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Dirichlet mean identities and laws of a class of subordinators

LANCELOT F. JAMES

Department of Information Systems, Business Statistics and Operations Management, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong. E-mail: lancelot@ust.hk

An interesting line of research is the investigation of the laws of random variables known as Dirichlet means. However, there is not much information on interrelationships between different Dirichlet means. Here, we introduce two distributional operations, one of which consists of multiplying a mean functional by an independent beta random variable, the other being an operation involving an exponential change of measure. These operations identify relationships between different means and their densities. This allows one to use the often considerable analytic work on obtaining results for one Dirichlet mean to obtain results for an entire family of otherwise seemingly unrelated Dirichlet means. Additionally, it allows one to obtain explicit densities for the related class of random variables that have generalized gamma convolution distributions and the finite-dimensional distribution of their associated Lévy processes. The importance of this latter statement is that Lévy processes now commonly appear in a variety of applications in probability and statistics, but there are relatively few cases where the relevant densities have been described explicitly. We demonstrate how the technique allows one to obtain the finite-dimensional distribution of several interesting subordinators which have recently appeared in the literature.

Keywords: beta–gamma algebra; Dirichlet means and processes; exponential tilting; generalized gamma convolutions; Lévy processes

1. Introduction

In this work, we present two distributional operations which identify relationships between seemingly different classes of random variables which are representable as linear functionals of a Dirichlet process, otherwise known as *Dirichlet means*. Specifically, the first operation consists of multiplication of a Dirichlet mean by an independent beta random variable and the second operation involves an exponential change of measure to the density of a related infinitely divisible random variable having a generalized gamma convolution distribution (GGC). This latter operation is often referred to in the statistical literature as *exponential tilting* or in mathematical finance as an *Esscher transform*. We believe our results add a significant component to the foundational work of Cifarelli and Regazzini [5,6]. In particular, our results allow one to use the often considerable analytic work on obtaining results for one Dirichlet mean to obtain results for an entire family of otherwise seemingly unrelated mean functionals. It also allows one to obtain explicit densities for the related class of infinitely divisible random variables which are generalized gamma convolutions and an explicit description of the finite-dimensional distribution of their associated Lévy processes (see Bertoin [1] for the formalities of general Lévy processes). The importance of this latter statement is that Lévy processes now commonly appear in a variety

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of applications in probability and statistics, but there are relatively few cases where the relevant densities have been described explicitly. A detailed summary and outline of our results may be found in Section 1.2. Some background information on, and notation for, Dirichlet processes and Dirichlet means, their connection with GGC random variables, recent references and some motivation for our work are given in the next section.

1.1. Background and motivation

Let X be a non-negative random variable with cumulative distribution function F_X. Note, furthermore, that for a measurable set C, we use the notation $F_X(C)$ to mean the probability that X is in C. One may define a Dirichlet process random probability measure (see Freedman [17] and Ferguson [15,16]), say P_θ, on $[0,\infty)$ with total mass parameter θ and prior parameter F_X, via its finite-dimensional distribution as follows: for any disjoint partition on $[0,\infty)$, say $(C_1,\ldots,C_k)$, the distribution of the random vector $(P_\theta(C_1),\ldots,P_\theta(C_k))$ is a k-variate Dirichlet distribution with parameters $(\theta F_X(C_1),\ldots,\theta F_X(C_k))$. Hence, for each C,

$$P_\theta(C)=\int_0^\infty \mathbb{I}(x\in C)P_\theta(\mathrm{d}x)$$

has a beta distribution with parameters $(\theta F_X(C),\theta(1-F_X(C)))$. Equivalently, setting $\theta F_X(C_i)=\theta_i$ for $i=1,\ldots,k$,

$$(P_\theta(C_1),\ldots,P_\theta(C_k))\stackrel{\mathrm{d}}{=}\left(\frac{G_{\theta_i}}{G_\theta};i=1,\ldots,k\right),$$

where (G_{θ_i}) are independent random variables with gamma$(\theta_i,1)$ distributions and $G_\theta=G_{\theta_1}+\cdots+G_{\theta_k}$ has a gamma$(\theta,1)$ distribution. This means that one can define the Dirichlet process via the normalization of an independent increment gamma process on $[0,\infty)$, say $\gamma_\theta(\cdot)$, as

$$P_\theta(\cdot)=\frac{\gamma_\theta(\cdot)}{\gamma_\theta([0,\infty))},$$

where $\gamma_\theta(C_i)\stackrel{\mathrm{d}}{=}G_{\theta_i}$, whose almost surely finite total random mass is $\gamma_\theta([0,\infty))\stackrel{\mathrm{d}}{=}G_\theta$. A very important aspect of this construction is the fact that G_θ is independent of P_θ and hence of any functional of P_θ. This is a natural generalization of Lukacs' [35] characterization of beta and gamma random variables, which is fundamental to what is now referred to as the *beta–gamma algebra* (for more on this, see Chaumont and Yor ([4], Section 4.2); see also Emery and Yor [11] for some interesting relationships between gamma processes, Dirichlet processes and Brownian bridges). Hereafter, for a random probability measure P on $[0,\infty)$, we write

$$P\sim\Pi_{\theta,F_X},$$

to indicate that P is a Dirichlet process with parameters (θ,F_X).

These simple representations and other nice features of the Dirichlet process have, since the important work of Ferguson [15,16], contributed greatly to the relevance and practical utility of the field of Bayesian non- and semi-parametric statistics. Naturally, owing to the ubiquity of the gamma and beta random variables, the Dirichlet process also arises in other areas. One of the more interesting and, we believe, quite important topics related to the Dirichlet process is the study of the laws of random variables called *Dirichlet mean functionals*, or simply Dirichlet means, which we denote as

$$M_\theta(F_X) \stackrel{\mathrm{d}}{=} \int_0^\infty x P_\theta(\mathrm{d}x),$$

as initiated in the works of Cifarelli and Regazzini [5,6]. In [6], the authors obtained an important identity for the Cauchy–Stieltjes transform of order θ. This identity is often referred to as the *Markov–Krein identity*, as can be seen in, for example, Diaconis and Kemperman [9], Kerov [28] and Vershik, Yor and Tsilevich [40], where these authors highlight its importance to, for instance, the study of the Markov moment problem, continued fraction theory and exponential representation of analytic functions. This identity is later called the *Cifarelli–Regazzini identity* in [21]. Cifarelli and Regazzini [6], owing to their primary interest, used this identity to then obtain explicit density and cdf formulae for $M_\theta(F_X)$. The density formulae may be seen as Abel-type transforms and hence do not always have simple forms, although we stress that they are still useful for some analytic calculations. The general exception is the case $\theta = 1$, which has a nice form. Some examples of works that have proceeded along these lines are Cifarelli and Melilli [7], Regazzini, Guglielmi and di Nunno [38], Regazzini, Lijoi and Prünster [39], Hjort and Ongaro [20], Lijoi and Regazzini [32], and Epifani, Guglielmi and Melilli [12,13]. Moreover, the recent works of Bertoin *et al.* [2] and James, Lijoi and Prünster [25] (see also [23], which is a preliminary version of this work) show that the study of mean functionals is relevant to the analysis of phenomena related to Bessel and Brownian processes. In fact, the work of James, Lijoi and Prünster [25] identifies many new explicit examples of Dirichlet means which have interesting interpretations.

Related to these last points, Lijoi and Regazzini [32] have highlighted a close connection to the theory of generalized gamma convolutions (see [3]). Specifically, it is known that a rich subclass of random variables having generalized gamma convolutions (GGC) distributions may be represented as

$$T_\theta \stackrel{\mathrm{d}}{=} G_\theta M_\theta(F_X) \stackrel{\mathrm{d}}{=} \int_0^\infty x \gamma_\theta(\mathrm{d}x). \tag{1.1}$$

We call these random variables GGC(θ, F_X). In addition, we see from (1.1) that T_θ is a random variable derived from a weighted gamma process and, hence, the calculus discussed in Lo [33] and Lo and Weng [34] applies. In general, GGC random variables are an important class of infinitely divisible random variables whose properties have been extensively studied by [3] and others. We note further that although we have written a GGC(θ, F_X) random variable as $G_\theta M_\theta(F_X)$, this representation is not unique and, in fact, it is quite rare to see T_θ represented in this way. We will show that one can, in fact, exploit this non-uniqueness to obtain explicit densities for T_θ, even when it is not so easy to do so for $M_\theta(F_X)$. While the representation $G_\theta M_\theta(F_X)$ is not unique, it helps one to understand the relationship between the Laplace transform of T_θ and the Cauchy–Stieltjes transform of order θ of $M_\theta(F_X)$, which, indeed, characterize respectively the laws of T_θ and $M_\theta(F_X)$. Specifically, using the independence property of G_θ and $M_\theta(F_X)$

leads to, for $\lambda \geq 0$,

$$\mathbb{E}[\mathrm{e}^{-\lambda T_\theta}] = \mathbb{E}\big[\big(1+\lambda M_\theta(F_X)\big)^{-\theta}\big] = \mathrm{e}^{-\theta\psi_{F_X}(\lambda)}, \tag{1.2}$$

where

$$\psi_{F_X}(\lambda) = \int_0^\infty \log(1+\lambda x)F_X(\mathrm{d}x) = \mathbb{E}[\log(1+\lambda X)] \tag{1.3}$$

is the *Lévy exponent* of T_θ. We note that T_θ and $M_\theta(F_X)$ exist if and only if $\psi_{F_X}(\lambda) < \infty$ for $\lambda > 0$ (see, e.g., [8] and [3]). The expressions in (1.2) equate with the aforementioned identity obtained by Cifarelli and Regazzini [6].

Despite these interesting results, there is very little work on the relationship between different mean functionals. Suppose, for instance, that for each fixed value of $\theta > 0$, $M_\theta(F_X)$ denotes a Dirichlet mean and $(M_\theta(F_{Z_c}); c > 0)$ denotes a collection of Dirichlet mean random variables indexed by a family of distributions $(F_{Z_c}; c > 0)$. One may then ask the following question: for what choices of X and Z_c are these mean functionals related, and in what sense? In particular, one may wish to know how their densities are related. The rationale here is that if such a relationship is established, then the effort that one puts forth to obtain results such as the explicit density of $M_\theta(F_X)$ can be applied to an entire family of Dirichlet means $(M_\theta(F_{Z_c}); c > 0)$. Furthermore, since Dirichlet means are associated with GGC random variables, this would establish relationships between a GGC(θ, F_X) random variable and a family of GGC(θ, F_{Z_c}) random variables. Simple examples are, of course, the choices $Z_c = X + c$ and $Z_c = cX$, which, due to the linearity properties of mean functionals, result easily in the identities in law

$$M_\theta(F_{X+c}) = c + M_\theta(F_X) \quad \text{and} \quad M_\theta(F_{cX}) = cM_\theta(F_X).$$

Naturally, we are going to discuss more complex relationships, but with the same goal. That is, we will identify non-trivial relationships so that the often considerable efforts that one makes in the study of one mean functional $M_\theta(F_X)$ can then be used to obtain more easily results for other mean functionals, their corresponding GGC random variables and Lévy processes. In this paper, we will describe two such operations which we elaborate on in the next subsection.

