Bernoulli **16**(4), 2010, 1224–1239
DOI: 10.3150/10-BEJ249

Testing composite hypotheses via convex duality

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We study the problem of testing composite hypotheses versus composite alternatives, using a convex duality approach. In contrast to classical results obtained by Krafft and Witting (*Z. Wahrsch. Verw. Gebiete* **7** (1967) 289–302), where sufficient optimality conditions are derived via Lagrange duality, we obtain necessary and sufficient optimality conditions via Fenchel duality under compactness assumptions. This approach also differs from the methodology developed in Cvitanić and Karatzas (*Bernoulli* **7** (2001) 79–97).

Keywords: composite hypotheses; convex duality; generalized Neyman–Pearson lemma; randomized test

1. Introduction

The problem of hypothesis testing is well understood in the classical case of testing a simple hypothesis versus a simple alternative. Suppose one wants to discriminate between two probability measures P (the "null hypothesis") and Q (the "alternative hypothesis"). In the classical Neyman–Pearson formulation, one seeks a randomized test $\varphi:\Omega\to[0,1]$ which is *optimal*, in that it minimizes the overall probability $\mathbb{E}^Q[1-\varphi]$ of not rejecting P when this hypothesis is false, while keeping below a given significance level $\alpha\in(0,1)$ the overall probability $\mathbb{E}^P[\varphi]$ of rejecting the hypothesis P when in fact it is true.

In this classical framework, an optimal randomized test $\widetilde{\varphi}$ always exists and can be calculated explicitly in terms of a reference probability measure R, with respect to which both measures are absolutely continuous (for instance, $R=(P+Q)/2$). This test has the randomized 0–1 structure

$$\widetilde{\varphi}=1_{\{L>\mathfrak{z}\}}+\delta\cdot 1_{\{L=\mathfrak{z}\}} \tag{1.1}$$

which involves the likelihood ratio $L=(\mathrm{d}Q/\mathrm{d}R)/(\mathrm{d}P/\mathrm{d}R)$ of the densities of the null and the alternative hypotheses, the quantile $\mathfrak{z}=\inf\{z\ge 0\colon\ P(L>z)\le\alpha\}$ and the number $\delta\in[0,1]$ which enforces the significance-level requirement without slackness, that is, $\mathbb{E}^P[\widetilde{\varphi}]=\alpha$.

The problem becomes considerably more involved when the hypotheses are composite, that is, when one has to discriminate between two entire *families of probability measures;* likelihood ratios of mixed strategies then have to be considered. This type of problem also arises in the financial mathematics context of minimizing the expected hedging loss in incomplete or constrained markets; see, for example Cvitanić [7], Schied [22] and Rudloff [21]. It was shown by

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Lehmann [15], Krafft and Witting [14], Baumann [4], Huber and Strassen [13], Österreicher [18], Witting [25], Vajda [24] and Cvitanić and Karatzas [8], that duality plays a crucial role in solving the testing problem. Most of these papers deal with Lagrange duality; they prove that the typical 0–1 structure of (1.1) is sufficient for optimality and that it is both necessary and sufficient if a dual solution exists. An important question then is to decide when a dual solution will exist and to describe it when it does.

The most recent of these papers, Cvitanić and Karatzas [8], takes a different duality approach. Methods from non-smooth convex analysis are employed and the set of densities in the null hypothesis is enlarged in order to obtain the existence of a dual solution, which again plays a crucial role.

In the present paper, we shall use Fenchel duality. One advantage of this approach is that as soon as one can prove the validity of strong duality, the existence of a dual solution follows. We shall show that strong duality holds under certain compactness assumptions. This generalizes previous results insofar as no need to enlarge the set of densities arises, a dual solution is obtained and thus necessary and sufficient conditions for optimality ensue.

In Section 2, we introduce the problem of testing composite hypotheses. Section 3 gives an overview of the duality results, which are established and explained in detail in Section 4. In Section 5, the imposed assumptions are discussed and possible extensions are given. A comparison of the results and methods of this paper with those in the existing literature can be found in the last sections, notably Section 6.

2. Testing of composite hypotheses

Let $(\Omega, \mathcal{F})$ be a measurable space. A central problem in the theory of hypothesis testing is to discriminate between a given family $\mathcal{P}$ of probability measures (composite "null hypothesis") and another given family $\mathcal{Q}$ of probability measures (composite "alternative hypothesis") on this space.

Suppose that there exists a *reference probability measure* R on $(\Omega, \mathcal{F})$, that is, a probability measure with respect to which all probability measures $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$ are absolutely continuous. We shall use the notation $Z_\Pi \equiv \mathrm{d}\Pi/\mathrm{d}R$ for the Radon–Nikodym derivative of a finite measure Π which is absolutely continuous with respect to the reference measure and $\mathbb{E}^\Pi[Y] := \int_\Omega Y\,\mathrm{d}\Pi = \int_\Omega Z_\Pi Y\,\mathrm{d}R$ for the integral with respect to such Π of an $\mathcal{F}$-measurable function $Y : \Omega \to [0, \infty)$. Finally, we shall denote the sets of these Radon–Nikodym derivatives for the composite null hypothesis and for the composite alternative hypothesis, respectively, by

$$\mathfrak{Z}_\mathcal{P} := \{Z_P \mid P \in \mathcal{P}\} \quad \text{and} \quad \mathfrak{Z}_\mathcal{Q} := \{Z_Q \mid Q \in \mathcal{Q}\}.$$

Both $\mathfrak{Z}_\mathcal{P}$ and $\mathfrak{Z}_\mathcal{Q}$ are subsets of the non-negative cone $\mathbb{L}^1_+$ and of the unit ball in the Banach space $\mathbb{L}^1 \equiv \mathbb{L}^1(\Omega, \mathcal{F}, R)$. We assume that $\Omega \times \mathfrak{Z}_\mathcal{P} \ni (\omega, Z) \longmapsto Z(\omega) \in [0, \infty)$ is measurable with respect to the product σ-algebra $\mathcal{F} \otimes \mathcal{B}$, where $\mathcal{B}$ is the σ-algebra of Borel subsets of $\mathfrak{Z}_\mathcal{P}$.

We shall denote by Φ the set of all *randomized tests,* that is, of all Borel-measurable functions $\varphi : \Omega \to [0, 1]$ on $(\Omega, \mathcal{F})$. The interpretation is as follows: if the outcome $\omega \in \Omega$ is observed and the randomized test φ is used, then the null hypothesis $\mathcal{P}$ is rejected with probability $\varphi(\omega)$. Thus,

$\mathbb{E}^P[\varphi] = \int_\Omega \varphi(\omega) P(\mathrm{d}\omega)$ is the overall probability of *type I error* (of rejecting the null hypothesis, when in fact it is true) under a scenario $P \in \mathcal{P}$, whereas $\mathbb{E}^Q[1-\varphi]$ is the overall probability of *type II error* (of not rejecting the null hypothesis, when in fact it is false) under the scenario $Q \in \mathcal{Q}$.

We shall adopt the Neyman–Pearson point of view, whereby a type I error is viewed as the more severe one and is not allowed to occur with probability that exceeds a given acceptable *significance level* $\alpha \in (0,1)$, regardless of which scenario $P \in \mathcal{P}$ might materialize. Among all randomized tests that observe this constraint,

$$\mathfrak{s}(\varphi) := \sup_{P \in \mathcal{P}} \mathbb{E}^P[\varphi] \le \alpha, \tag{2.1}$$

we then try to minimize the highest probability $\sup_{Q \in \mathcal{Q}}(1 - \mathbb{E}^Q[\varphi])$ of type II error over all scenarios in the alternative hypothesis. In other words, we look for a randomized test $\widetilde{\varphi}$ that maximizes the smallest *power* with respect to all alternative scenarios,

$$\pi(\varphi) := \inf_{Q \in \mathcal{Q}} \mathbb{E}^Q[\varphi],$$

over all randomized tests φ whose 'size' $\mathfrak{s}(\varphi)$, the quantity defined in (2.1), does not exceed a given significance level α.

