On lower limits and equivalences for distribution tails of randomly stopped sums

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For a distribution $F^{*\tau}$ of a random sum $S_\tau = \xi_1 + \cdots + \xi_\tau$ of i.i.d. random variables with a common distribution F on the half-line $[0, \infty)$, we study the limits of the ratios of tails $\overline{F^{*\tau}}(x)/\overline{F}(x)$ as $x \to \infty$ (here, τ is a counting random variable which does not depend on $\{\xi_n\}_{n \geq 1}$). We also consider applications of the results obtained to random walks, compound Poisson distributions, infinitely divisible laws, and subcritical branching processes.

Keywords: convolution tail; convolution equivalence; lower limit; randomly stopped sums; subexponential distribution

1. Introduction

Let $\xi, \xi_1, \xi_2, \ldots,$ be independent identically distributed non-negative random variables. We assume that their common distribution F on the half-line $[0, \infty)$ has an unbounded support, that is, $\overline{F}(x) \equiv F(x, \infty) > 0$ for all x. Put $S_0 = 0$ and $S_n = \xi_1 + \cdots + \xi_n$, $n = 1, 2, \ldots$.

Let τ be a counting random variable which does not depend on $\{\xi_n\}_{n \geq 1}$ and which has finite mean. Denote by $F^{*\tau}$ the distribution of a randomly stopped sum $S_\tau = \xi_1 + \cdots + \xi_\tau$.

In this paper, we discuss how the tail behavior of $F^{*\tau}$ relates to that of F and, in particular, under what conditions

$$\liminf_{x \to \infty} \frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)} = \mathbf{E}\tau. \tag{1}$$

Relations on lower limits of ratios of tails were first discussed by Rudin [21]. Theorem 2* of that paper states (for an integer p) the following.

Theorem 1. *Suppose there exists a positive $p \in [1, \infty)$ such that $\mathbf{E}\xi^p = \infty$, but $\mathbf{E}\tau^p < \infty$. Then* (1) *holds.*

Rudin's studies were motivated by Chover, Ney and Wainger [7] who considered, in particular, the problem of existence of a limit for the ratio

$$\frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)} \qquad \text{as } x \to \infty. \tag{2}$$

From Theorem 1, it follows that if F and τ satisfy its conditions and if a limit of (2) exists, then that limit must equal $\mathbf{E}\tau$.

Rudin proved Theorem 1 via probability generating function techniques. Below, we give an alternative and more direct proof of Theorem 1 in the case of any positive p (i.e., not necessarily integer). Our method is based on truncation arguments; in this way, we propose a general scheme (see Theorem 4 below) which may also be applied to distributions having all moments finite.

The condition $\mathbf{E}\xi^p = \infty$ rules out many distributions of interest in, say, the theory of subexponential distributions. For example, log-normal and Weibull-type distributions have all moments finite. Our first result presents a natural condition on a stopping time τ guaranteeing relation (1) for the whole class of heavy-tailed distributions.

Recall that a random variable ξ has a *light-tailed* distribution F on $[0, \infty)$ if $\mathbf{E}e^{\gamma\xi} < \infty$ with some $\gamma > 0$. Otherwise, F is called a *heavy-tailed* distribution; this happens if and only if $\mathbf{E}e^{\gamma\xi} = \infty$ for all $\gamma > 0$.

Theorem 2. *Let F be a heavy-tailed distribution and τ have a light-tailed distribution. Then* (1) *holds.*

The proof of Theorem 2 is based on a new technical tool (see Lemma 2) and significantly differs from the proof of Theorem 1 in Foss and Korshunov [15], where the particular case $\tau = 2$ was considered. Theorem 2 is restricted to the case of light-tailed τ, but here, we extend Rudin's result to the class of all heavy-tailed distributions. The reasons for the restriction to $\mathbf{E}e^{\gamma\tau} < \infty$ come from the proof of Theorem 2, but are, in fact, rather natural: the tail of τ should be lighter than the tail of any heavy-tailed distribution. Indeed, if $\xi_1 \geq 1$, then $\overline{F^{*\tau}}(x) \geq \mathbf{P}\{\tau > x\}$. This shows that the tail of $F^{*\tau}$ is at least as heavy as that of τ. Note that in Theorem 1, in some sense, the tail of F is heavier than the tail of τ.

Theorem 2 may be applied in various areas where randomly stopped sums appear; see Sections 8–11 (random walks, compound Poisson distributions, infinitely divisible laws and branching processes) and, for instance, Kalashnikov [17] for further examples.