1.2. Outline and summary of results

Section 1.3 reviews some of the existing formulae for the densities and cdfs of Dirichlet means. In Section 2, we will describe the operation of multiplying a mean functional $M_{\theta\sigma}(F_X)$ by an independent beta random variable with parameters $(\theta\sigma, \theta(1-\sigma))$, say, $\beta_{\theta\sigma,\theta(1-\sigma)}$, where $0 < \sigma < 1$. We call this operation *beta scaling*. Theorem 2.1 shows that the resulting random variable $\beta_{\theta\sigma,\theta(1-\sigma)}M_{\theta\sigma}(F_X)$ is again a mean functional, but now of order θ. In addition, the GGC$(\theta\sigma, F_X)$ random variable $G_{\theta\sigma}M_{\theta\sigma}(F_X)$ is equivalently a GGC random variable of order θ. Now, keeping in mind that tractable densities of mean functionals of order $\theta = 1$ are the easiest to obtain, Theorem 2.1 shows that by setting $\theta = 1$, the densities of the uncountable collection of random variables $(\beta_{\sigma,1-\sigma}M_\sigma(F_X); 0 < \sigma \leq 1)$ are all mean functionals of order $\theta = 1$. Theorem 2.2 then shows that efforts used to calculate the explicit density of any one of these random variables, via the formulae of [6], lead to the explicit calculation of the densities of all of them. Additionally, Theorem 2.2 shows that the corresponding GGC random variables may all be expressed as GGC random variables of order $\theta = 1$, representable in distribution as $G_1\beta_{\sigma,1-\sigma}M_\sigma(F_X)$. A key point here is that Theorem 2.2 gives a tractable density for $\beta_{\sigma,1-\sigma}M_\sigma(F_X)$ without requiring knowledge of the density of $M_\sigma(F_X)$, which is usually expressed in a complicated manner. These results will also yield some non-obvious integral identities. Furthermore, noting that a GGC(θ, F_X) random variable, T_θ, is infinitely divisible, we associate it with an independent increment process $(\zeta_\theta(t); t \geq 0)$, known as a *subordinator* (a non-decreasing non-negative Lévy process), where, for each fixed t,

$$\mathbb{E}\big[\mathrm{e}^{-\lambda\zeta_\theta(t)}\big] = \mathbb{E}[\mathrm{e}^{-\lambda T_{\theta t}}] = \mathrm{e}^{-t\theta\psi_{F_X}(\lambda)}.$$

That is, marginally, $\zeta_\theta(1) \stackrel{\mathrm{d}}{=} T_\theta$ and $\zeta_\theta(t) \stackrel{\mathrm{d}}{=} \zeta_{\theta t}(1) \stackrel{\mathrm{d}}{=} T_{\theta t}$. In addition, for $s < t$, $\zeta_\theta(t) - \zeta_\theta(s) \stackrel{\mathrm{d}}{=} \zeta_\theta(t-s)$ is independent of $\zeta_\theta(s)$. We say that the process $(\zeta_\theta(t); t \geq 0)$ is a *GGC*(θ, F_X) *subordinator*. Proposition 2.1 shows how Theorems 2.1 and 2.2 can be used to address the usually difficult problem of explicitly describing the densities of the finite-dimensional distribution of a subordinator (see [29]). This has implications in, for instance, the explicit description of densities of Bayesian nonparametric prior and posterior models, but is clearly of wider interest in terms of the distribution theory of infinitely divisible random variables and associated processes.

In Section 3, we describe how the operation of exponentially tilting the density of a GGC(θ, F_X) random variable leads to a relationship between the densities of the mean functional $M_\theta(F_X)$ and yet another family of mean functionals. This is summarized in Theorem 3.1. Section 3.1 then discusses a combination of the two operations. Proposition 3.1 describes the density of beta-scaled and tilted mean functionals of order 1. Using this, Proposition 3.2 describes a method to calculate a key quantity in the explicit description of the densities and cdfs of mean functionals. In Section 4, we show how the results in Sections 2 and 3 are used to derive the finite-dimensional distribution and related quantities for classes of subordinators suggested by the recent work of James, Lijoi and Prünster [25] and Bertoin *et al.* [2].

1.3. Preliminaries

Suppose that X is a positive random variable with distribution F_X and define the function

$$\Phi_{F_X}(t) = \int_0^\infty \log(|t-x|)\mathbb{I}(t \neq x)F_X(\mathrm{d}x) = \mathbb{E}[\log(|t-X|)\mathbb{I}(t \neq X)].$$

Furthermore, define

$$\Delta_\theta(t|F_X) = \frac{1}{\pi}\sin(\pi\theta F_X(t))\mathrm{e}^{-\theta\Phi_{F_X}(t)},$$

where, using a Lebesgue–Stieltjes integral, $F_X(t) = \int_0^t F_X(\mathrm{d}x)$. Cifarelli and Regazzini [6] (see also [7]) apply an inversion formula to obtain the distributional formula for $M_\theta(F_X)$ as follows. For all $\theta > 0$, the cdf can be expressed as

$$\int_0^x (x-t)^{\theta-1}\Delta_\theta(t|F_X)\,\mathrm{d}t, \tag{1.4}$$

provided that θF_X possesses no jumps of size greater than or equal to one. If we let $\xi_{\theta F_X}(\cdot)$ denote the density of $M_\theta(F_X)$, then it takes its simplest form for $\theta = 1$, which is

$$\xi_{F_X}(x) = \Delta_1(x|F_X) = \frac{1}{\pi}\sin(\pi F_X(x))\mathrm{e}^{-\Phi(x)}. \tag{1.5}$$

Density formulae for $\theta > 1$ are described as

$$\xi_{\theta F_X}(x) = (\theta - 1)\int_0^x (x-t)^{\theta-2}\Delta_\theta(t|F_X)\,\mathrm{d}t. \tag{1.6}$$

An expression for the density, which holds for all $\theta > 0$, was recently obtained by James, Lijoi and Prünster [25] as follows:

$$\xi_{\theta F_X}(x) = \frac{1}{\pi}\int_0^x (x-t)^{\theta-1} d_\theta(t|F_X)\,\mathrm{d}t, \tag{1.7}$$

where

$$d_\theta(t|F_X) = \frac{\mathrm{d}}{\mathrm{d}t}\sin(\pi\theta F_X(t))\mathrm{e}^{-\theta\Phi(t)}.$$

For additional formulae, see [6,32,38].

Remark 1.1. Throughout, for random variables R and X, when we write the product RX, we will assume, unless otherwise mentioned, that R and X are independent. This convention will also apply to the multiplication of the special random variables that are expressed as mean functionals. That is, the product $M_\theta(F_X)M_\theta(F_Z)$ is understood to be a product of independent Dirichlet means.

Remark 1.2. Throughout, we will be using the fact that if R is a gamma random variable, then the independent random variables R, X, Z satisfying $RX \stackrel{\mathrm{d}}{=} RZ$ imply that $X \stackrel{\mathrm{d}}{=} Z$. This is true because gamma random variables are simplifiable. For the precise meaning of this term and associated conditions, see Chaumont and Yor [4], Sections 1.12 and 1.13. This fact also applies to the case where R is a positive stable random variable.

2. Beta scaling

In this section, we investigate the simple operation of multiplying a Dirichlet mean functional $M_\theta(F_X)$ by certain beta random variables. Note, first, that if M denotes an arbitrary positive random variable with density f_M, then, by elementary arguments, it follows that the random variable $W \stackrel{\mathrm{d}}{=} \beta_{a,b}M$, where $\beta_{a,b}$ is beta(a,b) independent of M, has density expressible as

$$f_W(w) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 f_M(w/u)u^{a-2}(1-u)^{b-1}\,\mathrm{d}u.$$

However, it is only in special cases that the density f_W can be expressed in even simpler terms. That is to say, it is not obvious how to carry out the integration. In the next results, we show how remarkable simplifications can be achieved when $M = M_\theta(F_X)$, in particular, for the range $0 < \theta \le 1$, and when $\beta_{a,b}$ is a symmetric beta random variable. First, we will need to introduce some additional notation. Let Y_σ denote a Bernoulli random variable with success probability $0 < \sigma \le 1$. Then, if X is a random variable with distribution F_X, independent of Y_σ, it follows that the random variable XY_σ has distribution

$$F_{XY_\sigma}(\mathrm{d}x) = \sigma F_X(\mathrm{d}x) + (1-\sigma)\delta_0(\mathrm{d}x) \tag{2.1}$$

and cdf

$$F_{XY_\sigma}(x) = \sigma F_X(x) + (1-\sigma)\mathbb{I}(x \ge 0). \tag{2.2}$$

Hence, there exists the mean functional

$$M_\theta(F_{XY_\sigma}) \stackrel{\mathrm{d}}{=} \int_0^\infty y \tilde{P}_\theta(\mathrm{d}y),$$

where $\tilde{P}_\theta(\mathrm{d}y)$ denotes a Dirichlet process with parameters (θ, F_{XY_σ}). In addition, we have, for $x > 0$,

$$\Phi_{F_{XY_\sigma}}(x) = \mathbb{E}[\log(|x - XY_\sigma|)\mathbb{I}(XY_\sigma \ne x)] = \sigma \Phi_{F_X}(x) + (1-\sigma)\log(x). \tag{2.3}$$

When $\sigma = 1$, $Y_\sigma = 1$ and hence $F_{XY_1}(x) = F_X(x)$. Let E_σ denote a set such that $\mathbb{E}[P_\theta(E_\sigma)] = \sigma$. Note, now, that every beta random variable, $\beta_{a,b}$, where a, b are arbitrary positive constants, can be represented as the simple mean functional

$$P_\theta(E_\sigma) \stackrel{\mathrm{d}}{=} \beta_{\theta\sigma,\theta(1-\sigma)} \stackrel{\mathrm{d}}{=} M_\theta(F_{Y_\sigma}),$$

by choosing

$$\sigma = \frac{a}{a+b} \quad \text{and} \quad \theta = a + b.$$

We note, however, that there are other choices of F_X that will also yield beta random variables as mean functionals. Throughout, we will use the convention that $\beta_{\theta,0} := 1$, that is, the case where $\sigma = 1$. We now present our first result.

Theorem 2.1. *For $\theta > 0$ and $0 < \sigma \le 1$, let $\beta_{\theta\sigma,\theta(1-\sigma)}$ denote a beta random variable with parameters $(\theta\sigma, \theta(1-\sigma))$, independent of the mean functional $M_{\theta\sigma}(F_X)$. Then*:

(i) $\beta_{\theta\sigma,\theta(1-\sigma)} M_{\theta\sigma}(F_X) \stackrel{d}{=} M_\theta(F_{XY_\sigma})$;
(ii) *equivalently*, $M_\theta(F_{Y_\sigma}) M_{\theta\sigma}(F_X) \stackrel{d}{=} M_\theta(F_{XY_\sigma})$;
(iii) $G_{\theta\sigma} M_{\theta\sigma}(F_X) \stackrel{d}{=} G_\theta M_\theta(F_{XY_\sigma})$;
(iv) *that is*, $GGC(\theta\sigma, F_X) = GGC(\theta, F_{XY_\sigma})$.