Equivalently, we look for a test $\widetilde{\varphi} \in \Phi$ that attains the supremum

$$\underline{V} := \sup_{\varphi \in \Phi_\alpha} \pi(\varphi) = \sup_{\varphi \in \Phi_\alpha} \Big(\inf_{Q \in \mathcal{Q}} \mathbb{E}^Q[\varphi] \Big) \tag{2.2}$$

of the power $\pi(\varphi)$ over all generalized tests in the class

$$\Phi_\alpha := \Big\{ \varphi \in \Phi \Big| \sup_{P \in \mathcal{P}} \mathbb{E}^P[\varphi] \le \alpha \Big\}. \tag{2.3}$$

When such a randomized test $\widetilde{\varphi}$ exists, it will be called (max–min) *optimal*.

3. Duality

We shall denote by Λ_+ the set of finite measures on the measurable space $(\mathfrak{Z}_\mathcal{P}, \mathcal{B})$. We shall then associate to the maximization problem of (2.2) the dual minimization problem

$$V^* := \inf_{\substack{Q \in \mathcal{Q} \\ \lambda \in \Lambda_+}} \mathcal{D}(Q, \lambda), \tag{3.1}$$

where

$$\mathcal{D}(Q,\lambda) := \mathbb{E}^R\bigg[\bigg(Z_Q - \int_{\mathfrak{Z}_\mathcal{P}} Z_P \, \mathrm{d}\lambda\bigg)^+\bigg] + \alpha\lambda(\mathfrak{Z}_\mathcal{P}). \tag{3.2}$$

Here, and in the sequel, we view $\int_{\mathfrak{Z}_\mathcal{P}} Z_P(\omega)\,\mathrm{d}\lambda$ as the integral with respect to the measure λ of the continuous functional $\mathfrak{Z}_\mathcal{P} \ni Z \longmapsto \ell(Z;\omega) := Z(\omega) \in \mathbb{R}$, for fixed $\omega \in \Omega$; see (4.10) below for an amplification of this point.

The idea behind (3.1) and (3.2) is simple: we regard $\lambda \in \Lambda_+$ as a 'Bayesian prior' distribution on the set $\mathfrak{Z}_\mathcal{P}$ of densities for the null hypothesis, whose effect is to reduce the composite null hypothesis $\mathcal{P}$ to a simple one $\{P_*\}$ with $Z_{P_*} = \int_{\mathfrak{Z}_\mathcal{P}} Z_P\,\mathrm{d}\lambda$ and whose total mass $\lambda(\mathfrak{Z}_\mathcal{P}) < \infty$ is a variable that enforces the constraint in (2.1). More precisely: for any given $Q \in \mathcal{Q}$ and any $\varphi \in \Phi_\alpha$, we have the *weak duality*

$$\begin{aligned}\mathbb{E}^Q[\varphi] = \mathbb{E}^R[\varphi Z_Q] &= \mathbb{E}^R\left[\varphi\left(Z_Q - \int_{\mathfrak{Z}_\mathcal{P}} Z_P\,\mathrm{d}\lambda\right)\right] + \mathbb{E}^R\left[\varphi\int_{\mathfrak{Z}_\mathcal{P}} Z_P\,\mathrm{d}\lambda\right] \\ &\le \mathbb{E}^R\left[\left(Z_Q - \int_{\mathfrak{Z}_\mathcal{P}} Z_P\,\mathrm{d}\lambda\right)^+\right] + \alpha\lambda(\mathfrak{Z}_\mathcal{P}) = \mathcal{D}(Q,\lambda), \qquad \lambda \in \Lambda_+,\end{aligned} \tag{3.3}$$

since

$$\begin{aligned}\mathbb{E}^R\left[\varphi\int_{\mathfrak{Z}_\mathcal{P}} Z_P\,\mathrm{d}\lambda\right] &= \int_\Omega \varphi(\omega)\left(\int_{\mathfrak{Z}_\mathcal{P}} Z_P(\omega)\,\mathrm{d}\lambda\right)\mathrm{d}R(\omega) \\ &= \int_{\mathfrak{Z}_\mathcal{P}}\left(\int_\Omega \varphi(\omega) Z_P(\omega)\,\mathrm{d}R(\omega)\right)\mathrm{d}\lambda\end{aligned}$$

holds by the Fubini–Tonelli theorems, and this last quantity is dominated by $\alpha\lambda(\mathfrak{Z}_\mathcal{P})$ on the strength of (2.1). We now observe that equality holds in (3.3) if and only if both

$$\varphi(\omega) = \begin{cases} 1, & \text{if } Z_Q(\omega) > \int_{\mathfrak{Z}_\mathcal{P}} Z_P(\omega)\,\mathrm{d}\lambda, \\ 0, & \text{if } Z_Q(\omega) < \int_{\mathfrak{Z}_\mathcal{P}} Z_P(\omega)\,\mathrm{d}\lambda, \end{cases} \qquad \text{for } R\text{-a.e. } \omega \in \Omega \tag{3.4}$$

and

$$\mathbb{E}^R[\varphi Z_P] = \alpha, \qquad \text{for } \lambda\text{-a.e. } Z_P \in \mathfrak{Z}_\mathcal{P} \tag{3.5}$$

hold. It follows from (3.3) that the inequality $\sup_{\varphi\in\Phi_\alpha} \mathbb{E}^Q[\varphi] \le \mathcal{D}(Q,\lambda)$ holds for all $\lambda \in \Lambda_+$ and $Q \in \mathcal{Q}$ so that in the notation of (2.2) and (3.1), we obtain

$$\underline{V} \le \overline{V} := \inf_{Q\in\mathcal{Q}}\left(\sup_{\varphi\in\Phi_\alpha} \mathbb{E}^Q[\varphi]\right) \le V^*. \tag{3.6}$$

The challenge, then, it to turn this 'weak' duality into 'strong' duality. That is, to show that: equalities $\underline{V} = \overline{V} = V^*$ prevail in (3.6); the infimum in (3.1) is attained by some $(\widetilde{Q},\widetilde{\lambda}) \in \mathcal{Q} \times \Lambda_+$; there exists a $\widetilde{\varphi} \in \Phi_\alpha$ for which the triple $(\widetilde{\varphi},\widetilde{Q},\widetilde{\lambda})$ satisfies (3.4), (3.5); for this triple equality prevails in (3.3); and the first element $\widetilde{\varphi}$ of this triple is optimal for the generalized hypothesis-testing problem, that is, attains the supremum in (2.2). We shall carry out this program, under appropriate conditions, throughout the remainder of the paper.

When it exists, the measure $\widetilde{\lambda} \in \Lambda_+$ is called the "least favorable distribution". For explicit computations of least favorable distributions in testing composite hypotheses against simple alternatives (with $\mathcal{Q}$ a singleton), see Lehmann and Stein [16], Lehmann [15] and Reinhardt [19], as well as Witting [25], pages 276–281 and Lehmann and Romano [17], Chapter 3. Here is an example abridged from these last two sources.

Example 3.1. Consider random variables $X_1, X_2, \ldots, X_n$ on $(\Omega, \mathcal{F})$ and probability measures on this space under which these variables are independent with common Gaussian distribution $\mathcal{N}(\xi, \sigma^2)$. Thus, the random variables $\overline{X} = (1/n)\sum_{i=1}^n X_i$ and $U = \sum_{i=1}^n (X_i - \overline{X})^2$ are sufficient for the vector of parameters (ξ, σ^2). For some given real numbers ξ_1 and $\sigma_1^2 > 0$, $\sigma_0^2 > 0$, we shall consider testing each of two composite hypotheses $\mathcal{P}^{(1)}$, $\mathcal{P}^{(2)}$ versus the simple alternative $\mathcal{Q} = \{Q\}$; this latter corresponds to $(\xi, \sigma^2) = (\xi_1, \sigma_1^2)$. On the other hand, the hypothesis $\mathcal{P}^{(1)}$ corresponds to $(\xi, \sigma^2) \in \mathbb{R} \times [\sigma_0^2, \infty)$, whereas the hypothesis $\mathcal{P}^{(2)}$ corresponds to $(\xi, \sigma^2) \in \mathbb{R} \times (0, \sigma_0^2]$.

