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Absolute continuity for some one-dimensional processes

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We introduce an elementary method for proving the absolute continuity of the time marginals of one-dimensional processes. It is based on a comparison between the Fourier transform of such time marginals with those of the one-step Euler approximation of the underlying process. We obtain some absolute continuity results for stochastic differential equations with Hölder continuous coefficients. Furthermore, we allow such coefficients to be random and to depend on the whole path of the solution. We also show how it can be extended to some stochastic partial differential equations and to some Lévy-driven stochastic differential equations. In the cases under study, the Malliavin calculus cannot be used, because the solution in generally not Malliavin differentiable.

Keywords: absolute continuity; Hölder coefficients; Lévy processes; random coefficients; stochastic differential equations; stochastic partial differential equations

1. Introduction

In this paper, we introduce a new method for proving the absolute continuity of the time marginals of some one-dimensional processes. The main idea is elementary and quite rough. It is based on the explicit law of the associated one-step Euler scheme and related to an estimate which says that the process and its Euler scheme remain very close to each other during one step.

As we will see, this method is quite robust and applies to many processes for which the use of the Malliavin calculus (see Nualart [23], Malliavin [21]) is not possible because the processes do not have Malliavin derivatives: examples of this include SDEs with Hölder coefficients and SDEs with random coefficients.

However, we are not able, for the moment, to extend it to multidimensional processes. The difficulty seems to be that we use some integrability properties of some Fourier transforms which depend heavily on the dimension.

To illustrate this method, we will consider four types of one-dimensional processes. Let us summarize roughly the results we obtain and compare them to existing results.

Brownian SDEs with Hölder coefficients

To introduce our method in a simple way, we consider a process satisfying an SDE of the form $\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}B_t + b(X_t)\,\mathrm{d}t$. We assume that b is measurable with at most linear growth and that σ is Hölder continuous with exponent $\theta > 1/2$. We show that X_t has a density on $\{\sigma \neq 0\}$ whenever $t > 0$. The proof is very short.

Such a result is probably not far from being already known. In the case where σ is bounded below, Aronson [1] obtains some absolute continuity results assuming only that σ and b are measurable (together with some growth conditions) by analytical methods. Our result might be deduced from [1] by a localization argument, however, we did not succeed in this direction. In any case, our proof is much simpler.

Let us observe that, to our knowledge, all of the probabilistic papers on this topic assume at least that σ, b are Lipschitz continuous; see the paper by Bouleau and Hirsch [8] (the case where b is measurable can also be treated by using Girsanov's theorem).

Finally, let us mention that in [8], one gets the absolute continuity of the law of X_t for all $t > 0$ provided $\sigma(x_0) \neq 0$, if $X_0 = x_0$. Such a result cannot hold in full generality for Hölder continuous coefficients: choose $x_0 > 0$, $\sigma(x) = x$ and $b(x) = -\operatorname{sign}(x)|x|^\alpha$ for some $\alpha \in (0, 1)$. Let $\tau_\varepsilon = \inf\{t \geq 0, X_t = \varepsilon\}$ for $\varepsilon \in \mathbb{R}_+$. One can check, using Itô's formula, that for $\varepsilon \in (0, x_0)$, $\mathbb{E}[X_{t\wedge\tau_\varepsilon}^{1-\alpha}] = x_0^{1-\alpha} - \mathbb{E}[\int_0^{t\wedge\tau_\varepsilon} (\frac{\alpha(1-\alpha)}{2} X_s^{1-\alpha} + (1-\alpha))\,\mathrm{d}s] \leq x_0^{1-\alpha} - (1-\alpha)\mathbb{E}[\tau_\varepsilon \wedge t]$, whence $\mathbb{E}[\tau_\varepsilon] \leq x_0^{1-\alpha}/(1-\alpha)$. As a consequence, $\mathbb{E}[\tau_0] \leq x_0^{1-\alpha}/(1-\alpha)$. But it also holds that $X_{\tau_0+t} = 0$ a.s. for all $t \geq 0$. Thus, $\Pr[X_t = 0] > 0$, at least for sufficiently large t.

Brownian SDEs with random coefficients depending on the paths

We consider here a process solving an SDE of the form $\mathrm{d}X_t = \sigma(X_t)\kappa(t, (X_u)_{u\leq t}, H_t)\,\mathrm{d}B_t + b(t, (X_u)_{u\leq t}, H_t)\,\mathrm{d}t$ for some auxiliary adapted process H. We assume some Hölder conditions on $\sigma\kappa$, some growth conditions and that κ is bounded below. We prove the absolute continuity of the law of X_t on the set $\{\sigma \neq 0\}$ for all $t > 0$.

Observe that we do not assume that H is Malliavin differentiable, which would, of course, be needed if we wanted to use Malliavin calculus.

SDEs with random coefficients arise, for example, in finance. Indeed, stochastic volatility models are now widely used; see, for example, Heston [14], Fouque, Papanicolaou and Sircar [11]. SDEs with coefficients depending on the paths of the solutions arise in random mechanics: if one writes an SDE satisfied by the velocity of a particle, the coefficients will often depend on its position, which is nothing but the integral of its velocity. One can also imagine a particle with position X_t whose diffusion and drift coefficients depend on the distance covered by the particle at time t, that is, $\sup_{[0,t]} X_s - \inf_{[0,t]} X_s$.

Here, again, the result is not far from being known: if $\sigma\kappa$ is bounded below, one may use the result of Gyongy [13] which says that the solution of an SDE (with random coefficients depending on the whole paths of the solution) has the same time marginals as the solution of an SDE with deterministic coefficients depending only on time and position. These coefficients being measurable and uniformly elliptic, one may then use the result of Aronson [1]. However, our method is extremely simple and we do not have to assume that σ is bounded below.

Stochastic heat equation

We also study the heat equation $\partial_t U = \partial_{xx} U + b(U) + \sigma(U)\dot{W}$ on $\mathbb{R}_+ \times [0, 1]$, with Neumann boundary conditions, where W is a space–time white noise; see Walsh [26]. We prove that $U(t,x)$ has a density on $\{\sigma \neq 0\}$ for all $t>0$ and all $x \in [0,1]$, provided that σ is Hölder continuous with exponent $\theta > 1/2$ and that b is measurable and has at most linear growth.

This result shows the robustness of our method: the best absolute continuity result was due to Pardoux and Zhang [24], who assume that b and σ are Lipschitz continuous. Let us, however, mention that their non-degeneracy condition is very sharp since they obtain the absolute continuity of $U(t,x)$ for all $t>0$ and all $x \in [0,1]$, assuming only that $\sigma(U(0,x_0)) \neq 0$ for some $x_0 \in [0,1]$ (if $U(0,\cdot)$ is continuous).

Lévy-driven SDEs

We finally consider the SDE $\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}L_t + b(X_t)\,\mathrm{d}t$, where $(L_t)_{t\geq 0}$ is a Lévy martingale process without Brownian part and with Lévy measure ν. Roughly, we assume that $\int_{|z|\leq\varepsilon} z^2\nu(\mathrm{d}z) \simeq \varepsilon^{2-\lambda}$ for all $\varepsilon \in (0,1]$ and some $\lambda \in (3/4,2)$. We obtain that the law of X_t has a density on $\{\sigma \neq 0\}$ for all $t>0$, under the following assumptions:

(a) if $\lambda \in (3/2,2)$, then b is measurable and has at most linear growth and σ is Hölder continuous with exponent $\theta > 1/2$;

(b) if $\lambda \in [1,3/2]$, then b and σ are Hölder continuous with exponents $\alpha > 3/2-\lambda$ and $\theta > 1/2$;

(c) if $\lambda \in (3/4,1)$, then b,σ are Hölder continuous with exponent $\theta > 3/(2\lambda)-1$.

This result appears to be the first absolute continuity result for jumping SDEs with non-Lipschitz coefficients. Observe that, in some cases, we allow the drift coefficient to be only measurable, even when the driving Lévy process has no Brownian part. Such a result cannot be obtained using a trick like Girsanov's theorem (because even the law of such a Lévy process $(L_t)_{t\in[0,1]}$ and that of $(L_t+t)_{t\in[0,1]}$ are clearly not equivalent). To our knowledge, this gives the first absolute continuity result for Lévy-driven SDEs with measurable drift.

