x_2\,\mathrm{d}y_2. \tag{3.4}$$

Define $\Omega_3 = \{x_1 > x_2 > 0, 0 \geq y_1 \geq y_2\}$ and $\Omega_4 = \{x_1 > x_2 > 0, y_1 < y_2 \leq 0\}$. Then, by (2.4), $I(n, x_{1,2}, y_{1,2})$ equals

$$\begin{aligned} &\mathbb{E}\big[(-1)^{N_n(x_1,|y_1|)+N_n(x_2,|y_2|)}\big] \\ &\quad= \mathbb{E}\big[(-1)^{\Delta_{(0,0)}N_n(x_1,|y_1|)+\Delta_{(0,0)}N_n(x_2,|y_2|)}\big]. \end{aligned} \tag{3.5}$$

Note that $\sum_{i=1}^{2}\Delta_{(0,0)}N_n(x_i,|y_i|)$ is equal to the sum of the increments of the Poisson process over some disjoint rectangles, and the rectangles which contribute to the value of $I(n,x_{1,2},y_{1,2})$ are those which appear only once. Since two-parameter Poisson processes have independent increments, after some simple calculation, we obtain that on Ω_1,

$$\begin{aligned} &I(n,x_{1,2},y_{1,2}) \\ &\quad=\mathbb{E}\big[(-1)^{\Delta_{(0,|y_1|)}N_n(x_1,|y_2|)+\Delta_{(x_1,|y_1|)}N_n(x_2,|y_2|)+\Delta_{(x_1,0)}N_n(x_2,|y_1|)}\big] \\ &\quad=\mathbb{E}\big[(-1)^{\Delta_{(0,|y_1|)}N_n(x_1,|y_2|)}\big]\mathbb{E}\big[(-1)^{\Delta_{(x_1,|y_1|)}N_n(x_2,|y_2|)}\big]\mathbb{E}\big[(-1)^{\Delta_{(x_1,0)}N_n(x_2,|y_1|)}\big] \\ &\quad=\exp\{-2n(x_1y_1-x_2y_2)\}. \end{aligned} \tag{3.6}$$

Using the same method as above, we obtain that

$$I(n,x_{1,2},y_{1,2})=\exp\{-2n[x_1(y_2-y_1)-(x_2-x_1)y_2]\} \qquad \text{on } \Omega_2, \tag{3.7}$$

$$I(n,x_{1,2},y_{1,2})=\exp\{-2n[x_2(y_1-y_2)-(x_1-x_2)y_1]\} \qquad \text{on } \Omega_3, \tag{3.8}$$

$$I(n,x_{1,2},y_{1,2})=\exp\{-2n(x_2y_2-x_1y_1)\} \qquad \text{on } \Omega_4. \tag{3.9}$$

Substituting (3.6)–(3.9) into (3.4) and using a change of the integration variables if necessary, we can easily obtain (3.1). □

Lemma 3.2. *If $\int_0^\infty\int_{-\infty}^0 f^2(x,y)\,\mathrm{d}x\,\mathrm{d}y<\infty$, then $I_1(n,f)$ defined by (3.2) is such that*

$$I_1(n,f)\le\frac{37}{8}\int_0^\infty\int_{-\infty}^0 f^2(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.10}$$

Proof. Define $B=\{0<x_1\le x_2\le 2x_1\}$, $C=\{2y_1\le y_2\le y_1\le 0\}$ and $A=B\cap C$. Let $I_1^A(n)$, $I_1^B(n)$ and $I_1^C(n)$ denote the integral (3.2), where Ω_1 is replaced by A, $\Omega_1\setminus B$ and $\Omega_1\setminus C$, respectively. Then,

$$I_1(n,f)\le I_1^A(n,f)+I_1^B(n,f)+I_1^C(n,f). \tag{3.11}$$

Using the elementary inequality $2ab\le a^2+b^2$, from (3.2), we have that

$$I_1^A(n,f)\le\tfrac{1}{2}\big(I_{11}(n)+I_{12}(n)\big), \tag{3.12}$$

where $I_{11}(n)$ is

$$\begin{aligned} &n^2\int_A f^2(x_1,y_1)x_1|y_1|\mathrm{e}^{-2n(x_1y_1-x_2y_2)}\,\mathrm{d}x_1\,\mathrm{d}y_1\,\mathrm{d}x_2\,\mathrm{d}y_2 \\ &\quad=n^2\int_0^\infty\int_{-\infty}^0\int_{x_1}^{2x_1}\int_{2y_1}^{y_1} f^2(x_1,y_1)x_1|y_1|\mathrm{e}^{-2n(x_1y_1-x_2y_2)}\,\mathrm{d}x_1\,\mathrm{d}y_1\,\mathrm{d}x_2\,\mathrm{d}y_2 \end{aligned}$$

and $I_{12}(n)$ is

$$n^2 \int_A f^2(x_2, y_2) x_2 |y_2| \mathrm{e}^{-2n(x_1 y_1 - x_2 y_2)} \,\mathrm{d}x_1 \,\mathrm{d}y_1 \,\mathrm{d}x_2 \,\mathrm{d}y_2$$
$$= n^2 \int_0^\infty \int_{-\infty}^0 \int_{x_1}^{2x_1} \int_{2y_1}^{y_1} f^2(x_2, y_2) x_2 |y_2| \mathrm{e}^{-2n(x_1 y_1 - x_2 y_2)} \,\mathrm{d}x_1 \,\mathrm{d}y_1 \,\mathrm{d}x_2 \,\mathrm{d}y_2.$$

Since, for any $(x_1, y_1, x_2, y_2) \in \Omega_1$,

$$x_1 y_1 - x_2 y_2 \geq (x_2 - x_1)|y_1| + (|y_2| - |y_1|) x_1, \tag{3.13}$$

$I_{11}(n)$ is bounded by

$$n^2 \int_0^\infty \int_{-\infty}^0 \int_{x_1}^{2x_1} \int_{2y_1}^{y_1} f^2(x_1, y_1) x_1 |y_1| \mathrm{e}^{-2n[(x_2 - x_1)|y_1| + (|y_2| - |y_1|) x_1]} \,\mathrm{d}x_1 \,\mathrm{d}y_1 \,\mathrm{d}x_2 \,\mathrm{d}y_2.$$