It is clear that the least favorable measure $\widetilde{\lambda} \in \Lambda_+$ should correspond to a distribution on $\mathcal{B}(\mathbb{R} \times (0, \infty))$ of the form $\mu \otimes \delta_{\sigma_0^2}$, where μ is a measure on $\mathcal{B}(\mathbb{R})$; thus, under the measure P_* with $Z_{P_*} = \int_{\mathfrak{Z}_\mathcal{P}} Z_P \, \mathrm{d}\lambda$, the distribution of $\overline{X}$ has probability density

$$\int_{\mathbb{R}} \bigl(2\pi(\sigma_0^2/n)\bigr)^{-1/2} \exp\left\{-\frac{n(x-\xi)^2}{2\sigma_0^2}\right\} \mu(\mathrm{d}\xi), \qquad x \in \mathbb{R},$$

whereas, under the alternative $\mathcal{Q}$, the distribution of $\overline{X}$ has probability density

$$\bigl(2\pi(\sigma_1^2/n)\bigr)^{-1/2} \exp\left\{-\frac{n(x-\xi_1)^2}{2\sigma_1^2}\right\}, \qquad x \in \mathbb{R}.$$

- *Testing $\mathcal{P}^{(1)}$ versus $\mathcal{Q}$, with $\sigma_0^2 > \sigma_1^2$*: The least favorable $\widetilde{\lambda} \in \Lambda_+$ corresponds to $\mu = \delta_{\xi_1}$. This way, the distribution of $\overline{X}$ is normal under both P_* and Q, with the same mean and with the smallest possible difference in the variances.
- *Testing $\mathcal{P}^{(2)}$ versus $\mathcal{Q}$, with $\sigma_0^2 < \sigma_1^2$*: In this case, the least favorable $\widetilde{\lambda} \in \Lambda_+$ corresponds to μ with Gaussian $\mathcal{N}(\xi_1, (\sigma_1^2 - \sigma_0^2)/n)$ density. This guarantees that the distribution of $\overline{X}$ is the same under both P_* and Q.

4. Results

In order to carry out the program outlined in the previous section, we shall impose the following assumptions. A discussion of their role can be found in Remark 5.1.

Assumption 4.1.

(i) $\mathfrak{Z}_\mathcal{Q}$ *is a weakly compact, convex subset of* $\mathbb{L}^1$.
(ii) $\mathfrak{Z}_\mathcal{P}$ *is a weakly compact subset of* $\mathbb{L}^1$.

Our main result reads as follows.

Theorem 4.2 (Generalized Neyman–Pearson lemma). *Let $\mathcal{P}, \mathcal{Q}$ be families of probability measures on $(\Omega, \mathcal{F})$, as in Sections 2 and 3, that satisfy Assumption 4.1. For a given constant $\alpha \in (0,1)$, recall the subclass Φ_α of randomized tests in (2.3).*

There then exists a randomized test $\widetilde{\varphi} \in \Phi_\alpha$ which attains the supremum in (2.2). There also exists a solution to the dual problem of (3.1), namely, a pair $(\widetilde{Q}, \widetilde{\lambda}) \in \mathcal{Q} \times \Lambda_+$ which attains the infimum there.

Furthermore, strong duality is satisfied, in the sense that:

- *the optimal test for (2.2) has the structure of (3.4), (3.5), namely*

$$\widetilde{\varphi}(\omega) = \begin{cases} 1, & \text{if } Z_{\widetilde{Q}}(\omega) > \displaystyle\int_{\mathfrak{Z}_\mathcal{P}} Z_P(\omega)\,\mathrm{d}\widetilde{\lambda}, \\ 0, & \text{if } Z_{\widetilde{Q}}(\omega) < \displaystyle\int_{\mathfrak{Z}_\mathcal{P}} Z_P(\omega)\,\mathrm{d}\widetilde{\lambda}, \end{cases} \qquad \text{for } R\text{-a.e. } \omega \in \Omega \tag{4.1}$$

  and

$$\mathbb{E}^R[\widetilde{\varphi} Z_P] = \alpha \qquad \text{for } \widetilde{\lambda}\text{-a.e. } Z_P \in \mathfrak{Z}_\mathcal{P}, \text{ whereas} \tag{4.2}$$

- *$(\widetilde{\varphi}, \widetilde{Q})$ is a saddlepoint in $\Phi_\alpha \times \mathcal{Q}$ of the functional $(\varphi, Q) \mapsto \mathbb{E}^Q[\varphi]$, namely,*

$$\mathbb{E}^{\widetilde{Q}}[\varphi] \le \mathbb{E}^{\widetilde{Q}}[\widetilde{\varphi}] \le \mathbb{E}^{Q}[\widetilde{\varphi}] \qquad \forall (\varphi, Q) \in \Phi_\alpha \times \mathcal{Q}. \tag{4.3}$$

The theorem will be proven in several steps, using the following lemmata. The first of these, Lemma 4.3, seems to be well known (cf. the Appendix in Lehmann and Romano [17]). We could not, however, find in the literature a result in exactly the form we needed that we could cite directly, so we provide a proof for the sake of completeness. We shall freely use the convention of denoting by "max" (resp., "min") a supremum (resp., infimum) which is attained.

Lemma 4.3. *The supremum in (2.2) is attained by some randomized test $\widetilde{\varphi} \in \Phi_\alpha$ and there exists a $\widetilde{Q} \in \mathcal{Q}$ such that the saddlepoint property (4.3) holds. In particular, the lower- and upper-values $\underline{V}$ and $\overline{V}$ of (2.2) and (3.6), respectively, are the same:*

$$\max_{\varphi \in \Phi_\alpha} \Bigl(\min_{Q \in \mathcal{Q}} \mathbb{E}^Q[\varphi] \Bigr) = \min_{Q \in \mathcal{Q}} \Bigl(\max_{\varphi \in \Phi_\alpha} \mathbb{E}^Q[\varphi] \Bigr). \tag{4.4}$$

Proof. It is well known that the set Φ of all randomized tests is weakly-* compact (this follows from weak sequential compactness; cf. [25], page 270 or [17], Theorem A.5.1). We give a short proof for the sake of completeness. The subset Φ of the Banach space $\mathbb{L}^\infty \equiv \mathbb{L}^\infty(\Omega, \mathcal{F}, R)$ is weakly-* compact, as it is a weakly-* closed subset of the weakly-* compact unit ball in $\mathbb{L}^\infty$ (Alaoglu's theorem; see, e.g., Dunford and Schwartz [9], Theorem V.4.2 and Corollary V.4.3). To see that Φ is weakly-* closed, consider a net $\{\varphi_\alpha\}_{\alpha \in D} \subseteq \Phi$ that converges to φ with respect to the weak-* topology in $\mathbb{L}^\infty$. This means that for all $X \in \mathbb{L}^1$, we have $\mathbb{E}^R[\varphi_\alpha X] \to \mathbb{E}^R[\varphi X]$. If there existed an event $\Omega_1 \in \mathcal{F}$ with $R(\Omega_1) > 0$ and $\{\varphi > 1\} \subseteq \Omega_1$, then we could choose $\widehat{X}(\omega) = 1_{\Omega_1}(\omega) \in \mathbb{L}^1$ and obtain $\mathbb{E}^R[\varphi\widehat{X}] > R(\Omega_1)$. However, this contradicts $\mathbb{E}^R[\varphi\widehat{X}] = \lim_\alpha \mathbb{E}^R[\varphi_\alpha \widehat{X}] \le R(\Omega_1)$, which follows from $\varphi_\alpha \le 1$ for all $\alpha \in D$ since $\varphi_\alpha \in \Phi$. Hence, $\varphi \le 1$ holds R-a.e. It can be similarly shown that $\varphi \ge 0$ also holds R-a.e.