Also, observe that we allow the intensity measure of the Poissonian part to be singular: even without Brownian part and without drift, our result yields some absolute continuity for Lévy-driven SDEs, even when the Lévy measure of the driving process is completely singular. Such cases are not included in the famous works of Bichteler and Jacod [7] or Bichteler, Gravereaux and Jacod [6]. Picard [25] obtained some very complete results in that direction for SDEs with smooth coefficients. Note that Picard obtained his results for any $\lambda \in (0,2)$: our assumption is quite heavy since we have to restrict our study to the case where $\lambda > 3/4$.

Ishikawa and Kunita [15] have obtained some regularity results under some very simple assumptions for a different type of jumping SDE, namely *canonical SDEs with jumps*; see [15], formula (6.1).

Let us finally mention a completely different approach developed by Denis [10], Nourdin and Simon [22], Bally [2], Kulik [19,20] and others, where singular Lévy measures are allowed when the drift coefficient is sufficiently non-constant. The case under study is truly different since we allow the drift coefficient to be completely degenerate.

We will frequently use the following classical lemma.

Lemma 1.1. *For μ a non-negative finite measure on $\mathbb{R}$, we denote by $\widehat{\mu}(\xi) = \int_{\mathbb{R}} e^{i\xi x} \mu(dx)$ its Fourier transform (for all $\xi \in \mathbb{R}$). If $\int_{\mathbb{R}} |\widehat{\mu}(\xi)|^2 \, d\xi < \infty$, then μ has a density with respect to the Lebesgue measure.*

Proof. For $n \geq 1$, consider $\mu_n = \mu \star g_n$, where g_n is the centered Gaussian distribution with variance $1/n$. Then, of course, $|\widehat{\mu}_n(\xi)| \leq |\widehat{\mu}(\xi)|$. Furthermore, μ_n has a density $f_n \in L^1 \cap L^\infty(\mathbb{R}, dx)$ (for each fixed $n \geq 1$), so we may apply the Plancherel equality, which yields $\int_{\mathbb{R}} f_n^2(x) \, dx = (2\pi)^{-1} \int_{\mathbb{R}} |\widehat{\mu}_n(\xi)|^2 \, d\xi \leq (2\pi)^{-1} \int_{\mathbb{R}} |\widehat{\mu}(\xi)|^2 \, d\xi =: C < \infty$. Due to the weak compactness of the balls of $L^2(\mathbb{R}, dx)$, we may extract a subsequence n_k and find a function $f \in L^2(\mathbb{R}, dx)$ such that f_{n_k} goes weakly in $L^2(\mathbb{R}, dx)$ to f. But, on the other hand, $\mu_n(dx) = f_n(x) \, dx$ tends weakly (in the sense of measures) to μ. As a consequence, μ is nothing but $f(x) \, dx$. □

Observe here that this lemma is optimal. Indeed, the fact that $\widehat{\mu}$ belongs to L^p with $p > 2$ does not imply that μ has a density; see counterexamples in Kahane and Salem [17]. The following localization argument will also be of constant use.

Lemma 1.2. *For $\delta > 0$, we introduce a function $f_\delta : \mathbb{R}_+ \mapsto [0, 1]$, vanishing on $[0, \delta]$, positive on (δ, ∞) and globally Lipschitz continuous (with Lipschitz constant 1).*

Consider a probability measure μ on $\mathbb{R}$ and a function $\sigma : \mathbb{R} \mapsto \mathbb{R}_+$. Assume that for each $\delta > 0$, the measure $\mu_\delta(dx) = f_\delta(\sigma(x))\mu(dx)$ has a density. Thus, μ has a density on $\{x \in \mathbb{R}, \sigma(x) > 0\}$.

Proof. Let $A \subset \mathbb{R}$ be a Borel set with Lebesgue measure 0. We have to prove that $\mu(A \cap \{\sigma > 0\}) = 0$. For each $\delta > 0$, the measures $\mathbf{1}_{\{\sigma(x) > \delta\}} \mu(dx)$ and $\mu_\delta(dx)$ are clearly equivalent. By assumption, $\mu_\delta(A) = 0$ for each $\delta > 0$, whence $\mu(A \cap \{\sigma > \delta\}) = 0$. Hence, $\mu(A \cap \{\sigma > 0\}) = \lim_{\delta \to 0} \mu(A \cap \{\sigma > \delta\}) = 0$. □

The sections of this paper are almost independent. In Section 2, we consider the case of simple Brownian SDEs. Section 3 is devoted to Brownian SDEs with random coefficients depending on the whole path of the solution. The stochastic heat equation is treated in Section 4. Finally, we consider some Lévy-driven SDEs in Section 5.

2. Simple Brownian SDEs

We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ and an $(\mathcal{F}_t)_{t \geq 0}$-Brownian motion $(B_t)_{t \geq 0}$. For $x \in \mathbb{R}$ and $\sigma, b : \mathbb{R} \mapsto \mathbb{R}$, we consider the one-dimensional SDE

$$X_t = x + \int_0^t \sigma(X_s) \, dB_s + \int_0^t b(X_s) \, ds. \tag{2.1}$$

Our aim in this section is to prove the following result.

Theorem 2.1. *Assume that σ is Hölder continuous with exponent $\theta \in (1/2, 1]$ and that b is measurable and has at most linear growth. Consider a continuous $(\mathcal{F}_t)_{t \geq 0}$-adapted solution $(X_t)_{t \geq 0}$ to (2.1). Then, for all $t > 0$, the law of X_t has a density on the set $\{x \in \mathbb{R}, \sigma(x) \neq 0\}$.*

Observe that the (weak or strong) existence of solutions to (2.1) does not hold under the assumptions of Theorem 2.1. However, at least weak existence holds if one additionally assumes that b is continuous or that σ is bounded below; see Karatzas and Shreve [18].

Proof. By a scaling argument, it suffices to consider the case $t = 1$. We divide the proof into three parts.

Step 1. For every $\varepsilon \in (0, 1)$, we consider the random variable

$$Z_\varepsilon := X_{1-\varepsilon} + \int_{1-\varepsilon}^{1} \sigma(X_{1-\varepsilon})\,\mathrm{d}B_s = X_{1-\varepsilon} + \sigma(X_{1-\varepsilon})(B_1 - B_{1-\varepsilon}).$$

Conditioning with respect to $\mathcal{F}_{1-\varepsilon}$, we get, for all $\xi \in \mathbb{R}$,

$$|\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi Z_\varepsilon}|\mathcal{F}_{1-\varepsilon}]| = \big|\exp\big(\mathrm{i}\xi X_{1-\varepsilon} - \varepsilon\sigma^2(X_{1-\varepsilon})\xi^2/2\big)\big| = \exp\big(-\varepsilon\sigma^2(X_{1-\varepsilon})\xi^2/2\big).$$

Step 2. Using classical arguments (Doob's inequality and Gronwall's lemma) and the fact that σ and b have at most linear growth, one may show that there exists a constant C such that for all $0 \le s \le t \le 1$,

$$\mathbb{E}\Big[\sup_{[0,1]} X_t^2\Big] \le C, \qquad \mathbb{E}[(X_t - X_s)^2] \le C(t-s). \tag{2.2}$$

Next, since σ is Hölder continuous with index $\theta \in (1/2, 1]$ and since b has at most linear growth, we get, for all $\varepsilon \in (0, 1)$,

$$\begin{aligned}
\mathbb{E}[(X_1 - Z_\varepsilon)^2] &\le 2\int_{1-\varepsilon}^{1} \mathbb{E}\big[\big(\sigma(X_s) - \sigma(X_{1-\varepsilon})\big)^2\big]\,\mathrm{d}s + 2\mathbb{E}\bigg[\bigg(\int_{1-\varepsilon}^{1} b(X_s)\,\mathrm{d}s\bigg)^2\bigg] \\
&\le C\int_{1-\varepsilon}^{1} \mathbb{E}[|X_s - X_{1-\varepsilon}|^{2\theta}]\,\mathrm{d}s + 2\varepsilon\int_{1-\varepsilon}^{1} \mathbb{E}[b^2(X_s)]\,\mathrm{d}s \\
&\le C\int_{1-\varepsilon}^{1} \mathbb{E}[|X_s - X_{1-\varepsilon}|^2]^\theta\,\mathrm{d}s + C\varepsilon\int_{1-\varepsilon}^{1} \mathbb{E}[1 + X_s^2]\,\mathrm{d}s \\
&\le C\varepsilon^{1+\theta} + C\varepsilon^2 \le C\varepsilon^{1+\theta},
\end{aligned}$$

where we have used (2.2).