Integrating with respect to x_2 and then with respect to y_2 in the above integral, we obtain the following bound:

$$I_{11}(n) \leq \frac{1}{4} \int_0^\infty \int_{-\infty}^0 f^2(x_1, y_1) \,\mathrm{d}x_1 \,\mathrm{d}y_1. \tag{3.14}$$

Furthermore, in the region A, it follows from (3.13) that

$$x_1 y_1 - x_2 y_2 \geq (x_2 - x_1)|y_2|/2 + (|y_2| - |y_1|) x_2/2.$$

By the same argument as above, we then have that

$$I_{12}(n) \leq \int_0^\infty \int_{-\infty}^0 f^2(x_2, y_2) \,\mathrm{d}x_2 \,\mathrm{d}y_2. \tag{3.15}$$

Therefore, from (3.12), (3.14) and (3.15), it follows that

$$I_1^A(n, f) \leq \frac{5}{8} \int_0^\infty \int_{-\infty}^0 f^2(x, y) \,\mathrm{d}x \,\mathrm{d}y. \tag{3.16}$$

We now consider $I_1^B(n, f)$. Note that

$$\begin{aligned} 2(x_1 y_1 - x_2 y_2) &= 2(x_2 - x_1)|y_1| + 2(|y_2| - |y_1|) x_2 \\ &\geq (x_2 - x_1)|y_1| + (|y_2| - |y_1|) x_1 + x_1 |y_1| \\ &= (x_2 - x_1)|y_1| + |y_2| x_1 \\ &\geq \tfrac{1}{2} x_2 |y_1| + |y_2| x_1 \end{aligned}$$

for any $(x_1, y_1, x_2, y_2) \in \Omega_1 \setminus B = \{x_2 > 2x_1 > 0, y_2 \leq y_1 \leq 0\}$. Define $\tilde{I}_{11}(n)$ and $\tilde{I}_{12}(n)$ as

$I_{11}(n)$ and $I_{12}(n)$, respectively, with $\Omega_1 \setminus B$ instead of A. Then,

$$\tilde{I}_{11}(n) \le n^2 \int_0^{\infty}\int_{-\infty}^{0}\int_{2x_1}^{\infty}\int_{-\infty}^{y_1} f^2(x_1, y_1)x_1|y_1|\mathrm{e}^{-n(|y_2|x_1+(1/2)x_2|y_1|)}\,\mathrm{d}x_1\,\mathrm{d}y_1\,\mathrm{d}x_2\,\mathrm{d}y_2,$$
$$\tilde{I}_{12}(n) \le n^2 \int_0^{\infty}\int_{-\infty}^{0}\int_{2x_1}^{\infty}\int_{-\infty}^{y_1} f^2(x_2, y_2)x_2|y_2|\mathrm{e}^{-n(|y_2|x_1+(1/2)x_2|y_1|)}\,\mathrm{d}x_1\,\mathrm{d}y_1\,\mathrm{d}x_2\,\mathrm{d}y_2.$$

Integrating in $\tilde{I}_{11}(n)$ with respect to x_2 and y_2, and in $\tilde{I}_{12}(n)$ with respect to x_1 and y_1, respectively, we obtain that

$$\tilde{I}_{11}(n) + \tilde{I}_{12}(n) \le 4\int_0^{\infty}\int_{-\infty}^{0} f^2(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

Since $I_1^B(n, f) \le \frac{1}{2}(\tilde{I}_{11}(n) + \tilde{I}_{12}(n))$, we have

$$I_1^B(n, f) \le 2\int_0^{\infty}\int_{-\infty}^{0} f^2(x, y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.17}$$

Similarly, for any $(x_1, y_1, x_2, y_2) \in \Omega_1 \setminus C = \{x_2 \ge x_1 > 0,\ y_2 < 2y_1 \le 0\}$, we have that

$$\begin{aligned}2(x_1y_1 - x_2y_2) &= 2(x_2|y_2| - |y_1|x_1) \ge 2(|y_2| - |y_1|)x_2\\ &\ge (|y_2| - |y_1|)x_2 + |y_1|x_2 \ge (|y_2| - |y_1|)x_1 + |y_1|x_2\\ &\ge \tfrac{1}{2}|y_2|x_1 + x_2|y_1|.\end{aligned}$$

Therefore, using a similar approach to the one above, we have

$$I_1^C(n, f) \le 2\int_0^{\infty}\int_{-\infty}^{0} f^2(x, y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.18}$$

Combining (3.11) with (3.16)–(3.18), we get (3.10). □

From (1.8) and Lemma 3.2, we can immediately get the following corollary.

Corollary 3.1. *For each $n > 0$, $I_1(n, \phi_t) \le \frac{37}{16}t^{2H}$.*

Observe that for every non-negative measurable function $f(x, y)$, by (2.5) and (3.3),

$$I_2(n, f) = \int_0^{\infty}\mathrm{d}x_1\int_{-\infty}^{0} f(x_1, y_1)F_{n,f}(x_1, y_1)\,\mathrm{d}y_1. \tag{3.19}$$

The following lemmas concern $I_2(n, f)$. We focus on the case where $f = \phi_t$.