Thus, Φ is indeed weakly-* closed, hence weakly-* compact. Since the mapping $\varphi \mapsto \sup_{P\in\mathcal{P}} \mathbb{E}^P[\varphi]$ is lower-semicontinuous in the weak-* topology, the set Φ_α in (2.3) is weakly-* closed, hence weakly-* compact. Because of the upper-semicontinuity of the mapping $\varphi \mapsto \pi(\varphi) = \inf_{Q\in\mathcal{Q}} \mathbb{E}^Q[\varphi]$ in the weak-* topology, there exists a $\widetilde{\varphi} \in \Phi_\alpha$ that attains the supremum in (2.2).

The weak-* compactness and convexity of Φ_α together with the weak compactness and convexity of $\mathfrak{Z}_\mathcal{Q}$ (Assumption 4.1(i)) enable us to apply the von Neumann/Sion minimax theorem (see, for instance, [3], Theorem 7, Section 7.1, or [2], Section 2.7, pages 39–45); the assertions follow. □

Let us now fix an arbitrary $Q \in \mathcal{Q}$ and consider as our primal problem the inner maximization in the middle term of (3.6), namely,

$$\mathfrak{p}(Q) := \sup_{\varphi\in\Phi_\alpha} \mathbb{E}^Q[\varphi]. \tag{4.5}$$

This supremum is always attained since Φ_α is weakly-* compact. We want to show that strong duality holds between (4.5) and its *Fenchel dual problem* which, we claim, is of the form

$$\mathfrak{d}(Q) = \inf_{\lambda\in\Lambda_+} \mathcal{D}(Q,\lambda) = \inf_{\lambda\in\Lambda_+} \left[\int_\Omega \left(Z_Q - \int_{\mathfrak{Z}_\mathcal{P}} Z_P \, \mathrm{d}\lambda \right)^+ \mathrm{d}R + \alpha\lambda(\mathfrak{Z}_\mathcal{P}) \right]. \tag{4.6}$$

Note that, from (3.3), we have $\mathfrak{p}(Q) \le \mathfrak{d}(Q)$.

In this setting, the typical 0–1 structure of the randomized test $\widetilde{\varphi}_Q \in \Phi_\alpha$ that attains the supremum in (4.5) is necessary and sufficient for optimality.

Lemma 4.4. *Strong duality holds for problems* (4.5) *and* (4.6), *that is*,

$$\forall Q \in \mathcal{Q}, \qquad \mathfrak{d}(Q) = \mathfrak{p}(Q).$$

Moreover, for each $Q \in \mathcal{Q}$, there exists a measure $\widetilde{\lambda}_Q \in \Lambda_+$ which attains the infimum in (4.6), *whereas an optimal test $\widetilde{\varphi}_Q \in \Phi_\alpha$ that attains the supremum in* (4.5) *exists and has the structure of* (3.4) *and* (3.5), *namely*,

$$\widetilde{\varphi}_Q(\omega) = \begin{cases} 1, & \text{if } Z_Q(\omega) > \int_{\mathfrak{Z}_\mathcal{P}} Z_P(\omega)\, \mathrm{d}\widetilde{\lambda}_Q, \\ 0, & \text{if } Z_Q(\omega) < \int_{\mathfrak{Z}_\mathcal{P}} Z_P(\omega)\, \mathrm{d}\widetilde{\lambda}_Q, \end{cases} \qquad \text{for } R\text{-a.e. } \omega \in \Omega \tag{4.7}$$

and

$$\mathbb{E}^R[\widetilde{\varphi}_Q Z_P] = \alpha \qquad \text{for } \widetilde{\lambda}_Q\text{-a.e. } Z_P \in \mathfrak{Z}_\mathcal{P}. \tag{4.8}$$

Proof. Let $\mathcal{L}$ be the linear space of all continuous functionals $\ell : \mathfrak{Z}_{\mathcal{P}} \to \mathbb{R}$ on the weakly compact subset $\mathfrak{Z}_{\mathcal{P}}$ of $\mathbb{L}^1$ (Assumption 4.1(ii)) with pointwise addition and multiplication by real numbers and pointwise partial order

$$\ell_1 \le \ell_2 \quad \Longleftrightarrow \quad \ell_2 - \ell_1 \in \mathcal{L}_+ := \{\ell \in \mathcal{L} \mid \ell(Z_P) \ge 0, \forall P \in \mathcal{P}\}. \tag{4.9}$$

We endow $\mathcal{L}$ with the supremum norm $\|\ell\|_{\mathcal{L}} = \sup_{P \in \mathcal{P}} |\ell(Z_P)|$, which ensures that $\mathcal{L}$ is a Banach space (Dunford and Schwartz [9], Section IV.6).

Similarly, we let Λ be the space of regular finite signed measures $\lambda = \lambda^+ - \lambda^-$ on $(\mathfrak{Z}_{\mathcal{P}}, \mathcal{B})$, with $\lambda^\pm \in \Lambda_+$. The space Λ is sometimes denoted $\mathfrak{ca}_r(\mathfrak{Z}_{\mathcal{P}}, \mathcal{B})$ in the literature, in order to stress that it consists of *r*egular *c*ountably *a*dditive signed measures of finite variation ([1], Definition 12.11). We regard this space as the norm-dual of $\mathcal{L}$ with the bilinear form

$$\langle \ell, \lambda \rangle = \int_{\mathfrak{Z}_{\mathcal{P}}} \ell \, d\lambda \qquad \text{for } \ell \in \mathcal{L}, \lambda \in \Lambda; \tag{4.10}$$

see Aliprantis and Border [1], Corollary 14.15. (Intuitively speaking, the elements of Λ are generalized Bayesian priors that may assign negative mass to certain null hypotheses.) We also note the clear identification $\Lambda_+ \equiv \{\lambda \in \Lambda \mid \lambda(B) \ge 0, \forall B \in \mathcal{B}\}$.

Let us define a linear operator $A : (\mathbb{L}^\infty, \|\cdot\|_{\mathbb{L}^\infty}) \to (\mathcal{L}, \|\cdot\|_{\mathcal{L}})$ by

$$\mathbb{L}^\infty \ni \varphi \quad \longmapsto \quad (A\varphi)(Z_P) := -\mathbb{E}^P[\varphi] = -\mathbb{E}^R[\varphi Z_P] \in \mathbb{R} \tag{4.11}$$

for $Z_P \in \mathfrak{Z}_{\mathcal{P}}$. This operator is bounded, thus continuous. We also introduce the constant functionals $\mathbf{1}, \mathbf{0} \in \mathcal{L}$ by

$$\forall Z_P \in \mathfrak{Z}_{\mathcal{P}}, \qquad \mathbf{1}(Z_P) = 1 \in \mathbb{R}, \qquad \mathbf{0}(Z_P) = 0 \in \mathbb{R}.$$

The constraint of (2.1) can then be rewritten as

$$\alpha\mathbf{1} + A\varphi \ge \mathbf{0} \quad \Longleftrightarrow \quad A\varphi \in \mathcal{L}_+ - \alpha\mathbf{1}.$$