For any distribution on $[0, \infty)$, let

$$\varphi(\gamma) = \int_0^\infty e^{\gamma x} F(\mathrm{d}x) \in (0, \infty], \qquad \gamma \in \mathbf{R},$$

and

$$\widehat{\gamma} = \sup\{\gamma : \varphi(\gamma) < \infty\} \in [0, \infty].$$

Note that the moment generating function $\varphi(\gamma)$ is increasing and continuous in the interval $(-\infty, \widehat{\gamma})$ and that $\varphi(\widehat{\gamma}) = \lim_{\gamma \uparrow \widehat{\gamma}} \varphi(\gamma) \in [1, \infty]$. The following result was proven in Foss and

Korshunov [15], Theorem 3. Let

$$\frac{\overline{F * F}(x)}{\overline{F}(x)} \to c \qquad \text{as } x \to \infty,$$

where $c \in (0, \infty]$. Then, necessarily, $c = 2\varphi(\widehat{\gamma})$. We state now a generalization to τ-fold convolution.

Theorem 3. *Let $\varphi(\widehat{\gamma}) < \infty$ and $\mathbf{E}(\varphi(\widehat{\gamma}) + \varepsilon)^{\tau} < \infty$ for some $\varepsilon > 0$. Assume that*

$$\frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)} \to c \qquad \text{as } x \to \infty,$$

where $c \in (0, \infty]$. Then $c = \mathbf{E}(\tau \varphi^{\tau-1}(\widehat{\gamma}))$.

For (comments on) earlier partial results in the case $\tau = 2$, see, for example, Chover, Ney and Wainger [6,7], Cline [8], Embrechts and Goldie [10], Foss and Korshunov [15], Pakes [19], Rogozin [20], Teugels [23] and further references therein. The proof of Theorem 3 follows from Lemmas 3 and 4 in Section 7.

2. Preliminary result

We start with the following result.

Theorem 4. *Assume that there exists a non-decreasing concave function $h : \mathbf{R}^+ \to \mathbf{R}^+$ such that*

$$\mathbf{E}e^{h(\xi)} < \infty \quad and \quad \mathbf{E}\xi e^{h(\xi)} = \infty. \tag{3}$$

For any $n \geq 1$, put $A_n = \mathbf{E}e^{h(\xi_1 + \cdots + \xi_n)}$. Assume that F is heavy-tailed and that

$$\mathbf{E}\tau A_{\tau-1} < \infty. \tag{4}$$

Then, for any light-tailed distribution G on $[0, \infty)$,

$$\liminf_{x \to \infty} \frac{\overline{G * F^{*\tau}}(x)}{\overline{F}(x)} = \mathbf{E}\tau. \tag{5}$$

By considering G concentrated at 0, we get the following.

Corollary 1. *In the conditions of Theorem 4, (1) holds.*

In order to prove Theorem 4, first we restate Theorem 1* of Rudin [21] (in Lemma 1 below) in terms of probability distributions and stopping times.

Lemma 1. *For any distribution F on $[0, \infty)$ with unbounded support and any counting random variable τ,*

$$\liminf_{x \to \infty} \frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)} \geq \mathbf{E}\tau.$$

Proof. For any two distributions F_1 and F_2 on $[0, \infty)$ with unbounded supports,

$$\overline{F_1 * F_2}(x) \geq (F_1 \times F_2)\big((x, \infty) \times [0, x]\big) + (F_1 \times F_2)\big([0, x] \times (x, \infty)\big)$$
$$\sim \overline{F}_1(x) + \overline{F}_2(x) \qquad \text{as } x \to \infty.$$

By induction arguments, this implies that, for any $n \geq 1$,

$$\liminf_{x \to \infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} \geq n.$$

Applying Fatou's lemma to the representation

$$\frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)} = \sum_{n=1}^{\infty} \mathbf{P}\{\tau = n\} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)},$$

completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 4. It follows from Lemma 1 that it is sufficient to prove the following inequality:

$$\liminf_{x \to \infty} \frac{\overline{G * F^{*\tau}}(x)}{\overline{F}(x)} \leq \mathbf{E}\tau.$$

Assume the contrary, that is, that there exist $\delta > 0$ and x_0 such that

$$\overline{G * F^{*\tau}}(x) \geq (\mathbf{E}\tau + \delta)\overline{F}(x) \qquad \text{for all } x > x_0. \tag{6}$$

For any positive $b > 0$, consider a concave function

$$h_b(x) \equiv \min\{h(x), bx\}, \tag{7}$$

which is non-negative because $h \geq 0$. Since F is heavy-tailed, $h(x) = o(x)$ as $x \to \infty$. Therefore, for any fixed b, there exists x_1 such that $h_b(x) = h(x)$ for all $x > x_1$. Hence, by condition (3),

$$\mathbf{E}e^{h_b(\xi)} < \infty \quad \text{and} \quad \mathbf{E}\xi e^{h_b(\xi)} = \infty. \tag{8}$$

For any x, we have the convergence $h_b(x) \downarrow 0$ as $b \downarrow 0$. Then, for any fixed n,

$$A_{n,b} \equiv \mathbf{E}e^{h_b(\xi_1 + \cdots + \xi_n)} \downarrow 1 \qquad \text{as } b \downarrow 0.$$

