Bernoulli **16**(4), 2010, 953–970
DOI: 10.3150/09-BEJ241

Ball throwing on spheres

ANNE ESTRADE[1] and JACQUES ISTAS[2]

[1]*MAP5 UMR CNRS 8145, Université Paris Descartes, 45 rue des Saints-Pères, F-75270 Paris cedex 06 & ANR-05-BLAN-0017. E-mail: anne.estrade@parisdescartes.fr*
[2]*Laboratoire Jean Kuntzmann, Université de Grenoble et CNRS, F-38041 Grenoble cedex 9. E-mail: Jacques.Istas@imag.fr*

Ball throwing on Euclidean spaces has been considered for some time. A suitable renormalization leads to a fractional Brownian motion as limit object. In this paper, we investigate ball throwing on spheres. A different behavior is exhibited: we still get a Gaussian limit, but it is no longer a fractional Brownian motion. However, the limit is locally self-similar when the self-similarity index H is less than $1/2$.

Keywords: fractional Brownian motion; overlapping balls; scaling; self-similarity; spheres

1. Introduction

Random balls models have been studied for a long time and are known as *germ-grain models*, *shot noise* or *micropulses*. The common feature of those models consists in throwing balls that eventually overlap at random in an n-dimensional space. Many random phenomena can be modelled through this procedure and there are many fields of application: Internet traffic in one dimension, communication networks or imaging in two dimensions and biology or material sciences in three dimensions. A pioneering work is due to Wicksell [24], dealing with the study of corpuscles. The literature on germ-grain models involves two main currents: the research either focuses on the geometrical or morphological aspect of the union of random balls (see [20] or [21] and references therein), or it is concerned with the number of balls covering each point. This second approach is currently known as *shot noise* or *spot noise* (see [6], for instance). In three dimensions, the shot noise process is a natural candidate for modelling porous or granular media, and, more generally, heterogeneous media with irregularities at any scale. The idea is to build a microscopic model which yields a macroscopic field with self-similar properties. The same idea is expected in one dimension for Internet traffic, for instance [25]. A common method for finding self-similarity is to deal with scaling limits. Roughly speaking, the balls are dilated with a scaling parameter λ and one lets λ go either to 0 or to infinity. We quote, for instance, [4] and [12] for the case $\lambda \to 0^+$, [11] and [2] for the case $\lambda \to +\infty$, and [5] and [3] where both cases are considered.

In the present paper, we follow a procedure which is similar to [2] and [3]. Let us describe it precisely. A collection of random balls in $\mathbb{R}^n$ whose centers and radii are chosen according to a random Poisson measure on $\mathbb{R}^n \times \mathbb{R}^+$ is considered. The Poisson intensity is given as follows:

$$\nu(\mathrm{d}x, \mathrm{d}r) = r^{-n-1+2H}\,\mathrm{d}x\,\mathrm{d}r,$$

for some real parameter H. Since the Lebesgue measure $\mathrm{d}x$ is invariant with respect to isometry, so is the random balls model, and so will be any (eventual) limit. As the distribution of the radii follows a homogeneous distribution, a self-similar scaling limit may be expected. Indeed, with additional technical conditions, the scaling limits of such random balls models are isometry-invariant self-similar Gaussian fields. The self-similarity index depends on the parameter H. When $0 < H < 1/2$, the Gaussian field is nothing but the well-known fractional Brownian motion [16,18,19].

Manifold-indexed fields that share properties with Euclidean self-similar fields have been, and still are, extensively studied (for example, [8,9,13–15,17,22,23]). In this paper, we consider what happens when balls are thrown onto a sphere, rather than a Euclidean space. More precisely, is there a scaling limit of random balls models and, if it exists, is this scaling limit a fractional Brownian field indexed by the surface for $0 < H < 1/2$?

The random field is still obtained by throwing overlapping balls in a Poissonian way. The Poisson intensity is chosen as follows:

$$\nu(\mathrm{d}x, \mathrm{d}r) = f(r)\sigma(\mathrm{d}x)\,\mathrm{d}r.$$

The Lebesgue measure $\mathrm{d}x$ has been replaced by the surface measure $\sigma(\mathrm{d}x)$. The function f, which controls the distribution of the radii, is still equivalent to $r^{-n-1+2H}$, at least for small r, where n stands for the surface dimension. It turns out that the results are completely different in the two cases (Euclidean, spherical). In the spherical case, there is a Gaussian scaling limit for any H, but it is no longer a fractional Brownian field, as defined by [13]. We then investigate the local behavior, in the tangent bundle, of this scaling limit, in the spirit of local self-similarity [1,7,15]. It is locally asymptotically self-similar with a Euclidean fractional Brownian field as tangent field. Our microscopic model has led to a local self-similar macroscopic model.

The paper is organized as follows. In Section 2, the spherical model is introduced and we prove the existence of a scaling limit. In Section 3, we study the locally self-similar property of the asymptotic field. Section 4 is devoted to a comparative analysis between the Euclidean and spherical cases. Eventually, some technical computations are presented in the Appendix.

2. Scaling limit

We work on $\mathbb{S}_n$, the n-dimensional unit sphere, $n \geq 1$:

$$\mathbb{S}_n = \left\{(x_i)_{1\leq i\leq n+1} \in \mathbb{R}^{n+1};\ \sum_{1\leq i\leq n+1} x_i^2 = 1\right\}.$$

2.1. Spherical caps

For $x, y \in \mathbb{S}_n$, let $d(x, y)$ denote the distance between x and y on $\mathbb{S}_n$, that is, the non-oriented angle between Ox and Oy, where O denotes the origin of $\mathbb{R}^{n+1}$. For $r \geq 0$, $B(x, r)$ denotes the closed ball on S centered at x with radius r:

$$B(x, r) = \{y \in \mathbb{S}_n; d(x, y) \leq r\}.$$

Let us note that for $r < \pi$, $B(x,r)$ is a spherical cap on the unit sphere $\mathbb{S}_n$, centered at x with opening angle r and that for $r \geq \pi$, $B(x,r) = \mathbb{S}_n$.

Denoting by $\sigma(\mathrm{d}x)$ the surface measure on $\mathbb{S}_n$, we define $\phi(r)$ as the surface of any ball on $\mathbb{S}_n$ with radius r:

$$\phi(r) := \sigma(B(x,r)), \qquad x \in \mathbb{S}_n, r \geq 0.$$

We also introduce the following function defined for z and z', two points in $\mathbb{S}_n$ and $r \in \mathbb{R}^+$:

$$\Psi(z, z', r) := \int_{\mathbb{S}_n} \mathbf{1}_{d(x,z)<r} \mathbf{1}_{d(x,z')<r} \sigma(\mathrm{d}x). \tag{1}$$

Actually, $\Psi(z, z', r)$ denotes the surface measure of the set of all points in $\mathbb{S}_n$ that belong to both balls $B(z,r)$ and $B(z',r)$. Clearly, $\Psi(z,z',r)$ depends only on the distance $d(z,z')$ between z and z'. We write

$$\psi(d(z,z'), r) = \Psi(z, z', r) \tag{2}$$

and note that it satisfies the following: $\forall (u,r) \in [0,\pi] \times \mathbb{R}^+$,

- $0 \leq \psi(u,r) \leq \sigma(\mathbb{S}_n) \wedge \phi(r)$;
- if $r < u/2$, then $\psi(u,r) = 0$ and if $r > \pi$, then $\psi(u,r) = \sigma(\mathbb{S}_n)$;
- $\psi(0,r) = \phi(r) \sim cr^n$ as $r \to 0^+$.