Proof. Since $M_\theta(F_{Y_\sigma}) \stackrel{\mathrm{d}}{=} \beta_{\theta\sigma,\theta(1-\sigma)}$, statements (i) and (ii) are equivalent. We proceed by first establishing (iii) and (iv). Note that, using (1.3),

$$\mathbb{E}[\log(1+\lambda X Y_\sigma)] = \sigma \mathbb{E}[\log(1+\lambda X)] = \sigma \int_0^\infty \log(1+\lambda x) F_X(\mathrm{d}x).$$

Hence,

$$\mathbb{E}\big[\mathrm{e}^{-\lambda G_\theta M_\theta(F_{XY_\sigma})}\big] = \mathrm{e}^{-\theta\sigma \int_0^\infty \log(1+\lambda x) F_X(\mathrm{d}x)} = \mathbb{E}\big[\mathrm{e}^{-\lambda G_{\theta\sigma} M_{\theta\sigma}(F_X)}\big],$$

which means that $G_\theta M_\theta(F_{XY_\sigma}) \stackrel{\mathrm{d}}{=} G_{\theta\sigma} M_{\theta\sigma}(F_X)$, establishing statements (iii) and (iv). Now, writing $G_{\theta\sigma} = G_\theta \beta_{\theta\sigma,\theta(1-\sigma)}$, it follows that

$$G_\theta M_\theta(F_{XY_\sigma}) \stackrel{\mathrm{d}}{=} G_\theta \beta_{\theta\sigma,\theta(1-\sigma)} M_{\theta\sigma}(F_X).$$

Hence, $\beta_{\theta\sigma,\theta(1-\sigma)} M_{\theta\sigma}(F_X) \stackrel{\mathrm{d}}{=} M_\theta(F_{XY_\sigma})$, by the fact that gamma random variables are simplifiable. □

Remark 2.1. We note that parts (i) and (ii) of Theorem 2.1 also follow as consequences of Ethier and Griffiths [14], Lemma 1. We now state their interesting result for clarity.

Lemma 2.1 (Ethier and Griffiths [14]). *Let ν_1 and ν_2 denote two probability measures. Now, for $\theta_1, \theta_2 > 0$, define the probability measure*

$$\nu_{(\theta_1,\theta_2)}(\mathrm{d}x) = \frac{\theta_1}{\theta_1+\theta_2}\nu_1(\mathrm{d}x) + \frac{\theta_2}{\theta_1+\theta_2}\nu_2(\mathrm{d}x).$$

Then, for independent Dirichlet processes $\mu_1 \sim \Pi_{\theta_1,\nu_1}$ and $\mu_2 \sim \Pi_{\theta_2,\nu_2}$,

$$\mu_{1,2}(\cdot) \stackrel{d}{=} \beta_{\theta_1,\theta_2}\mu_1(\cdot) + (1-\beta_{\theta_1,\theta_2})\mu_2(\cdot),$$

where $\mu_{1,2}$ is a Dirichlet process with parameters $(\theta_1+\theta_2, \nu_{(\theta_1,\theta_2)})$.

Hence, as a general consequence,

$$M_{\theta_1+\theta_2}\big(\nu_{(\theta_1,\theta_2)}\big) \stackrel{d}{=} \beta_{\theta_1,\theta_2} M_{\theta_1}(\nu_1) + (1-\beta_{\theta_1,\theta_2}) M_{\theta_2}(\nu_2).$$

Now, from (2.1), we see that setting $\nu_1 = F_X$, $\nu_2 = \delta_0$, $\theta_1 = \theta\sigma$ and $\theta_2 = \theta(1-\sigma)$ yields statements (i) and (ii). This is because $M_{\theta(1-\sigma)}(\delta_0) = 0$.

When $\theta = 1$, we obtain results for random variables $\beta_{\sigma,1-\sigma} M_\sigma(F_X)$. The symmetric beta random variables $\beta_{\sigma,1-\sigma}$ arise in a variety of important contexts and are often referred to as generalized arcsine laws with density expressible as

$$\frac{\sin(\pi\sigma)}{\pi} u^{\sigma-1}(1-u)^{-\sigma} \qquad \text{for } 0 < u < 1.$$

Now, using (2.1) and (2.2), let $\mathcal{C}(F_X) = \{x : F_X(x) > 0\}$. Then, for $x > 0$,

$$\sin(\pi F_{XY_\sigma}(x)) = \begin{cases} \sin\bigl(\pi\sigma[1 - F_X(x)]\bigr), & \text{if } x \in \mathcal{C}(F_X), \\ \sin(\pi(1-\sigma)), & \text{if } x \notin \mathcal{C}(F_X). \end{cases} \tag{2.4}$$

Also, note that $\sin(\pi[1 - F_X(x)]) = \sin(\pi F_X(x))$. The next result yields another surprising property of these random variables.

Theorem 2.2. *Consider the setting in Theorem 2.1. Then, when $\theta = 1$, it follows that for each fixed $0 < \sigma \le 1$, the random variable $M_1(F_{XY_\sigma}) \stackrel{d}{=} \beta_{\sigma,1-\sigma} M_\sigma(F_X)$ has density*

$$\xi_{F_{XY_\sigma}}(x) = \frac{x^{\sigma-1}}{\pi} \sin(\pi F_{XY_\sigma}(x)) \mathrm{e}^{-\sigma \Phi_{F_X}(x)} \qquad \textit{for } x > 0, \tag{2.5}$$

specified by (2.4). Since GGC(σ, F_X) =GGC($1, F_{XY_\sigma}$), this implies that the random variable $G_\sigma M_\sigma(F_X) \stackrel{d}{=} G_1 M_1(F_{XY_\sigma})$ has density

$$g_{\sigma,F_X}(x) = \frac{1}{\pi} \int_0^\infty \mathrm{e}^{-x/y} y^{\sigma-2} \sin(\pi F_{XY_\sigma}(y)) \mathrm{e}^{-\sigma \Phi_{F_X}(y)} \,\mathrm{d}y. \tag{2.6}$$

Proof. Since $M_1(F_{XY_\sigma}) \stackrel{\mathrm{d}}{=} \beta_{\sigma,1-\sigma} M_\sigma(F_X)$, the density is of the form (1.5) for each fixed $\sigma \in (0, 1]$. Furthermore, we use the identity in (2.3). □

Remark 2.2. It is worthwhile to mention that transforming to the random variable $1/\beta_{\sigma,1-\sigma}$, (2.5) is equivalent to the otherwise non-obvious integral identity

$$\frac{\sin(\pi\sigma)}{\pi} \int_1^\infty \frac{\xi_{\sigma F_X}(xy)}{(y-1)^\sigma} \,\mathrm{d}y = \frac{x^{\sigma-1}}{\pi} \sin(\pi F_{XY_\sigma}(x)) \mathrm{e}^{-\sigma \Phi(x)}.$$

This leads to interesting results when the density $\xi_{\sigma F_X}(x)$ has a known form. On the other hand, we see that one does not need the explicit density of $M_\sigma(F_X)$ to obtain the density of $M_1(F_{XY_\sigma}) \stackrel{\mathrm{d}}{=} \beta_{\sigma,1-\sigma} M_\sigma(F_X)$. In fact, owing to our goal of yielding simple densities for many Dirichlet means from one mean, we see that the effort to calculate the density of $M_1(F_{XY_\sigma})$ for each $0 < \sigma \le 1$ is no more than what is needed to calculate the density of $M_1(F_X)$.

We now see how this translates into the usually difficult problem of explicitly describing the density of the finite-dimensional distribution of a subordinator. In the next result, we write, for some set C,

$$\zeta_\theta(C) := \int_0^\infty \mathbb{I}(s \in C) \zeta_\theta(\mathrm{d}s).$$

Proposition 2.1. *Let $(\zeta_\theta(t); t \le 1/\theta)$ denote a GGC(θ, F_X) subordinator on $[0, 1/\theta]$. Furthermore, let $(C_1, \ldots, C_k)$ denote an arbitrary disjoint partition of the interval $[0, 1/\theta]$. The* *finite-dimensional distribution* $(\zeta_\theta(C_1), \ldots, \zeta_\theta(C_k))$ *then has a joint density*

$$\prod_{i=1}^{k} g_{\sigma_i, F_X}(x_i), \tag{2.7}$$

where each $\sigma_i = \theta|C_i| > 0$ *and* $\sum_{i=1}^{k} \sigma_i = 1$. *The density* g_{σ_i, F_X} *is given by* (2.6). *That is,* $\zeta_\theta(C_i) \stackrel{d}{=} G_1 M_1(F_{XY_{\sigma_i}})$ *and these are independent for* $i = 1, \ldots, k$, *where* $M_1(F_{XY_{\sigma_i}}) \stackrel{d}{=} \beta_{\sigma_i, 1-\sigma_i} M_{\sigma_i}(F_X)$ *has density*

$$\frac{1}{\pi} x^{\sigma_i - 1} \sin(\pi F_{XY_{\sigma_i}}(x)) \mathrm{e}^{-\sigma_i \Phi_{F_X}(x)}.$$

Proof. First, since $(C_1, \ldots, C_k)$ partitions the interval $[0, 1/\theta]$, it follows that their sizes satisfy $0 < |C_k| \le 1/\theta$ and $\sum_{i=1}^{k} |C_k| = 1/\theta$. Since ζ_θ is a subordinator, the independence of the $\zeta_\theta(C_i)$ is a consequence of its independent increment property. In fact, these are essentially equivalent statements. Hence, we can isolate each $\zeta_\theta(C_i)$. It follows that for each i, the Laplace transform is given by

$$\mathbb{E}\big[\mathrm{e}^{-\lambda \zeta_\theta(C_i)}\big] = \mathrm{e}^{-\theta |C_i| \psi_{F_X}(\lambda)} = \mathrm{e}^{-\sigma_i \psi_{F_X}(\lambda)},$$

which shows that each $\zeta_\theta(C_i)$ is $\mathrm{GGC}(\sigma_i, F_X)$ for $0 < \sigma_i \le 1$. Hence, the result follows from Theorem 2.2. $\square$

3. Exponential tilting/Esscher transform

In this section, we describe how the operation of *exponential tilting* of the density of a $\mathrm{GGC}(\theta, F_X)$ random variable leads to a non-trivial relationship between a mean functional determined by F_X and θ, and an entire family of mean functionals indexed by an arbitrary constant $c > 0$. Additionally, this will identify a non-obvious relationship between two classes of mean functionals. Exponential tilting is merely a convenient phrase for the operation of applying an exponential change of measure to a density or more general measure. In mathematical finance and other applications, it is known as an *Esscher transform* and is a key tool for option pricing. We mention that there is much known about exponential tilting of infinitely divisible random variables and, in fact, Bondesson [3], Example 3.2.5, explicitly discusses the case of GGC random variables, albeit not in the way we shall describe it. In addition, examining the gamma representation in (1.1), one can see a relationship to Lo and Weng [34], Proposition 3.1 (see also Küchler and Sorensen [30] and James [22], Proposition 2.1), for results on exponential tilting of Lévy processes). However, our focus here is on the properties of related mean functionals, which leads to genuinely new insights.