Lemma 3.3. *For any $0 \le t \le 1$, let $S = \{x > 0, -t - s < y < -s\}$ and $\tilde{\phi}_t(x, y) = \phi_t(x, y)\mathbf{1}_S(x, y)$. There then exist a non-negative function $\tilde{\Psi}_t(x, y)$ on $\{x > 0, y \le 0\}$ such that for all $n > 0$,*

$$F_{n,\tilde{\phi}_t}(x_1, y_1)\mathbf{1}_S(x_1, y_1) \le \tilde{\Psi}_t(x_1, y_1), \tag{3.20}$$

and a positive constant K_1 which depends only on H, such that for all $n > 0$,

$$I_2(n, \tilde{\phi}_t) \le \int_S \phi_t(x_1, y_1)\tilde{\Psi}_t(x_1, y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1 \le K_1\sqrt{\frac{s+1}{s}}t^{2H}. \tag{3.21}$$

Proof. By (1.5) and (1.6),

$$\phi_t(x, y) = \begin{cases} t/x^{2-H}, & -y - s < x,\ y < -t - s, \\ (x + t + s + y)/x^{2-H}, & -t - s - y < x \le -y - s,\ y < -t - s, \\ -(y + s)/x^{2-H}, & -y - s < x,\ -t - s \le y < -s, \\ 1/x^{1-H}, & 0 < x \le -y - s,\ -t - s \le y < -s, \\ 0, & \text{otherwise}. \end{cases} \tag{3.22}$$

Let $S_1 = S \cap \{0 < x < -y - s\}$ and $S_2 = S \cap \{x > -y - s\}$. Then,

$$F_{n,\tilde{\phi}_t}(x_1, y_1)\mathbf{1}_{S_1}(x_1, y_1) = \mathbf{1}_{S_1}(x_1, y_1)\sqrt{x_1}(I_{21} + I_{22} + I_{23}), \tag{3.23}$$

where

$$I_{21} = n^2 \int_{y_1}^{-s-x_1} \int_{x_1}^{-s-y_2} \phi_t(x_2, y_2)\sqrt{x_2|y_1 y_2|}\mathrm{e}^{-2n[x_1(y_2-y_1)-(x_2-x_1)y_2]}\,\mathrm{d}x_2\,\mathrm{d}y_2,$$

$$I_{22} = n^2 \int_{y_1}^{-s-x_1} \int_{-s-y_2}^{\infty} \phi_t(x_2, y_2)\sqrt{x_2|y_1 y_2|}\mathrm{e}^{-2n[x_1(y_2-y_1)-(x_2-x_1)y_2]}\,\mathrm{d}x_2\,\mathrm{d}y_2,$$

$$I_{23} = n^2 \int_{-s-x_1}^{-s} \int_{x_1}^{\infty} \phi_t(x_2, y_2)\sqrt{x_2|y_1 y_2|}\mathrm{e}^{-2n[x_1(y_2-y_1)-(x_2-x_1)y_2]}\,\mathrm{d}x_2\,\mathrm{d}y_2.$$

For $(x_1, y_1) \in S_1$, from (3.22), we have that

$$\begin{aligned} I_{21} &= n^2 \int_{y_1}^{-s-x_1} \int_{x_1}^{-s-y_2} \frac{1}{x_2^{1/2-H}}\sqrt{|y_1 y_2|}\mathrm{e}^{-2n[x_1(y_2-y_1)-(x_2-x_1)y_2]}\,\mathrm{d}x_2\,\mathrm{d}y_2 \\ &\le \sqrt{\frac{t+s}{s}}n^2 \int_{y_1}^{-s-x_1} \int_{x_1}^{-s-y_2} \frac{1}{x_2^{1/2-H}}|y_2|\mathrm{e}^{-2n[x_1(y_2-y_1)-(x_2-x_1)y_2]}\,\mathrm{d}x_2\,\mathrm{d}y_2 \\ &\le t^{H-1/2}\sqrt{\frac{t+s}{s}}\frac{n}{2}\int_{y_1}^{-s-x_1} \mathrm{e}^{-2nx_1(y_2-y_1)}\,\mathrm{d}y_2 \\ &\le \frac{t^{H-1/2}}{4x_1}C_s =: \psi_1(x_1, y_1), \end{aligned} \tag{3.24}$$

where $C_s = \sqrt{(1+s)/s}$ is a constant. Similarly, for $(x_1, y_1) \in S_1$,

$$\begin{aligned} I_{22} &= n^2 \int_{y_1}^{-s-x_1} \int_{-s-y_2}^{\infty} \frac{-s-y_2}{x_2^{3/2-H}} \sqrt{|y_1 y_2|} \mathrm{e}^{-2n[x_1(y_2-y_1)-(x_2-x_1)y_2]} \, \mathrm{d}x_2 \, \mathrm{d}y_2 \\ &\le C_s n^2 \int_{y_1}^{-s-x_1} \int_{-s-y_2}^{\infty} (-s-y_2)^{H-1/2} |y_2| \mathrm{e}^{-2n[x_1(y_2-y_1)-(x_2-x_1)y_2]} \, \mathrm{d}x_2 \, \mathrm{d}y_2 \\ &\le (-s-y_1)^{H-1/2} C_s \frac{n}{2} \int_{y_1}^{-s-x_1} \mathrm{e}^{-2nx_1(y_2-y_1)} \, \mathrm{d}y_2 \\ &\le \frac{(-s-y_1)^{H-1/2}}{4x_1} C_s =: \psi_2(x_1, y_1) \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} I_{23} &= n^2 \int_{-s-x_1}^{-s} \int_{x_1}^{\infty} \frac{-s-y_2}{x_2^{3/2-H}} \sqrt{|y_1||y_2|} \mathrm{e}^{-2n[x_1(y_2-y_1)-(x_2-x_1)y_2]} \, \mathrm{d}x_2 \, \mathrm{d}y_2 \\ &\le C_s n^2 \int_{-s-x_1}^{-s} \int_{x_1}^{\infty} \frac{x_1}{x_1^{3/2-H}} |y_2| \mathrm{e}^{-2n[x_1(y_2-y_1)-(x_2-x_1)y_2]} \, \mathrm{d}x_2 \, \mathrm{d}y_2 \\ &\le C_s \frac{1}{4x_1^{3/2-H}} =: \psi_3(x_1, y_1). \end{aligned} \tag{3.26}$$