Step 3. Let $\delta > 0$ be fixed, consider the function f_δ defined in Lemma 1.2 and the measure $\mu_{\delta,X_1}(\mathrm{d}x) = f_\delta(|\sigma(x)|)\mu_{X_1}(\mathrm{d}x)$, where μ_{X_1} is the law of X_1. Then, for all $\xi \in \mathbb{R}$, all $\varepsilon \in (0, 1)$, we may write

$$\begin{aligned}
|\widehat{\mu_{\delta,X_1}}(\xi)| &= |\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi X_1} f_\delta(|\sigma(X_1)|)]| \\
&\le |\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi X_1} f_\delta(|\sigma(X_{1-\varepsilon})|)]| + \mathbb{E}\big[\big|f_\delta(|\sigma(X_1)|) - f_\delta(|\sigma(X_{1-\varepsilon})|)\big|\big] \\
&\le |\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi Z_\varepsilon} f_\delta(|\sigma(X_{1-\varepsilon})|)]| + |\xi|\mathbb{E}[|X_1 - Z_\varepsilon|] \\
&\quad + \mathbb{E}\big[\big|f_\delta(|\sigma(X_1)|) - f_\delta(|\sigma(X_{1-\varepsilon})|)\big|\big],
\end{aligned}$$

where we used the inequality $|e^{i\xi x} - e^{i\xi z}| \le |\xi| \cdot |x - z|$ and the fact that f_δ is bounded by 1. First, Step 1 implies that

$$\begin{aligned}|\mathbb{E}[e^{i\xi Z_\varepsilon} f_\delta(|\sigma(X_{1-\varepsilon})|)]| &\le \mathbb{E}[|\mathbb{E}[e^{i\xi Z_\varepsilon} f_\delta(|\sigma(X_{1-\varepsilon})|)|\mathcal{F}_{1-\varepsilon}]|] \\ &\le \mathbb{E}\big[f_\delta(|\sigma(X_{1-\varepsilon})|)e^{-\varepsilon\sigma^2(X_{1-\varepsilon})\xi^2/2}\big] \le \exp(-\varepsilon\delta^2\xi^2/2)\end{aligned}$$

since f_δ is bounded by 1 and vanishes on $[0,\delta]$. Step 2 implies that $|\xi|\mathbb{E}[|X_1 - Z_\varepsilon|] \le C|\xi|\varepsilon^{(1+\theta)/2}$. Since f_δ is Lipschitz continuous and σ is Hölder continuous with index $\theta \in (1/2, 1]$, we deduce from (2.2) that $\mathbb{E}[|f_\delta(|\sigma(X_1)|) - f_\delta(|\sigma(X_{1-\varepsilon})|)|] \le C\mathbb{E}[|X_1 - X_{1-\varepsilon}|^\theta] \le C\varepsilon^{\theta/2}$.

As a conclusion, we deduce that for all $\xi \in \mathbb{R}$ and all $\varepsilon \in (0,1)$,

$$|\widehat{\mu_{\delta,X_1}}(\xi)| \le \exp(-\varepsilon\delta^2\xi^2/2) + C|\xi|\varepsilon^{(1+\theta)/2} + C\varepsilon^{\theta/2}.$$

For each $|\xi| \ge 1$ fixed, we apply this formula with the choice $\varepsilon := (\log|\xi|)^2/\xi^2 \in (0,1)$. This gives

$$|\widehat{\mu_{\delta,X_1}}(\xi)| \le \exp\big(-\delta^2(\log|\xi|)^2/2\big) + C(\log|\xi|)^{1+\theta}/|\xi|^\theta + C(\log|\xi|)^\theta/|\xi|^\theta.$$

This holding for all $|\xi| \ge 1$, and $\widehat{\mu_{\delta,X_1}}$ being bounded by 1, we get that $\int_{\mathbb{R}} |\widehat{\mu_{\delta,X_1}}(\xi)|^2\,d\xi < \infty$ since $\theta > 1/2$, by assumption. Lemma 1.1 implies that the measure μ_{δ,X_1} has a density for each $\delta > 0$. Lemma 1.2 allows us to conclude that μ_{X_1} has a density on $\{|\sigma| > 0\}$. □

3. Brownian SDEs with random coefficients

We again start with a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, P)$ and a $(\mathcal{F}_t)_{t\ge0}$-Brownian motion $(B_t)_{t\ge0}$.

To model the randomness of the coefficients, we consider an auxiliary predictable process $(H_t)_{t\ge0}$, with values in some normed space $(\mathcal{S}, \|\cdot\|)$. We then consider $\sigma : \mathbb{R} \mapsto \mathbb{R}$ and two measurable maps $\kappa, b : \mathcal{A} \mapsto \mathbb{R}$, where

$$\mathcal{A} := \{(s, (x_u)_{u\le s}, h), s \ge 0, (x_u)_{u\ge0} \in C(\mathbb{R}_+, \mathbb{R}), h \in \mathcal{S}\},$$

and the following one-dimensional SDE:

$$X_t = x + \int_0^t \sigma(X_s)\kappa(s, (X_u)_{u\le s}, H_s)\,dB_s + \int_0^t b(s, (X_u)_{u\le s}, H_s)\,ds. \tag{3.1}$$

Here, again, the existence of solutions to such a general equation does not, of course, always hold, even under the assumptions below. However, there are many particular cases for which the (weak or strong) existence can be proven by classical methods (Picard iteration, martingale problems, change of probability, change of time, etcetera).

Theorem 3.1. *Assume that the auxiliary process H satisfies, for some $\eta > 1/2$ and all $0 \le s \le t \le T$,*

$$\mathbb{E}[\|H_t\|^2] \le C_T \quad \text{and} \quad \mathbb{E}[\|H_t - H_s\|^2] \le C_T (t-s)^\eta. \tag{3.2}$$

Assume, also, that $\kappa\sigma$ and b have at most linear growth, that is, for all $0 \le t \le T$, all $(x_u)_{u \ge 0} \in C(\mathbb{R}_+, \mathbb{R})$ and all $h \in \mathcal{S}$,

$$|\sigma(x_t)\kappa(t,(x_u)_{u\le t},h)| + |b(t,(x_u)_{u\le t},h)| \le C_T \Big(1 + \sup_{[0,t]} |x_u| + \|h\|\Big), \tag{3.3}$$

that σ is Hölder continuous with index $\alpha \in (1/2, 1]$ and that for some $\theta_1 \in (1/4, 1]$, $\theta_2 \in (1/2, 1]$, $\theta_3 \in (1/2\eta, 1]$, all $0 \le s \le t \le T$, all $(x_u)_{u\ge 0} \in C(\mathbb{R}_+, \mathbb{R})$ and all $h, h' \in \mathcal{S}$, we have

$$\begin{aligned} &|\sigma(x_t)\kappa(t,(x_u)_{u\le t},h) - \sigma(x_s)\kappa(s,(x_u)_{u\le s},h')| \\ &\quad \le C_T \Big((t-s)^{\theta_1} + \sup_{u\in[s,t]} |x_u - x_s|^{\theta_2} + \|h-h'\|^{\theta_3}\Big). \end{aligned} \tag{3.4}$$

Finally, assume that κ is bounded below by some constant $\kappa_0 > 0$. Consider a continuous $(\mathcal{F}_t)_{t\ge 0}$-adapted solution $(X_t)_{t\ge 0}$ to (3.1). The law of X_t then has a density on $\{x \in \mathbb{R}, \sigma(x) \ne 0\}$ whenever $t > 0$.

Note that (3.2) does not imply that H is a.s. continuous: it is just a type of L^2-continuity. Also, observe that we assume no regularity for the drift coefficient b. This is not so surprising, if we consider Girsanov's theorem. However, Girsanov's theorem might be difficult to use in such a context due to the randomness of the coefficients (a change of probability also changes the law of the auxiliary process). Let us briefly illustrate (3.4).

Example 3.2. (a) Let $\sigma(x_s)\kappa(s,(x_u)_{u\le s},h) = \phi(s, x_s, \sup_{[0,s]}\varphi(x_u), h)$ with $\phi: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathcal{S} \mapsto \mathbb{R}$ satisfying $|\phi(s,x,m,h) - \phi(s',x',m',h')| \le C(|s-s'|^{\theta_1} + |x-x'|^{\theta_2} + |m-m'|^{\zeta} + \|h-h'\|^{\theta_3})$ and $\varphi: \mathbb{R} \mapsto \mathbb{R}$ satisfying $|\varphi(x) - \varphi(x')| \le C|x-x'|^r$ with $\zeta r \ge \theta_2$. Then, $\sigma\kappa$ satisfies (3.4).