In what follows, we consider a family of balls $B(X_j, R_j)$ generated at random, following a strategy described in the next section.

2.2. Poisson point process

We consider a Poisson point process $(X_j, R_j)_j$ in $\mathbb{S}_n \times \mathbb{R}^+$ or, equivalently, $N(\mathrm{d}x, \mathrm{d}r)$, a Poisson random measure on $\mathbb{S}_n \times \mathbb{R}^+$ with intensity

$$\nu(\mathrm{d}x, \mathrm{d}r) = f(r)\sigma(\mathrm{d}x)\,\mathrm{d}r,$$

where f satisfies the following assumptions $\mathbf{A}(H)$ for some $H > 0$:

- $\mathrm{supp}(f) \subset [0,\pi)$;
- f is bounded on any compact subset of $(0,\pi)$;
- $f(r) \sim r^{-n-1+2H}$ as $r \to 0^+$.

Remarks.

(1) The first condition ensures that no balls of radius R_j on the sphere self-intersect.

(2) Since $\phi(r) \sim cr^n$, $r \to 0^+$, the last condition implies that $\int_{\mathbb{R}^+} \phi(r) f(r)\,\mathrm{d}r < +\infty$, which means that the mean surface, with respect to f, of the balls $B(X_j, R_j)$ is finite.

2.3. Random field

Let $\mathcal{M}$ denote the space of signed measures μ on $\mathbb{S}_n$ with finite total variation $|\mu|(\mathbb{S}_n)$, with $|\mu|$ the total variation measure of μ. For any $\mu \in \mathcal{M}$, we define

$$X(\mu) = \int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x, r)) N(\mathrm{d}x, \mathrm{d}r). \tag{3}$$

Note that the stochastic integral in (3) is well defined since

$$\begin{aligned}\int_{\mathbb{S}_n \times \mathbb{R}^+} |\mu(B(x, r))| f(r)\sigma(\mathrm{d}x)\,\mathrm{d}r &\le \int_{\mathbb{S}_n}\int_{\mathbb{S}_n}\int_{\mathbb{R}^+} \mathbf{1}_{d(x,y)<r} f(r)\sigma(\mathrm{d}x)|\mu|(\mathrm{d}y)\,\mathrm{d}r \\ &= |\mu|(\mathbb{S}_n)\left(\int_{\mathbb{R}^+} \phi(r) f(r)\,\mathrm{d}r\right) < +\infty.\end{aligned}$$

In the particular case where μ is a Dirac measure δ_z for some point $z \in \mathbb{S}_n$, we simply write

$$X(z) = X(\delta_z) = \int_{\mathbb{S}_n \times \mathbb{R}^+} \mathbf{1}_{B(x,r)}(z) N(\mathrm{d}x, \mathrm{d}r). \tag{4}$$

The pointwise field $\{X(z); z \in \mathbb{S}_n\}$ corresponds to the number of random balls (X_j, R_j) covering each point of $\mathbb{S}_n$. Each random variable $X(z)$ has a Poisson distribution with mean $\int_{\mathbb{R}^+} \phi(r) f(r)\,\mathrm{d}r$.

Furthermore, for any $\mu \in \mathcal{M}$,

$$\mathbb{E}(X(\mu)) = \mu(\mathbb{S}_n)\left(\int_{\mathbb{R}^+} \phi(r) f(r)\,\mathrm{d}r\right)$$

and

$$\mathrm{Var}(X(\mu)) = \int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x, r))^2 f(r)\sigma(\mathrm{d}x)\,\mathrm{d}r \in (0, +\infty].$$

2.4. Key lemma

For $H > 0$, we would like to compute the integral

$$\int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x, r))^2 r^{-n-1+2H}\sigma(\mathrm{d}x)\,\mathrm{d}r,$$

which is a candidate for the variance of an eventual scaling limit. We first introduce $\mathcal{M}^H$, the set of measures for which the above integral does converge:

$$\mathcal{M}^H = \mathcal{M} \qquad \text{if } 2H < n; \qquad \mathcal{M}^H = \{\mu \in \mathcal{M}; \mu(\mathbb{S}_n) = 0\} \qquad \text{if } 2H > n.$$

The following lemma deals with the function ψ defined by (2).

Lemma 2.1. *Let $H > 0$ with $2H \neq n$. We introduce*

$$\psi^{(H)} = \psi \qquad \text{if } 0 < 2H < n, \qquad \psi^{(H)} = \psi - \sigma(\mathbb{S}_n) \qquad \text{if } 2H > n.$$

Then, for all $u \in [0, \pi]$,

$$\int_{\mathbb{R}^+} \left|\psi^{(H)}(u, r)\right| r^{-n-1+2H}\, \mathrm{d}r < +\infty.$$

Furthermore, letting

$$K_H(u) = \int_{\mathbb{R}^+} \psi^{(H)}(u, r) r^{-n-1+2H}\, \mathrm{d}r \tag{5}$$

for any u in $[0, \pi]$, we have, for all $\mu \in \mathcal{M}^H$,

$$0 \le \int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x, r))^2 r^{-n-1+2H} \sigma(\mathrm{d}x)\, \mathrm{d}r = \int_{\mathbb{S}_n \times \mathbb{S}_n} K_H(d(z, z'))\mu(\mathrm{d}z)\mu(\mathrm{d}z') < +\infty.$$

Remark 2.2.

(1) For x, y in $\mathbb{S}_n$, the difference of Dirac measures $\delta_x - \delta_y$ belongs to $\mathcal{M}^H$ for any H.

(2) In the case $2H > n$, since any $\mu \in \mathcal{M}^H$ is centered, the integral on the right-hand side is not changed when a constant is added to the kernel K_H.

(3) This lemma proves that the kernel K_H defines a covariance function on $\mathcal{M}^H$.

Proof of Lemma 2.1. Using the properties of ψ, we get, in the case $0 < 2H < n$, that

$$\begin{aligned} 0 &\le \int_{\mathbb{R}^+} \psi(u, r) r^{-n-1+2H}\, \mathrm{d}r \\ &\le \int_{(0,\pi)} \phi(r) r^{-n-1+2H}\, \mathrm{d}r + \sigma(\mathbb{S}_n) \int_{(\pi,\infty)} r^{-n-1+2H}\, \mathrm{d}r \\ &< +\infty. \end{aligned}$$

In the same vein, in the case $2H > n$, we get

$$\begin{aligned} 0 &\le \int_{\mathbb{R}^+} \bigl(\sigma(\mathbb{S}_n) - \psi(u, r)\bigr) r^{-n-1+2H}\, \mathrm{d}r \\ &\le \sigma(\mathbb{S}_n) \int_{(0,\pi)} r^{-n-1+2H}\, \mathrm{d}r \\ &< +\infty. \end{aligned}$$

We have just established that there exists a finite constant C_H such that

$$\forall u \in [0, \pi], \qquad \int_{\mathbb{R}^+} \left|\psi^{(H)}(u, r)\right| r^{-n-1+2H}\, \mathrm{d}r \le C_H. \tag{6}$$

The first statement is proved.