In addition, for $(x_1, y_1) \in S_2$,

$$\begin{aligned} F_{n,\tilde{\phi}_t}(x_1, y_1) &= n^2 \int_{y_1}^{-s} \int_{x_1}^{\infty} \frac{-s-y_2}{x_2^{3/2-H}} \sqrt{x_1 |y_1 y_2|} \mathrm{e}^{-2n[x_1(y_2-y_1)-(x_2-x_1)y_2]} \, \mathrm{d}x_2 \, \mathrm{d}y_2 \\ &\le C_s \frac{-s-y_1}{4x_1^{2-H}} =: \psi_4(x_1, y_1). \end{aligned} \tag{3.27}$$

Define $\tilde{\Psi}_t(x_1, y_1)$ as

$$\sqrt{x_1}\big(\psi_1(x_1, y_1) + \psi_2(x_1, y_1) + \psi_3(x_1, y_1)\big) \mathbf{1}_{S_1}(x_1, y_1) + \psi_4(x_1, y_1) \mathbf{1}_{S_2}(x_1, y_1).$$

Obviously, $\tilde{\Psi}_t(x_1, y_1)$ is positive and (3.20) follows from (3.23)–(3.27). Note that

$$\begin{aligned} I_2(n, \tilde{\phi}_t) &= \int_S \tilde{\phi}_t(x_1, y_1) F_{n,\tilde{\phi}_t}(x_1, y_1) \, \mathrm{d}x_1 \, \mathrm{d}y_1 \\ &\le \int_S \tilde{\phi}_t(x_1, y_1) \tilde{\Psi}_t(x_1, y_1) \, \mathrm{d}x_1 \, \mathrm{d}y_1. \end{aligned}$$

With some basic calculations, we obtain the following results:

$$\int_{S_1} \phi_t(x,y)\sqrt{x_1}\psi_1(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1 = \frac{C_s}{4}\int_{-t-s}^{-s}\int_0^{-y_1-s} \frac{t^{H-1/2}}{x_1^{3/2-H}}\,\mathrm{d}x_1\,\mathrm{d}y_1 = \frac{C_s t^{2H}}{4H^2-1};$$

$$\begin{aligned}\int_{S_1} \phi_t(x,y)\sqrt{x_1}\psi_2(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1 &= \frac{C_s}{4}\int_{-t-s}^{-s}\int_0^{-y_1-s} \frac{(-s-y_1)^{H-1/2}}{x_1^{3/2-H}}\,\mathrm{d}x_1\,\mathrm{d}y_1 \\ &= \frac{C_s t^{2H}}{4(2H-1)H};\end{aligned}$$

$$\int_{S_1} \phi_t(x,y)\sqrt{x_1}\psi_3(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1 = \frac{C_s}{4}\int_{-t-s}^{-s}\int_0^{-y_1-s} \frac{1}{x_1^{2-2H}}\,\mathrm{d}x_1\,\mathrm{d}y_1 = \frac{C_s t^{2H}}{8(2H-1)H};$$

$$\begin{aligned}\int_{S_2} \phi_t(x,y)\psi_4(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1 &= \frac{C_s}{4}\int_{-t-s}^{-s}\int_{-y_1-s}^{\infty} \frac{(-y_1-s)^2}{x_1^{4-2H}}\,\mathrm{d}x_1\,\mathrm{d}y_1 \\ &= \frac{C_s t^{2H}}{8(3-2H)H}.\end{aligned}$$

From the above integrals, (3.21) follows with $K_1 = \frac{3}{8H(2H-1)} + \frac{1}{8H(3-2H)} + \frac{1}{4H^2-1}$. □

Lemma 3.4. *For $0 \le t \le 1$ and each constant $M < -t-s$, let $G(M) = \{-t-s-y < x, M < y < -t-s\}$ and $\hat{\phi}_t(x,y) = \phi_t(x,y)\mathbf{1}_{G(M)}(x,y)$. There then exist a non-negative function $\hat{\Psi}_t(x,y)$ on $\{x > 0, y \le 0\}$ such that for all $n > 0$,*

$$F_{n,\hat{\phi}_t}(x_1,y_1)\mathbf{1}_{G(M)}(x_1,y_1) \le \hat{\Psi}_t(x_1,y_1), \tag{3.28}$$

and a positive constant K_2 which only depends on H, such that

$$I_2(n,\hat{\phi}_t) \le \int_{G(M)} \phi_t(x_1,y_1)\hat{\Psi}_t(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1 \le K_2\sqrt{\frac{|M|}{s}}t^{2H}. \tag{3.29}$$

Proof. From (3.22), we have that

$$F_{n,\hat{\phi}_t}(x_1,y_1) = \hat{I}_{21} + \hat{I}_{22}, \tag{3.30}$$

where

$$\hat{I}_{21} = n^2\int_{y_1}^{-t-s}\int_{-s-y_2}^{\infty} \frac{t}{x_2^{3/2-H}}\sqrt{x_1|y_1y_2|}\mathrm{e}^{-2n[x_1(y_2-y_1)-(x_2-x_1)y_2]}\,\mathrm{d}x_2\,\mathrm{d}y_2,$$

$$\hat{I}_{22} = n^2\int_{y_1}^{-t-s}\int_{-x_1}^{-s-y_2} \frac{x_2+t+s+y_2}{x_2^{3/2-H}}\sqrt{x_1|y_1y_2|}\mathrm{e}^{-2n[x_1(y_2-y_1)-(x_2-x_1)y_2]}\,\mathrm{d}x_2\,\mathrm{d}y_2.$$

For $(x_1, y_1) \in G(M)$, since $|y_1| < |M|$,

$$\begin{aligned}\hat{I}_{21} \vee \hat{I}_{22} &\leq \sqrt{\frac{|M|}{s}} \frac{t}{x_1^{1-H}} \int_{y_1}^{-t-s} \mathrm{d}y_1 \int_{-s-y_2}^{\infty} n^2 |y_2| \mathrm{e}^{-2n[x_1(y_2-y_1)-(x_2-x_1)y_2]} \,\mathrm{d}x_2 \,\mathrm{d}y_2 \\ &\leq \sqrt{\frac{|M|}{s}} \frac{t}{4x_1^{2-H}}.\end{aligned} \tag{3.31}$$