(b) Let $\sigma(x_s)\kappa(s,(x_u)_{u\le s},h) = \phi(s, x_s, \int_0^s \varphi(x_u)\,\mathrm{d}u, h)$ with $\phi: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathcal{S} \mapsto \mathbb{R}$ satisfying the condition $|\phi(s,x,m,h) - \phi(s',x',m',h')| \le C(|s-s'|^{\theta_1} + |x-x'|^{\theta_2} + |m-m'|^{\theta_1} + \|h-h'\|^{\theta_3})$ and with $\varphi: \mathbb{R} \mapsto \mathbb{R}$ bounded. Then, $\sigma\kappa$ satisfies (3.4).

Proof of Theorem 3.1. The scheme of the proof is exactly the same as that of Theorem 2.1. For the sake of simplicity, we show the result only when $t = 1$.

Step 1. For $\varepsilon \in (0,1)$, we consider the random variable

$$Z_\varepsilon := X_{1-\varepsilon} + \int_{1-\varepsilon}^1 \sigma(X_{1-\varepsilon})\kappa\big(1-\varepsilon, (X_u)_{u\le 1-\varepsilon}, H_{1-\varepsilon}\big)\,\mathrm{d}B_s.$$

Conditioning with respect to $\mathcal{F}_{1-\varepsilon}$ and using the fact that $\kappa \geq \kappa_0$, we get, for all $\xi \in \mathbb{R}$,

$$|\mathbb{E}[e^{i\xi Z_\varepsilon}|\mathcal{F}_{1-\varepsilon}]| = \left|\exp\left(i\xi X_{1-\varepsilon} - \varepsilon\sigma^2(X_{1-\varepsilon})\kappa^2\left(1-\varepsilon, (X_u)_{u\leq 1-\varepsilon}, H_{1-\varepsilon}\right)\xi^2/2\right)\right|$$
$$\leq \exp\left(-\varepsilon\kappa_0^2\sigma^2(X_{1-\varepsilon})\xi^2/2\right).$$

Step 2. Using Doob's inequality, Gronwall's lemma, (3.3) and (3.2), one easily shows that for all $0 \leq s \leq t \leq 1$,

$$\mathbb{E}\Big[\sup_{[0,1]} X_t^2\Big] \leq C, \qquad \mathbb{E}\Big[\sup_{u\in[s,t]} (X_u - X_s)^2\Big] \leq C(t-s). \tag{3.5}$$

Next, using (3.2)–(3.5), we get, for all $\varepsilon \in (0,1)$,

$$\begin{aligned}
&\mathbb{E}[(X_1 - Z_\varepsilon)^2] \\
&\quad\leq 2\int_{1-\varepsilon}^1 \mathbb{E}\Big[\big(\sigma(X_s)\kappa(s,(X_u)_{u\leq s}, H_s) - \sigma(X_{1-\varepsilon})\kappa\big(1-\varepsilon,(X_u)_{u\leq 1-\varepsilon}, H_{1-\varepsilon}\big)\big)^2\Big]\,\mathrm{d}s \\
&\qquad + 2\mathbb{E}\Big[\Big(\int_{1-\varepsilon}^1 b(s,(X_u)_{u\leq s}, H_s)\,\mathrm{d}s\Big)^2\Big] \\
&\quad\leq C\int_{1-\varepsilon}^1 \mathbb{E}\Big[\big(s-(1-\varepsilon)\big)^{2\theta_1} + \sup_{u\in[1-\varepsilon,s]} |X_u - X_{1-\varepsilon}|^{2\theta_2} + \|H_s - H_{1-\varepsilon}\|^{2\theta_3}\Big]\,\mathrm{d}s \\
&\qquad + 2\varepsilon\int_{1-\varepsilon}^1 \mathbb{E}[b^2(s,(X_u)_{u\leq s}, H_s)]\,\mathrm{d}s \\
&\quad\leq C\varepsilon^{1+2\theta_1} + C\varepsilon\mathbb{E}\Big[\sup_{u\in[1-\varepsilon,1]} |X_u - X_{1-\varepsilon}|^2\Big]^{\theta_2} \\
&\qquad + C\varepsilon \sup_{u\in[1-\varepsilon,1]} \mathbb{E}[\|H_u - H_{1-\varepsilon}\|^2]^{\theta_3} \\
&\qquad + C\varepsilon\int_{1-\varepsilon}^1 \mathbb{E}\Big[1 + \sup_{u\in[0,s]} X_u^2 + \|H_s\|^2\Big]\,\mathrm{d}s \\
&\quad\leq C\varepsilon^{1+2\theta_1} + C\varepsilon^{1+\theta_2} + C\varepsilon^{1+\eta\theta_3} + C\varepsilon^2 \leq C\varepsilon^{1+\theta},
\end{aligned}$$

where $\theta := \min(2\theta_1, \theta_2, \eta\theta_3, 1) \in (1/2, 1]$, by assumption.

Step 3. Let $\delta > 0$ be fixed and consider the function f_δ of Lemma 1.2 and the measure $\mu_{\delta,X_1}(\mathrm{d}x) = f_\delta(|\sigma(x)|)\mu_{X_1}(\mathrm{d}x)$, where μ_{X_1} is the law of X_1. Then, as in the proof of Theorem 2.1, we may write, for all $\xi \in \mathbb{R}$ and all $\varepsilon \in (0,1)$,

$$|\widehat{\mu_{\delta,X_1}}(\xi)| \leq |\mathbb{E}[e^{i\xi Z_\varepsilon} f_\delta(|\sigma(X_{1-\varepsilon})|)]| + |\xi|\mathbb{E}[|X_1 - Z_\varepsilon|]$$
$$+ \mathbb{E}\big[\big|f_\delta(|\sigma(X_1)|) - f_\delta(|\sigma(X_{1-\varepsilon})|)\big|\big].$$

Exactly as in the proof of Theorem 2.1, using the facts that σ is Hölder continuous with exponent $\alpha \in (1/2, 1]$ and that f_δ is bounded by 1, Lipschitz continuous and vanishes on $[0, \delta]$, we obtain from Steps 1 and 2 that for all $\varepsilon \in (0,1)$,

$$|\widehat{\mu_{\delta,X_1}}(\xi)| \le \exp(-\varepsilon\kappa_0^2\delta^2\xi^2/2) + C|\xi|\varepsilon^{(1+\theta)/2} + C\varepsilon^{\alpha/2}.$$

For each $|\xi| \ge 1$ fixed, we apply this formula with the choice $\varepsilon := (\log|\xi|)^2/\xi^2 \in (0,1)$ and deduce, as in the proof of Theorem 2.1, that $\int_{\mathbb{R}} |\widehat{\mu_{\delta,X_1}}(\xi)|^2\,\mathrm{d}\xi < \infty$ because $\theta > 1/2$ and $\alpha > 1/2$. Due to Lemma 1.1, this implies that μ_{δ,X_1} has a density for each $\delta > 0$. Thus, μ_{X_1} has a density on $\{|\sigma| > 0\}$ thanks to Lemma 1.2. □

4. Stochastic heat equation

On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, P)$, we consider an $(\mathcal{F}_t)_{t\ge0}$ space–time white noise $W(\mathrm{d}t,\mathrm{d}x)$ on $\mathbb{R}_+ \times [0,1]$, based on $\mathrm{d}t\,\mathrm{d}x$; see Walsh [26]. For two functions $\sigma, b : \mathbb{R} \mapsto \mathbb{R}$, we consider the stochastic heat equation with Neumann boundary conditions

$$\begin{aligned} &\partial_t U(t,x) = \partial_{xx} U(t,x) + b(U(t,x)) + \sigma(U(t,x))\dot{W}(t,x), \\ &\partial_x U(t,0) = \partial_x U(t,1) = 0, \end{aligned} \tag{4.1}$$

with some initial condition $U(0,x) = U_0(x)$ for some deterministic $U_0 \in L^\infty([0,1])$.