Let us define, for $\mu \in \mathcal{M}^H$,

$$I_H(\mu) = \int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x,r))^2 r^{-n-1+2H} \sigma(\mathrm{d}x)\,\mathrm{d}r$$

and start by proving that $I_H(\mu)$ is finite. We will essentially use Fubini's theorem in the following lines:

$$\begin{aligned} I_H(\mu) &= \int_{\mathbb{R}^+} \left(\int_{\mathbb{S}_n} \mu(B(x,r))^2 \sigma(\mathrm{d}x) \right) r^{-n-1+2H}\,\mathrm{d}r \\ &= \int_{\mathbb{R}^+} \left(\int_{\mathbb{S}_n \times \mathbb{S}_n} \Psi(z,z',r)\mu(\mathrm{d}z)\mu(\mathrm{d}z') \right) r^{-n-1+2H}\,\mathrm{d}r. \end{aligned}$$

Since $\mu \in \mathcal{M}^H$ is centered in the case $2H > n$, one can change ψ into $\psi^{(H)}$ inside the previous integral. Then

$$\begin{aligned} I_H(\mu) &\le \int_{\mathbb{R}^+} \left(\int_{\mathbb{S}_n \times \mathbb{S}_n} \left|\psi^{(H)}(d(z,z'),r)\right| |\mu|(\mathrm{d}z)|\mu|(\mathrm{d}z') \right) r^{-n-1+2H}\,\mathrm{d}r \\ &\le \int_{\mathbb{S}_n \times \mathbb{S}_n} \left(\int_{\mathbb{R}^+} \left|\psi^{(H)}(d(z,z'),r)\right| r^{-n-1+2H}\,\mathrm{d}r \right) |\mu|(\mathrm{d}z)|\mu|(\mathrm{d}z') \\ &\le C_H |\mu|(\mathbb{S}_n)^2 < +\infty. \end{aligned}$$

Following the same lines (except for the last one) without the '$|\cdot|$' allows the computation of $I_H(\mu)$ and completes the proof. $\square$

An explicit value for the kernel K_H is available, starting from its definition. The point is to compute $\psi^{(H)}$. In the Appendix, we provide a recurrence formula for $\psi^{(H)}$, based on the dimension n of the unit sphere $\mathbb{S}_n$ (see Lemma 4.1).

2.5. Scaling

Let $\rho > 0$ and λ be any positive function on $(0, +\infty)$. We consider the scaled Poisson measure N_ρ obtained from the original Poisson measure N by taking the image under the map $(x,r) \in S \times \mathbb{R}^+ \mapsto (x, \rho r)$ and multiplying the intensity by $\lambda(\rho)$. Hence, N_ρ is still a Poisson random measure with intensity

$$\nu_\rho(\mathrm{d}x, \mathrm{d}r) = \lambda(\rho)\rho^{-1} f(\rho^{-1}r)\sigma(\mathrm{d}x)\,\mathrm{d}r.$$

We also introduce the scaled random field X_ρ defined on $\mathcal{M}$ by

$$X_\rho(\mu) = \int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x,r)) N_\rho(\mathrm{d}x, \mathrm{d}r). \tag{7}$$

Theorem 2.3. *Let $H > 0$ with $2H \neq n$ and let f satisfy* $\mathbf{A}(H)$. *For all positive functions λ such that $\lambda(\rho)\rho^{n-2H} \underset{\rho\to+\infty}{\longrightarrow} +\infty$, the limit*

$$\left\{\frac{X_\rho(\mu) - \mathbb{E}(X_\rho(\mu))}{\sqrt{\lambda(\rho)\rho^{n-2H}}}; \mu \in \mathcal{M}^H\right\} \underset{\rho\to+\infty}{\overset{fdd}{\longrightarrow}} \{W_H(\mu); \mu \in \mathcal{M}^H\}$$

holds in the sense of finite-dimensional distributions of the random functionals. Here, W_H is the centered Gaussian random linear functional on $\mathcal{M}^H$ with

$$\operatorname{Cov}(W_H(\mu), W_H(\nu)) = \int_{\mathbb{S}_n\times\mathbb{S}_n} K_H(d(z,z'))\mu(\mathrm{d}z)\nu(\mathrm{d}z'), \tag{8}$$

where K_H is the kernel introduced in Lemma 2.1.

The theorem can be rephrased in term of the pointwise field $\{X(z); z \in \mathbb{S}_n\}$ defined in (4).

Corollary 2.4. *Let $H > 0$ with $2H \neq n$ and let f satisfy* $\mathbf{A}(H)$. *For all positive functions λ such that $\lambda(\rho)\rho^{n-2H} \underset{\rho\to+\infty}{\longrightarrow} +\infty$,*

- *if $0 < 2H < n$, then*

$$\left\{\frac{X_\rho(z) - \mathbb{E}(X_\rho(z))}{\sqrt{\lambda(\rho)\rho^{n-2H}}}; z \in \mathbb{S}_n\right\} \underset{\rho\to+\infty}{\overset{fdd}{\longrightarrow}} \{W_H(z); z \in \mathbb{S}_n\},$$

  where W_H is the centered Gaussian random field on $\mathbb{S}_n$ with

$$\operatorname{Cov}(W_H(z), W_H(z')) = K_H(d(z,z'));$$

- *if $2H > n$, then for any fixed point $z_0 \in \mathbb{S}_n$,*

$$\left\{\frac{X_\rho(z) - X_\rho(z_0)}{\sqrt{\lambda(\rho)\rho^{n-2H}}}; z \in \mathbb{S}_n\right\} \underset{\rho\to+\infty}{\overset{fdd}{\longrightarrow}} \{W_{H,z_0}(z); z \in \mathbb{S}_n\},$$

where W_{H,z_0} is the centered Gaussian random field on $\mathbb{S}_n$ with

$$\operatorname{Cov}(W_{H,z_0}(z), W_{H,z_0}(z')) = K_H(d(z,z')) - K_H(d(z,z_0)) - K_H(d(z',z_0)) + K_H(0).$$

Proof of Theorem 2.3. Let us define $n(\rho) := \sqrt{\lambda(\rho)\rho^{n-2H}}$. The characteristic function of the normalized field $(X_\rho(\cdot) - \mathbb{E}(X_\rho(\cdot)))/n(\rho)$ is then given by

$$\mathbb{E}\left(\exp\left(\mathrm{i}\frac{X_\rho(\mu) - \mathbb{E}(X_\rho(\mu))}{n(\rho)}\right)\right) = \exp\left(\int_{\mathbb{S}_n\times\mathbb{R}^+} G_\rho(x,r)\,\mathrm{d}r\,\sigma(\mathrm{d}x)\right), \tag{9}$$

where

$$G_\rho(x,r) = \left(\mathrm{e}^{\mathrm{i}\mu(B(x,r))/n(\rho)} - 1 - \mathrm{i}\frac{\mu(B(x,r))}{n(\rho)}\right)\lambda(\rho)\rho^{-1} f(\rho^{-1}r). \tag{10}$$

We will make use of Lebesgue's theorem in order to get the limit of $\int_{\mathbb{S}_n\times\mathbb{R}^+} G_\rho(x,r)\,\mathrm{d}r\,\sigma(\mathrm{d}x)$ as $\rho\to+\infty$.