With this notation, for any given $Q \in \mathcal{Q}$, the primal problem (4.5) is cast as

$$\begin{aligned} -\mathfrak{p}(-Q) &= \inf_{\varphi \in \mathbb{L}^\infty} \big(\mathbb{E}^Q[\varphi] + \mathcal{I}_\Phi(\varphi) + \mathcal{I}_{\mathcal{L}_+ - \alpha\mathbf{1}}(A\varphi)\big) \\ &= \inf_{\varphi \in \mathbb{L}^\infty} \big(f(\varphi) + g(A\varphi)\big), \end{aligned} \tag{4.12}$$

where $-Q$ is interpreted as a finite, signed measure on $(\Omega, \mathcal{F})$ (cf. the discussion preceding (4.10)). Here, and for the remainder of this proof, we use the notation $\mathcal{I}_C(\varphi) := 0$ for $\varphi \in C$, $\mathcal{I}_C(\varphi) := \infty$ for $\varphi \notin C$, as well as

$$f(\varphi) := \mathbb{E}^Q[\varphi] + \mathcal{I}_\Phi(\varphi), \qquad g(A\varphi) := \mathcal{I}_{\mathcal{L}_+ - \alpha\mathbf{1}}(A\varphi). \tag{4.13}$$

• We claim that the *Fenchel dual* of the primal problem in (4.5) has the form (4.6). We shall begin the proof of this claim by recalling from Ekeland and Temam [10], Proposition III.1.1, Theorem III.4.1 and Remark III.4.2 that the Fenchel dual of the problem (4.12) is given by

$$-\mathfrak{d}(-Q) = \sup_{\lambda \in \Lambda} \big(-f^*(A^*\lambda) - g^*(-\lambda) \big), \tag{4.14}$$

where $A^* : \Lambda \to \mathfrak{ba}(\Omega, \mathcal{F}, R)$ is the adjoint of the operator A in (4.11). Here, and in the sequel, $\mathfrak{ba}(\Omega, \mathcal{F}, R)$ is the space of $\mathfrak{b}$ounded, (finitely-) $\mathfrak{a}$dditive set-functions on $(\Omega, \mathcal{F})$ which are absolutely continuous with respect to R; see, for instance, Yosida [26], Chapter IV, Section 9, Example 5.

The function $g^*(\cdot)$ is the conjugate of the function $g(\cdot)$, namely,

$$\begin{aligned} g^*(\lambda) &= \sup_{\widetilde{\ell} \in \mathcal{L}} \big(\langle \widetilde{\ell}, \lambda \rangle - \mathcal{I}_{\mathcal{L}_+ - \alpha \mathbf{1}}(\widetilde{\ell}) \big) = \sup_{\widetilde{\ell} \in \mathcal{L}_+ - \alpha \mathbf{1}} \langle \widetilde{\ell}, \lambda \rangle = \sup_{\ell \in \mathcal{L}_+} \langle \ell - \alpha \mathbf{1}, \lambda \rangle \\ &= \sup_{\ell \in \mathcal{L}_+} \langle \ell, \lambda \rangle - \alpha \int_{\mathfrak{Z}_\mathcal{P}} \mathrm{d}\lambda = \mathcal{I}_{\mathcal{L}_+^*}(\lambda) - \alpha \lambda(\mathfrak{Z}_\mathcal{P}), \end{aligned}$$

where

$$\mathcal{L}_+^* := \{ \lambda \in \Lambda \mid \langle \ell, \lambda \rangle \le 0, \forall \ell \in \mathcal{L}_+ \} \tag{4.15}$$

is the negative dual cone of $\mathcal{L}_+$. The last equality in the above string holds because the set $\mathcal{L}_+$ defined in (4.9) is a cone containing the origin $\mathbf{0} \in \mathcal{L}$.

To determine the conjugate $f^*(\cdot)$ of the function $f(\cdot)$ at $A^*\lambda$, namely

$$f^*(A^*\lambda) = \sup_{\varphi \in \mathbb{L}^\infty} \{ \langle A^*\lambda, \varphi \rangle - \mathbb{E}^Q[\varphi] - \mathcal{I}_\Phi(\varphi) \},$$

we have to calculate $\langle A^*\lambda, \varphi \rangle$. By the definition of A^*, the equation $\langle A^*\lambda, \varphi \rangle = \langle \lambda, A\varphi \rangle$ has to be satisfied for all $\varphi \in \mathbb{L}^\infty$, $\lambda \in \Lambda$ (see [1], Chapter 6.8). Thus,

$$\forall \varphi \in \mathbb{L}^\infty, \forall \lambda \in \Lambda, \qquad \langle A^*\lambda, \varphi \rangle = -\int_{\mathfrak{Z}_\mathcal{P}} \mathbb{E}^R[\varphi Z_P] \, \mathrm{d}\lambda$$

and the conjugate of the function $f(\cdot)$ at $A^*\lambda$ is evaluated as

$$f^*(A^*\lambda) = \sup_{\varphi \in \Phi} \bigg(-\int_{\mathfrak{Z}_\mathcal{P}} \mathbb{E}^R[\varphi Z_P] \, \mathrm{d}\lambda - \mathbb{E}^Q[\varphi] \bigg).$$

The dual problem (4.14) therefore becomes

$$\begin{aligned} -\mathfrak{d}(-Q) &= \sup_{\lambda \in \Lambda} \bigg[-\sup_{\varphi \in \Phi} \bigg(-\int_{\mathfrak{Z}_\mathcal{P}} \mathbb{E}^R[\varphi Z_P] \, \mathrm{d}\lambda - \mathbb{E}^Q[\varphi] \bigg) - \mathcal{I}_{-\mathcal{L}_+^*}(\lambda) - \alpha \lambda(\mathfrak{Z}_\mathcal{P}) \bigg] \\ &= \sup_{\lambda \in -\mathcal{L}_+^*} \bigg[-\sup_{\varphi \in \Phi} \bigg(-\int_{\mathfrak{Z}_\mathcal{P}} \mathbb{E}^R[\varphi Z_P] \, \mathrm{d}\lambda - \mathbb{E}^Q[\varphi] \bigg) - \alpha \lambda(\mathfrak{Z}_\mathcal{P}) \bigg]. \end{aligned} \tag{4.16}$$

It is not hard to show the property $-\mathcal{L}_+^* = \Lambda_+$ for the set in (4.15), so the expression of (4.16) can be recast in the form

$$-\mathfrak{d}(-Q) = \sup_{\lambda\in\Lambda_+}\left[-\sup_{\varphi\in\Phi}\left(-\int_{\mathfrak{Z}_\mathcal{P}} \mathbb{E}^R[Z_P\varphi]\,\mathrm{d}\lambda - \mathbb{E}^R[Z_Q\varphi]\right) - \alpha\lambda(\mathfrak{Z}_\mathcal{P})\right]. \tag{4.17}$$

• Now, both $(\Omega, \mathcal{F}, R)$ and $(\mathfrak{Z}_\mathcal{P}, \mathcal{B}, \lambda)$ for $\lambda \in \Lambda_+$ are positive, finite measure spaces. The mapping $\Omega \times \mathfrak{Z}_\mathcal{P} \ni (\omega, Z_P) \longmapsto f(\omega, Z_P) := Z_P(\omega)\varphi(\omega) \in \mathbb{R}$ is measurable with respect to the product σ-algebra $\mathcal{F} \otimes \mathcal{B}$ for every $\varphi \in \Phi$, thanks to the measurability assumption of Section 2, whereas

$$\int_{\mathfrak{Z}_\mathcal{P}}\int_\Omega |Z_P\varphi|\,\mathrm{d}R\,\mathrm{d}\lambda \le \Big(\sup_{P\in\mathcal{P}} \|Z_P\|_{\mathbb{L}^1}\Big)\lambda(\mathfrak{Z}_P) = \lambda(\mathfrak{Z}_\mathcal{P}) < \infty$$

holds for every $\lambda \in \Lambda_+$ and $\varphi \in \Phi$ since $\|\varphi\|_{\mathbb{L}_\infty} \le 1$. Thus, we can apply the Fubini–Tonelli theorem (see [9], Corollary III.11.15) and deduce that the order of integration can be interchanged, that is, for all $\lambda \in \Lambda_+$ and all $\varphi \in \Phi$, we have

$$\int_{\mathfrak{Z}_\mathcal{P}}\int_\Omega Z_P\varphi\,\mathrm{d}R\,\mathrm{d}\lambda = \int_\Omega\int_{\mathfrak{Z}_\mathcal{P}} Z_P\varphi\,\mathrm{d}\lambda\,\mathrm{d}R < \infty.$$