Before we elaborate on this, we describe generically what we mean by exponential tilting. Suppose that T denotes an arbitrary positive random variable with density, say, f_T. It follows that for each positive c, the random variable cT is well defined and has density

$$\frac{1}{c} f_T(t/c).$$

Exponential tilting refers to the exponential change of measure resulting in a random variable, say $\tilde{T}_c$, defined by the density

$$f_{\tilde{T}_c}(t) = \frac{\mathrm{e}^{-t}(1/c) f_T(t/c)}{\mathbb{E}[\mathrm{e}^{-cT}]}.$$

Thus, from the random variable T, one gets a family of random variables $(\tilde{T}_c; c > 0)$. Obviously, the density for each $\tilde{T}_c$ does not differ much. However, something interesting happens when T is a scale mixture of a gamma random variables, that is, $T = G_\theta M$ for some random positive random variable M independent of G_θ. In that case, one can show, see [23], that $T_c = G_\theta \tilde{M}_c$, where $\tilde{M}_c$ is sufficiently distinct for each value of c. We demonstrate this for the case where $M = M_\theta(F_X)$.

First, note that, obviously, $cM_\theta(F_X) = M_\theta(F_{cX})$ for each $c > 0$, which, in itself, is not a very interesting transformation. Now, setting $T_\theta = G_\theta M_\theta(F_X)$ with density denoted g_{θ,F_X}, the corresponding random variable $\tilde{T}_{\theta,c}$ resulting from exponential tilting has density

$$\mathrm{e}^{-t}(1/c) g_{\theta,F_X}(t/c) \mathrm{e}^{\theta\psi_{F_X}(c)} \tag{3.1}$$

and Laplace transform

$$\frac{\mathbb{E}[\mathrm{e}^{-c(1+\lambda)G_\theta M_\theta(F_X)}]}{\mathbb{E}[\mathrm{e}^{-cG_\theta M_\theta(F_X)}]} = \mathrm{e}^{-\theta[\psi_{F_X}(c(1+\lambda)) - \psi_{F_X}(c)]}. \tag{3.2}$$

Now, for each $c > 0$, define the random variable

$$A_c \stackrel{\mathrm{d}}{=} \frac{cX}{(cX+1)}.$$

That is, the cdf of the random variable A_c can be expressed as

$$F_{A_c}(y) = F_X\left(\frac{y}{c(1-y)}\right) \qquad \text{for } 0 < y < 1.$$

In the next theorem, we will show that $M_\theta(F_X)$ relates to the family of mean functionals $(M_\theta(F_{A_c}); c > 0)$ by the tilting operation described above. Moreover, we will describe the relationship between their densities.

Theorem 3.1. *Suppose that X has distribution F_X and for each $c > 0$, $A_c \stackrel{d}{=} cX/(cX+1)$ is a random variable with distribution F_{A_c}. For each $\theta > 0$, let $T_\theta = G_\theta M_\theta(F_X)$ denote a GGC (θ, F_X) random variable having density g_{θ,F_X}. Let $\tilde{T}_{\theta,c}$ denote a random variable with density and Laplace transform described by (3.1) and (3.2), respectively. $\tilde{T}_{\theta,c}$ is then a GGC(θ, F_{A_c}) random variable and hence representable as $G_\theta M_\theta(F_{A_c})$. Furthermore, the following relationships exist between the densities of the mean functionals $M_\theta(F_X)$ and $M_\theta(F_{A_c})$:*

(i) *supposing that the density of* $M_\theta(F_X)$, *say* $\xi_{\theta F_X}$, *is known, then the density of* $M_\theta(F_{A_c})$ *is expressible as*

$$\xi_{\theta F_{A_c}}(y) = \frac{1}{c}\mathrm{e}^{\theta\psi_{F_X}(c)}(1-y)^{\theta-2}\xi_{\theta F_X}\left(\frac{y}{c(1-y)}\right)$$

for $0 < y < 1$;

(ii) *conversely, if the density of* $M_\theta(F_{A_c})$, $\xi_{\theta F_{A_c}}(y)$, *is known, then the density of* $M_\theta(F_X)$ *is given by*

$$\xi_{\theta F_X}(x) = (1+x)^{\theta-2}\xi_{\theta F_{A_1}}\left(\frac{x}{1+x}\right)\mathrm{e}^{-\theta\psi_{F_X}(1)}.$$

Proof. We proceed by first examining the Lévy exponent $[\psi_{F_X}(c(1+\lambda)) - \psi_{F_X}(c)]$ associated with $\tilde{T}_{\theta,c}$ as described in (3.2). Note that

$$\psi_{F_X}\big(c(1+\lambda)\big) = \int_0^\infty \log\big(1+c(1+\lambda)x\big)F_X(\mathrm{d}x)$$

and $\psi_{F_X}(c)$ is of the same form with $\lambda = 0$. Hence, isolating the logarithmic terms, we can focus on the difference

$$\log\big(1+c(1+\lambda)x\big) - \log(1+cx).$$

This is equivalent to

$$\log\left(1+\frac{cx}{1+cx}\lambda\right) = \log\left(\frac{1}{1+cx} + \frac{cx}{1+cx}(1+\lambda)\right),$$

showing that $\tilde{T}_{\theta,c}$ is GGC(θ, F_{A_c}). This fact can also be deduced from Proposition 3.1 in Lo and Weng [34]. The next step is to identify the density of $M_\theta(F_{A_c})$ in terms of the density of $M_\theta(F_X)$. Using the fact that $T_\theta = G_\theta M_\theta(F_X)$, one may write the density of T_θ in terms of a gamma mixture as

$$g_{\theta,F_X}(t) = \frac{t^{\theta-1}}{\Gamma(\theta)}\int_0^\infty \mathrm{e}^{-t/m}m^{-\theta}\xi_{\theta F_X}(m)\,\mathrm{d}m.$$

Hence, rearranging terms in (3.1), it follows that the density of $\tilde{T}_{\theta,c}$ can be written as

$$\mathrm{e}^{\theta\psi_{F_X}(c)}\frac{t^{\theta-1}}{\Gamma(\theta)}\int_0^\infty \mathrm{e}^{-t(cm+1)/(cm)}(cm)^{-\theta}\xi_{\theta F_X}(m)\,\mathrm{d}m.$$

Now, further algebraic manipulation makes this look like a mixture of a gamma$(\theta, 1)$ random variables, as follows,

$$\frac{t^{\theta-1}}{\Gamma(\theta)}\int_0^\infty \mathrm{e}^{-t(cm+1)/(cm)}\left[\frac{cm+1}{cm}\right]^\theta \frac{\mathrm{e}^{\theta\psi_{F_X}(c)}\xi_{\theta F_X}(m)}{(1+cm)^\theta}\,\mathrm{d}m.$$

Hence, it is evident that $M_\theta(F_{A_c})$ has the same distribution as a random variable $cM/(cM+1)$, where M has density

$$\mathrm{e}^{\theta\psi_{F_X}(c)}(1+cm)^{-\theta}\xi_{\theta F_X}(m).$$

Thus, statements (i) and (ii) follow. □

3.1. Tilting and beta scaling

This section describes what happens when one applies the exponential tilting operation relative to a mean functional resulting from beta scaling. Recall that the tilting operation applied to $G_\theta M_\theta(F_X)$ described in the previous section sets up a relationship between $M_\theta(F_X)$ and $M_\theta(F_{Ac})$. Consider the random variable $\beta_{\theta\sigma,\theta(1-\sigma)}M_{\theta\sigma}(F_X) \stackrel{\mathrm{d}}{=} M_\theta(F_{XY_\sigma})$. Then, tilting $G_\theta M_\theta(F_{XY_\sigma})$ as in the previous section leads to the random variable $G_\theta M_\theta(F_{cXY_\sigma/(cXY_\sigma+1)})$ and hence relates

$$\beta_{\theta\sigma,\theta(1-\sigma)}M_{\theta\sigma}(F_X) \stackrel{\mathrm{d}}{=} M_\theta(F_{XY_\sigma})$$

to the Dirichlet mean of order θ,

$$M_\theta\bigl(F_{cXY_\sigma/(cXY_\sigma+1)}\bigr).$$

Now, letting $F_{A_cY_\sigma}$ denote the distribution of A_cY_σ, one has

$$A_cY_\sigma \stackrel{\mathrm{d}}{=} \frac{cXY_\sigma}{(cXY_\sigma+1)}$$

and hence

$$M_\theta\bigl(F_{cXY_\sigma/(cXY_\sigma+1)}\bigr) \stackrel{\mathrm{d}}{=} M_\theta(F_{A_cY_\sigma}) \stackrel{\mathrm{d}}{=} \beta_{\theta\sigma,\theta(1-\sigma)}M_{\theta\sigma}(F_{A_c}). \tag{3.3}$$

In a way, this shows that the order of beta scaling and tilting can be interchanged. We now derive a result for the cases of $M_1(F_{XY_\sigma}) = \beta_{\sigma,1-\sigma}M_\sigma(F_X)$ and $M_1(F_{A_cY_\sigma}) = \beta_{\sigma,1-\sigma}M_\sigma(F_{A_c})$, related by the tilting operation described above. Combining Theorem 2.2 with Theorem 3.1 leads to the following result.

Proposition 3.1. *For each* $0<\sigma\le 1$, *the random variables* $M_1(F_{XY_\sigma}) = \beta_{\sigma,1-\sigma}M_\sigma(F_X)$ *and* $M_1(F_{A_cY_\sigma}) = \beta_{\sigma,1-\sigma}M_\sigma(F_{A_c})$ *satisfy the following*:

(i) *the density of* $M_1(F_{A_cY_\sigma})$ *is expressible as*

$$\xi_{F_{A_cY_\sigma}}(y) = \frac{\mathrm{e}^{\sigma\psi_{F_X}(c)}y^{\sigma-1}}{\pi c^\sigma(1-y)^\sigma}\sin\biggl[\pi F_{XY_\sigma}\biggl(\frac{y}{c(1-y)}\biggr)\biggr]\mathrm{e}^{-\sigma\Phi_{F_X}(y/(c(1-y)))}$$

for $0<y<1$;

(ii) *conversely, the density of* $M_1(F_{XY_\sigma})$ *is given by*

$$\xi_{F_{XY_\sigma}}(x) = \frac{\mathrm{e}^{-\sigma\psi_{F_X}(1)}x^{\sigma-1}}{\pi(1+x)}\sin\left[\pi F_{A_1 Y_\sigma}\left(\frac{x}{1+x}\right)\right]\mathrm{e}^{-\sigma\Phi_{F_{A_1}}(x/(1+x))}.$$

Proof. For clarity, statement (i) is obtained by first using Theorem 3.1, which gives

$$\xi_{F_{A_c Y_\sigma}}(y) = \frac{1}{c}\mathrm{e}^{\psi_{F_{XY_\sigma}}(c)}(1-y)^{-1}\xi_{F_{XY_\sigma}}\left(\frac{y}{c(1-y)}\right)$$

for $0 < y < 1$. It then remains to substitute the form of the density (2.5) given in Theorem 2.2. Statement (ii) proceeds in the same way, using (2.6). □

Note that even if one can calculate $\Phi_{F_{A_c}}$ for some fixed value of c, it may not be so obvious how to calculate it for another value. The previous results allow us to relate their calculation to that of Φ_{F_X}, as described next.