This and condition (4) together imply that there exists b such that

$$\mathbf{E}\tau A_{\tau-1,b} \le \mathbf{E}\tau + \delta/8. \tag{9}$$

Let η be a random variable with distribution G which does not depend on $\{\xi_n\}_{n\ge 1}$ and τ. Since G is light-tailed,

$$\mathbf{E}\eta e^{h_b(\eta)} < \infty. \tag{10}$$

In addition, we may choose $b > 0$ sufficiently small that

$$\mathbf{E}e^{h_b(\eta)}(\mathbf{E}\tau + \delta/8) \le \mathbf{E}\tau + \delta/4. \tag{11}$$

For any real a and t, put $a^{[t]} = \min\{a, t\}$. Then

$$\frac{\mathbf{E}(\eta + \xi_1^{[t]} + \cdots + \xi_\tau^{[t]})e^{h_b(\eta+\xi_1+\cdots+\xi_\tau)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}} = \sum_{n=1}^{\infty} \frac{\mathbf{E}\eta e^{h_b(\eta+\xi_1+\cdots+\xi_n)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}}\mathbf{P}\{\tau = n\}$$

$$+ \sum_{n=1}^{\infty} n\frac{\mathbf{E}\xi_1^{[t]}e^{h_b(\eta+\xi_1+\cdots+\xi_n)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}}\mathbf{P}\{\tau = n\}.$$

By the concavity of the function h_b,

$$\sum_{n=1}^{\infty} \frac{\mathbf{E}\eta e^{h_b(\eta+\xi_1+\cdots+\xi_n)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}}\mathbf{P}\{\tau = n\} \le \sum_{n=1}^{\infty} \frac{\mathbf{E}\eta e^{h_b(\eta)+h_b(\xi_1+\cdots+\xi_n)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}}\mathbf{P}\{\tau = n\}$$

$$= \frac{\mathbf{E}\eta e^{h_b(\eta)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}}\mathbf{E}A_{\tau,b}$$

$$\to 0 \qquad \text{as } t \to \infty,$$

due to (10), (9) and (8). Again, by the concavity of the function h_b,

$$\sum_{n=1}^{\infty} n\frac{\mathbf{E}\xi_1^{[t]}e^{h_b(\eta+\xi_1+\cdots+\xi_n)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}}\mathbf{P}\{\tau = n\} \le \sum_{n=1}^{\infty} n\frac{\mathbf{E}\xi_1^{[t]}e^{h_b(\eta)+h_b(\xi_1)+h_b(\xi_2+\cdots+\xi_n)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}}\mathbf{P}\{\tau = n\}$$

$$= \mathbf{E}e^{h_b(\eta)}\sum_{n=1}^{\infty} nA_{n-1,b}\mathbf{P}\{\tau = n\}$$

$$\le \mathbf{E}\tau + \delta/4,$$

by (9) and (11). Hence, for sufficiently large t,

$$\frac{\mathbf{E}(\eta + \xi_1^{[t]} + \cdots + \xi_\tau^{[t]})e^{h_b(\eta+\xi_1+\cdots+\xi_\tau)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}} \le \mathbf{E}\tau + \delta/2. \tag{12}$$

On the other hand, since $(\eta + \xi_1 + \cdots + \xi_\tau)^{[t]} \le \eta + \xi_1^{[t]} + \cdots + \xi_\tau^{[t]}$,

$$\frac{\mathbf{E}(\eta + \xi_1^{[t]} + \cdots + \xi_\tau^{[t]})e^{h_b(\eta+\xi_1+\cdots+\xi_\tau)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}} \ge \frac{\mathbf{E}(\eta + \xi_1 + \cdots + \xi_\tau)^{[t]}e^{h_b(\eta+\xi_1+\cdots+\xi_\tau)}}{\mathbf{E}\xi_1^{[t]}e^{h_b(\xi_1)}}$$

$$= \frac{\int_0^\infty x^{[t]}e^{h_b(x)}(G * F^{*\tau})(\mathrm{d}x)}{\int_0^\infty x^{[t]}e^{h_b(x)}F(\mathrm{d}x)}. \tag{13}$$

The right-hand side, after integration by parts, is equal to

$$\frac{\int_0^\infty \overline{G * F^{*\tau}}(x)\,\mathrm{d}(x^{[t]}e^{h_b(x)})}{\int_0^\infty \overline{F}(x)\,\mathrm{d}(x^{[t]}e^{h_b(x)})}.$$

Since $\mathbf{E}\xi_1 e^{h_b(\xi_1)} = \infty$, both integrals in this fraction tend to infinity as $t \to \infty$. For the *non-decreasing* function $h_b(x)$, the latter fact and assumption (6) together imply that

$$\liminf_{t\to\infty} \frac{\int_0^\infty \overline{G * F^{*\tau}}(x)\,\mathrm{d}(x^{[t]}e^{h_b(x)})}{\int_0^\infty \overline{F}(x)\,\mathrm{d}(x^{[t]}e^{h_b(x)})} = \liminf_{t\to\infty} \frac{\int_{x_0}^\infty \overline{G * F^{*\tau}}(x)\,\mathrm{d}(x^{[t]}e^{h_b(x)})}{\int_{x_0}^\infty \overline{F}(x)\,\mathrm{d}(x^{[t]}e^{h_b(x)})} \ge \mathbf{E}\tau + \delta.$$

Substituting this into (13), we get a contradiction of (12) for sufficiently large t. The proof is thus complete. $\qquad\square$

3. Proof of Theorem 1

Take an integer $k \ge 0$ such that $p - 1 \le k < p$. Without loss of generality, we may assume that $\mathbf{E}\xi^k < \infty$ (otherwise, we may consider a smaller p).