On the one hand, $n(\rho)$ tends to $+\infty$ so that $(\mathrm{e}^{\mathrm{i}\mu(B(x,r))/n(\rho)}-1-\mathrm{i}\frac{\mu(B(x,r))}{n(\rho)})$ behaves like $-\frac{1}{2}(\frac{\mu(B(x,r))}{n(\rho)})^2$. Together with the assumption $\mathbf{A}(H)$, it yields the following asymptotic result: for all $(x,r)\in\mathbb{S}_n\times\mathbb{R}^+$,

$$G_\rho(x,r)\underset{\rho\to+\infty}{\longrightarrow}-\tfrac{1}{2}\mu(B(x,r))^2r^{-n-1+2H}. \tag{11}$$

On the other hand, since $\frac{|\mu|(B(x,r))}{n(\rho)}\le|\mu|(\mathbb{S}_n)$ for ρ large enough, we note that there exists some positive constant K such that for all x,r,ρ,

$$\left|\mathrm{e}^{\mathrm{i}\mu(B(x,r))/n(\rho)}-1-\mathrm{i}\frac{\mu(B(x,r))}{n(\rho)}\right|\le K\left(\frac{\mu(B(x,r))}{n(\rho)}\right)^2.$$

Therefore,

$$|G_\rho(x,r)|\le K\mu(B(x,r))^2\rho^{-n-1+2H}f(\rho^{-1}r).$$

There exists $C>0$ such that for all $r\in\mathbb{R}^+$, $f(r)\le Cr^{-n-1+2H}$. We then get

$$|G_\rho(x,r)|\le KC\mu(B(x,r))^2r^{-n-1+2H}, \tag{12}$$

where the right-hand side is integrable on $\mathbb{S}_n\times\mathbb{R}^+$, by Lemma 2.1.

Applying Lebesgue's theorem yields

$$\begin{aligned}\int_{\mathbb{S}_n\times\mathbb{R}^+} G_\rho(x,r)\sigma(\mathrm{d}x)\,\mathrm{d}r &\underset{\rho\to+\infty}{\longrightarrow} -\frac{1}{2}\int_{\mathbb{S}_n\times\mathbb{R}^+}\mu(B(x,r))^2r^{-n-1+2H}\sigma(\mathrm{d}x)\,\mathrm{d}r\\ &= -\frac{1}{2}\int_{\mathbb{S}_n\times\mathbb{S}_n}K_H(d(z,z'))\mu(\mathrm{d}z)\mu(\mathrm{d}z').\end{aligned}$$

Hence, $(X_\rho(\mu)-\mathbb{E}(X_\rho(\mu)))/n(\rho)$ converges in distribution to the centered Gaussian random variable $W(\mu)$ whose variance is equal to

$$\mathbb{E}(W(\mu)^2)=C\int_{\mathbb{S}_n\times\mathbb{S}_n}K_H(d(z,z'))\mu(\mathrm{d}z)\mu(\mathrm{d}z').$$

By linearity, the covariance of W satisfies (8). $\square$

Remark 2.5.

(1) The pointwise limit field $\{W_H(z);z\in\mathbb{S}_n\}$ in Corollary 2.4 is stationary, that is, its distribution is invariant under the isometry group of $\mathbb{S}_n$, whereas the increments of $\{W_{H,z_0}(z);z\in\mathbb{S}_n\}$ are distribution-invariant under the group of all isometries of $\mathbb{S}_n$ which keep the point z_0 invariant.

(2) When $0 < H < 1/2$, the Gaussian field W_H does not coincide with the field introduced in [13] as the spherical fractional Brownian motion on $\mathbb{S}_n$.

Indeed, let us consider the case $n = 1$. It is easy to obtain the following piecewise expression for $\psi = \psi_1$: $\forall (u, r) \in [0, \pi] \times \mathbb{R}^+$,

$$\begin{aligned}\psi_1(u, r) &= 0 \qquad \text{for } 0 \le r < u/2 \\ &= 2r - u \qquad \text{for } u/2 \le r \le \pi - u/2 \\ &= 4r - 2\pi \qquad \text{for } \pi - u/2 \le r \le \pi \\ &= 2\pi \qquad \text{for } r > \pi.\end{aligned}$$

It is also easy to compute

$$K_H(u) = \frac{1}{H(1-2H)2^{2H}}\bigl(2(2H)^{2H} - u^{2H} - (2\pi - u)^{2H}\bigr).$$

Actually, we compute the variance of the increments of W_H:

$$\begin{aligned}\mathbb{E}\bigl(W_H(z) - W_H(z')\bigr)^2 &= 2K_H(0) - 2K_H(d(z, z')) \\ &= \frac{2}{H(1-2H)2^{2H}}\bigl[d^{2H}(z, z') + \bigl(2\pi - d(z, z')\bigr)^{2H} - (2\pi)^{2H}\bigr].\end{aligned}$$

The spherical fractional Brownian motion B_H, introduced in [13], satisfies

$$\mathbb{E}\bigl(B_H(z) - B_H(z')\bigr)^2 = d^{2H}(z, z').$$

Even up to a constant, the processes W_H and B_H are clearly different. The Euclidean situation is therefore different. Indeed, [3], the variance of the increments of the corresponding field W_H is proportional to $|z - z'|^{2H}$.

3. Local self-similar behavior

We consider whether the limit field W_H obtained in the previous section satisfies a local asymptotic self-similar (LASS) property. More precisely, we will let a 'dilation' of order ε act on W_H near a fixed point A in $\mathbb{S}_n$ and, as in [15], up to a renormalization factor, we look for an asymptotic behavior as ε goes to 0. An H-self-similar tangent field T_H is expected. Recall that W_H is defined on a subspace $\mathcal{M}^H$ of measures on $\mathbb{S}_n$ so that T_H will be defined on a subspace of measures on the tangent space $\mathcal{T}_A\mathbb{S}_n$ of $\mathbb{S}_n$.

3.1. Dilation

Let us fix a point A in $\mathbb{S}_n$ and consider $\mathcal{T}_A\mathbb{S}_n$, the tangent space of $\mathbb{S}_n$ at A. It can be identified with $\mathbb{R}^n$ and A with the null vector of $\mathbb{R}^n$.

Let $1 < \delta < \pi$. The exponential map at point A, denoted by exp, is a diffeomorphism between the Euclidean ball $\{y \in \mathbb{R}^n, \|y\| < \delta\}$ and $\overset{\circ}{B}(A, \delta) \subset \mathbb{S}_n$, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$ and $\overset{\circ}{B}(A, \delta)$ the open ball with center A and radius δ in $\mathbb{S}_n$.

Furthermore, for all $y, y' \in \mathbb{R}^n$ such that $\|y\|, \|y'\| < \delta$,

$$d(A, \exp y) = \|y\| \quad \text{and} \quad d(\exp y, \exp y') \leq \|y - y'\|.$$

We refer to [10] for details on the exponential map.