In (4.17), only elements $\lambda \in \Lambda_+$ and $\varphi \in \Phi$ are considered, so we can interchange the order of integration and obtain

$$-\mathfrak{d}(-Q) = \sup_{\lambda\in\Lambda_+}\left(-\sup_{\varphi\in\Phi}\mathbb{E}^R\left[\varphi\left(-Z_Q - \int_{\mathfrak{Z}_\mathcal{P}} Z_P\,\mathrm{d}\lambda\right)\right] - \alpha\lambda(\mathfrak{Z}_\mathcal{P})\right). \tag{4.18}$$

Since $\varphi \in \Phi$ is a randomized test, the supremum over all $\varphi \in \Phi$ in (4.18) is attained by some $\overline{\varphi}_{\lambda,-Q} \in \Phi$ of a form similar to (3.4), namely,

$$\overline{\varphi}_{\lambda,-Q}(\omega) = \begin{cases} 1, & \text{if } -Z_Q(\omega) > \displaystyle\int_{\mathfrak{Z}_\mathcal{P}} Z_P(\omega)\,\mathrm{d}\lambda \\ 0, & \text{if } -Z_Q(\omega) < \displaystyle\int_{\mathfrak{Z}_\mathcal{P}} Z_P(\omega)\,\mathrm{d}\lambda \end{cases} \qquad \text{for } R\text{-a.e. } \omega \in \Omega. \tag{4.19}$$

Given any finite, signed measure $\Pi = \Pi^+ - \Pi^-$ on $(\Omega, \mathcal{F})$ with $\Pi^\pm \ll R$, let us write $Z_\Pi = Z_{\Pi^+} - Z_{\Pi^-}$, define

$$\Upsilon_{\lambda,\Pi} := Z_\Pi - \int_{\mathfrak{Z}_\mathcal{P}} Z_P\,\mathrm{d}\lambda \in \mathbb{L}^1 \tag{4.20}$$

and let $\Upsilon^+_{\lambda,\Pi}$ (resp., $\Upsilon^-_{\lambda,\Pi}$) be the positive (resp., negative) part of the random variable in (4.20). With this notation, and recalling (4.19), the value of the dual problem (4.18) becomes

$$-\mathfrak{d}(-Q) = \sup_{\lambda\in\Lambda_+}\{-\mathbb{E}^R[\Upsilon^+_{\lambda,-Q}] - \alpha\lambda(\mathfrak{Z}_\mathcal{P})\}$$

and thus

$$\mathfrak{d}(Q) = \inf_{\lambda \in \Lambda_+} \{\mathbb{E}^R[\Upsilon^+_{\lambda,Q}] + \alpha\lambda(\mathfrak{Z}_{\mathcal{P}})\}. \tag{4.21}$$

We deduce from this representation and (4.20) that the dual $\mathfrak{d}(Q)$ of the primal problem $\mathfrak{p}(Q)$ of (4.5) is indeed as claimed in equation (4.6).

• Strong duality now holds if both $f(\cdot)$ and $g(\cdot)$ are convex, $g(\cdot)$ is continuous at some $A\varphi_0$ with $\varphi_0 \in \mathrm{dom}(f)$ and $\mathfrak{p}(Q)$ is finite (see [10], Theorem III.4.1 and Remark III.4.2).

Indeed, $\mathfrak{p}(Q)$ is clearly finite. In (4.13), the function $f(\cdot)$ is convex since Φ is a convex set; the function $g(\cdot)$ is convex since the set $\mathcal{L}_+ - \alpha\mathbf{1}$ is convex, as well as continuous at some $A\varphi_0$ with $\varphi_0 \in \mathrm{dom}(f)$, provided that $A\varphi_0 \in \mathrm{int}(\mathcal{L}_+ - \alpha\mathbf{1})$. If we take $\varphi_0 \equiv 0$, then $\varphi_0 \in \mathrm{dom}(f)$ since $\varphi_0 \in \Phi$, and we see that $A\varphi_0 = \mathbf{0} \in \mathrm{int}(\mathcal{L}_+ - \alpha\mathbf{1})$ since $\mathrm{int}(\mathcal{L}_+) \neq \varnothing$ in the norm topology and $\alpha > 0$. Hence, we have strong duality.

• The existence of a solution to the primal problem $\mathfrak{p}(Q)$ (i.e., of a generalized test $\widetilde{\varphi}_Q \in \Phi_\alpha$ which attains the supremum in (4.5)) follows from the weak-* compactness of Φ_α. With strong duality established, the existence of a solution to the dual problem, that is, of an element $\widetilde{\lambda}_Q \in \Lambda_+$ which attains the infimum in (4.6), follows ([10], Theorem III.4.1 and Remark III.4.2), whereas the values of the primal (resp., the dual) objective functions at $\widetilde{\varphi}_Q$ (resp., $\widetilde{\lambda}_Q$) coincide. To indicate the dependence of these quantities on the selected $Q \in \mathcal{Q}$, we have used the notation $\widetilde{\varphi}_Q$ and $\widetilde{\lambda}_Q$ for the primal and dual solutions, respectively.

• *These considerations lead to a necessary and sufficient condition for optimality*. Indeed, let us write the expression for $\mathbb{E}^Q[\varphi]$ from (3.3) as

$$\mathbb{E}^Q[\varphi] = \mathbb{E}^R[\varphi\Upsilon^+_{\lambda,Q}] - \mathbb{E}^R[\varphi\Upsilon^-_{\lambda,Q}] + \mathbb{E}^R\left[\varphi\int_{\mathfrak{Z}_{\mathcal{P}}} Z_P \,\mathrm{d}\lambda\right],$$

in the notation of (4.20), and subtract it from the dual objective function $\mathbb{E}^R[\Upsilon^+_{\lambda,Q}] + \alpha\lambda(\mathfrak{Z}_{\mathcal{P}})$, as in (4.21). Because of strong duality, this difference has to be zero when evaluated at $(\varphi, \lambda) = (\widetilde{\varphi}_Q, \widetilde{\lambda}_Q)$, namely,

$$\mathbb{E}^R[\Upsilon^+_{\widetilde{\lambda}_Q,Q}(1-\widetilde{\varphi}_Q)] + \mathbb{E}^R[\Upsilon^-_{\widetilde{\lambda}_Q,Q}\widetilde{\varphi}_Q] + \int_{\mathfrak{Z}_{\mathcal{P}}} (\alpha - \mathbb{E}^R[Z_P\widetilde{\varphi}_Q])\,\mathrm{d}\widetilde{\lambda}_Q = 0.$$

Each of these three integrals is non-negative, so their sum is zero if and only if $\widetilde{\varphi}_Q \in \Phi_\alpha$ satisfies the condition (4.8) of Lemma 4.4 and is of the form (4.7) or, equivalently, of the form $\widetilde{\varphi}_Q \equiv \overline{\varphi}_{\widetilde{\lambda}_Q,-Q}$ of (4.19). □

We are ready now to prove our main result.