Consider a concave non-decreasing function $h(x) = (p-1)\ln x$. Then $\mathbf{E}e^{h(\xi_1)} < \infty$ and $\mathbf{E}\xi_1 e^{h(\xi_1)} = \infty$. Thus,

$$A_n \equiv \mathbf{E}e^{h(\xi_1+\cdots+\xi_n)} = \mathbf{E}(\xi_1 + \cdots + \xi_n)^{p-1}$$

$$\le \left(\mathbf{E}(\xi_1 + \cdots + \xi_n)^k\right)^{(p-1)/k}$$

since $(p-1)/k \le 1$. Further,

$$\mathbf{E}(\xi_1 + \cdots + \xi_n)^k = \sum_{i_1,\ldots,i_k=1}^n \mathbf{E}(\xi_{i_1} \cdot \ldots \cdot \xi_{i_k}) \le cn^k,$$

where

$$c \equiv \sup_{1 \le i_1,\ldots,i_k \le n} \mathbf{E}(\xi_{i_1} \cdot \ldots \cdot \xi_{i_k}) < \infty,$$

due to the fact that $\mathbf{E}\xi^k < \infty$. Hence, $A_n \le c^{(p-1)/k}n^{p-1}$ for all n. Therefore, we get $\mathbf{E}\tau A_{\tau-1} \le c^{(p-1)/k}\mathbf{E}\tau^p < \infty$. All conditions of Theorem 4 are met and the proof is complete.

4. Characterization of heavy-tailed distributions

In the sequel, we need the following existence result which strengthens a lemma in Rudin [21], page 989; and Lemma 1 in Foss and Korshunov [15]. Fix any $\delta \in (0, 1]$.

Lemma 2. *If a random variable $\xi \geq 0$ has a heavy-tailed distribution, then there exists a non-decreasing concave function $h : \mathbf{R}^+ \to \mathbf{R}^+$ such that $\mathbf{E}e^{h(\xi)} \leq 1 + \delta$ and $\mathbf{E}\xi e^{h(\xi)} = \infty$.*

Proof. Without loss of generality, assume that $\xi > 0$ a.s., that is, that $\overline{F}(0) = 1$. We will construct a piecewise linear function $h(x)$. For that, we introduce two positive sequences, $x_n \uparrow \infty$ and $\varepsilon_n \downarrow 0$ as $n \to \infty$, and let

$$h(x) = h(x_{n-1}) + \varepsilon_n(x - x_{n-1}) \qquad \text{if } x \in (x_{n-1}, x_n], \ n \geq 1.$$

This function is non-decreasing since $\varepsilon_n > 0$. Moreover, this function is concave due to the monotonicity of ε_n.

Put $x_0 = 0$ and $h(0) = 0$. Since ξ is heavy-tailed, we can choose $x_1 \geq 2$ so that

$$\mathbf{E}\{e^\xi; \xi \in (x_0, x_1]\} + e^{x_1}\overline{F}(x_1) > 1 + \delta.$$

Choose $\varepsilon_1 > 0$ so that

$$\mathbf{E}\{e^{\varepsilon_1\xi}; \xi \in (x_0, x_1]\} + e^{\varepsilon_1 x_1}\overline{F}(x_1) = e^{h(x_0)}\overline{F}(0) + \delta/2 = 1 + \delta/2,$$

which is equivalent to

$$\mathbf{E}\{e^{h(\xi)}; \xi \in (x_0, x_1]\} + e^{h(x_1)}\overline{F}(x_1) = e^{h(x_0)}\overline{F}(0) + \delta/2.$$

By induction, we construct an increasing sequence x_n and a decreasing sequence $\varepsilon_n > 0$ such that $x_n \geq 2^n$ and

$$\mathbf{E}\{e^{h(\xi)}; \xi \in (x_{n-1}, x_n]\} + e^{h(x_n)}\overline{F}(x_n) = e^{h(x_{n-1})}\overline{F}(x_{n-1}) + \delta/2^n$$

for any $n \geq 2$. For $n = 1$, this is already done. Make the induction hypothesis for some $n \geq 2$. Due to heavy-tailedness, there exists $x_{n+1} \geq 2^{n+1}$ sufficiently large that

$$\mathbf{E}\{e^{\varepsilon_n(\xi - x_n)}; \xi \in (x_n, x_{n+1}]\} + e^{\varepsilon_n(x_{n+1} - x_n)}\overline{F}(x_{n+1}) > 1 + \delta.$$