Consider the heat kernel $G_t(x,y) := \frac{1}{\sqrt{4\pi t}} \sum_{n\in\mathbb{Z}} [\mathrm{e}^{-(y-x-2n)^2/(4t)} + \mathrm{e}^{-(y+x-2n)^2/(4t)}]$. Following the ideas of Walsh [26], we say that a continuous $(\mathcal{F}_t)_{t\ge0}$-adapted process $(U(t,x))_{t>0,x\in[0,1]}$ is a weak solution to (4.1) if a.s., for all $t > 0$ and all $x \in [0,1]$,

$$\begin{aligned} U(t,x) = {} & \int_0^1 G_t(x,y)U_0(y)\,\mathrm{d}y + \int_0^t \int_0^1 G_{t-s}(x,y)b(U(s,y))\,\mathrm{d}y\,\mathrm{d}s \\ & + \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(U(s,y))W(\mathrm{d}s,\mathrm{d}y). \end{aligned} \tag{4.2}$$

In this section, we will show the following result.

Theorem 4.1. *Assume that b is measurable and has at most linear growth, and that σ is Hölder continuous with exponent $\theta \in (1/2,1]$. Consider a continuous $(\mathcal{F}_t)_{t\ge0}$-adapted weak solution $(U(t,x))_{t>0,x\in[0,1]}$ to (4.1). Then, for all $x \in [0,1]$ and all $t > 0$, the law of $U(t,x)$ has a density on $\{u \in \mathbb{R}, \sigma(u) \ne 0\}$.*

Again, the existence of solutions is not proved under the assumptions of Theorem 4.1 alone. We mention Gatarek and Goldys [12], from which we obtain the weak existence of a solution by additionally assuming that b is continuous. On the other hand, Bally, Gyongy and Pardoux [3] have proven the existence of a solution for a (locally) Lipschitz continuous diffusion coefficient σ bounded below and a (locally) bounded measurable drift coefficient b.

We will use the following estimates relating to the heat kernel, which can be found in the Appendix of Bally and Pardoux [4] and Bally, Millet and Sanz-Solé [5], Lemma B1.

For some constants $0 < c < C$, all $\varepsilon \in (0,1)$, all $x, y \in [0,1]$ and all $0 \le s \le t \le 1$,

$$\begin{aligned} c\sqrt{\varepsilon} \le \kappa_\varepsilon(x) &:= \int_{1-\varepsilon}^1 \int_{0\vee(x-\sqrt{\varepsilon})}^{1\wedge(x+\sqrt{\varepsilon})} G_{1-u}^2(x,z)\,\mathrm{d}z\,\mathrm{d}u \\ &\le \int_{1-\varepsilon}^1 \int_0^1 G_{1-u}^2(x,z)\,\mathrm{d}z\,\mathrm{d}u \le C\sqrt{\varepsilon}, \end{aligned} \tag{4.3}$$

$$\int_0^t \int_0^1 \big(G_{t-u}(x,z) - G_{t-u}(y,z)\big)^2\,\mathrm{d}z\,\mathrm{d}u \le C|x-y|, \tag{4.4}$$

$$\int_0^s \int_0^1 \big(G_{t-u}(x,z) - G_{s-u}(x,z)\big)^2\,\mathrm{d}z\,\mathrm{d}u + \int_s^t \int_0^1 G_{t-u}^2(x,z)\,\mathrm{d}z\,\mathrm{d}u \le C|t-s|^{1/2}. \tag{4.5}$$

Proof of Theorem 4.1. We assume that $t = 1$ for simplicity and we fix $x \in [0,1]$.

Step 1. For $\varepsilon \in (0,1)$, let

$$\begin{aligned} Z_\varepsilon &:= \int_0^1 G_1(x,y)U_0(y)\,\mathrm{d}y + \int_0^{1-\varepsilon}\int_0^1 G_{1-s}(x,y)b(U(s,y))\,\mathrm{d}y\,\mathrm{d}s \\ &\quad + \int_0^{1-\varepsilon}\int_0^1 G_{1-s}(x,y)\sigma(U(s,y))W(\mathrm{d}s,\mathrm{d}y) \\ &\quad + \int_{1-\varepsilon}^1\int_0^1 G_{1-s}(x,y)\sigma\big(U(1-\varepsilon,y)\big)W(\mathrm{d}s,\mathrm{d}y). \end{aligned}$$

As before, we observe that

$$\begin{aligned} \big|\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi Z_\varepsilon}|\mathcal{F}_{1-\varepsilon}]\big| &= \exp\biggl(-\frac{|\xi|^2}{2}\int_{1-\varepsilon}^1\int_0^1 G_{1-s}^2(x,y)\sigma^2\big(U(1-\varepsilon,y)\big)\,\mathrm{d}y\,\mathrm{d}s\biggr) \\ &\le \exp\big(-\kappa_\varepsilon(x)Y_\varepsilon|\xi|^2/2\big), \end{aligned}$$

where, recalling (4.3),

$$Y_\varepsilon := \frac{1}{\kappa_\varepsilon(x)}\int_{1-\varepsilon}^1\int_{0\vee(x-\sqrt{\varepsilon})}^{1\wedge(x+\sqrt{\varepsilon})} G_{1-s}^2(x,y)\sigma^2\big(U(1-\varepsilon,y)\big)\,\mathrm{d}y\,\mathrm{d}s.$$

Step 2. Using some classical computations involving (4.3)–(4.5), as well as the fact that $t, x \mapsto \int_0^1 G_t(x,y)U_0(y)\,\mathrm{d}y$ is of class C_b^∞ on $(t_0, 1] \times [0,1]$ for all $t_0 \in (0,1)$, we get, for some constant C,

$$\forall t \in [0,1], \forall y \in [0,1], \qquad \mathbb{E}[U^2(t,y)] \le C; \tag{4.6}$$

$$\forall s,t \in [1/2,1], \forall y \in [0,1], \qquad \mathbb{E}\big[\big(U(t,y) - U(s,y)\big)^2\big] \le C|t-s|^{1/2}; \tag{4.7}$$

$$\forall t \in [1/2,1], \forall y,z \in [0,1], \qquad \mathbb{E}\big[\big(U(t,y) - U(t,z)\big)^2\big] \le C|y-z|. \tag{4.8}$$

Step 2.1. We now prove that for all $\varepsilon \in (0, 1/2)$,

$$\mathbb{E}\big[\big(U(1,x) - Z_\varepsilon\big)^2\big] \le C\varepsilon^{(1+\theta)/2}.$$

Since σ is Hölder continuous and since b has at most linear growth, using (4.6) and (4.7), we obtain

$$\begin{aligned}
&\mathbb{E}\big[\big(U(1,x) - Z_\varepsilon\big)^2\big] \\
&\quad \le 2\mathbb{E}\bigg[\bigg(\int_{1-\varepsilon}^1 \int_0^1 G_{1-s}(x,y) b(U(s,y))\,\mathrm{d}y\,\mathrm{d}s\bigg)^2\bigg] \\
&\qquad + 2\int_{1-\varepsilon}^1 \int_0^1 G_{1-s}^2(x,y)\mathbb{E}\big[\big(\sigma(U(s,y)) - \sigma\big(U(1-\varepsilon,y)\big)\big)^2\big]\,\mathrm{d}y\,\mathrm{d}s \\
&\quad \le 2\varepsilon \int_{1-\varepsilon}^1 \int_0^1 G_{1-s}^2(x,y)\mathbb{E}[b^2(U(s,y))]\,\mathrm{d}y\,\mathrm{d}s \\
&\qquad + C\int_{1-\varepsilon}^1 \int_0^1 G_{1-s}^2(x,y)\mathbb{E}[|U(s,y) - U(1-\varepsilon,y)|^{2\theta}]\,\mathrm{d}y\,\mathrm{d}s \\
&\quad \le C\varepsilon \int_{1-\varepsilon}^1 \int_0^1 G_{1-s}^2(x,y)\mathbb{E}[1 + U^2(s,y)]\,\mathrm{d}y\,\mathrm{d}s \\
&\qquad + C\int_{1-\varepsilon}^1 \int_0^1 G_{1-s}^2(x,y)\mathbb{E}[|U(s,y) - U(1-\varepsilon,y)|^2]^\theta\,\mathrm{d}y\,\mathrm{d}s \\
&\quad \le C\varepsilon \int_{1-\varepsilon}^1 \int_0^1 G_{1-s}^2(x,y)\,\mathrm{d}y\,\mathrm{d}s + C\varepsilon^{\theta/2}\int_{1-\varepsilon}^1 \int_0^1 G_{1-s}^2(x,y)\,\mathrm{d}y\,\mathrm{d}s \\
&\quad \le C\varepsilon^{3/2} + C\varepsilon^{(1+\theta)/2} \le C\varepsilon^{(1+\theta)/2},
\end{aligned}$$

where, in the final inequality, we have used (4.3).