Proof of Theorem 4.2. Lemma 4.4 guarantees that

$$\min_{Q\in\mathcal{Q}}\Bigl(\max_{\varphi\in\Phi_\alpha} \mathbb{E}^Q[\varphi]\Bigr) = \min_{Q\in\mathcal{Q}} \mathfrak{p}(Q) = \min_{Q\in\mathcal{Q}} \mathfrak{d}(Q) = \min_{(Q,\lambda)\in\mathcal{Q}\times\Lambda_+} \mathcal{D}(Q,\lambda) = V^*$$

in the notation of (3.1) and (4.5), (4.6). From Lemma 4.3, it follows that there exists an element $\widetilde{Q}$ of $\mathcal{Q}$ which attains the infimum in (3.6). For this $\widetilde{Q}$, Lemma 4.4 shows the existence of an element $\widetilde{\lambda}_{\widetilde{Q}}$ of Λ_+ that attains the infimum in (4.6). Thus, there exists a pair $(\widetilde{Q}, \widetilde{\lambda})$ that attains the infimum in (3.1) and Lemma 4.4 gives the required structural result. □

Corollary. *It follows that the optimal randomized test has the form*

$$\widetilde{\varphi}(\omega) = 1_{\{Z_{\widetilde{Q}} > \int_{\mathfrak{Z}_\mathcal{P}} Z_P \mathrm{d}\widetilde{\lambda}\}}(\omega) + \delta(\omega) \cdot 1_{\{Z_{\widetilde{Q}} = \int_{\mathfrak{Z}_\mathcal{P}} Z_P \mathrm{d}\widetilde{\lambda}\}}(\omega), \tag{4.22}$$

reminiscent of (1.1), *where the random variable* $\delta : \Omega \to [0, 1]$ *is chosen so that* (4.2) *is satisfied by this* $\widetilde{\varphi}$.

5. Extensions and ramifications

Remark 5.1. The weak compactness of the set of alternative densities $\mathfrak{Z}_\mathcal{Q}$ in Assumption 4.1(i) seems to be crucial. Without it, we can still get, by Fenchel duality,

$$\max_{\varphi \in \Phi_\alpha} \Big(\inf_{Q \in \mathcal{Q}} \mathbb{E}^Q[\varphi] \Big) = \inf_{Q \in \mathcal{Q}} \Big(\max_{\varphi \in \Phi_\alpha} \mathbb{E}^Q[\varphi] \Big),$$

endowing $\mathbb{L}^\infty$ with the norm topology. There is no guarantee anymore, however, that the infimum will be attained in $\mathcal{Q}$. The infimum will be attained at some element $\widehat{\mu}$ of the set $\mathfrak{M} \subseteq \{\mu \in \mathfrak{ba}(\Omega, \mathcal{F})_+ \mid \mu(\Omega) = 1\}$, which contains $\mathcal{Q}$; however, a Hahn decomposition might not exist for this $\widehat{\mu}$, so we do not generally obtain the 0–1 structure in (4.22) of the primal solution with respect to the dual solution.

It seems reasonable to endow $\mathbb{L}^\infty$ with the weak-* topology and to apply Fenchel duality. But it then becomes difficult to show that a suitable constraint qualification (e.g., that $\rho(\varphi) = \sup_{Q \in \mathcal{Q}} \mathbb{E}^Q[\varphi]$ be weakly-* continuous at some $\varphi_0 \in \Phi_\alpha$) is satisfied, which is needed to obtain strong duality.

Assuming $\mathfrak{Z}_\mathcal{Q}$ to be weakly compact and convex, as we have done throughout the present work, has enabled us to apply a min–max theorem and to ensure that the infimum in the dual problem is attained within $\mathfrak{Z}_\mathcal{Q} \subseteq \mathbb{L}^1$.

• If we were to drop the weak compactness Assumption 4.1(ii) on the set of densities $\mathfrak{Z}_\mathcal{P}$, then the norm-dual of $\mathcal{L}$ would be $\mathfrak{ba}(\mathfrak{Z}_\mathcal{P}, \mathcal{B})$ instead of Λ (recall the definitions of the spaces $\mathcal{L}$ and Λ from the start of the proof of Lemma 4.4). The elements of the space $\mathfrak{ba}(\mathfrak{Z}_\mathcal{P}, \mathcal{B})$ are the ultimate generalized Bayesian priors: they are allowed to assign negative weights to sets of possible scenarios and to be just finitely (as opposed to countably) additive. However, in such a setting, the Fubini–Tonelli theorem can no longer be applied. It is possible to endow $\mathcal{L}$ with the Mackey topology, to ensure that Λ is the topological dual of $\mathcal{L}$, but proving strong duality under this topology is a challenge. Throughout this paper, Assumption 4.1(ii) is imposed to ensure that the norm-dual of the space $\mathcal{L}$ is Λ and that a strong duality result can be obtained.

The weak compactness Assumption 4.1(ii) on $\mathfrak{Z}_\mathcal{P}$ can be dropped when using a different duality approach as in Cvitanić and Karatzas [8] (cf. Section 6), or a utility maximization duality approach similar to Föllmer and Leukert [12], Section 7. In both cases, however, one would lose some information about the structure of the optimal randomized test since $\widetilde{\varphi}$ will no longer be expressed with respect to the original family of probability measures $\mathcal{P}$, but with respect to some enlarged set (see Section 6).

Remark 5.2. The results in this paper can be extended in several directions. For instance, our proofs have not used the assumption that $\mathcal{P}$ and $\mathcal{Q}$ are families of probability measures. The results still hold if we instead consider two arbitrary subsets of $\mathbb{L}^1$, namely, $\mathcal{G}$ (in lieu of $\mathfrak{Z}_{\mathcal{P}}$) and $\mathcal{H}$ (in lieu of $\mathfrak{Z}_{\mathcal{Q}}$), that satisfy Assumption 4.1, as well as $\sup_{G\in\mathcal{G}} \|G\|_{\mathbb{L}^1} < \infty$.

Furthermore, instead of a constant $\alpha \in (0,1)$, we may consider a positive continuous functional $\alpha : \mathcal{G} \to \mathbb{R}_+$. The corresponding optimization problem is then

$$\sup_{\varphi\in\Phi}\Big(\inf_{H\in\mathcal{H}} \mathbb{E}[\varphi H]\Big), \tag{5.1}$$

subject to

$$\mathbb{E}^R[\varphi G] \le \alpha(G) \qquad \forall G \in \mathcal{G}. \tag{5.2}$$

The problem (5.1)–(5.2) is no longer of hypothesis-testing form, in the classical sense, but its structure is similar to that of testing composite hypotheses. Such so-called ‘generalized hypothesis-testing problems’ also arise in the context of hedging contingent claims in incomplete or constrained markets, for instance, when one tries to minimize the expected hedging loss (see Cvitanić [7], Rudloff [21] or, in a related context, Schied [22]). This kind of generalized hypothesis-testing problem was studied for the case of a simple alternative (i.e., $\mathcal{H}$ being a singleton) and a positive, bounded and measurable function $\alpha(\cdot)$, by Witting [25], Section 2.5.1. For this case, it was shown with Lagrange duality that the generalized 0–1 structure (4.22) of a test is sufficient for optimality. Furthermore, it was shown in [25] that for a finite set $\mathcal{G}$, the conditions (4.7), (4.8) are necessary and sufficient for optimality. The proof of Lemma 4.4 shows that a generalization of these results is even possible when both the ‘hypothesis’ set $\mathcal{G}$ and the ‘alternative hypothesis’ set $\mathcal{H}$ are infinite, if they satisfy the above conditions (Assumption 4.1 and $\sup_{G\in\mathcal{G}} \|G\|_{\mathbb{L}^1} < \infty$) and $\alpha(\cdot)$ is a positive, continuous function.

6. Comparisons and conclusion

The problem of Neyman–Pearson-type testing of a composite null hypothesis against a simple alternative has a long history–it has been considered in a variety of papers and in several books: Ferguson [11], Witting [25], Strasser [23], Vajda [24], Lehmann [17].