Therefore, from (3.30) and (3.31), it follows that

$$F_{n,\hat{\phi}_t}(x_1, y_1)\mathbf{1}_{G(M)}(x_1, y_1) \leq \sqrt{\frac{|M|}{s}} \frac{t}{4x_1^{2-H}}.$$

Let $\hat{\Psi}_t(x, y) = \sqrt{\frac{|M|}{s}} t x^{H-2}/2$. Then, (3.28) holds. Furthermore, by some basic calculations,

$$\begin{aligned}&\int_{G(M)} \phi_t(x_1, y_1)\hat{\Psi}_t(x_1, y_1) \,\mathrm{d}x_1 \,\mathrm{d}y_1 \\ &\quad\leq \sqrt{\frac{|M|}{s}} \int_M^{-t-s} \int_{-t-s-y_1}^{\infty} \phi_t(x_1, y_1) \frac{t}{2x_1^{2-H}} \,\mathrm{d}x_1 \,\mathrm{d}y_1 \\ &\quad\leq \sqrt{\frac{|M|}{s}} \int_{-\infty}^{-t-s} \left[\int_{-s-y_1}^{\infty} \frac{t^2}{2x_1^{4-2H}} \,\mathrm{d}x_1 + \int_{-t-s-y_1}^{-s-y_1} \frac{t(x_1+t+s+y_1)}{2x_1^{4-2H}} \,\mathrm{d}x_1 \right] \mathrm{d}y_1 \\ &\quad= \sqrt{\frac{|M|}{s}} \left[\frac{t^{2H}}{4(1-H)(3-2H)} + \int_0^t \mathrm{d}x_1 \int_{-t-s-x_1}^{-t-s} \frac{t(x_1+t+s+y_1)}{2x_1^{4-2H}} \,\mathrm{d}y_1 \right. \\ &\qquad\left. + \int_t^{\infty} \mathrm{d}x_1 \int_{-t-s-x_1}^{-s-x_1} \frac{t(x_1+t+s+y_1)}{2x_1^{4-2H}} \mathrm{d}y_1 \right] \\ &\quad= \frac{1}{4(1-H)(3-2H)(2H-1)} \sqrt{\frac{|M|}{s}} t^{2H}.\end{aligned} \tag{3.32}$$

Hence, (3.29) holds for $K_2 = \frac{1}{4(1-H)(3-2H)(2H-1)}$. □

Lemma 3.5. *For $0 \leq t \leq 1$, $M \leq -2-s$, let $\bar{G}(M) = \{-t-s-y < x, y \leq M\}$ and $\bar{\phi}_t(x, y) = \phi_t(x, y)\mathbf{1}_{\bar{G}(M)}(x, y)$. There then exist a non-negative function $\bar{\Psi}_{s,t}(x, y)$ on $\{x > 0, y \leq 0\}$ such that for all $n > 0$,*

$$F_{n,\bar{\phi}_t}(x_1, y_1)\mathbf{1}_{\bar{G}(M)}(x_1, y_1) \leq \bar{\Psi}_t(x_1, y_1), \tag{3.33}$$

and a constant $C > 0$ which is independent of t, M and H, such that

$$I_2(n, \bar{\phi}_t) \leq \int_{\bar{G}(M)} \phi_t(x_1, y_1)\bar{\Psi}_t(x_1, y_1) \,\mathrm{d}x_1 \,\mathrm{d}y_1 \leq K_2 C t^{2H}, \tag{3.34}$$

where K_2 is the constant in Lemma 3.4.

Proof. From (3.22), it follows that

$$\begin{aligned} F_{n,\bar{\phi}_t}(x_1,y_1) &= n^2 \int_{y_1}^{M}\int_{x_1}^{\infty} \bar{\phi}_t(x_2,y_2)\prod_{i=1}^{2}\sqrt{x_i|y_i|}\mathrm{e}^{-2n[x_1(y_2-y_1)-(x_2-x_1)y_2]}\,\mathrm{d}x_2\,\mathrm{d}y_2 \\ &= \frac{tn^2}{x_1^{3/2-H}}\int_{y_1}^{M}\sqrt{x_1}|y_1|\int_{x_1}^{\infty}\mathrm{e}^{-2n[x_1(y_2-y_1)-(x_2-x_1)y_2]}\,\mathrm{d}x_2\,\mathrm{d}y_2 \\ &= \frac{t}{2x_1^{2-H}}\int_{|M|}^{|y_1|}\frac{nx_1|y_1|}{|y_2|}\mathrm{e}^{2nx_1|y_2|}\,\mathrm{d}|y_2|\mathrm{e}^{-2nx_1|y_1|} \\ &= \frac{t}{2x_1^{2-H}}\mathrm{e}^{-2nx_1|y_1|}nx_1|y_1|\int_{nx_1|M|}^{nx_1|y_1|}\frac{1}{w}\mathrm{e}^{2w}\,\mathrm{d}w. \end{aligned} \tag{3.35}$$

Let $Q(z)=\mathrm{e}^{-2z}z\int_1^z \frac{1}{w}\mathrm{e}^{2w}\,\mathrm{d}w$ for $z\ge 1$. Then, $Q(z)$ is continuous on $[1,\infty)$ and $Q(1)=0$, $\lim_{z\to\infty} Q(z)=1/2$. Hence, there is a constant $C>0$, which is independent of M,t and H, such that $Q(z)$ is bounded by C. Note that on $\bar{G}(M)$, because $|M|>1$ and $x_1\ge -t-s-M>1$, we have $nx_1|M|>1$ for all $n>0$. Therefore, (3.35) yields

$$F_{n,\bar{\phi}_t}(x_1,y_1)\mathbf{1}_{\bar{G}(M)}(x_1,y_1)\le \frac{t}{2x_1^{2-H}}C.$$

Let $\bar{\Psi}_t(x,y)=Ctx^{H-2}/2$. By calculations similar to those in (3.32), (3.34) holds for $K_2=\frac{1}{4(1-H)(3-2H)(2H-1)}$. □