Let τ be a signed measure on $\mathbb{R}^n$. We define the dilated measure τ_ε by

$$\forall B \in \mathcal{B}(\mathbb{R}^n) \qquad \tau_\varepsilon(B) = \tau(B/\varepsilon)$$

and then map it by the application of exp, defining the measure $\mu_\varepsilon = \exp^* \tau_\varepsilon$ on $\overset{\circ}{B}(A, \delta)$ by

$$\forall C \in \mathcal{B}(\overset{\circ}{B}(A, \delta)) \qquad \mu_\varepsilon(C) = \exp^* \tau_\varepsilon(C) = \tau_\varepsilon(\exp^{-1}(C)). \tag{13}$$

We then consider the measure μ_ε as a measure on the whole sphere $\mathbb{S}_n$ with support included in $\overset{\circ}{B}(A, \delta)$.

Finally, we define the dilation of W_H within a 'neighborhood of A' by the following procedure. For any finite measure τ on $\mathbb{R}^n$, we consider $\mu_\varepsilon = \exp^* \tau_\varepsilon$, as defined by (13), and compute $W_H(\mu_\varepsilon)$. We will establish the convergence in distribution of $\varepsilon^{-H} W_H(\exp^* \tau_\varepsilon)$ for any τ in an appropriate space of measures on $\mathbb{R}^n$. Since $W_H(\mu_\varepsilon)$ is Gaussian, we will focus on its variance.

3.2. Asymptotics of the kernel K_H

For $0 < H < 1/2$, we have already mentioned that the kernel $K_H(0) - K_H(u)$ is not proportional to u^{2H}. As a consequence, one cannot expect W_H to be self-similar. Nevertheless, as we are looking for an asymptotic local self-similarity, only the behavior of K_H near zero is relevant. Actually, we will establish that, up to a constant, $K_H(0) - K_H(u)$ behaves like u^{2H} when $u \to 0^+$.

Lemma 3.1. *Let $0 < H < 1/2$. The kernel K_H defined by (5) satisfies*

$$K_H(u) = K_1 - K_2 u^{2H} + o(u^{2H}), \qquad u \to 0^+,$$

where $K_1 = K_H(0)$ and K_2 are non-negative constants.

Proof. Let us state that the assumption $H < 1/2$ implies that $H < n/2$ so that, in that case, K_H is given by

$$K_H(u) = \int_{\mathbb{R}^+} \psi(u, r) r^{-n-1+2H} \, \mathrm{d}r, \qquad u \in [0, \pi].$$

We note that $K_H(0) < +\infty$ since $\psi(0,r) \sim cr^n$ as $r \to 0^+$ and $\psi(0,r) = \sigma(\mathbb{S}_n)$ for $r > \pi$. Then, subtracting $K_H(0)$ and observing that $\psi(0,r) = \psi(u,r) = \sigma(\mathbb{S}_n)$ for $r > \pi$, we write

$$\begin{aligned}K_H(0) - K_H(u) &= \int_0^\pi \bigl(\psi(0,r) - \psi(u,r)\bigr) r^{-n-1+2H}\,\mathrm{d}r \\ &= \int_0^\delta \bigl(\psi(0,r) - \psi(u,r)\bigr) r^{-n-1+2H}\,\mathrm{d}r \\ &\quad + \int_\delta^\pi \bigl(\psi(0,r) - \psi(u,r)\bigr) r^{-n-1+2H}\,\mathrm{d}r,\end{aligned}$$

where we recall that $\delta \in (1, \pi)$ is such that the exponential map is a diffeormorphism between $\{\|y\| < \delta\} \subset \mathbb{R}^n$ and $\overset{\circ}{B}(A,\delta) \subset \mathbb{S}_n$.

The second term is of order u and is therefore negligible with respect to u^{2H} since ψ is clearly Lipschitz on the compact interval $[\delta, \pi]$.

We now focus on the first term. Performing the change of variable $r \mapsto r/u$, we write it as

$$\begin{aligned}&\int_0^\delta \bigl(\psi(0,r) - \psi(u,r)\bigr) r^{-n-1+2H}\,\mathrm{d}r \\ &\quad = u^{2H} \int_{\mathbb{R}^+} \Delta(u,r) r^{-n-1+2H}\,\mathrm{d}r,\end{aligned}$$

where

$$\Delta(u,r) := \mathbf{1}_{ur<\delta} u^{-n} \bigl(\psi(u,ur) - \psi(0,ur)\bigr).$$

It only now remains to prove that $\int_{\mathbb{R}^+} \Delta(u,r) r^{-n-1+2H}\,\mathrm{d}r$ admits a finite limit K_2 as $u \to 0^+$. We will use Lebesgue's theorem and start by establishing the simple convergence of $\Delta(u,r)$ for any given $r \in \mathbb{R}^+$.

We fix a unit vector $\mathbf{v}$ in $\mathbb{R}^n$ and a point $A' = \exp \mathbf{v}$ in $\mathbb{S}_n$. We then consider, for any $u \in (0,\delta)$, the point $A'_u := \exp(u\mathbf{v}) \in \mathbb{S}_n$ located on the geodesic between A and A' such that $d(A, A'_u) = \|u\mathbf{v}\| = u$. We can then use (1) and (2) to write

$$\psi(u,\cdot) = \Psi(A, A'_u, \cdot) = \int_{\mathbb{S}_n} \mathbf{1}_{d(A,z)<\cdot} \mathbf{1}_{d(A'_u,z)<\cdot}\,\mathrm{d}\sigma(z)$$

and

$$\psi(0,\cdot) = \Psi(A, A, \cdot) = \int_{\mathbb{S}_n} \mathbf{1}_{d(A,z)<\cdot}\,\mathrm{d}\sigma(z)$$

in order to express $\Delta(u,r)$ as

$$\Delta(u,r) = \mathbf{1}_{ur<\delta} u^{-n} \int_{\mathbb{S}_n} \mathbf{1}_{d(A,z)<ur} \mathbf{1}_{d(A'_u,z)>ur}\,\mathrm{d}\sigma(z).$$

Since $ur < \delta$, the above integral runs on $\overset{\circ}{B}(A, ur) \subset \overset{\circ}{B}(A, \delta)$ and we can perform the exponential change of variable to get

$$\Delta(u,r) = \mathbf{1}_{ur<\delta} u^{-n} \int_{\mathbb{R}^n} \mathbf{1}_{\|y\|<ur} \mathbf{1}_{d(\exp(u\mathbf{v}),\exp(y))>ur} \, \mathrm{d}\sigma(\exp(y))$$
$$= \mathbf{1}_{ur<\delta} \int_{\mathbb{R}^n} \mathbf{1}_{\|y\|<r} \mathbf{1}_{d(\exp(u\mathbf{v}),\exp(uy))>ur} \tilde{\sigma}(uy) \, \mathrm{d}y.$$

In the last integral, the image under exp of the surface measure $\mathrm{d}\sigma(\exp(y))$ is written as $\tilde{\sigma}(y)\,\mathrm{d}y$, where $\mathrm{d}y$ denotes the Lebesgue measure on $\mathbb{R}^n$.