Proposition 3.2. *Set* $A_c = cX/(cX+1)$ *and define* $\Phi_{F_{A_c}}(y) = \mathbb{E}[\log(|y - A_c|)\mathbb{I}(A_c \neq y)]$. *Then, for* $0 < y < 1$,

$$\Phi_{F_{A_c}}(y) = \Phi_{F_X}\left(\frac{y}{c(1-y)}\right) - \psi_{F_X}(c) + \log\bigl(c(1-y)\bigr).$$

Proof. The result can be deduced by using Proposition 3.1 in the case $\sigma = 1$. First, note that $\sin(\pi F_X(\frac{y}{c(1-y)})) = \sin(\pi F_{A_c}(y))$. Now, equating the form of the density of $M_1(F_{A_c})$ given by (1.5) with the expression given in Proposition 3.1, it follows that

$$\mathrm{e}^{-\Phi_{F_{A_c}}(y)} = \frac{\mathrm{e}^{\psi_{F_X}(c)}}{c(1-y)}\mathrm{e}^{-\Phi_{F_X}(y/(c(1-y)))},$$

which yields the result. □

Remark 3.1. We point out that if G_κ represents a gamma random variable for $\kappa \neq \theta$, independent of $M_\theta(F_X)$, then it is not necessarily true that $G_\kappa M_\theta(F_X)$ is a GGC random variable. For this to be true, $M_\theta(F_X)$ would need to be equivalent in distribution to some $M_\kappa(F_R)$. In that case, our results above would be applied for a GGC(κ, F_R) model.

4. Examples

In this section, we will demonstrate how our results in Sections 2 and 3 can be applied to extend results for two random processes recently studied in the literature. The first involves a class of GGC subordinators that can be derived from a random mean of a two-parameter Poisson–Dirichlet process with a uniform base measure, which was studied as a special case in James, Lijoi and Prünster [25]; see Pitman and Yor [37] for more details of the two parameter Poisson–Dirichlet distribution. The second involves a class of processes recently studied in Bertoin *et* *al.* [2]; see also Maejima [36] for some discussion of this process. A key component will be the ability to obtain an explicit expression for the respective Φ_{F_X}. In the first example, we do not have much explicit information on the relevant density, $\xi_{\theta F_X}$; however, we can rely on a general theorem of James, Lijoi and Prünster [25] to obtain Φ_{F_X}. In the second case of the models discussed in Bertoin *et al.* [2], this theorem apparently does not apply. However, we will be able to use an explicit form of the density, obtained for a particular value of θ by Bertoin *et al.* [2], to obtain Φ_{F_X}.

As we shall show, both of these processes are connected to a random variable Z_α, whose properties we now describe. For $0 < \alpha < 1$, let S_α denote a positive α-stable random variable specified by its Laplace transform

$$\mathbb{E}[\mathrm{e}^{-\lambda S_\alpha}] = \mathrm{e}^{-\lambda^\alpha}.$$

In addition, define

$$Z_\alpha = \left(\frac{S_\alpha}{S'_\alpha}\right)^\alpha,$$

where S'_α is independent of S_α and has the same distribution. The density of this random variable was obtained by Lamperti [31] (see also Chaumont and Yor [4], Exercise 4.2.1) and has the remarkably simple form

$$f_{Z_\alpha}(y) = \frac{\sin(\pi\alpha)}{\pi\alpha}\frac{1}{y^2 + 2y\cos(\pi\alpha) + 1} \qquad \text{for } y > 0.$$

Furthermore (see Fujita and Yor [18] and (James [24], Proposition 2.1), it follows that the cdf of Z_α satisfies, for $z > 0$,

$$\begin{aligned} F_{Z_\alpha}(1/z) &= 1 - \frac{1}{\pi\alpha}\cot^{-1}\left(\frac{\cos(\pi\alpha) + 1/z}{\sin(\pi\alpha)}\right) \\ &= \frac{1}{\pi\alpha}\cot^{-1}\left(\frac{\cos(\pi\alpha) + z}{\sin(\pi\alpha)}\right) \\ &= 1 - F_{Z_\alpha}(z), \end{aligned}$$

$$\sin(\pi\alpha F_{Z_\alpha}(z)) = z\sin\bigl(\pi\alpha\bigl(1 - F_{Z_\alpha}(z)\bigr)\bigr) = \frac{z\sin(\pi\alpha)}{[z^2 + 2z\cos(\pi\alpha) + 1]^{1/2}} \tag{4.1}$$

and

$$\begin{aligned} \sin\bigl(2\pi\alpha[1 - F_{Z_\alpha}(z)]\bigr) &= \frac{\sin(2\pi\alpha) + 2z\sin(\pi\alpha)}{1 + 2z\cos(\pi\alpha) + z^2} \\ &= \frac{2\sin(\pi\alpha)[\cos(\pi\alpha) + z]}{1 + 2z\cos(\pi\alpha) + z^2}. \end{aligned} \tag{4.2}$$

When $\alpha = 1/2$,

$$\sin\bigl(\pi[1 - F_{Z_{1/2}}(z)]\bigr) = \frac{z}{z^2 + 1}.$$

4.1. Subordinators derived from an example in James, Lijoi and Prünster

For $0 < \alpha < 1$ and $\theta > -\alpha$, we define the special case of a two-parameter Poisson–Dirichlet random probability measures as

$$\tilde{P}_{\alpha,\theta}(\cdot) = \sum_{k=1}^{\infty} V_k \prod_{i=1}^{k-1} (1 - V_i)\delta_{U_k}(\cdot),$$

where U_k are i.i.d. Uniform[0,1] random variables and the V_k are a sequence of independent $\beta_{\alpha,\theta+k\alpha}$ random variables, independent of (U_k). So, in particular, these random variables satisfy $\mathbb{E}[\tilde{P}_{\alpha,\theta}(\cdot)] = F_U(\cdot)$, where U denotes a Uniform[0, 1] random variable. In addition, $\tilde{P}_{0,\theta}$ is a Dirichlet process. Then, consider the random means given as

$$\tilde{M}_{\alpha,\theta}(F_U) := \mathbb{U}_{\alpha,\theta} = \sum_{k=1}^{\infty} U_k V_k \prod_{i=1}^{k-1} (1 - V_i) = \int_0^1 u\, \tilde{P}_{\alpha,\theta}(\mathrm{d}u).$$

The $\mathbb{U}_{\alpha,\theta}$ represent a special case of random variables representable as mean functionals of the class of two-parameter (α, θ) Poisson–Dirichlet random probability measures – that is to say, random variables $\tilde{M}_{\alpha,\theta}(F_X)$, where F_X is a general distribution. An extensive study of this larger class was conducted by James, Lijoi and Prünster [25]. In regards to $\mathbb{U}_{\alpha,\theta}$, they show that $\mathbb{U}_{\alpha,0}$ has an explicit density

$$\frac{\sin(\pi\alpha)}{\alpha\pi} \frac{(\alpha + 1)t^{\alpha}(1 - t)^{\alpha}}{[t^{2\alpha+2} + 2t^{\alpha+1}(1 - t)^{\alpha+1} \cos(\pi\alpha) + (1 - t)^{2\alpha+2}]}.$$

Furthermore, from James, Lijoi and Prünster [25], Theorem 2.1, for $\theta > 0$,

$$\mathbb{U}_{\alpha,\theta} \stackrel{\mathrm{d}}{=} M_\theta(F_{\mathbb{U}_{\alpha,0}}).$$

This implies that

$$G_\theta \mathbb{U}_{\alpha,\theta} \stackrel{\mathrm{d}}{=} G_\theta M_\theta(F_{\mathbb{U}_{\alpha,0}})$$

are GGC$(\theta, F_{\mathbb{U}_{\alpha,0}})$. Now, from Vershik, Yor and Tslevich [40] (see also James, Lijoi and Prünster [25], equation (16)), it follows that

$$\begin{aligned}
\mathbb{E}[\mathrm{e}^{-\lambda G_\theta \mathbb{U}_{\alpha,\theta}}] &= \left(\frac{\lambda(\alpha + 1)}{(\lambda + 1)^{\alpha+1} - 1}\right)^{\theta/\alpha} \\
&= \exp\bigl(-\theta \mathbb{E}[\log(1 + \lambda \mathbb{U}_{\alpha,0})]\bigr),
\end{aligned}$$

where this expression follows from the generalized Stieltjes transform of order $-\alpha$ of a Uniform[0,1] random variable,

$$\mathbb{E}[(1+\lambda U)^{\alpha}] = \int_0^1 (1+\lambda x)^{\alpha}\,\mathrm{d}x = \frac{(\lambda+1)^{\alpha+1}-1}{\lambda(\alpha+1)}.$$

A description of the densities of $\mathbb{U}_{\alpha,\theta}$ for $\theta > -\alpha$ is available from the results of [25]. However, with the exceptions of $\mathbb{U}_{\alpha,1}$ and $\mathbb{U}_{\alpha,1-\alpha}$, their densities are generally expressed in terms of integrals with respect to functions that possibly take on negative values. Here, by focusing instead on random variables $\beta_{\theta,1-\theta}\mathbb{U}_{\alpha,\theta}$ for $0<\theta<1$, we can utilize the results in James, Lijoi and Prünster [25] to obtain explicit expressions for their densities and the corresponding GGC$(\theta, F_{\mathbb{U}_{\alpha,0}})$ random variables.

In particular, we will obtain explicit descriptions for the finite-dimensional distribution of a GGC$(\alpha, F_{\mathbb{U}_{\alpha,0}})$, say $(\Upsilon_\alpha(t), t>0)$, subordinator, where $\Upsilon_\alpha(1) \stackrel{\mathrm{d}}{=} G_\alpha \mathbb{U}_{\alpha,\alpha}$ and hence

$$\mathbb{E}\bigl[\mathrm{e}^{-\lambda\Upsilon_\alpha(1)}\bigr] = \frac{\lambda(\alpha+1)}{(\lambda+1)^{\alpha+1}-1}.$$

Although not immediately obvious, one can show that

$$\mathbb{U}_{\alpha,0} \stackrel{\mathrm{d}}{=} \frac{Z_\alpha^{1/(\alpha+1)}}{Z_\alpha^{1/(\alpha+1)}+1} \quad \text{and} \quad \text{hence } F_{\mathbb{U}_{\alpha,0}}(t) = F_{Z_\alpha}\biggl(\biggl(\frac{t}{1-t}\biggr)^{\alpha+1}\biggr).$$

From this, due to the tilting relationship discussed in Section 3, we see that we can also obtain results for the GGC$(\alpha, F_{Z_\alpha^{1/(\alpha+1)}})$ subordinator, say $(\Upsilon_\alpha^{\ddagger}(t), t>0)$. To the best of our knowledge, this process and its mean functionals $M_\theta(F_{Z_\alpha^{1/(\alpha+1)}})$ have not been studied. Now, from James, Lijoi and Prünster [25], Theorem 5.2(iii), it follows that

$$\mathrm{e}^{-\Phi_{F_{\mathbb{U}_{\alpha,0}}}(t)} = \frac{(\alpha+1)^{1/\alpha}}{[t^{2\alpha+2}+2t^{\alpha+1}(1-t)^{\alpha+1}\cos(\pi\alpha)+(1-t)^{2\alpha+2}]^{1/(2\alpha)}}. \tag{4.3}$$

This, combined with our results, leads to an explicit description of the finite-dimensional distribution of the relevant subordinators.