Note that

$$\mathbf{E}\{e^{\varepsilon_{n+1}(\xi - x_n)}; \xi \in (x_n, x_{n+1}]\} + e^{\varepsilon_{n+1}(x_{n+1} - x_n)}\overline{F}(x_{n+1})$$

as a function of ε_{n+1} is continuously decreasing to $\overline{F}(x_n)$ as $\varepsilon_{n+1} \downarrow 0$. Therefore, we can choose $\varepsilon_{n+1} \in (0, \varepsilon_n)$ so that

$$\mathbf{E}\{e^{\varepsilon_{n+1}(\xi - x_n)}; \xi \in (x_n, x_{n+1}]\} + e^{\varepsilon_{n+1}(x_{n+1} - x_n)}\overline{F}(x_{n+1})$$

$$= \overline{F}(x_n) + \delta/(2^{n+1}e^{h(x_n)}).$$

By definition of $h(x)$, this is equivalent to the following equality:

$$\mathbf{E}\{e^{h(\xi)}; \xi \in (x_n, x_{n+1}]\} + e^{h(x_{n+1})}\overline{F}(x_{n+1}) = e^{h(x_n)}\overline{F}(x_n) + \delta/2^{n+1}.$$

Our induction hypothesis now holds with $n + 1$ in place of n, as required.

Next,

$$\mathbf{E}e^{h(\xi)} = \sum_{n=0}^{\infty} \mathbf{E}\{e^{h(\xi)}; \xi \in (x_n, x_{n+1}]\}$$

$$= \sum_{n=0}^{\infty} \left(e^{h(x_n)}\overline{F}(x_n) - e^{h(x_{n+1})}\overline{F}(x_{n+1}) + \delta/2^{n+1}\right)$$

$$= e^{h(x_0)}\overline{F}(x_0) + \delta = 1 + \delta.$$

On the other hand, since $x_k \geq 2^k$,

$$\mathbf{E}\{\xi e^{h(\xi)}; \xi > x_n\} = \sum_{k=n}^{\infty} \mathbf{E}\{\xi e^{h(\xi)}; \xi \in (x_k, x_{k+1}]\}$$

$$\geq 2^n \sum_{k=n}^{\infty} \mathbf{E}\{e^{h(\xi)}; \xi \in (x_k, x_{k+1}]\}$$

$$\geq 2^n \sum_{k=n}^{\infty} \left(e^{h(x_k)}\overline{F}(x_k) - e^{h(x_{k+1})}\overline{F}(x_{k+1}) + \delta/2^{k+1}\right).$$

Then, for any n,

$$\mathbf{E}\{\xi e^{h(\xi)}; \xi > x_n\} \geq 2^n \left(e^{h(x_n)}\overline{F}(x_n) + \delta/2^n\right) \geq \delta,$$

which implies that $\mathbf{E}\xi e^{h(\xi)} = \infty$. Also note that, necessarily, $\lim_{n\to\infty} \varepsilon_n = 0$; otherwise, $\liminf_{x\to\infty} h(x)/x > 0$ and ξ is light-tailed. The proof of the lemma is thus complete. □

5. Proof of Theorem 2

Since τ has a light-tailed distribution,

$$\mathbf{E}\tau (1 + \varepsilon)^{\tau-1} < \infty$$

for some sufficiently small $\varepsilon > 0$. By Lemma 2, there exists a concave increasing function h, $h(0) = 0$, such that $\mathbf{E}e^{h(\xi_1)} \leq 1 + \varepsilon$ and $\mathbf{E}\xi_1 e^{h(\xi_1)} = \infty$. Then, by concavity,

$$A_n \equiv \mathbf{E}e^{h(\xi_1 + \cdots + \xi_n)} \leq \mathbf{E}e^{h(\xi_1) + \cdots + h(\xi_n)} \leq (1 + \varepsilon)^n.$$

Combining, we get $\mathbf{E}\tau A_{\tau-1} < \infty$. All conditions of Theorem 4 are met and the proof is thus complete.

6. Fractional exponential moments

One can go further and obtain various results on lower limits and equivalences for heavy-tailed distributions F which have all finite power moments (e.g., Weibull and log-normal distributions). For instance, we have the following result (see Denisov, Foss and Korshunov [9] for the proof).

Suppose there exists α, $0 < \alpha < 1$, such that $\mathbf{E}e^{c\xi^{\alpha}} = \infty$ for all $c > 0$. If $\mathbf{E}e^{\delta\tau^{\alpha}} < \infty$ for some $\delta > 0$, then (1) holds.

7. Tail equivalence for randomly stopped sums

The following auxiliary lemma compares the tail behavior of the convolution tail and that of the exponentially transformed distribution.