Step 2.2. We now check that there exists a constant C such that for all $\varepsilon \in (0, 1/2)$,

$$A_\varepsilon := \mathbb{E}[|\sigma^2(U(1,x)) - Y_\varepsilon|] \le C\varepsilon^{\theta/4}.$$

We have

$$\begin{aligned}
A_\varepsilon &= \frac{1}{\kappa_\varepsilon(x)}\mathbb{E}\bigg[\bigg|\int_{1-\varepsilon}^1 \int_{0\vee(x-\sqrt{\varepsilon})}^{1\wedge(x+\sqrt{\varepsilon})} G_{1-s}^2(x,y)\big[\sigma^2(U(1,x)) - \sigma^2\big(U(1-\varepsilon,y)\big)\big]\,\mathrm{d}y\,\mathrm{d}s\bigg|\bigg] \\
&\le \frac{1}{\kappa_\varepsilon(x)}\int_{1-\varepsilon}^1 \int_{0\vee(x-\sqrt{\varepsilon})}^{1\wedge(x+\sqrt{\varepsilon})} G_{1-s}^2(x,y)\mathbb{E}\big[\big|\sigma^2(U(1,x)) - \sigma^2\big(U(1-\varepsilon,y)\big)\big|\big]\,\mathrm{d}y\,\mathrm{d}s \\
&\le \sup_{y\in[x-\sqrt{\varepsilon},x+\sqrt{\varepsilon}],} \mathbb{E}\big[\big|\sigma^2(U(1,x)) - \sigma^2\big(U(1-\varepsilon,y)\big)\big|\big].
\end{aligned}$$

However, using the fact that σ is Hölder continuous and has at most linear growth, using (4.6)–(4.8), we deduce that for all $y \in [x-\sqrt{\varepsilon}, x+\sqrt{\varepsilon}]$,

$$\begin{aligned}
&\mathbb{E}\big[\big|\sigma^2(U(1,x)) - \sigma^2\big(U(1-\varepsilon,y)\big)\big|\big] \\
&\quad \le \mathbb{E}\big[\big|\sigma(U(1,x)) - \sigma\big(U(1-\varepsilon,y)\big)\big|^2\big]^{1/2}\mathbb{E}\big[\big|\sigma(U(1,x)) + \sigma\big(U(1-\varepsilon,y)\big)\big|^2\big]^{1/2} \\
&\quad \le C\mathbb{E}[|U(1,x) - U(1-\varepsilon,y)|^{2\theta}]^{1/2} \\
&\quad \le C\mathbb{E}[|U(1,x) - U(1-\varepsilon,y)|^2]^{\theta/2} \le C(\varepsilon^{1/2} + |x-y|)^{\theta/2} \le C\varepsilon^{\theta/4},
\end{aligned}$$

which concludes the step.

Step 3. Denote by $\mu_{U(1,x)}$ the law of $U(1,x)$. For $\delta > 0$, consider f_δ as in Lemma 1.2 and set $\mu_{\delta,U(1,x)}(\mathrm{d}u) = f_\delta(\sigma^2(u))\mu_{U(1,x)}(\mathrm{d}u)$. For all $\xi \in \mathbb{R}$ and all $\varepsilon \in (0,1/2)$, we may write, as in the proof of Theorem 2.1,

$$\begin{aligned}
|\widehat{\mu_{\delta,U(1,x)}}(\xi)| &= |\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi U(1,x)} f_\delta(\sigma^2((U(1,x)))]| \\
&\le |\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi Z_\varepsilon} f_\delta(Y_\varepsilon)]| + |\xi|\mathbb{E}[|U(1,x) - Z_\varepsilon|] + \mathbb{E}[|f_\delta(\sigma^2(U(1,x))) - f_\delta(Y_\varepsilon)|].
\end{aligned}$$

Using Steps 1, 2.1 and 2.2, observing that Y_ε is $\mathcal{F}_{1-\varepsilon}$-measurable and recalling that f_δ is bounded by 1 and vanishes on $[0,\delta]$, we get

$$|\widehat{\mu_{\delta,U(1,x)}}(\xi)| \le \mathrm{e}^{-\kappa_\varepsilon(x)\delta\xi^2/2} + C|\xi|\varepsilon^{(1+\theta)/4} + C\varepsilon^{\theta/4} \le \mathrm{e}^{-c\delta\sqrt{\varepsilon}\xi^2/2} + C|\xi|\varepsilon^{(1+\theta)/4} + C\varepsilon^{\theta/4},$$

using (4.3) for the last inequality. For each $|\xi| \ge 1$, we choose $\varepsilon := (\log|\xi|)^4/\xi^4 \in (0,1/2)$ and get

$$|\widehat{\mu_{\delta,U(1,x)}}(\xi)| \le \exp\big(-c\delta(\log|\xi|)^2/2\big) + C(\log|\xi|)^{1+\theta}/|\xi|^\theta + C(\log|\xi|)^\theta/|\xi|^\theta.$$

This holding for all $|\xi| \ge 1$ and $|\widehat{\mu_{\delta,U(1,x)}}(\xi)|$ being bounded by 1, we conclude, since $\theta > 1/2$, that $\int_{\mathbb{R}} |\widehat{\mu_{\delta,U(1,x)}}(\xi)|^2\,\mathrm{d}\xi < \infty$. Lemma 1.1 ensures that the law of $\mu_{\delta,U(1,x)}$ has a density for each $\delta > 0$. We conclude, using Lemma 1.2, that $\mu_{U(1,x)}$ has a density on $\{\sigma^2 > 0\}$. □

5. Lévy-driven SDEs

We conclude this paper by considering Lévy-driven SDEs. For simplicity, we restrict our study to the case of deterministic coefficients depending only on the position of the process. The result below extends without difficulty, as in the Brownian case, to SDEs with random coefficients depending on the whole paths, under some adequate conditions.

We thus consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, P)$ and a square-integrable compensated $(\mathcal{F}_t)_{t\ge0}$-Lévy process $(L_t)_{t\ge0}$ without drift, without Brownian part and with Lévy measure ν. Such a process is entirely characterized by its Fourier transform:

$$\mathbb{E}[\exp(\mathrm{i}\xi L_t)] = \exp\left(-t\int_{\mathbb{R}_*} (1 - \mathrm{e}^{\mathrm{i}\xi z} + \mathrm{i}\xi z)\nu(\mathrm{d}z)\right).$$

For $\sigma, b : \mathbb{R} \mapsto \mathbb{R}$, we consider the one-dimensional SDE

$$X_t = x + \int_0^t \sigma(X_{s-})\,\mathrm{d}L_s + \int_0^t b(X_s)\,\mathrm{d}s. \tag{5.1}$$

Our aim in this section is to prove the following result.

Theorem 5.1. *Assume that* $\int_{\mathbb{R}_*} z^2 \nu(\mathrm{d}z) < \infty$ *and that for some* $\lambda \in (3/4, 2)$, $c > 0$, $\xi_0 \geq 0$,

$$\forall |\xi| \geq \xi_0 \qquad \int_{\mathbb{R}_*} \big(1 - \cos(\xi z)\big)\nu(\mathrm{d}z) \geq c|\xi|^\lambda \tag{5.2}$$

and for some $\gamma \in [1, 2]$ (*with, necessarily,* $\gamma \geq \lambda$),

$$\int_{\mathbb{R}_*} |z|^\gamma \nu(\mathrm{d}z) < \infty. \tag{5.3}$$

We also assume that b is measurable with at most linear growth and that σ *is Hölder continuous with exponent* $\theta \in (3\gamma/(2\lambda) - 1, 1]$. *If* $\lambda \in (3/4, 3/2)$, *we additionally suppose that b is Hölder continuous with index* $\alpha \in (3\gamma/(2\lambda) - \gamma, 1]$.

Let $(X_t)_{t\geq 0}$ *be a cadlag* $(\mathcal{F}_t)_{t\geq 0}$*-adapted solution to* (5.1). *Then, for all* $t > 0$, *the law of* X_t *has a density on the set* $\{x \in \mathbb{R}, \sigma(x) \neq 0\}$.

Here, again, the (weak or strong) existence of solutions to (5.1) probably does not hold under the assumptions of Theorem 5.1 alone. See Jacod [16] for many existence results.

Let us comment on this result.