Theorem 4.1. *Consider the GGC*$(\alpha, F_{\mathbb{U}_{\alpha,0}})$ *subordinator* $(\Upsilon_\alpha(t), t\le 1/\alpha)$ *and the GGC*$(\alpha, F_{Z_\alpha^{1/(\alpha+1)}})$ *subordinator* $(\Upsilon_\alpha^{\ddagger}(t), t\le 1/\alpha)$. *Let* $(C_1,\ldots,C_k)$ *denote an arbitrary disjoint partition of the interval* $(0,1/\alpha]$ *with lengths* $|C_i|$ *and set* $\sigma_i=\alpha|C_i|$ *for* $i=1,\ldots,k$. *The following results then hold*:

(i) *The finite dimensional distribution of* $(\Upsilon_\alpha(C_1),\ldots,\Upsilon_\alpha(C_k))$ *is such that each* $\Upsilon_\alpha(C_i)$ *is independent and has distribution*

$$\Upsilon_\alpha(C_i) \stackrel{d}{=} G_1 M_1(F_{Y_{\sigma_i}\mathbb{U}_{\alpha,0}}),$$

where $M_1(F_{Y_{\sigma_i}\mathbb{U}_{\alpha,0}}) \stackrel{d}{=} \beta_{\sigma_i,1-\sigma_i}\mathbb{U}_{\alpha,\sigma_i}$. *Furthermore, for any fixed* $0 < \sigma \le 1$, *the density of* $M_1(F_{Y_\sigma\mathbb{U}_{\alpha,0}})$ *is given by, for* $0 < y < 1$,

$$\frac{(\alpha+1)^{\sigma/\alpha} y^{\sigma-1} \sin(\pi\sigma[1 - F_{\mathbb{U}_{\alpha,0}}(y)])}{[y^{2\alpha+2} + 2y^{\alpha+1}(1-y)^{\alpha+1}\cos(\pi\alpha) + (1-y)^{2\alpha+2}]^{\sigma/(2\alpha)}}.$$

(ii) *The finite-dimensional distribution of* $(\Upsilon^{\ddagger}_\alpha(C_1), \ldots, \Upsilon^{\ddagger}_\alpha(C_k))$ *is such that each* $\Upsilon^{\ddagger}_\alpha(C_i)$ *is independent and has distribution*

$$\Upsilon^{\ddagger}_\alpha(C_i) \stackrel{d}{=} G_1 M_1(F_{Y_{\sigma_i} Z_\alpha^{1/(\alpha+1)}}),$$

where $M_1(F_{Y_{\sigma_i} Z_\alpha^{1/(\alpha+1)}}) \stackrel{d}{=} \beta_{\sigma_i,1-\sigma_i} M_{\sigma_i}(F_{Z_\alpha^{1/(\alpha+1)}})$. *Furthermore, for any fixed* $0 < \sigma \le 1$, *the density of* $M_1(F_{Y_\sigma Z_\alpha^{1/(\alpha+1)}})$ *is given by, for* $x > 0$,

$$\frac{x^{\sigma-1}(x+1)^{\sigma(1+\alpha)/\alpha-1} \sin(\pi\sigma[1 - F_{Z_\alpha}(x^{\alpha+1})])}{[x^{2\alpha+2} + 2x^{\alpha+1}\cos(\pi\alpha) + 1]^{\sigma/(2\alpha)}}.$$

Proof. Statement (i) follows from Theorem 2.2 and Proposition 2.1 in combination with (4.3). Noting the relationship between $Z_\alpha^{1/(\alpha+1)}$ and $\mathbb{U}_{\alpha,0}$, statement (ii) follows from Theorem 3.1(ii). □

From this, combined with an application of (4.1), we obtain a description for the densities of $\Upsilon^{\ddagger}_\alpha(1)$ and $\Upsilon_\alpha(1)$. In addition, for $\alpha \le 1/2$, we obtain a description of the density of $\Upsilon_\alpha(2)$ using (4.3).

Proposition 4.1. *Let* $\Upsilon_\alpha(1)$ *and* $\Upsilon^{\ddagger}_\alpha(1)$ *denote GGC random variables with parameters* $(\alpha, F_{\mathbb{U}_{\alpha,0}})$ *and* $(\alpha, F_{Z_\alpha^{1/(\alpha+1)}})$, *respectively. Then*:

(i) $\Upsilon_\alpha(1) \stackrel{d}{=} G_1 M_1(F_{Y_\alpha\mathbb{U}_{\alpha,0}})$, *where* $M_1(F_{Y_\alpha\mathbb{U}_{\alpha,0}}) \stackrel{d}{=} \beta_{\alpha,1-\alpha}\mathbb{U}_{\alpha,\alpha}$ *has density, for* $0 < y < 1$,

$$\frac{\sin(\pi\alpha)}{\pi} \frac{(\alpha+1)y^{\alpha-1}(1-y)^{\alpha+1}}{[y^{2\alpha+2} + 2y^{\alpha+1}(1-y)^{\alpha+1}\cos(\pi\alpha) + (1-y)^{2\alpha+2}]}.$$

(ii) $\Upsilon^{\ddagger}_\alpha(1) \stackrel{d}{=} G_1 M_1(F_{Y_\alpha Z_\alpha^{1/(\alpha+1)}})$, *where* $M_1(F_{Y_\alpha Z_\alpha^{1/(\alpha+1)}})$ *has density*

$$\frac{\sin(\pi\alpha)}{\pi} \frac{x^{\alpha-1}(1+x)^\alpha}{[x^{2\alpha+2} + 2x^{\alpha+1}\cos(\pi\alpha) + 1]} \qquad \textit{for } x > 0.$$

(iii) *Supposing that* $\alpha \le 1/2$, *then the GGC*$(2\alpha, F_{Z_\alpha^{1/(\alpha+1)}})$ *random variable* $\Upsilon^{\ddagger}_{2\alpha}(1) \stackrel{d}{=} \Upsilon^{\ddagger}_\alpha(2)$ *is equivalent in distribution to* $G_1 M_1(F_{Y_{2\alpha} Z_\alpha^{1/(\alpha+1)}})$, *where* $M_1(F_{Y_{2\alpha} Z_\alpha^{1/(\alpha+1)}})$ *has density*

$$\frac{2x^{2\alpha-1}(x+1)^{2\alpha+1}\sin(\pi\alpha)[\cos(\pi\alpha) + x^{\alpha+1}]}{[x^{2\alpha+2} + 2x^{\alpha+1}\cos(\pi\alpha) + 1]^2} \qquad \textit{for } x > 0.$$

4.2. An example connected to Diaconis and Kemperman

Note that we have the following convergence in distribution results, as $\alpha \to 0$:

$$\tilde{M}_{\alpha,\theta}(F_U) = M_\theta(F_{\mathbb{U}_{\alpha,0}}) \stackrel{\mathrm{d}}{\to} M_\theta(F_U) \qquad \text{for } \theta > 0$$

and

$$\mathbb{U}_{\alpha,0} \stackrel{\mathrm{d}}{\to} 1 - U.$$

Furthermore, setting $W = (1-U)/U = G_1/G_1'$, we have

$$M_\theta(F_{Z_\alpha^{1/(\alpha+1)}}) \stackrel{\mathrm{d}}{\to} M_\theta(F_W) \quad \text{and} \quad Z_\alpha^{1/(\alpha+1)} \stackrel{\mathrm{d}}{\to} W,$$

where the last statement can be read from Chaumont and Yor [4], page 155 and page 169. It is then natural to investigate the laws of the random processes connected with the GGC(θ, F_U) and GGC(θ, F_W) random variables. It is known from Diaconis and Kemperman [9] that the density of $M_1(F_U)$ is

$$\frac{\mathrm{e}}{\pi} \sin(\pi y) y^{-y} (1-y)^{-(1-y)} \qquad \text{for } 0 < y < 1. \tag{4.4}$$

Note, furthermore, that $\tilde{T}_1 \stackrel{\mathrm{d}}{=} G_1 M_1(F_U)$ is GGC(1, F_U) and has Laplace transform

$$\mathbb{E}\big[\mathrm{e}^{-\lambda G_1 M_1(F_U)}\big] = \mathrm{e}^{-\psi_{F_U}(\lambda)} = \mathrm{e}(1+\lambda)^{-((\lambda+1)/\lambda)}.$$

Now, $G_1 M_1(F_W)$ is a GGC(1, F_W) with $\psi_{F_W}(\lambda) = \frac{\lambda}{\lambda-1}\log(\lambda)$. Theorem 3.1 shows that $M_1(F_U)$ arises from tilting the density of $G_1 M_1(F_W)$. The density of $M_1(F_W)$ is obtained by applying statement (ii) of Theorem 3.1 to (4.4), or by statement (ii) of Proposition 3.1, and is given by

$$\xi_{F_W}(x) = \frac{1}{\pi} \sin\left(\frac{\pi x}{1+x}\right) x^{-x/(1+x)} \qquad \text{for } x > 0.$$

We now apply Theorem 2.2 and Proposition 2.1 to give a description of the finite-dimensional distribution of the subordinators associated with the two random variables above.

Proposition 4.2. *Let U denote a uniform* [0,1] *random variable and let $W = G_1/G_1'$ denote a ratio of independent exponential* (1) *random variables.*

(i) *Suppose that $(\tilde{\zeta}_1(t); 0 < t < 1)$ is a GGC$(1, F_U)$ subordinator. Then, for $(C_1, \ldots, C_k)$, a disjoint partition of* $(0, 1)$, *the finite-dimensional distribution has joint density as in* (2.7), *with*

$$g_{\sigma_i, F_U}(x_i) = \int_0^1 \mathrm{e}^{-x_i/y} \frac{\mathrm{e}^{\sigma_i}}{\pi} \sin\big(\pi\sigma_i(1-y)\big) y^{\sigma_i(1-y)-2}(1-y)^{-\sigma_i(1-y)}\,\mathrm{d}y$$

for $i = 1, \ldots, k$.

(ii) *That is,* $\tilde{\zeta}_1(C_i) \stackrel{d}{=} G_1 M_1(F_{UY_{\sigma_i}})$ *and they are independent for* $i = 1, \ldots, k$. *Furthermore, the density of* $M_1(F_{UY_{\sigma_i}}) \stackrel{d}{=} \beta_{\sigma_i, 1-\sigma_i} M_{\sigma_i}(F_U)$ *is*

$$\frac{\mathrm{e}^{\sigma_i}}{\pi} \sin\bigl(\pi\sigma_i(1-y)\bigr) y^{\sigma_i(1-y)-1}(1-y)^{-\sigma_i(1-y)}$$

for $0 < y < 1$.

(iii) *If* $(\zeta_1(t); 0 < t < 1)$ *is a* $GGC(1, F_W)$ *subordinator, then the finite-dimensional distribution* $(\zeta_1(C_1), \ldots, \zeta_1(C_k))$ *is now described, with*

$$g_{\sigma_i, F_W}(x_i) = \int_0^\infty \mathrm{e}^{-x_i/w} \frac{1}{\pi} \sin\left(\frac{\pi\sigma_i}{1+w}\right) w^{\sigma_i/(1+w)-2}\,\mathrm{d}w.$$

(iv) *That is,* $\zeta_1(C_i) \stackrel{d}{=} G_1 M_1(F_{WY_{\sigma_i}})$ *and they are independent for* $i = 1, \ldots, k$. *Furthermore, the density of* $M_1(F_{WY_{\sigma_i}}) \stackrel{d}{=} \beta_{\sigma_i, 1-\sigma_i} M_{\sigma_i}(F_W)$ *is*

$$\frac{1}{\pi} \sin\left(\frac{\pi\sigma_i}{1+x}\right) x^{\sigma_i/(1+x)-1}$$

for $x > 0$.