Lemma 3. *Let the distribution F and the number $\gamma \geq 0$ be such that $\varphi(\gamma) < \infty$. Let the distribution G be the result of the exponential change of measure with parameter γ, that is, $G(\mathrm{d}u) = e^{\gamma u} F(\mathrm{d}u)/\varphi(\gamma)$. Let τ be any counting random variable such that $\mathbf{E}\varphi^{\tau}(\gamma) < \infty$ and let ν have the distribution $\mathbf{P}\{\nu = k\} = \varphi^{k}(\gamma)\mathbf{P}\{\tau = k\}/\mathbf{E}\varphi^{\tau}(\gamma)$. Then*

$$\liminf_{x\to\infty} \frac{\overline{G^{*\nu}}(x)}{\overline{G}(x)} \geq \frac{1}{\mathbf{E}\varphi^{\tau-1}(\gamma)} \liminf_{x\to\infty} \frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)}$$

and

$$\limsup_{x\to\infty} \frac{\overline{G^{*\nu}}(x)}{\overline{G}(x)} \leq \frac{1}{\mathbf{E}\varphi^{\tau-1}(\gamma)} \limsup_{x\to\infty} \frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)}.$$

Proof. Put

$$\widehat{c} \equiv \liminf_{x\to\infty} \frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)}.$$

By Lemma 1, $\widehat{c} \in [\mathbf{E}\tau, \infty]$. For any fixed $c \in (0, \widehat{c})$, there exists $x_0 > 0$ such that, for any $x > x_0$,

$$\overline{F^{*\tau}}(x) \geq c\overline{F}(x). \tag{14}$$

By the total probability law,

$$\overline{G^{*\nu}}(x) = \sum_{k=1}^{\infty} \mathbf{P}\{\nu = k\}\overline{G^{*k}}(x)$$

$$= \sum_{k=1}^{\infty} \frac{\varphi^{k}(\gamma)\mathbf{P}\{\tau = k\}}{\mathbf{E}\varphi^{\tau}(\gamma)} \int_{x}^{\infty} e^{\gamma y} \frac{F^{*k}(\mathrm{d}y)}{\varphi^{k}(\gamma)}$$

$$= \frac{1}{\mathbf{E}\varphi^{\tau}(\gamma)} \sum_{k=1}^{\infty} \mathbf{P}\{\tau = k\} \int_{x}^{\infty} e^{\gamma y} F^{*k}(\mathrm{d}y).$$

Integrating by parts, we obtain

$$\sum_{k=1}^{\infty} \mathbf{P}\{\tau = k\} \left[e^{\gamma x} \overline{F^{*k}}(x) + \int_x^{\infty} \overline{F^{*k}}(y) \, de^{\gamma y} \right] = e^{\gamma x} \overline{F^{*\tau}}(x) + \int_x^{\infty} \overline{F^{*\tau}}(y) \, de^{\gamma y}.$$

Also using (14) we get, for $x > x_0$,

$$\overline{G^{*\nu}}(x) \geq \frac{c}{\mathbf{E}\varphi^{\tau}(\gamma)} \left[e^{\gamma x} \overline{F}(x) + \int_x^{\infty} \overline{F}(y) \, de^{\gamma y} \right]$$

$$= \frac{c}{\mathbf{E}\varphi^{\tau}(\gamma)} \int_x^{\infty} e^{\gamma y} F(\mathrm{d}y) = \frac{c}{\mathbf{E}\varphi^{\tau-1}(\gamma)} \overline{G}(x).$$

Letting $c \uparrow \widehat{c}$, we obtain the first conclusion of the lemma. The proof of the second conclusion follows similarly. □

Lemma 4. *If* $0 < \widehat{\gamma} < \infty$, $\varphi(\widehat{\gamma}) < \infty$ *and* $\mathbf{E}(\varphi(\widehat{\gamma}) + \varepsilon)^{\tau} < \infty$ *for some* $\varepsilon > 0$, *then*

$$\liminf_{x \to \infty} \frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)} \leq \mathbf{E}\tau \varphi^{\tau-1}(\widehat{\gamma})$$

and

$$\limsup_{x \to \infty} \frac{\overline{F^{*\tau}}(x)}{\overline{F}(x)} \geq \mathbf{E}\tau \varphi^{\tau-1}(\widehat{\gamma}).$$

Proof. We apply the exponential change of measure with parameter $\widehat{\gamma}$ and consider the distribution $G(\mathrm{d}u) = e^{\widehat{\gamma}u} F(\mathrm{d}u)/\varphi(\widehat{\gamma})$ and the stopping time ν with the distribution $\mathbf{P}\{\nu = k\} = \varphi^k(\widehat{\gamma})\mathbf{P}\{\tau = k\}/\mathbf{E}\varphi^{\tau}(\widehat{\gamma})$. From the definition of $\widehat{\gamma}$, the distribution G is heavy-tailed. The distribution of ν is light-tailed because $\mathbf{E}e^{\kappa\nu} < \infty$ with $\kappa = \ln(\varphi(\widehat{\gamma}) + \varepsilon) - \ln \varphi(\widehat{\gamma}) > 0$. Hence,

$$\limsup_{x \to \infty} \frac{\overline{G^{*\nu}}(x)}{\overline{G}(x)} \geq \liminf_{x \to \infty} \frac{\overline{G^{*\nu}}(x)}{\overline{G}(x)} = \mathbf{E}\nu,$$

by Theorem 2. The result now follows from Lemma 3 with $\gamma = \widehat{\gamma}$, since $\mathbf{E}\nu = \mathbf{E}\tau \varphi^{\tau}(\widehat{\gamma})/\mathbf{E}\varphi^{\tau}(\widehat{\gamma})$. □

Proof of Theorem 3. In the case where F is heavy-tailed, we have $\widehat{\gamma} = 0$ and $\varphi(\widehat{\gamma}) = 1$. By Theorem 2, $c = \mathbf{E}\tau$, as required.