Proposition 3.2. *For any $0\le t\le 1$, there exists a constant K, independent of t but dependent on s, such that*

$$\mathbb{E}\left[\left(n\int_0^n\int_{-n}^0 \phi_t(x,y)\sqrt{x|y|}(-1)^{N_n(x,y)}\,\mathrm{d}x\,\mathrm{d}y\right)^2\right]\le Kt^{2H}. \tag{3.36}$$

Proof. Using the same notation as in Lemmas 3.1–3.5, taking $M=-2-s$ and observing that from (3.22), $\phi_t(x,y)=0$ if (x,y) is not in the set

$$\{(x,y): y<-s, x>0 \text{ and } x+y>-t-s\}=S\cup G(M)\cup\bar{G}(M),$$

we can rewrite the left-hand side of (3.36) by

$$\begin{aligned} I(n,\phi_t) &:= \mathbb{E}\left[\left(n\int_0^n\int_{-n}^0 \phi_t(x,y)\sqrt{x|y|}(-1)^{N_n(x,y)}\,\mathrm{d}x\,\mathrm{d}y\right)^2\right] \\ &= \mathbb{E}\left[\left(n\int_0^n\int_{-n}^0 \big(\tilde{\phi}_t(x,y)+\hat{\phi}_t(x,y)+\bar{\phi}_t(x,y)\big)\sqrt{x|y|}(-1)^{N_n(x,y)}\,\mathrm{d}x\,\mathrm{d}y\right)^2\right], \end{aligned} \tag{3.37}$$

which is bounded by

$$3I(n,\tilde{\phi}_t)+3I(n,\hat{\phi}_t)+3I(n,\bar{\phi}_t). \tag{3.38}$$

Note that $0\le\tilde{\phi}_t,\hat{\phi}_t,\bar{\phi}_t\le\phi_t$. Lemma 3.1, Corollary 3.1 and Lemma 3.3 imply that

$$\begin{aligned}I(n,\tilde{\phi}_t)&\le 2I_1(n,\tilde{\phi}_t)+2I_2(n,\tilde{\phi}_t)\\&\le\left(\frac{37}{8}+2K_1\sqrt{\frac{1+s}{s}}\right)t^{2H},\end{aligned} \tag{3.39}$$

and Lemma 3.1, Corollary 3.1 and Lemma 3.4 imply that

$$I(n,\hat{\phi}_t)\le 2I_1(n,\hat{\phi}_t)+2I_2(n,\hat{\phi}_t)\le\left(\frac{37}{8}+2K_2\sqrt{\frac{2+s}{s}}\right)t^{2H}. \tag{3.40}$$

Furthermore, from Lemma 3.1, Corollary 3.1 and Lemma 3.5, we have

$$I(n,\bar{\phi}_t)\le 2I_1(n,\bar{\phi}_t)+2I_2(n,\bar{\phi}_t)\le\left(\frac{37}{8}+2K_2C\right)t^{2H}. \tag{3.41}$$

Therefore, (3.37)–(3.41) yield that

$$I(n,\phi_t)\le 3\left[\frac{111}{8}+2K_1\sqrt{(1+s)/s}+2K_2\sqrt{(2+s)/s}+2K_2C\right]t^{2H}.$$

Taking $K=3[\frac{111}{8}+2K_1\sqrt{(1+s)/s}+2K_2\sqrt{(2+s)/s}+2K_2C]$ then completes the proof of Proposition 3.2. □

Finally, we prove Proposition 3.1.

Proof of Proposition 3.1. To prove the tightness of $\{Y_n\}_{n\ge 1}$ in $\mathcal{C}[0,1]$, it suffices to show that for some $r>0$ there exist two constants $\tilde{M}>0$ and $\eta>1$ such that for any $t,t'\in[0,1]$,

$$\begin{aligned}&\mathbb{E}\left[\left(n\int_0^n\int_{-n}^0\frac{g_s(t,x,y)-g_s(t',x,y)}{x^{2-H}}\sqrt{x|y|}(-1)^{N_n(x,y)}\,\mathrm{d}x\,\mathrm{d}y\right)^r\right]\\&\quad\le\tilde{M}(t-t')^\eta,\end{aligned}$$

which follows from the criterion given by Billingsley (see [5], Theorem 12.3) and the fact that our processes are null at the origin.

Without loss of generality, let $t>t'$. Note that from (1.6), we have

$$\begin{aligned}g_s(t,x,y)-g_s(t',x,y)&=h(t+s,x,y)-h(t'+s,x,y)\\&=g_{t'+s}(t-t',x,y).\end{aligned}$$

By Proposition 3.2, it is easy to check that the above inequality holds for $r = 2$, $\eta = 2H > 1$ and $\tilde{M} = 3[\frac{111}{8} + 2K_1\sqrt{(1+s)/s} + 2K_2\sqrt{(2+s)/s}) + 2K_2C]$, which completes the proof of Proposition 3.1. □

Remark 3.1. Compared with the proofs of tightness in Bardina et al. [2,4], we have found that under the condition that the kernel f can be separated by its arguments (x, y), the calculation of $I(n, f)$ is transformed to the calculation of $I_1(n, f)$, which is relatively simple; see the proofs of Lemmas 3.1 and 3.2 in [2] and the proof of Lemma 3.1 in [4]. However, in our case, the kernel f cannot be separated by the arguments (x, y), so we need to discuss $I_2(n, f)$. From our proof, we can see that the calculation of $I_2(n, f)$ is more complicated and delicate than that of $I_1(n, f)$. In addition, the fact that the kernel f cannot be separated by the arguments (x, y) also creates some difficulties in the identification of the limit law; see the proof of (4.7) in the next section.

Remark 3.2. By Proposition 10.3 in [10], page 149, Proposition 3.1 also shows that $\mathbb{P}(Y_n \in \mathcal{C}[0, 1], n \in \mathbb{N}) = 1$.

4. Limit law of Y_n

In this section, we proceed with the identification of the limit law. We will prove the following proposition.