(a) Observe that (5.3) implies $\int_{\mathbb{R}_*} (1 - \cos(\xi z))\nu(\mathrm{d}z) \leq C|\xi|^\gamma$ so that under (5.2), (5.3) can hold only for some $\gamma \geq \lambda$.

Indeed, since $0 \leq 1 - \cos x \leq 2(x^2 \wedge 1)$, we may write $\int_{\mathbb{R}_*} (1-\cos(\xi z))\nu(\mathrm{d}z) \leq 2\int_{|z|\leq 1/|\xi|} \xi^2 \times z^2 \nu(\mathrm{d}z) + 2\int_{|z|\geq 1/|\xi|} \nu(\mathrm{d}z) \leq 2\xi^2 \int_{|z|\leq 1/|\xi|} |z|^\gamma |\xi|^{\gamma-2}\nu(\mathrm{d}z) + 2\int_{|z|\geq 1/|\xi|} |z|^\gamma |\xi|^\gamma \nu(\mathrm{d}z) \leq 2|\xi|^\gamma \times \int_{\mathbb{R}_*} |z|^\gamma \nu(\mathrm{d}z)$.

(b) Using a standard localization procedure, one may easily eliminate large jumps, that is, replace the assumptions $\int_{\mathbb{R}_*} (|z|^2 + |z|^\gamma)\nu(\mathrm{d}z) < \infty$ by $\int_{\mathbb{R}_*} \min(1, |z|^\gamma)\nu(\mathrm{d}z) < \infty$.

(c) If (5.2) holds with $\lambda > 3/2$, we assume no regularity on the drift coefficient b. Observe, here, that no trick using Girsanov's theorem may allow us to remove the drift: there is a clear difference in nature between the paths of a Lévy process without Brownian part with and without drift.

(d) Assume that ν satisfies $\int_{\mathbb{R}_*} z^2 \nu(\mathrm{d}z) < \infty$ and that the following property holds for some $\lambda \in (3/4, 2)$: there exist $0 < c_0 < c_1$ such that for all $\varepsilon \in (0, 1]$,

$$c_0 \varepsilon^{2-\lambda} \leq \int_{|z|\leq\varepsilon} z^2 \nu(\mathrm{d}z) \leq c_1 \varepsilon^{2-\lambda}. \tag{5.4}$$

Then, (5.2) holds and (5.3) holds with any $\gamma \in (\lambda, 2]$. Indeed, since $1 - \cos x \geq x^2/2$ for $x \in [0, 1]$, we get, for $|\xi| > 1$, $\int_{\mathbb{R}_*} (1 - \cos(\xi z))\nu(\mathrm{d}z) \geq (\xi^2/4)\int_{|z|\leq 1/|\xi|} z^2 \nu(\mathrm{d}z) \geq c_0|\xi|^\lambda/4$, whence

(5.2). Next, let $\gamma \in (\lambda, 2)$ be fixed. To show that (5.3) holds, it clearly suffices to prove that $\int_{|z|<1} |z|^\gamma \nu(\mathrm{d}z) < \infty$. Let us, for example, show that $\int_0^1 z^\gamma \nu(\mathrm{d}z) < \infty$. Using an integration by parts, one easily gets $\int_0^1 z^\gamma \nu(\mathrm{d}z) = \int_0^1 z^{\gamma-2} z^2 \nu(\mathrm{d}z) = \int_0^1 (2-\gamma) z^{\gamma-3} [\int_0^z y^2 \nu(\mathrm{d}y)]\,\mathrm{d}z \le (2-\gamma) c_1 \int_0^1 z^{\gamma-3} z^{2-\lambda}\,\mathrm{d}z < \infty$ since $\gamma - \lambda > 0$.

Thus, our result holds in the following situations:

- $\lambda > 3/2$, σ is Hölder continuous with exponent $\theta > 1/2$;
- $\lambda \in [1, 3/2]$, σ is Hölder continuous with index $\theta > 1/2$, b is Hölder continuous with exponent $\alpha > 3/2 - \lambda$;
- $\lambda \in (3/4, 1]$, σ and b are Hölder continuous with exponent $\theta > 3/(2\lambda) - 1$.

(e) For example, $\nu(\mathrm{d}z) = z^{-1-\lambda} \mathbf{1}_{[0,1]}(z)\,\mathrm{d}z$ satisfies (5.4), as well as $\nu(\mathrm{d}z) = \sum_{n\ge1} n^{\lambda-1} \delta_{1/n}$, or, more generally, $\nu(\mathrm{d}z) = \sum_{n\ge1} n^{\lambda\alpha-1} \delta_{n^{-\alpha}}$ with $\alpha > 0$.

(f) Our assumption that $\lambda > 3/4$ might seem strange. However, our method does not seem to work for smaller values of λ, even if σ, b are Lipschitz continuous.

As noted by the anonymous referee, however, it is possible to obtain some results for $\lambda \in (1/2, 3/4]$ if there is no drift part ($b \equiv 0$).

Proof of Theorem 5.1. By scaling, it suffices to consider the case $t = 1$. We will often write the Lévy process as

$$L_t = \int_0^t \int_{\mathbb{R}_*} z \tilde{N}(\mathrm{d}s, \mathrm{d}z),$$

where $\tilde{N}(\mathrm{d}s, \mathrm{d}z)$ is a compensated Poisson measure on $\mathbb{R}_+ \times \mathbb{R}_*$ with intensity measure $\mathrm{d}s\,\nu(\mathrm{d}z)$. Thus, (5.1) can be rewritten as

$$X_t = x + \int_0^t \int_{\mathbb{R}_*} \sigma(X_{s-}) z \tilde{N}(\mathrm{d}s, \mathrm{d}z) + \int_0^t b(X_s)\,\mathrm{d}s. \tag{5.5}$$

Step 1. For $\varepsilon \in (0, 1)$, we consider the random variable

$$Z_\varepsilon := X_{1-\varepsilon} + \int_{1-\varepsilon}^1 \sigma(X_{1-\varepsilon})\,\mathrm{d}L_s + \int_{1-\varepsilon}^1 b(X_{1-\varepsilon})\,\mathrm{d}s.$$

For $\delta > 0$, consider the function f_δ of Lemma 1.2. Recall that f_δ is bounded and vanishes on $[0, \delta]$. Conditioning with respect to $\mathcal{F}_{1-\varepsilon}$ and using (5.2), we get, for all $|\xi| \ge \xi_0/\delta$,

$$\begin{aligned}
&|\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi Z_\varepsilon} f_\delta(|\sigma(X_{1-\varepsilon})|) | \mathcal{F}_{1-\varepsilon}]| \\
&\quad = f_\delta(|\sigma(X_{1-\varepsilon})|) \bigg| \exp\bigg(\mathrm{i}\xi X_{1-\varepsilon} + \mathrm{i}\xi\varepsilon b(X_{1-\varepsilon}) \\
&\qquad\qquad - \varepsilon \int_{\mathbb{R}_*} \big(1 - \mathrm{e}^{\mathrm{i}\xi\sigma(X_{1-\varepsilon})z} + \mathrm{i}\xi\sigma(X_{1-\varepsilon})z\big) \nu(\mathrm{d}z)\bigg)\bigg|
\end{aligned}$$

$$= f_\delta(|\sigma(X_{1-\varepsilon})|)\exp\biggl(-\varepsilon\int_{\mathbb{R}_*}\bigl(1-\cos(\xi\sigma(X_{1-\varepsilon})z)\bigr)\nu(\mathrm{d}z)\biggr)$$
$$\leq f_\delta(|\sigma(X_{1-\varepsilon})|)\exp(-c\varepsilon\delta^\lambda|\xi|^\lambda)\leq\exp(-c\varepsilon\delta^\lambda|\xi|^\lambda).$$

We have used the fact that f_δ is bounded by 1 and vanishes on $[0,\delta]$ to obtain the two last inequalities.