In the case $\widehat{\gamma} \in (0, \infty)$ and $\varphi(\widehat{\gamma}) < \infty$, the desired conclusion follows from Lemma 4. □

8. Supremum of a random walk

Hereafter, we need the notion of subexponential distributions. A distribution F on $\mathbf{R}^+$ is called *subexponential* if $\overline{F * F}(x) \sim 2\overline{F}(x)$ as $x \to \infty$.

Let $\{\xi_n\}$ be a sequence of independent random variables with a common distribution F on $\mathbf{R}$ and $\mathbf{E}\xi_1 = -m < 0$. Put $S_0 = 0$, $S_n = \xi_1 + \cdots + \xi_n$. By the strong law of large numbers (SLLN), $M = \sup_{n \geq 0} S_n$ is finite with probability 1.

Let F^I be the integrated tail distribution on $\mathbf{R}^+$, that is,

$$\overline{F^I}(x) \equiv \min\left(1, \int_x^\infty \overline{F}(y)\,dy\right), \qquad x > 0.$$

It is well known (see, e.g., Asmussen [1], Embrechts, Klüppelberg and Mikosch [12], Embrechts and Veraverbeke [13] and references therein) that if F^I is subexponential, then

$$\mathbf{P}\{M > x\} \sim \frac{1}{m}\overline{F^I}(x) \qquad \text{as } x \to \infty. \tag{15}$$

Korshunov [18] proved the converse: (15) implies subexponentiality of F^I. We now supplement this assertion with the following result.

Theorem 5. *Let F^I be long-tailed, that is, $\overline{F^I}(x+1) \sim \overline{F^I}(x)$ as $x \to \infty$. If, for some $c > 0$,*

$$\mathbf{P}\{M > x\} \sim c\overline{F^I}(x) \qquad \text{as } x \to \infty,$$

then $c = 1/m$ and F^I is subexponential.

Proof. Consider the defective stopping time

$$\eta = \inf\{n \geq 1 : S_n > 0\} \leq \infty$$

and let $\{\psi_n\}$ be i.i.d. random variables with common distribution function

$$G(x) \equiv \mathbf{P}\{\psi_n \leq x\} = \mathbf{P}\{S_\eta \leq x \mid \eta < \infty\}.$$

It is well known (see, e.g., Feller [14], Chapter XII) that the distribution of the maximum M coincides with the distribution of the randomly stopped sum $\psi_1 + \cdots + \psi_\tau$, where the counting random variable τ is independent of the sequence $\{\psi_n\}$ and is geometrically distributed with parameter $p = \mathbf{P}\{M > 0\} < 1$, that is, $\mathbf{P}\{\tau = k\} = (1 - p)p^k$ for $k = 0, 1, \ldots$. Equivalently,

$$\mathbf{P}\{M \in B\} = G^{*\tau}(B).$$

It follows from Borovkov [4], Chapter 4, Theorem 10, that if F^I is long-tailed, then

$$\overline{G}(x) \sim \frac{1-p}{pm}\overline{F^I}(x). \tag{16}$$

The theorem hypothesis then implies that

$$\overline{G^{*\tau}}(x) \sim \frac{cpm}{1-p}\overline{G}(x) \qquad \text{as } x \to \infty.$$

Therefore, by Theorem 3 with $\widehat{\gamma} = 0$, $c = \mathbf{E}\tau(1-p)/pm = 1/m$. It then follows from Korshunov [18] that F^I is subexponential. The proof is now complete. $\qquad\square$

9. The compound Poisson distribution

Let F be a distribution on $\mathbf{R}_+$ and t a positive constant. Let G be the compound Poisson distribution

$$G = \mathrm{e}^{-t} \sum_{n \geq 0} \frac{t^n}{n!} F^{*n}.$$

Considering τ in Theorem 3 with distribution $\mathbf{P}\{\tau = n\} = t^n \mathrm{e}^{-t}/n!$, we get the following result.

Theorem 6. *Let* $\varphi(\widehat{\gamma}) < \infty$. *If, for some* $c > 0$, $\overline{G}(x) \sim c\overline{F}(x)$ *as* $x \to \infty$, *then* $c = t \mathrm{e}^{t(\varphi(\widehat{\gamma})-1)}$.