We use the fact that $d(\exp(ux), \exp(ux')) \sim u\|x - x'\|$ as $u \to 0^+$ to obtain the following limit for the integrand:

$$\mathbf{1}_{d(\exp(u\mathbf{v}),\exp(uy))<ur} \tilde{\sigma}(uy) \longrightarrow \mathbf{1}_{\|\mathbf{v}-y\|>r} \tilde{\sigma}(0).$$

Since the integrand is clearly dominated by

$$\|\sigma\|_\infty := \sup\{\tilde{\sigma}(y), \|y\| \le \delta\},$$

Lebesgue's theorem yields, for all $r \in \mathbb{R}^+$,

$$\Delta(u,r) \longrightarrow \tilde{\sigma}(0) \int_{\mathbb{R}^n} \mathbf{1}_{\|y\|<r} \mathbf{1}_{\|\mathbf{v}-y\|>r} \, \mathrm{d}y.$$

We recall that $d(\exp x, \exp x') \le \|x - x'\|$ for all $x, x' \in \mathbb{R}^n$ with norm less than δ. Therefore, for all u,

$$\Delta(u,r) \le \|\sigma\|_\infty \int_{\mathbb{R}^n} \mathbf{1}_{\|y\|<r} \mathbf{1}_{\|\mathbf{v}-y\|>r} \, \mathrm{d}y,$$

where the right-hand side belongs to $L^1(\mathbb{R}^+, r^{-n-1+2H}\,\mathrm{d}r)$ (see [2], Lemma A.2).

Using Lebesgue's theorem for the last time, we obtain

$$\int_{\mathbb{R}^+} \Delta(u,r) r^{-n-1+2H} \, \mathrm{d}r \underset{u\to 0^+}{\longrightarrow} K_2,$$

where

$$K_2 = \tilde{\sigma}(0) \int_{\mathbb{R}^n \times \mathbb{R}^+} \mathbf{1}_{\|y\|<r} \mathbf{1}_{\|\mathbf{v}-y\|>r} r^{-n-1+2H} \, \mathrm{d}y \, \mathrm{d}r \in (0, +\infty). \qquad \square$$

Let us remark that the proof makes it clear that the case $H > 1/2$ is dramatically different. The kernel $K_H(0) - K - H(u)$ behaves like u near zero and loses its $2H$ power.

3.3. Main result

Let $0 < H < 1/2$. We consider the following space of measures on $\mathcal{T}_A\mathbb{S}_n \cong \mathbb{R}^n$:

$$\mathfrak{M}^H = \Big\{ \text{measures } \tau \text{ on } \mathbb{R}^n \text{ with finite total variation such that}$$
$$\tau(\mathbb{R}^n) = 0 \text{ and } \int_{\mathbb{R}^n\times\mathbb{R}^n} \|x - x'\|^{2H} |\tau|(\mathrm{d}x)|\tau|(\mathrm{d}x') < +\infty \Big\}.$$

For any measure $\tau \in \mathfrak{M}^H$, we compute the variance of $W_H(\mu_\varepsilon)$, where $\mu_\varepsilon = \exp^* \tau_\varepsilon$ is defined by (13).

By Lemma 2.1, since μ_ε belongs to $\mathcal{M} = \mathcal{M}^H$ in the case $H < 1/2$,

$$\operatorname{var}(W_H(\mu_\varepsilon)) = \int_{B(A,\delta)\times B(A,\delta)} K_H(d(z,z'))\mu_\varepsilon(\mathrm{d}z)\mu_\varepsilon(\mathrm{d}z').$$

Performing an exponential change of variable followed by a dilation in $\mathbb{R}^n$, we get

$$\begin{aligned}\operatorname{var}(W_H(\mu_\varepsilon)) &= \int_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\|y\|<\delta}\mathbf{1}_{\|y'\|<\delta} K_H(d(\exp(y),\exp(y')))\tau_\varepsilon(\mathrm{d}y)\tau_\varepsilon(\mathrm{d}y') \\ &= \int_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\|x\|<\delta/\varepsilon}\mathbf{1}_{\|x'\|<\delta/\varepsilon} K_H(d(\exp(\varepsilon x),\exp(\varepsilon x')))\tau(\mathrm{d}x)\tau(\mathrm{d}x').\end{aligned}$$

Defining $\widetilde{K_H}(u) = K_H(u) - K_H(0)$, we have

$$\begin{aligned}\operatorname{var}(W_H(\mu_\varepsilon)) = &\int_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\|x\|<\delta/\varepsilon}\mathbf{1}_{\|x'\|<\delta/\varepsilon} \widetilde{K_H}(d(\exp(\varepsilon x),\exp(\varepsilon x')))\tau(\mathrm{d}x)\tau(\mathrm{d}x') \\ &+ K_H(0)\tau(\{\|x\| < \delta/\varepsilon\})^2.\end{aligned}$$

Let us temporarily accept that

$$\frac{\tau(\{\|x\| < \delta/\varepsilon\})^2}{\varepsilon^{2H}} \underset{\varepsilon\to 0^+}{\longrightarrow} 0. \tag{14}$$

Then, applying Lebesgue's theorem with the convergence argument on $\widetilde{K_H}$ obtained in Lemma 3.1 yields

$$\frac{\operatorname{var}(W_H(\mu_\varepsilon))}{\varepsilon^{2H}} \underset{\varepsilon\to 0^+}{\longrightarrow} -K_2 \int_{\mathbb{R}^n\times\mathbb{R}^n} \|x - x'\|^{2H} \tau(\mathrm{d}x)\tau(\mathrm{d}x'). \tag{15}$$

Let us now establish (14), where we recall that τ is any measure in $\mathfrak{M}^H$. In particular, the total mass of τ is zero so that

$$\frac{\tau(\{\|x\| < \delta/\varepsilon\})}{\varepsilon^H} = -\frac{\tau(\{\|x\| > \delta/\varepsilon\})}{\varepsilon^H} = -\int_{\mathbb{R}^n} \varepsilon^{-H}\mathbf{1}_{\|x\|>\delta/\varepsilon}\tau(\mathrm{d}x).$$

For any fixed $x \in \mathbb{R}^n$, $\varepsilon^{-H}\mathbf{1}_{\|x\|>\delta/\varepsilon}$ is zero when ε is small enough. Moreover, $\varepsilon^{-H}\mathbf{1}_{\|x\|>\delta/\varepsilon}$ is dominated by $\delta^{-H}\|x\|^H$, which belongs to $L^1(\mathbb{R}^n, |\tau|(\mathrm{d}x))$ since τ belongs to $\mathfrak{M}^H$. Lebesgue's theorem applies once more.

From the asymptotic result (15), we deduce the following theorem.

Theorem 3.2. *Let* $0 < H < 1/2$. *The limit*

$$\frac{W_H(\exp^* \tau_\varepsilon)}{\varepsilon^H} \xrightarrow[\varepsilon\to 0^+]{fdd} T_H(\tau)$$

holds for all $\tau \in \mathfrak{M}^H$, *in the sense of finite-dimensional distributions of the random functionals. Here*, T_H *is the centered Gaussian random linear functional on* $\mathfrak{M}^H$ *with*

$$\operatorname{Cov}(T_H(\tau), T_H(\tau')) = -K_2 \int_{\mathbb{R}^n\times\mathbb{R}^n} \|x - x'\|^{2H} \tau(\mathrm{d}x)\tau'(\mathrm{d}x'). \tag{16}$$

As for Theorem 2.3, Theorem 3.2 can be rephrased in terms of pointwise fields. Indeed, $\delta_x - \delta_O$ belongs to $\mathfrak{M}^H$ for all x in $\mathbb{R}^n$. Let us apply Theorem 3.2 with $\tau = \delta_x - \delta_O$. Then $T_H(\delta_x - \delta_O)$ has the covariance

$$\operatorname{Cov}\bigl(T_H(\delta_x - \delta_O), T_H(\delta_{x'} - \delta_O)\bigr) = K_2(\|x\|^{2H} + \|x'\|^{2H} - \|x - x'\|^{2H})$$

and the field $\{T_H(\delta_x - \delta_O); x \in \mathbb{R}^n\}$ is a Euclidean fractional Brownian field.