Proposition 4.1. *The finite-dimensional distributions of $Y_n = \{Y_n(t), t \in [0, 1]\}$ defined by (2.3) converge weakly, as n tends to ∞, to those of a fractional Brownian motion $B^H = \{B^H(t), t \in [0, 1]\}$ with Hurst index $H \in (1/2, 1)$.*

Proof. For each $k \in \mathbb{N}$, $a_1, \ldots, a_k \in \mathbb{R}$ and $0 \le t_1 < t_2 < \cdots < t_k \le 1$, we define

$$L_n = \sum_{j=1}^{k} a_j Y_n(t_j) \quad \text{and} \quad U = \sum_{j=1}^{k} a_j B^H(t_j).$$

It suffices to prove that for any $\xi \in \mathbb{R}$, as $n \to \infty$,

$$J(n) := |\mathbb{E}[\exp(\mathrm{i}\xi L_n)] - \mathbb{E}[\exp(\mathrm{i}\xi U)]| \to 0. \tag{4.1}$$

For $T > 1 + s$, let

$$L_n(T) := n \sum_{j=1}^{k} a_j \int_0^T \int_{-T}^0 \phi_{t_j}(x, y)\sqrt{x|y|}(-1)^{N_n(x,y)}\,\mathrm{d}x\,\mathrm{d}y$$

and

$$U(T) := \sum_{j=1}^{k} a_j \int_0^T \int_{-T}^0 \phi_{t_j}(x, y) B(\mathrm{d}x, \mathrm{d}y),$$

where $B(x, y)$ is given by Lemma 2.1. Let

$$J_1(n,T) = |\mathbb{E}[\exp(\mathrm{i}\xi L_n(T))] - \mathbb{E}[\exp(\mathrm{i}\xi U(T))]|,$$
$$J_2(n,T) = |\mathbb{E}[\exp(\mathrm{i}\xi L_n)] - \mathbb{E}[\exp(\mathrm{i}\xi L_n(T))]|,$$
$$J_3(T) = |\mathbb{E}[\exp(\mathrm{i}\xi U(T))] - \mathbb{E}[\exp(\mathrm{i}\xi U)]|.$$

Then,

$$J(n) \le J_1(n,T) + J_2(n,T) + J_3(T). \tag{4.2}$$

Below, we estimate $J_1(n,T)$, $J_2(n,T)$ and $J_3(T)$, respectively.

(1) We estimate $J_1(n,T)$.

Noting that $\phi_{t_j}(x, y)$ is a non-negative measurable function on $\{x > 0, y \le 0\}$, we can find a sequence of elementary functions $q^{m,j}(x, y)$ such that

$$0 \le q^{m,j}(x,y) \le \phi_{t_j}(x,y) \quad \text{and} \quad q^{m,j}(x,y) \to \phi_{t_j}(x,y) \qquad \text{a.e. as } m \to \infty. \tag{4.3}$$

Then, by the dominated convergence theorem, it follows from the fact that $\int_0^\infty \int_{-\infty}^0 [\phi_{t_j}(x, y)]^2 \,\mathrm{d}x\,\mathrm{d}y < \infty$ that as $m \to \infty$,

$$\int_0^T \int_{-T}^0 \big(\phi_{t_j}(x,y) - q^{m,j}(x,y)\big)^2 \,\mathrm{d}x\,\mathrm{d}y \to 0. \tag{4.4}$$

For any $m, j, n > 0$, define

$$Y_n^{m,j} = n \int_0^T \int_{-T}^0 q^{m,j}(x,y)\sqrt{x|y|}(-1)^{N_n(x,y)} \,\mathrm{d}x\,\mathrm{d}y,$$
$$B^{m,j} = \int_0^T \int_{-T}^0 q^{m,j}(x,y) B(\mathrm{d}x, \mathrm{d}y).$$

By Lemma 2.1, we can readily verify that for fixed $m \in \mathbb{N}$, as $n \to \infty$,

$$J_{11}(n,T,m) := \left| \mathbb{E}\left[\exp\left(\mathrm{i}\xi \sum_{j=1}^k a_j Y_n^{m,j}\right)\right] - \mathbb{E}\left[\exp\left(\mathrm{i}\xi \sum_{j=1}^k a_j B^{m,j}\right)\right]\right| \to 0 \tag{4.5}$$

because $Y_n^{m,j}$ is essentially a linear combination of increments of B_n defined by (2.2), and $B^{m,j}$ is the same linear combination of the corresponding limits of increments of B_n.

We further define

$$J_{12}(n,T,m) := \left| \mathbb{E}[\exp(\mathrm{i}\xi L_n(T))] - \mathbb{E}\left[\exp\left(\mathrm{i}\xi \sum_{j=1}^k a_j Y_n^{m,j}\right)\right]\right|,$$

$$J_{13}(T,m) := \left| \mathbb{E}[\exp(\mathrm{i}\xi U(T))] - \mathbb{E}\left[\exp\left(\mathrm{i}\xi \sum_{j=1}^k a_j B^{m,j}\right)\right]\right|.$$

Then, for any n, m,

$$J_1(n,T) \le J_{11}(n,T,m) + J_{12}(n,T,m) + J_{13}(n,T,m). \tag{4.6}$$

Below, we will show that for all fixed T, there exists some $\gamma_T(m) > 0$ such that for any $n > 0$,

$$J_{12}(n,T,m) \le \gamma_T(m) \to 0 \tag{4.7}$$

as $m \to \infty$. To this end, let $f_{m,j}(x,y) = \phi_{t_j}(x,y) - q^{m,j}(x,y)$. Define

$$\tilde{f}_{m,j}(x,y) := f_{m,j}(x,y)\mathbf{1}_{[0,T]\times[-t_j-s,0)}(x,y),$$
$$\hat{f}_{m,j}(x,y) := f_{m,j}(x,y)\mathbf{1}_{[0,T]\times[-T,-t_j-s)}(x,y).$$

By (4.3), as $m \to \infty$,

$$\phi_{t_j}(x,y) \ge f_{m,j}(x,y) \to 0 \qquad \text{a.e. in } [0,T]\times[-T,0]. \tag{4.8}$$