Proof. This now follows from Theorem 2.2, Proposition 2.1 and (4.4). Specifically, note that $\mathcal{C}(F_U) = (0, \infty)$, so for any $0 < \sigma < 1$,

$$\sin(\pi F_{UY_\sigma}(u)) = \sin\bigl(\pi\sigma(1-u)\bigr)$$

for $0 < u < 1$ and 0 otherwise. Furthermore, from (4.4), or by direct argument, it is easy to see that

$$\Phi_{F_U}(y) = -\log\bigl(y^{-y}(1-y)^{-(1-y)}\bigr) - 1.$$

This fact is also evident from Diaconis and Kemperman [9]. It follows that $M_1(F_{UY_\sigma})$ has density

$$\frac{\mathrm{e}^{\sigma}}{\pi} \sin\bigl(\pi\sigma(1-y)\bigr) y^{\sigma(1-y)-1}(1-y)^{-\sigma(1-y)} \qquad \text{for } 0 < y < 1.$$

The density for $M_1(F_{WY_\sigma})$ is obtained in a similar fashion by Proposition 3.1. □

Remark 4.1. Setting

$$A_c \stackrel{\mathrm{d}}{=} \frac{cG_1}{cG_1 + G_1'},$$

one can easily obtain the density of the random variable $M_1(F_{A_c})$ for each $c > 0$ by using statement (ii) of Theorem 3.1. Note, also, that one can deduce from the density of $M_1(F_W)$ that

$\Phi_{F_W}(x) = [x/(1+x)]\log(x)$. Hence, in this case, an application of Proposition 3.2 shows that

$$\Phi_{F_{A_c}}(y) = \frac{y}{c(1-y)+y}\log\left(\frac{y}{c(1-y)}\right) - \frac{c\log(c)}{c-1} + \log\bigl(c(1-y)\bigr).$$

We note that, otherwise, it is not easy to calculate Φ_{A_c}, in this case, by direct arguments.

4.3. The finite-dimensional distribution of subordinators of Bertoin *et al.*

Our final example shows how one can apply the results in Sections 2 and 3 to obtain new results for subordinators recently studied by Bertoin *et al.* [2]. In particular, they investigate properties of the random variables corresponding to the lengths of excursions of Bessel processes straddling an independent exponential time, which can be expressed as

$$d_{\mathbf{e}}^{(\alpha)} - g_{\mathbf{e}}^{(\alpha)},$$

where, for any $t > 0$,

$$g_t^{(\alpha)} := \sup\{s \le t, R_s = 0\}, \qquad d_t^{(\alpha)} := \inf\{s \ge t, R_s = 0\} \tag{4.5}$$

for $(R_t, t \ge 0)$ a Bessel process starting from 0 with dimension $d = 2(1-\alpha)$, with $0 < d < 2$ or, equivalently, $0 < \alpha < 1$. Additionally, $\mathbf{e} \stackrel{\mathrm{d}}{=} G_1$, an exponentially distributed random variable with mean 1. See also Fujita and Yor [19] for closely related work.

In order to avoid confusion, we will now denote relevant random variables appearing originally as Δ_α and G_α in Bertoin *et al.* [2] as Σ_α and $\mathbb{G}_\alpha$, respectively. From Bertoin *et al.* [2], let $(\Sigma_\alpha(t); t > 0)$ denote a subordinator such that

$$\begin{aligned}\mathbb{E}\bigl[\mathrm{e}^{-\lambda\Sigma_\alpha(t)}\bigr] &= \bigl((\lambda+1)^\alpha - \lambda^\alpha\bigr)^t \\ &= \exp\bigl(-t(1-\alpha)\mathbb{E}[\log(1+\lambda/\mathbb{G}_\alpha)]\bigr),\end{aligned}$$

where, from Bertoin *et al.* [2], Theorems 1.1 and 1.3, $\mathbb{G}_\alpha$ denotes a random variable such that

$$\mathbb{G}_\alpha \stackrel{\mathrm{d}}{=} \frac{Z_{1-\alpha}^{1/\alpha}}{1+Z_{1-\alpha}^{1/\alpha}}$$

and has density on (0, 1) given by

$$f_{\mathbb{G}_\alpha}(u) = \frac{\alpha\sin(\pi\alpha)}{(1-\alpha)\pi}\frac{u^{\alpha-1}(1-u)^{\alpha-1}}{u^{2\alpha} - 2(1-u)^\alpha u^\alpha\cos(\pi\alpha) + (1-u)^{2\alpha}}.$$

Hence, it follows that the random variable $1/\mathbb{G}_\alpha$ takes its values on $(1, \infty)$ with probability one and has cdf satisfying

$$1 - F_{1/\mathbb{G}_\alpha}(x) = F_{\mathbb{G}_\alpha}(1/x) = F_{Z_{1-\alpha}}\bigl((x-1)^{-\alpha}\bigr).$$

As noted by Bertoin *et al.* [2], $(\Sigma_\alpha(t); t > 0)$ is a GGC$(1-\alpha, F_{1/\mathbb{G}_\alpha})$ subordinator, where the GGC$(1-\alpha, F_{1/\mathbb{G}_\alpha})$ random variable $\Sigma_\alpha \stackrel{\mathrm{d}}{=} \Sigma_\alpha(1)$ satisfies

$$\Sigma_\alpha \stackrel{\mathrm{d}}{=} d_{\mathbf{e}}^{(\alpha)} - g_{\mathbf{e}}^{(\alpha)} \stackrel{\mathrm{d}}{=} \frac{G_{1-\alpha}}{\beta_{\alpha,1}} \stackrel{\mathrm{d}}{=} \frac{G_{1-\alpha}}{U^{1/\alpha}},$$

where U denotes a uniform[0, 1] random variable and, for clarity, $G_{1-\alpha}$ is a gamma$(1-\alpha, 1)$ random variable. This means that the density of Σ_α is

$$\frac{\alpha}{\Gamma(1-\alpha)} x^{-\alpha-1}(1-\mathrm{e}^{-x}) \qquad \text{for } x > 0.$$

It is evident, as investigated in Fujita and Yor [18], that

$$M_{1-\alpha}(F_{1/\mathbb{G}_\alpha}) \stackrel{\mathrm{d}}{=} \frac{1}{\beta_{\alpha,1}} \stackrel{\mathrm{d}}{=} U^{-1/\alpha}.$$

Remark 4.2. Note that when $\alpha = 1/2$, $\mathbb{G}_{1/2} \stackrel{\mathrm{d}}{=} \beta_{1/2,1/2}$. It is known that for each fixed t,

$$\Sigma_{1/2}(t) \stackrel{\mathrm{d}}{=} \frac{G_{t/2}}{\beta_{1/2,(1+t)/2}},$$

which implies that

$$M_{t/2}(F_{1/\mathbb{G}_{1/2}}) = M_{t/2}(F_{1/\beta_{1/2,1/2}}) \stackrel{\mathrm{d}}{=} \frac{1}{\beta_{1/2,(1+t)/2}}. \tag{4.6}$$

This result may be found in James and Yor [27]. Related to this fact, Cifarelli and Melilli [7] have shown that $M_{t/2}(F_{\beta_{1/2,1/2}}) \stackrel{\mathrm{d}}{=} \beta_{(t+1)/2,(t+1)/2}$ for $t > 0$.

In regards to exponentially tilting GGC$(1-\alpha, F_{1/\mathbb{G}_\alpha})$, note that for $c > 0$,

$$\frac{c/\mathbb{G}_\alpha}{c/\mathbb{G}_\alpha + 1} = \frac{c}{\mathbb{G}_\alpha + c}.$$

Thus, a GGC$(1-\alpha, F_{c/(\mathbb{G}_\alpha+c)})$ subordinator, say $(\Sigma^{\dagger}_{\alpha,c}(t), t \le 1/(1-\alpha))$, arises from exponential tilting. Naturally, the density of $\Sigma^{\dagger}_{\alpha,c}(1)/c$ is given by

$$\frac{\alpha x^{-\alpha-1}\mathrm{e}^{-cx}(1-\mathrm{e}^{-x})}{[(c+1)^\alpha - c^\alpha]\Gamma(1-\alpha)} \qquad \text{for } x > 0.$$

Equivalently, $\Sigma^{\dagger}_{\alpha,c}(1) \stackrel{\mathrm{d}}{=} G_{1-\alpha} M_{1-\alpha}(F_{c/(\mathbb{G}_\alpha+c)})$, where $M_{1-\alpha}(F_{c/(\mathbb{G}_\alpha+c)})$ has density

$$\frac{\alpha c^\alpha}{(c+1)^\alpha - c^\alpha} u^{-\alpha-1} \qquad \text{for } \frac{c}{c+1} < u < 1.$$

Now, using the facts discussed above, we will show how to use the results in Section 2 to explicitly describe the finite-dimensional distribution of the subordinators $(\Sigma_\alpha(t), t > 0)$ and $(\Sigma^\dagger_{\alpha,c}(t), t > 0)$ over the range $0 < t \le 1/(1-\alpha)$. Additionally, the analysis will also yield expressions for mean functionals based on $F_{1/\mathbb{G}_\alpha}$. First, note that, using (2.4), one has

$$\sin(\pi F_{Y_{1-\alpha}/\mathbb{G}_\alpha}(x)) = \begin{cases} \sin\bigl(\pi(1-\alpha)F_{\mathbb{G}_\alpha}(1/x)\bigr), & \text{if } x > 1, \\ \sin\bigl(\pi(1-\alpha)\bigr), & \text{if } 0 < x \le 1, \end{cases} \tag{4.7}$$

where, again using the properties of $F_{Z_{1-\alpha}}$, as deduced from James [24], Proposition 2.1(iii),

$$\sin\bigl(\pi(1-\alpha)F_{\mathbb{G}_\alpha}(1/x)\bigr) = \frac{\sin(\pi(1-\alpha))}{[(x-1)^{2\alpha} - 2(x-1)^\alpha \cos(\pi\alpha) + 1]^{1/2}}. \tag{4.8}$$

We now use this to calculate

$$\Phi_{F_{1/\mathbb{G}_\alpha}}(x) = \mathbb{E}[\log(|x - 1/\mathbb{G}_\alpha|)\mathbb{I}(x \ne 1/\mathbb{G}_\alpha)]. \tag{4.9}$$

Proposition 4.3. *For* $0 < \alpha < 1$, *consider* $\Phi_{F_{1/\mathbb{G}_\alpha}}(x)$ *as defined in* (4.9). *Then*,

$$\begin{aligned} &\Phi_{F_{1/\mathbb{G}_\alpha}}(x) \\ &\quad = \begin{cases} \dfrac{1}{2(1-\alpha)}\left[\log\left(\dfrac{x^2}{[(x-1)^{2\alpha} - 2(x-1)^\alpha\cos(\pi\alpha) + 1]}\right)\right], & \textit{if } x > 1, \\ \dfrac{1}{1-\alpha}\log\bigl(x/[1-(1-x)^\alpha]\bigr), & \textit{if } 0 < x \le 1. \end{cases} \end{aligned} \tag{4.10}$$

Proof. Using simple beta–gamma algebra, we have

$$\Sigma_\alpha \stackrel{d}{=} \frac{G_{1-\alpha}}{\beta_{\alpha,1}} \stackrel{d}{=} G_1 \frac{\beta_{1-\alpha,\alpha}}{U^{1/\alpha}}.$$