4. Comparative analysis

In this section, we will discuss the differences and similarities between the Euclidean and spherical cases.

Let us first consider the existence of a scaling limit random field. The variance of this limit field should be

$$\mathbf{V} = \int_{\mathbb{M}_n}\int_{\mathbb{R}^+} \mu(B(x,r))^2 \sigma(\mathrm{d}x) r^{-n-1+2H}\,\mathrm{d}r,$$

where $\mathbb{M}_n$ is the n-dimensional corresponding surface with its surface measure σ. When speaking of the Euclidean case $\mathbb{M}_n = \mathbb{R}^n$, we refer to [3]. In the present paper, we have studied the case $\mathbb{M}_n = \mathbb{S}_n$. Moreover, in this discussion, the hyperbolic case $\mathbb{M}_n = \mathbb{H}_n = \{(x_i)_{1\le i\le n+1} \in \mathbb{R}^{n+1}; x_{n+1}^2 - \sum_{1\le i\le n} x_i^2 = 1, x_{n+1} \ge 1\}$ is invoked.

In the Euclidean case, the random fields are defined on the space of measures with vanishing total mass. So, let us first consider measures μ such that $\mu(\mathbb{M}_n) = 0$. In this case, whatever the surface $\mathbb{M}_n$, the integral $\mathbf{V}$ involves the integral of the surface of the symmetric difference between two balls with the same radius r. As r goes to infinity, three different behaviors emerge:

- $\mathbb{M}_n = \mathbb{S}_n$: this surface vanishes;

- $\mathbb{M}_n = \mathbb{R}^n$: the order of magnitude of this surface is r^{n-1};
- $\mathbb{M}_n = \mathbb{H}_n$: the surface grows exponentially.

The consequences are the following:

- $\mathbb{M}_n = \mathbb{S}_n$: any positive H is admissible;
- $\mathbb{M}_n = \mathbb{R}^n$: the range of admissible H is $(0, 1/2)$;
- $\mathbb{M}_n = \mathbb{H}_n$: no H is admissible.

In the Euclidean case, the restriction $\mu(\mathbb{R}^n) = 0$ is mandatory, whereas it is unnecessary in the spherical case for $H < n/2$. Indeed, the integral $\mathbf{V}$ is clearly convergent.

Let us now discuss the (local) self-similarity of the limit field. Of course, we no longer consider the hyperbolic case.

- $\mathbb{M}_n = \mathbb{R}^n$: dilating a ball is a homogeneous operation. Therefore, the limit field is self-similar.
- $\mathbb{M}_n = \mathbb{S}_n$: dilation is no longer homogeneous. Only local self-similarity can be expected. The natural framework of this local self-similarity is the tangent bundle, where the situation is Euclidean. Therefore, we must return to the restricting condition $H < 1/2$.

Appendix

Recurrence formula for the ψ_n's

Recall that the functions ψ_n are defined by (1) and (2):

$$\begin{aligned}\psi_n(u,r) &= \Psi_n(M, M', r)\\ &= \int_{\mathbb{S}_n} \mathbf{1}_{d(M,N)<r} \mathbf{1}_{d(M',N)<r}\, \mathrm{d}\sigma_n(N), \qquad (u,r) \in [0,\pi] \times \mathbb{R}^+,\end{aligned}$$

for any pair (M, M') in $\mathbb{S}_n$ such that $d(M, M') = u$. Here, σ_n stands for the surface measure on $\mathbb{S}_n$.

Lemma 4.1. *The family of functions $\psi_n, n \geq 2$, satisfies the following recursion*: $\forall (u,r) \in [0,\pi] \times \mathbb{R}^+$,

$$\psi_n(u,r) = \int_{-\sin r}^{\sin r} (1-a^2)^{n/2} \psi_{n-1}\left(u, \arccos\left(\frac{\cos r}{\sqrt{1-a^2}}\right)\right) \mathrm{d}a.$$

Proof. An arbitrary point of $\mathbb{S}_n$ is parameterized either in Cartesian coordinates, $(x_i)_{1\leq i\leq n+1}$, or in spherical ones,

$$(\phi_i)_{1\leq i\leq n} \in [0,\pi)^{n-1} \times [0, 2\pi),$$

with

$$\begin{aligned} x_1 &= \cos\phi_1, \\ x_2 &= \sin\phi_1 \cos\phi_2, \\ x_3 &= \sin\phi_1 \sin\phi_2 \cos\phi_3, \\ &\vdots \\ x_n &= \sin\phi_1 \sin\phi_2 \cdots \sin\phi_{n-1} \cos\phi_n, \\ x_{n+1} &= \sin\phi_1 \sin\phi_2 \cdots \sin\phi_{n-1} \sin\phi_n. \end{aligned}$$

Let M be the point $(\phi_i)_{1\le i\le n} = (0, \ldots, 0)$. One can write the ball $B_n(M, r)$ of radius r, which is a spherical cap on $\mathbb{S}_n$ with opening angle r, as follows:

$$B_n(M, r) = \{(\phi_i)_{1\le i\le n} \in \mathbb{S}_n; \phi_1 \le r\};$$

or, in Cartesian coordinates,

$$B_n(M, r) = \{(x_i)_{1\le i\le n+1} \in \mathbb{S}_n; x_1 \ge \cos r\}.$$

Let $a \in (-1, 1)$ and let P_a be the hyperplane of $\mathbb{R}^{n+1}$ defined by $x_{n+1} = a$. Let us consider the intersection $P_a \cap B_n(M, r)$.

- If $1 - a^2 < \cos^2 r$, then $P_a \cap B_n(M, r) = \emptyset$.
- If $1 - a^2 \ge \cos^2 r$, then

$$\begin{aligned} P_a \cap B_n(M, r) &= \{(x_i)_{1\le i\le n+1} \in \mathbb{S}_n; x_1 \ge \cos r \text{ and } x_{n+1} = a\} \\ &= \left\{(x_i)_{1\le i\le n} \in \mathbb{R}^n; x_1 \ge \cos r \text{ and } \sum_{1\le i\le n} x_i^2 = 1 - a^2\right\} \times \{a\}. \end{aligned}$$

In other words, denoting by $\mathbb{S}_{n-1}(R)$ the $(n-1)$-dimensional sphere of radius R,

$$P_a \cap B_n(M, r) = B_{n-1,\sqrt{1-a^2}}(M(a), r(a)) \times \{a\},$$

where $B_{n-1,\sqrt{1-a^2}}(M(a), r(a))$ is the spherical cap on $\mathbb{S}_{n-1}(\sqrt{1-a^2})$, centered at $M(a) = (\sqrt{1-a^2}, 0, \ldots, 0)$ and with opening angle $r(a) = \arccos(\frac{\cos r}{\sqrt{1-a^2}})$.