Results on testing a composite hypothesis against a composite alternative have been obtained in Lehmann [15], in Krafft and Witting [14] (which is apparently the first work to introduce ideas of convex duality to the theory of hypothesis testing) and in Baumann [4], Huber and Strassen [13], Österreicher [18], Vajda [24], pages 361–362 and Schied [22]. Lehmann [15] works with a finite set $\mathfrak{Z}_{\mathcal{Q}}$, provides existence results and shows that the composite testing problem can be reduced to one with simple hypotheses (consisting of the optimal mixed strategy). Krafft and Witting [14] and Witting [25] use Lagrange duality and show that even without any compactness assumptions on the sets $\mathfrak{Z}_{\mathcal{P}}$ and $\mathfrak{Z}_{\mathcal{Q}}$, the generalized 0–1 structure (4.22) of $\widetilde{\varphi}$ is sufficient for optimality, as well as necessary and sufficient if a dual solution exists (e.g., when $\mathfrak{Z}_{\mathcal{P}}$ and $\mathfrak{Z}_{\mathcal{Q}}$ are both finite). In this paper, we show that under Assumption 4.1, the generalized 0–1 structure of (4.22) is necessary and sufficient for optimality, due to strong duality with respect to the Fenchel dual problem; the existence of a dual solution then follows from strong duality.

Baumann [4] proves the existence of a max–min optimal test using duality results from linear programming and weak compactness arguments. The problem is also studied for densities that are contents and not necessarily measures.

Huber and Strassen [13] dispense with the assumption that all measures in $\mathfrak{Z}_{\mathcal{P}}$ and $\mathfrak{Z}_{\mathcal{Q}}$ be absolutely continuous with respect to a reference measure R, at the expense of assuming that these two sets can be described in terms of "alternating capacities", in the sense of Choquet; for related results, see Rieder [20] and Bendarski [5,6].

Finally, using totally different methods and motivated by optimal investment problems in mathematical finance, Schied [22] studies variational problems of Neyman–Pearson type for convex risk measures and for law-invariant robust utility functionals, obtaining explicit solutions for quantile-based coherent risk measures that satisfy the Huber–Strassen–Choquet alternating capacity conditions.

One of the most recent works on this subject is the paper by Cvitanić and Karatzas [8]. These authors introduce the enlargement

$$\mathcal{W} := \{W \in \mathbb{L}^1_+ \mid \mathbb{E}^R[\varphi W] \le \alpha, \forall \varphi \in \Phi_\alpha\} \supseteq \mathrm{co}(\mathfrak{Z}_{\mathcal{P}}) \tag{6.1}$$

of the convex hull of the Radon–Nikodym densities of $\mathcal{P}$. This 'enlarged' set $\mathcal{W}$ is convex, bounded in $\mathbb{L}^1$ and closed under R-a.e. convergence. Furthermore, they assume that the set of densities of $\mathcal{Q}$ is convex and closed under R-a.e. convergence. The starting point of [8] is the observation that

$$\forall Q \in \mathcal{Q}, \forall W \in \mathcal{W}, \forall z > 0, \forall \varphi \in \Phi_\alpha, \qquad \mathbb{E}^Q[\varphi] \le \mathbb{E}^R[(Z_Q - zW)^+] + \alpha z. \tag{6.2}$$

The existence of a quadruple $(\widehat{Q}, \widehat{W}, \widehat{z}, \widehat{\varphi}) \in \mathcal{Q} \times \mathcal{W} \times (0, \infty) \times \Phi_\alpha$ which satisfies (6.2) as equality is then shown and the structure

$$\widehat{\varphi}(\omega) = 1_{\{\widehat{z}\widehat{W} < Z_{\widehat{Q}}\}}(\omega) + \delta(\omega) \cdot 1_{\{\widehat{z}\widehat{W} = Z_{\widehat{Q}}\}}(\omega) \tag{6.3}$$

for the optimal randomized test $\widehat{\varphi}$ is deduced. Here, the triple $(\widehat{Q}, \widehat{W}, \widehat{z})$ is a solution of the optimization problem

$$\inf_{\substack{z>0 \\ (Q,W)\in\mathcal{Q}\times\mathcal{W}}} \left(\alpha z + \mathbb{E}^R[(Z_Q - zW)^+]\right) \tag{6.4}$$

and the random variable $\delta : \Omega \to [0, 1]$ is chosen so that $\sup_{P\in\mathcal{P}} \mathbb{E}^P[\widetilde{\varphi}] = \alpha$.

The methodology of the present paper obviates the need to introduce the enlargement set $\mathcal{W}$ of (6.1). Thus, we provide a result about the structure of the solution $\widetilde{\varphi}$ in terms of the original families of probability measures $\mathcal{P}$ and $\mathcal{Q}$. We do, however, need the set $\mathfrak{Z}_{\mathcal{Q}}$ to be weakly compact.

• Let us study the relationship between Theorem 4.2 and the results of Cvitanić and Karatzas [8]. From the Fubini–Tonelli theorem, it is easy to show that

$$k(\lambda) \int_{\mathfrak{Z}_{\mathcal{P}}} Z_P \, \mathrm{d}\lambda \in \mathcal{W} \qquad \forall \lambda \in \Lambda_+, \tag{6.5}$$

where $k(\lambda) = (\lambda(\mathfrak{Z}_{\mathcal{P}}))^{-1}$ if $\lambda(\mathfrak{Z}_{\mathcal{P}}) > 0$, and $k(\lambda) = 0$ if $\lambda(\mathfrak{Z}_{\mathcal{P}}) = 0$. The case $\lambda(\mathfrak{Z}_{\mathcal{P}}) = 0$ implies that $\lambda(B) = 0$ for all $B \in \mathcal{B}$ and thus $\int_{\mathfrak{Z}_{\mathcal{P}}} Z_P \,\mathrm{d}\lambda = 0$. If, in (6.2), we consider only elements W of the form $k \int_{\mathfrak{Z}_{\mathcal{P}}} Z_P \,\mathrm{d}\lambda \in \mathcal{W}$, then the inequality (6.2) coincides with the weak duality between the primal and dual objective functions $\mathfrak{p}(Q)$ and $\mathfrak{d}(Q)$, and reduces to the inequality in (3.3), whereas Problem (6.4) reduces to (3.1).

To summarize, Cvitanić and Karatzas [8] proved the existence of primal and dual solutions that satisfy (6.2) as equality. In order to do this, strong closure assumptions had to be imposed. In the methodology of the present paper, the validity of strong duality, hence also the equality in (3.3), are shown directly via Fenchel duality; the existence of a dual solution then follows. Both methods lead to a result about the structure of an optimal test. However, it is now possible, as in Theorem 4.2, to show the impact of the original family $\mathcal{P}$ on the sets that define the solution $\widehat{\varphi}$ in [8], as in (6.3):

$$\widehat{z}\widehat{W} = \int_{\mathfrak{Z}_{\mathcal{P}}} Z_P \,\mathrm{d}\widetilde{\lambda}, \tag{6.6}$$

where $(\widetilde{Q}, \widetilde{\lambda})$ is the pair that attains the infimum in (3.1), $\widehat{z} = \widetilde{\lambda}(\mathfrak{Z}_{\mathcal{P}})$ and

$$\widehat{W} = k(\widetilde{\lambda}) \int_{\mathfrak{Z}_{\mathcal{P}}} Z_P \,\mathrm{d}\widetilde{\lambda}, \qquad \text{in the notation of (6.5).}$$

No embedding of $\mathfrak{Z}_{\mathcal{P}}$ into the larger set $\mathcal{W}$ of (6.1) is any longer necessary. However, instead of assuming that $\mathfrak{Z}_{\mathcal{Q}} = \{Z_Q \mid Q \in \mathcal{Q}\}$ is closed under R-a.e. convergence, we need to assume here that this set is weakly compact in $\mathbb{L}^1$.

Acknowledgements

This research was supported in part by the National Science Foundation, under Grants DMS-06-01774 and DMS-09-05754.

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Received June 2009 and revised November 2009