Hence, applying Theorem 2.1, with $\theta = 1$ and $\sigma = 1 - \alpha$, it follows that Σ_α is also $\mathrm{GGC}(1, F_{Y_{1-\alpha}/\mathbb{G}_\alpha})$ and

$$B_\alpha := \frac{\beta_{1-\alpha,\alpha}}{\beta_{\alpha,1}} \stackrel{d}{=} \frac{\beta_{1-\alpha,\alpha}}{U^{1/\alpha}} \stackrel{d}{=} M_1(F_{Y_{1-\alpha}/\mathbb{G}_\alpha}). \tag{4.11}$$

By standard calculations, the density of $B_\alpha = \beta_{1-\alpha,\alpha}/\beta_{\alpha,1}$ is given by

$$f_{B_\alpha}(x) = \frac{\sin(\pi(1-\alpha))}{\pi} x^{-\alpha-1}[1 - (1-x)^\alpha \mathbb{I}(x \le 1)].$$

However, we see from (4.11) that $B_\alpha \stackrel{d}{=} M_1(F_{Y_{1-\alpha}/\mathbb{G}_\alpha})$. Hence, Theorem 2.2 applies and the density of B_α can be written as

$$f_{B_\alpha}(x) = \frac{x^{-\alpha}}{\pi} \sin(\pi F_{Y_{1-\alpha}/\mathbb{G}_\alpha}(x)) \mathrm{e}^{-(1-\alpha)\Phi_{F_{1/\mathbb{G}_\alpha}}(x)}.$$

Now, equating the two forms of the density of B_α and using (4.7) and (4.8), one then obtains the expression for $\Phi_{F_{1/\mathbb{G}_\alpha}}$. □

Now, for $z > 0$, define the function

$$\mathcal{S}_{\alpha,\sigma}(z) = \sin(\pi\sigma F_{Z_{1-\alpha}}(z^{-\alpha}))[z^{2\alpha} - 2z^\alpha \cos(\pi\alpha) + 1]^{\sigma/(2(1-\alpha))}$$

and define,

$$\mathcal{D}_{\alpha,\sigma}(x) = \begin{cases} \sin(\pi\sigma)[1 - (1-x)^\alpha]^{\sigma/(1-\alpha)}, & \text{if } x \le 1, \\ \mathcal{S}_{\alpha,\sigma}(x-1), & \text{if } x > 1. \end{cases}$$

Hereafter, $(C_1, \ldots, C_k)$ will denote an arbitrary disjoint partition of the interval $(0, 1/(1-\alpha)]$ with lengths $|C_i|$, and $\sigma_i = (1-\alpha)|C_i|$ for $i = 1, \ldots, k$.

Theorem 4.2. *Consider the GGC*$(1-\alpha, F_{1/\mathbb{G}_\alpha})$ *subordinator* $(\Sigma_\alpha(t), t \le 1/(1-\alpha))$ *and, for each fixed* $c > 0$, *the GGC*$(1-\alpha, F_{c/(\mathbb{G}_\alpha+c)}$ *subordinator* $(\Sigma^\dagger_{\alpha,c}(t), t \le 1/(1-\alpha))$. *The following results then hold*:

(i) *The finite-dimensional distribution of* $(\Sigma_\alpha(C_1), \ldots, \Sigma_\alpha(C_k))$ *is such that each* $\Sigma_\alpha(C_i)$ *is independent and has distribution*

$$\Sigma_\alpha(C_i) \stackrel{d}{=} G_1 M_1(F_{Y_{\sigma_i}/\mathbb{G}_\alpha}),$$

where $M_1(F_{Y_{\sigma_i}/\mathbb{G}_\alpha}) \stackrel{d}{=} \beta_{\sigma_i,1-\sigma_i} M_{\sigma_i}(F_{1/\mathbb{G}_\alpha})$. *Furthermore, for any fixed* $0 < \sigma \le 1$, *the density of* $M_1(F_{Y_\sigma/\mathbb{G}_\alpha})$ *is given by*

$$\frac{1}{\pi} x^{-(\sigma\alpha/(1-\alpha)+1)} \mathcal{D}_{\alpha,\sigma}(x) \qquad \text{for } x > 0.$$

(ii) *For the GGC*$(1-\alpha, F_{c/(\mathbb{G}_\alpha+c)})$ *process*, $\Sigma^\dagger_{\alpha,c}$, *it follows that each*

$$\Sigma^\dagger_{\alpha,c}(C_i) \stackrel{d}{=} G_1 M_1\big(F_{Y_{\sigma_i} c/(\mathbb{G}_\alpha+c)}\big),$$

where for each $0 < \sigma \le 1$, $M_1(F_{Y_\sigma c/(\mathbb{G}_\alpha+c)})$ *has density*

$$\frac{[c(1-y)]^{\sigma\alpha/(1-\alpha)} \mathcal{D}_\alpha(y/(c(1-y)))}{\pi[(c+1)^\alpha - c^\alpha]^\sigma y^{\sigma\alpha/(1-\alpha)+1}} \qquad \text{for } 0 < y < 1.$$

Proof. From Theorem 2.2, we have that the general form of the density of $M_1(F_{Y_\sigma/\mathbb{G}_\alpha})$ is given by

$$\frac{x^{\sigma-1}}{\pi} \sin(\pi F_{Y_\sigma/\mathbb{G}_\alpha}(x)) e^{-\sigma\Phi_{F_{1/\mathbb{G}_\alpha}}(x)}.$$

The proof is completed by applying Proposition 4.3 and (4.7) and (4.8). □

Remark 4.3. The process $\Sigma_{\alpha,c}(t)/c$ is well defined for $c \geq 0$ and $0 \leq \alpha < 1$, and presents itself as an interesting class worthy of further investigation. Letting $c \to 0$, it is evident that $\Sigma^{\dagger}_{\alpha,c}(1)/c$ converges to $\Sigma_\alpha(1)$. As shown by Bertoin *et al.* [2], Section 3.6.3, $\Sigma_{0,c}(1)/c$, for $c > 0$, has a similar interpretation as $\Sigma_\alpha(1)$, but where the Bessel process $(R_t, t > 0)$ is now replaced by a diffusion process whose inverse local time at 0 is distributed as a gamma subordinator $(\gamma_l/c; l > 0)$. Furthermore, albeit not explicitly addressed in Bertoin *et al.* [2], the random variable $\Sigma_{\alpha,c}(1)/c \stackrel{\mathrm{d}}{=} d^{(\alpha,c)}_{\mathbf{e}} - g^{(\alpha,c)}_{\mathbf{e}}$ has a similar interpretation where $(R_t, t > 0)$ is now replaced by a process $(R^{(\alpha,c)}_t, t > 0)$ whose inverse local time is distributed as a generalized gamma subordinator, that is, a subordinator whose Lévy density is specified by $Cy^{-\alpha-1}\mathrm{e}^{-cy}$ for $y > 0$. This interpretation may be deduced from Donati-Martin and Yor ([10], see page 880 (1.c)), where $R^{(\alpha,c)}$ equates with a downwards Bessel process with drift c.

Bertoin *et al.* [2] also show that a GGC $(1-\alpha, F_{\mathbb{G}_\alpha})$ random variable satisfies

$$G_{1-\alpha} M_{1-\alpha}(F_{\mathbb{G}_\alpha}) = G_{1-\alpha} U.$$

Hence, the Laplace transform of the GGC $(1-\alpha, F_{\mathbb{G}_\alpha})$ subordinator, say $(\mathcal{Z}^{\dagger}_{\alpha,1}(t), t > 0)$, is given by

$$\left(\frac{1}{\alpha\lambda}[(\lambda+1)^\alpha - 1]\right)^t.$$

Additionally, using the fact that

$$\frac{1}{\mathbb{G}_\alpha} \stackrel{\mathrm{d}}{=} \frac{1}{Z^{1/\alpha}_{1-\alpha}} + 1 \stackrel{\mathrm{d}}{=} Z^{1/\alpha}_{1-\alpha} + 1 \tag{4.12}$$

leads to

$$M_{1-\alpha}(F_{Z^{1/\alpha}_{1-\alpha}}) \stackrel{\mathrm{d}}{=} M_{1-\alpha}(F_{1/\mathbb{G}_\alpha}) - 1 \stackrel{\mathrm{d}}{=} \frac{G_1}{G_\alpha},$$

which leads to a description of a GGC$(1-\alpha, F_{Z^{1/\alpha}_{1-\alpha}})$ subordinator. The above points may also be found in the survey paper of James, Roynette and Yor [26].

Theorem 4.3. *Consider the GGC*$(1-\alpha, F_{Z^{1/\alpha}_{1-\alpha}})$ *subordinator* $(\mathcal{Z}_\alpha(t), t \leq 1/(1-\alpha))$ *and the GGC*$(1-\alpha, F_{\mathbb{G}_\alpha})$ *subordinator* $(\mathcal{Z}^{\dagger}_{\alpha,1}(t), t \leq 1/(1-\alpha))$. *The following results then hold*:

(i) *The finite-dimensional distribution of* $(\mathcal{Z}_\alpha(C_1), \ldots, \mathcal{Z}_\alpha(C_k))$ *is such that each* $\mathcal{Z}_\alpha(C_i)$ *is independent and is equivalent in distribution to*

$$\mathcal{Z}_\alpha(C_i) \stackrel{d}{=} G_1 M_1(F_{Y_{\sigma_i} Z^{1/\alpha}_{1-\alpha}}).$$

Furthermore, for any fixed $0 < \sigma \leq 1$, *the density of* $M_1(F_{Y_\sigma Z_{1-\alpha}^{1/\alpha}}) \stackrel{d}{=} \beta_{\sigma,1-\sigma} M_\sigma(F_{Z_{1-\alpha}^{1/\alpha}})$ *is given by, for* $z > 0$,

$$\frac{z^{\sigma-1}}{\pi(1+z)^{\sigma/(1-\alpha)}} \mathcal{S}_{\alpha,\sigma}(z).$$

(ii) *Similarly, each* $\mathcal{Z}_\alpha^\dagger(C_i) \stackrel{d}{=} G_1 M_1(F_{Y_{\sigma_i}\mathbb{G}_\alpha})$ *and, for each fixed* $0 < \sigma \leq 1$, $M_1(F_{Y_\sigma \mathbb{G}_\alpha})$ *has density*

$$\frac{\alpha^{\sigma/(1-\alpha)}}{\pi} y^{\sigma-1}(1-y)^{\sigma\alpha/(1-\alpha)} \mathcal{S}_{\alpha,\sigma}\left(\frac{y}{1-y}\right).$$

Proof. Apply Theorem 2.2 and Theorem 3.1, where, from (4.12),

$$\Phi_{F_{Z_{1-\alpha}^{1/\alpha}}}(z) = \Phi_{F_{1/\mathbb{G}_\alpha}}(z+1).$$

□

Remark 4.4. Note that as $\alpha \to 1$,

$$M_\theta(F_{\mathbb{G}_\alpha}) \stackrel{\mathrm{d}}{\to} M_\theta(F_U) \quad \text{and} \quad M_\theta(F_{Z_{1-\alpha}^{1/\alpha}}) \stackrel{\mathrm{d}}{\to} M_\theta(F_W).$$

Hence, they have the same limiting behavior, described in Section 4.2, as the random variables in Section 4.1.

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