Now, let M' be defined in spherical coordinates by $(\phi_i)_{1\le i\le n} = (u, 0, \ldots, 0)$ so that $d(M, M') = u$. The intersection $P_a \cap B_n(M', r)$ is the map of $P_a \cap B_n(M, r)$ by the rotation of angle u and center C in the plane $x_3 = \cdots = x_{n+1} = 0$. So,

- if $1 - a^2 < \cos^2 r$, then $P_a \cap B_n(M', r) = \emptyset$;

- if $1 - x_0^2 \geq \cos^2 r$, then

$$P_a \cap B_n(M', r) = B_{n-1,\sqrt{1-a^2}}(M'(a), r(a)) \times \{a\},$$

where the $(n-1)$-dimensional spherical cap $B_{n-1,\sqrt{1-a^2}}(M'(a), r(a))$ is now centered at $M'(a) = (\sqrt{1-a^2}\cos u, \sqrt{1-a^2}\sin u, 0, \dots, 0)$.

We define $\psi_{n-1,R}(u,r)$ as the intersection surface of two spherical caps on $\mathbb{S}_{n-1}(R)$, whose centers are at a distance Ru and with the same opening angle r.

By homogeneity, this leads to

$$\psi_{n-1,R}(u,r) = R^{n-1}\psi_{n-1,1}(u,r) = R^n \psi_{n-1}(u,r).$$

The surface measure σ_n of $\mathbb{S}_n$ can be written as

$$\mathrm{d}\sigma_n(x_1, \dots, x_n, a) = \sqrt{1-a^2}\,\mathrm{d}\sigma_{n-1,\sqrt{1-a^2}}(x_1, \dots, x_n) \times \mathrm{d}a,$$

where $\sigma_{n-1,R}$ is the surface measure of $\mathbb{S}_{n-1}(R)$.

We then obtain

$$\begin{aligned}\psi_n(u,r) &= \int_{-1}^{1} \mathbf{1}_{1-a^2 \geq \cos^2 r}\, \psi_{n-1,\sqrt{1-a^2}}\left(u, \arccos\left(\frac{\cos r}{\sqrt{1-a^2}}\right)\right)\sqrt{1-a^2}\,\mathrm{d}a \\ &= \int_{-\sin r}^{\sin r} \left(\sqrt{1-a^2}\right)^n \psi_{n-1}\left(u, \arccos\left(\frac{\cos r}{\sqrt{1-a^2}}\right)\right)\mathrm{d}a,\end{aligned}$$

and Lemma 4.1 is proved. □

Acknowledgement

The authors are very grateful to the referee for careful reading of the paper and useful comments.

References

[1] Benassi, A., Jaffard, S. and Roux, D. (1997). Gaussian processes and Pseudodifferential Elliptic operators. *Rev. Mat. Iberoamericana* **13** 19–90. MR1462329

[2] Biermé, H. and Estrade, A. (2006). Poisson random balls: Self-similarity and X-ray images. *Adv. in Appl. Probab.* **38** 853–872. MR2285684

[3] Biermé, H., Estrade, A. and Kaj, I. (2010). Self-similar random fields and rescaled random balls model. *J. Theoret. Probab.* To appear. Available at http://hal.archives-ouvertes.fr/hal-00161614_v2/.

[4] Cioczek-Georges, R. and Mandelbrot, B.B. (1995). A class of micropulses and antipersistent fractional Brownian motion. *Stochastic Process. Appl.* **60** 1–18. MR1362316

[5] Cohen, S. and Taqqu, M. (2004). Small and large scale behavior of the Poissonized Telecom Process. *Methodol. Comput. Appl. Probab.* **6** 363–379. MR2108557

[6] Dalay, D.J. (1971). The definition of multi-dimensional generalization of shot-noise. *J. Appl. Probab.* **8** 128–135. MR0278408
[7] Dobrushin, R.L. (1980). Automodel generalized random fields and their renorm group. In *Multi-component Random Systems* (R.L. Dobrushin and Y.G. Sinai, eds.) 153–198. New York: Dekker. MR0599535
[8] Faraut, J. (1973). Fonction brownienne sur une variété riemannienne. *Séminaire de probabilités de Strasbourg* **7** 61–76. MR0391284
[9] Gangolli, R. (1967). Positive definite kernels on homogeneous spaces and certain stochastic processes related to Lévy's Brownian motion of several parameters. *Ann. Inst. H. Poincaré* **3** 121–226. MR0215331
[10] Helgason, S. (1962). *Differential Geometry and Symmetric Spaces*. New York: Academic Press. MR0145455
[11] Heinrich, L. and Schmidt, V. (1985). Normal convergence of multidimensional shot noise and rate of this convergence. *Adv. in Appl. Probab.* **17** 709–730. MR0809427
[12] Kaj, I., Leskelä, L., Norros, I. and Schmidt, V. (2007). Scaling limits for random fields with long-range dependence. *Ann. Probab.* **35** 528–550. MR2308587
[13] Istas, J. (2005). Spherical and hyperbolic fractional Brownian motion. *Electron. Commun. Probab.* **10** 254–262. MR2198600
[14] Istas, J. (2006). On fractional stable fields indexed by metric spaces. *Electron. Commun. Probab.* **11** 242–251. MR2266715
[15] Istas, J. and Lacaux, C. (2009). On locally self-similar fractional random fields indexed by a manifold. Preprint.
[16] Kolmogorov, A. (1940). Wienersche Spiralen und einige andere interessante Kurven im Hilbertsche Raum (German). *C. R. (Dokl.) Acad. Sci. URSS* **26** 115–118. MR0003441
[17] Lévy, P. (1965). *Processus stochastiques et mouvement brownien*. Paris: Gauthier-Villars. MR0190953
[18] Mandelbrot, B.B. and Van Ness, J.W. (1968). Fractional Brownian motions, fractional noises and applications. *SIAM Review* **10** 422–437. MR0242239
[19] Samorodnitsky, G. and Taqqu, M. (1994). *Stable Non-Gaussian Random Processes: Stochastic Models With Infinite Variance*. New York: Chapman & Hall. MR1280932
[20] Serra, J. (1984). *Image Analysis and Mathematical Morphology*. London: Academic Press. MR0753649
[21] Stoyan, D., Kendall, W.S. and Mecke, J. (1995). *Stochastic Geometry and Its Applications*, 2nd ed. Chichester: Wiley. MR0895588
[22] Takenaka, S. (1991). Integral-geometric construction of self-similar stable processes. *Nagoya Math. J.* **123** 1–12. MR1126180
[23] Takenaka, S., Kubo, I. and Urakawa, H. (1981). Brownian motion parametrized with metric space of constant curvature. *Nagoya Math. J.* **82** 131–140. MR0618812
[24] Wicksell, S.D. (1925). The corpuscle problem: A mathematical study of a biometric problem. *Biometrika* **17** 84–99.
[25] Willingsler, W., Taqqu, M.S., Sherman, R. and Wilson, D.V. (1997). Self-similarity through high variability: Statistical analysis of Ethernet LAN traffic at source level. *IEE/ACM Transactions in Networking* **5** 71–86.

Received October 2009 and revised June 2009