Corollary 2. *The following statements are equivalent*:

  (i) *F is subexponential*;
 (ii) *G is subexponential*;
(iii) $\overline{G}(x) \sim t\overline{F}(x)$ *as* $x \to \infty$;
(iv) *F is heavy-tailed and* $\overline{G}(x) \sim c\overline{F}(x)$ *as* $x \to \infty$, *for some* $c > 0$.

Proof. Equivalence of (i), (ii) and (iii) was proven in Embrechts, Goldie and Veraverbeke [11], Theorem 3. The implication (iv) $\Rightarrow$ (iii) follows from Theorem 3 with $\widehat{\gamma} = 0$. $\quad\square$

Some local aspects of this problem for heavy-tailed distributions were discussed in Asmussen, Foss and Korshunov [2], Theorem 6.

10. Infinitely divisible laws

Let H be an infinitely divisible law on $[0, \infty)$. The Laplace transform of an infinitely divisible law F can be expressed as

$$\int_0^\infty \mathrm{e}^{-\lambda x} H(\mathrm{d}x) = \mathrm{e}^{-a\lambda - \int_0^\infty (1-\mathrm{e}^{-\lambda x})\nu(\mathrm{d}x)}$$

(see, e.g., Feller [14], Chapter XVII). Here, $a \geq 0$ is a constant and the Lévy measure ν is a Borel measure on $(0, \infty)$ with the properties $\mu = \nu(1, \infty) < \infty$ and $\int_0^1 x\nu(\mathrm{d}x) < \infty$. Put $F(B) = \nu(B \cap (1, \infty))/\mu$.

Relations between the tail behavior of measure H and of the corresponding Lévy measure ν were considered in Embrechts, Goldie and Veraverbeke [11], Pakes [19] and Shimura and Watanabe [22]. The local analog of that result was proven in Asmussen, Foss and Korshunov [2]. We strengthen the corresponding result of Embrechts, Goldie and Veraverbeke [11] in the following way.

Theorem 7. *The following assertions are equivalent*:

  (i) *H is subexponential*;
 (ii) *F is subexponential*;
(iii) $\overline{\nu}(x) \sim \overline{H}(x)$ *as* $x \to \infty$;

(iv) H is heavy-tailed and $\overline{v}(x) \sim c\overline{H}(x)$ as $x \to \infty$, for some $c > 0$.

Proof. Equivalence of (i), (ii) and (iii) was proven in Embrechts, Goldie and Veraverbeke [11], Theorem 1.

It remains to prove the implication (iv) $\Rightarrow$ (iii). It is pointed out in Embrechts, Goldie and Veraverbeke [11] that the distribution H admits the representation $H = G * F^{*\tau}$, where $\overline{G}(x) = O(\mathrm{e}^{-\varepsilon x})$ for some $\varepsilon > 0$ and τ has a Poisson distribution with parameter μ. Since H is heavy-tailed and G is light-tailed, F is necessarily heavy-tailed. Then, by Theorem 4, we get

$$\liminf_{x \to \infty} \frac{\overline{H}(x)}{\overline{F}(x)} \equiv \liminf_{x \to \infty} \frac{\overline{G * F^{*\tau}}(x)}{\overline{F}(x)} = \mathbf{E}\tau = \mu.$$

On the other hand, for $x > 1$,

$$\frac{\overline{H}(x)}{\overline{F}(x)} = \mu \frac{\overline{H}(x)}{\overline{v}(x)} \to \mu c \qquad \text{as } x \to \infty,$$

by assumption (iv). Hence, $c = 1$. $\qquad\square$

11. Branching processes

In this section, we consider the limit behavior of subcritical, age-dependent branching processes for which the Malthusian parameter does not exist.

Let $h(z)$ be the particle production generating function of an age-dependent branching process with particle lifetime distribution F (see Athreya and Ney [3], Chapter IV, Harris [16], Chapter VI for background). We take the process to be subcritical, that is, $A \equiv h'(1) < 1$. Let $Z(t)$ denote the number of particles at time t. It is known (see, e.g., Athreya and Ney [3], Chapter IV, Section 5, or Chistyakov [5]) that $\mathbf{E}Z(t)$ admits the representation

$$\mathbf{E}Z(t) = (1 - A) \sum_{n=1}^{\infty} A^{n-1} \overline{F^{*n}}(t).$$

It was proven in Chistyakov [5] for sufficiently small values of A and then in Chover, Ney and Wainger [6,7] for any $A < 1$ that $\mathbf{E}Z(t) \sim \overline{F}(t)/(1 - A)$ as $t \to \infty$, provided F is subexponential. The local asymptotics were considered in Asmussen, Foss and Korshunov [2].

Applying Theorem 3 with τ geometrically distributed and $\widehat{\gamma} = 0$, we deduce the following.

Theorem 8. *Let F be heavy-tailed, and, for some $c > 0$, $\mathbf{E}Z(t) \sim c\overline{F}(t)$ as $t \to \infty$. Then $c = 1/(1 - A)$ and F is subexponential.*

Acknowledgements

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