Define

$$R_j(n,m,T) = \mathbb{E}\left[\left| n \int_0^T \int_{-T}^0 f_{m,j}(x,y)\sqrt{x|y|}(-1)^{N_n(x,y)}\,\mathrm{d}x\,\mathrm{d}y \right|\right].$$

Then, by Lemma 3.1,

$$\begin{aligned}
[R_j(n,m,T)]^2 &\le n^2 E\left[\left(\int_0^T \int_{-T}^0 \big(\hat{f}_{m,j}(x,y) + \tilde{f}_{m,j}(x,y)\big)\sqrt{x|y|}(-1)^{N_n(x,y)}\,\mathrm{d}x\,\mathrm{d}y\right)^2\right] \\
&\le 2\big(I(n,\hat{f}_{m,j}) + I(n,\tilde{f}_{m,j})\big) \qquad (4.9)\\
&\le 4\big(I_1(n,\hat{f}_{m,j}) + I_2(n,\hat{f}_{m,j}) + I_1(n,\tilde{f}_{m,j}) + I_2(n,\tilde{f}_{m,j})\big).
\end{aligned}$$

Lemma 3.2 and (4.4) show that for any $n > 0$, as $m \to \infty$,

$$I_1(n,\tilde{f}_{m,j}) \le \frac{37}{8}\int_0^\infty \int_{-\infty}^0 \tilde{f}_{m,j}^2(x,y)\,\mathrm{d}x\,\mathrm{d}y =: \tilde{\alpha}_j(m,T) \to 0. \tag{4.10}$$

Note that $f_{m,j}(x_1,y_1)\tilde{\Psi}_{t_j}(x_1,y_1) \to 0$ a.e. as $m \to \infty$. From Lemma 3.3 we know that

$$F_{n,\tilde{f}_{m,j}}(x_1,y_1) \le F_{n,\tilde{\phi}_{t_j}}(x_1,y_1) \le \tilde{\Psi}_{t_j}(x_1,y_1)$$

and that

$$\begin{aligned}
&\int_0^T \int_{-t_j-s}^0 f_{m,j}(x_1,y_1)F_{n,\tilde{f}_{m,j}}(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1 \\
&\quad \le \int_0^T \int_{-t_j-s}^0 \phi_{t_j}(x_1,y_1)\tilde{\Psi}_{t_j}(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1 < \infty.
\end{aligned}$$

By the dominated convergence theorem, as $m \to \infty$,

$$I_2(n, \tilde{f}_{m,j}) \le \tilde{\beta}_j(m, T) := \int_0^T \int_{-t_j-s}^0 f_{m,j}(x_1, y_1)\tilde{\Psi}_{t_j}(x_1, y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1 \to 0. \tag{4.11}$$

In a similar way, we know there are $\hat{\alpha}_j(m, T)$, $\hat{\beta}_j(m, T)$ such that for all $n > 0$, as $m \to \infty$,

$$I_1(n, \hat{f}_{m,j}) \le \hat{\alpha}_j(m, T) \to 0 \quad \text{and} \quad I_2(n, \hat{f}_{m,j}) \le \hat{\beta}_j(m, T) \to 0. \tag{4.12}$$

On the other hand, using the mean value theorem, we obtain that

$$\begin{aligned} J_{12}(n, T, m) &\le k|\xi| \max_{1\le j\le k} \mathbb{E}\left[\left|a_j n \int_0^T \int_{-T}^0 f_{m,j}(x, y)\sqrt{x|y|}(-1)^{N_n(x,y)}\,\mathrm{d}x\,\mathrm{d}y\right|\right] \\ &= k|\xi| \max_{1\le j\le k} (|a_j| R_j(n, m, T)). \end{aligned} \tag{4.13}$$

Hence, (4.7) follows from (4.9)–(4.13) with

$$\gamma_T(m) = 2k|\xi| \max_{1\le j\le k} \left(|a_j|\sqrt{\tilde{\alpha}_j(m, T) + \tilde{\beta}_j(m, T) + \hat{\alpha}_j(m, T) + \hat{\beta}_j(m, T)}\right).$$

For $J_{13}(m, T)$, we apply the mean value theorem again. Then, as $m \to \infty$,

$$\begin{aligned} J_{13}(m, T) &\le \xi \mathbb{E}\left[\left|a_j \sum_{j=1}^k \int_0^T \int_{-T}^0 \big(\phi_{t_j}(x, y) - q^{m,j}(x, j)\big) B(\mathrm{d}x, \mathrm{d}y)\right|\right] \\ &\le k\xi \max_{1\le j\le k} \mathbb{E}\left[\left|a_j \int_0^T \int_{-T}^0 \big(\phi_{t_j}(x, y) - q^{m,j}(x, j)\big) B(\mathrm{d}x, \mathrm{d}y)\right|\right] \\ &\le k\xi \max_{1\le j\le k} \left\{\left[\int_0^T \int_{-T}^0 \big(\phi_{t_j}(x, y) - q^{m,j}(x, y)\big)^2 \mathrm{d}x\,\mathrm{d}y\right]^{1/2}\right\} \to 0. \end{aligned} \tag{4.14}$$

From (4.5)–(4.7) and (4.14), we obtain that for any fixed T, as $n \to \infty$,

$$J_1(n, T) \to 0. \tag{4.15}$$

(2) In a similar way as was used to prove (4.7), there exists some $\zeta(T) > 0$ such that for $n > T$, as $T \to \infty$,

$$J_2(n, T) \le \zeta(T) \to 0. \tag{4.16}$$

(3) In a similar way as was used to prove (4.14), we obtain that there exists some $\theta(T) > 0$ such that as $T \to \infty$,

$$J_3(T) \le \theta(T) \to 0. \tag{4.17}$$

Therefore, combining (4.2) and (4.15)–(4.17), we can obtain that $J(n) \to 0$ as $n \to \infty$, completing the proof of Proposition 4.1. □

Note that Theorem 2.1 is an immediate result of Propositions 3.1 and 4.1. Therefore, the proof of Theorem 2.1 is complete.

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