Step 2. Recall that σ and b are Hölder continuous with exponent $\theta\in(0,1]$ and $\alpha\in[0,1]$ (when there is no regularity assumption on b, we say that it is Hölder with exponent 0). The goal of this step is to show that for all $\varepsilon\in(0,1)$,

$$I_\varepsilon:=\mathbb{E}[|X_1-Z_\varepsilon|^\gamma]\leq C\varepsilon^{1+\theta}+C\varepsilon^{\gamma+\alpha}\leq C\varepsilon^{1+\zeta},\tag{5.6}$$

where $\zeta:=\min(\theta,\gamma+\alpha-1)\in(3\gamma/2\lambda-1,1]$, by assumption. We first show that for all $0\leq s\leq t\leq 1$,

$$\mathbb{E}\Bigl[\sup_{[0,1]}|X_s|^\gamma\Bigr]\leq C,\qquad \mathbb{E}[|X_t-X_s|^\gamma]\leq C|t-s|.\tag{5.7}$$

First, using (5.5), the Burkholder–Davies–Gundy inequality (see Dellacherie and Meyer [9]), the subadditivity of $x\mapsto x^{\gamma/2}$, the Hölder inequality, (5.3) and the fact that b,σ have at most linear growth, we obtain, for all $t\in[0,1]$,

$$\begin{aligned}
&\mathbb{E}\Bigl[\sup_{u\in[0,t]}|X_u|^\gamma\Bigr]\\
&\quad\leq C|x|^\gamma+C\mathbb{E}\biggl[\sup_{u\in[0,t]}\biggl|\int_0^u\int_{\mathbb{R}_*}\sigma(X_{s-})z\tilde{N}(\mathrm{d}s,\mathrm{d}z)\biggr|^\gamma\biggr]+C\mathbb{E}\biggl[\biggl(\int_0^t|b(X_s)|\,\mathrm{d}s\biggr)^\gamma\biggr]\\
&\quad\leq C|x|^\gamma+C\mathbb{E}\biggl[\biggl(\int_0^t\int_{\mathbb{R}_*}|\sigma(X_{s-})z|^2N(\mathrm{d}s,\mathrm{d}z)\biggr)^{\gamma/2}\biggr]+C\mathbb{E}\biggl[\biggl(\int_0^t|b(X_s)|\,\mathrm{d}s\biggr)^\gamma\biggr]\\
&\quad\leq C|x|^\gamma+C\mathbb{E}\biggl[\int_0^t\int_{\mathbb{R}_*}|\sigma(X_{s-})z|^\gamma N(\mathrm{d}s,\mathrm{d}z)\biggr]+Ct^{\gamma-1}\mathbb{E}\biggl[\int_0^t|b(X_s)|^\gamma\,\mathrm{d}s\biggr]\\
&\quad\leq C|x|^\gamma+C\int_0^t\int_{\mathbb{R}_*}\mathbb{E}[|\sigma(X_{s-})|^\gamma]|z|^\gamma\nu(\mathrm{d}z)\,\mathrm{d}s+Ct^{\gamma-1}\int_0^t\mathbb{E}[|b(X_s)|^\gamma]\,\mathrm{d}s\\
&\quad\leq C|x|^\gamma+C\int_0^t\mathbb{E}[1+|X_s|^\gamma]\,\mathrm{d}s
\end{aligned}$$

and Gronwall's lemma allows us to conclude that $\mathbb{E}[\sup_{[0,1]}|X_s|^\gamma]\leq C$. The same arguments ensure that for $0\leq s\leq t\leq 1$, $\mathbb{E}[|X_t-X_s|^\gamma]\leq C\int_s^t\mathbb{E}[1+|X_u|^\gamma]\,\mathrm{d}u$, whence the second inequality of (5.7). We may now check (5.6). Using similar arguments and the Hölder continuity assumptions, we obtain

$$
\begin{aligned}
I_\varepsilon &\le C\mathbb{E}\bigg[\bigg(\int_{1-\varepsilon}^1 \int_{\mathbb{R}_*} \big|\big(\sigma(X_{s-}) - \sigma(X_{1-\varepsilon})\big)z\big|^2 N(\mathrm{d}s, \mathrm{d}z)\bigg)^{\gamma/2}\bigg] \\
&\quad + C\mathbb{E}\bigg[\bigg(\int_{1-\varepsilon}^1 |b(X_s) - b(X_{1-\varepsilon})|\,\mathrm{d}s\bigg)^{\gamma}\bigg] \\
&\le C\int_{1-\varepsilon}^1 \mathbb{E}[|\sigma(X_{s-}) - \sigma(X_{1-\varepsilon})|^\gamma]\,\mathrm{d}s + C\varepsilon^{\gamma-1}\int_{1-\varepsilon}^1 \mathbb{E}[|b(X_{s-}) - b(X_{1-\varepsilon})|^\gamma]\,\mathrm{d}s \\
&\le C\int_{1-\varepsilon}^1 \mathbb{E}[|X_s - X_{1-\varepsilon}|^{\gamma\theta}]\,\mathrm{d}s + C\varepsilon^{\gamma-1}\int_{1-\varepsilon}^1 \mathbb{E}[|X_s - X_{1-\varepsilon}|^{\alpha\gamma}]\,\mathrm{d}s \\
&\le C\int_{1-\varepsilon}^1 \mathbb{E}[|X_s - X_{1-\varepsilon}|^{\gamma}]^\theta\,\mathrm{d}s + C\varepsilon^{\gamma-1}\int_{1-\varepsilon}^1 \mathbb{E}[|X_s - X_{1-\varepsilon}|^{\gamma}]^\alpha\,\mathrm{d}s \\
&\le C\varepsilon^{1+\theta} + C\varepsilon^{\gamma+\alpha},
\end{aligned}
$$

where, in the final inequality, we have used (5.7).

Step 3. Let $\delta > 0$ be fixed and consider the measure $\mu_{\delta,X_1}(\mathrm{d}x) = f_\delta(|\sigma(x)|)\mu_{X_1}(\mathrm{d}x)$, where μ_{X_1} is the law of X_1. Then, as before, for all $\xi \in \mathbb{R}$ and all $\varepsilon \in (0, 1)$, we may write

$$
|\widehat{\mu_{\delta,X_1}}(\xi)| \le |\mathbb{E}[e^{i\xi Z_\varepsilon} f_\delta(|\sigma(X_{1-\varepsilon})|)]| + |\xi|\mathbb{E}[|X_1 - Z_\varepsilon|] + \mathbb{E}\big[\big|f_\delta(|\sigma(X_1)|) - f_\delta(|\sigma(X_{1-\varepsilon})|)\big|\big].
$$

Using the Hölder continuity of σ and (5.7), one easily gets (recall that $0 < \theta \le 1 \le \gamma$, by assumption) $\mathbb{E}[|f_\delta(|\sigma(X_1)|) - f_\delta(|\sigma(X_{1-\varepsilon})|)|] \le C\mathbb{E}[|X_1 - X_{1-\varepsilon}|^\theta] \le C\varepsilon^{\theta/\gamma}$. Next, using Steps 1 and 2, we obtain, for all $\varepsilon \in (0, 1)$ and all $|\xi| \ge \xi_0/\delta$,

$$
|\widehat{\mu_{\delta,X_1}}(\xi)| \le \exp(-c\delta^\lambda \varepsilon|\xi|^\lambda) + C|\xi|\varepsilon^{(1+\zeta)/\gamma} + C\varepsilon^{\theta/\gamma}.
$$

For each $|\xi| \ge \xi_1 \vee (\xi_0/\delta)$, we choose $\varepsilon := (\log|\xi|)^2/|\xi|^\lambda \in (0, 1)$ (this holds if ξ_1 is large enough). This gives

$$
\begin{aligned}
|\widehat{\mu_{\delta,X_1}}(\xi)| &\le \exp(-c\delta^\lambda(\log|\xi|)^2) + C(\log|\xi|)^{2(1+\zeta)/\gamma}/|\xi|^{\lambda(1+\zeta)/\gamma-1} \\
&\quad + C(\log|\xi|)^{2\theta/\gamma}/|\xi|^{\lambda\theta/\gamma}.
\end{aligned}
$$

This holding for all $|\xi| \ge \xi_1 \vee (\xi_0/\delta)$ and $\widehat{\mu_{\delta,X_1}}$ being bounded by 1, we get that $\int_{\mathbb{R}} |\widehat{\mu_{\delta,X_1}}(\xi)|^2\,\mathrm{d}\xi < \infty$. Indeed, $\lambda(1+\zeta)/\gamma - 1 > 1/2$ (because $\zeta > 3\gamma/2\lambda - 1$) and $\lambda\theta/\gamma > 1/2$ (because $\theta > 3\gamma/2\lambda - 1$ and $\lambda \le \gamma$). Lemma 1.1 implies that the measure μ_{δ,X_1} has a density (for $\delta > 0$ fixed) and we conclude using Lemma 1.2 that μ_{X_1} has a density on $\{|\sigma| > 0\}$. □

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