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One-dimensional backward stochastic differential equations whose coefficient is monotonic in y and non-Lipschitz in z

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In this paper we study one-dimensional BSDE's whose coefficient f is monotonic in y and non-Lipschitz in z. We obtain a general existence result when f has at most quadratic growth in z and ξ is bounded. We study the special case $f(t, y, z) = |z|^p$ where $p \in (1, 2]$. Finally, we study the case f has a linear growth in z, general growth in y and ξ is not necessarily bounded.

Keywords: backward stochatic differential equations; monotonic non-Lipschitz coefficient

1. Introduction

Nonlinear backward stochastic differential equations (BSDE) were first introduced by Pardoux and Peng (1990). They proved that when the terminal value ξ is square integrable and the coefficient f is uniformly Lipschitz in (y, z) there exists a unique solution with smooth square integrability properties. In the one-dimensional case, thanks to comparison properties, it is possible to obtain better existence results. For example, when f is only continuous in (y, z) there is a solution to the BSDE under the following assumptions:

- ξ is square integrable, and f has a uniform linear growth in y, z (see Lepeltier and San Martín 1996),

$$|f(t, y, z)| \leqslant C(1 + |y| + |z|);$$

- ξ is bounded and f has a superlinear growth in y and quadratic growth in z (see Lepeltier and San Martín 1998; Kobylanski 2000)

$$|f(t, y, z)| \leqslant l(y) + C|z|^2,$$

where $l > 0$ satisfies $\int_0^\infty \mathrm{d}x/l(x) = \int_{-\infty}^0 \mathrm{d}x/l(x) = \infty$.

We mention also the recent results obtained by Briand and Hu (2005), who assume superlinear growth in y, quadratic growth in z and an unbounded terminal random variable which should satisfy an exponential integrability condition. We also use in section 4 a localization method taken from that paper.

When the generator f has general growth in y there is no guarantee that a solution exists, even in the case where ξ is bounded. This is most easily seen when ξ is non-random and $f(t, \omega, y, z) = f(y)$, because in this case the BSDE is just an ordinary differential equation. We would like to mention the existence of local solutions for BSDE, proved in Lepeltier and San Martín (2002), where explosions may occur and a global solution does not exist.

This paper is related to the work of Pardoux (1999), who he studied the multidimensional case where f is assumed to be monotonic in y,

$$\langle f(t, y, z) - f(t, \hat{y}, z), y - \hat{y}\rangle \leqslant \mu|y - \hat{y}|^2, \tag{1.1}$$

for some $\mu \geqslant 0$. He also assumed f is Lipschitz in z and

$$|f(t, y, z)| \leqslant \varphi(|y|) + C|z| \tag{1.2}$$

for some non-decreasing continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$.

One of our objectives is to extend this (in the one-dimensional case) when f has at most quadratic growth in z and ξ is bounded (see Section 2). This result can also be seen as a generalization of the results in Lepeltier and San Martín (1998) where the monotone condition on f allows us to extend the superlinear condition in y to a quite general growth condition.

The purpose of Section 3 is to study, in a very particular setting, the situation when ξ is not bounded. For this purpose we study the special case

$$f(t, y, z) = |z|^2.$$

Interestingly, it turns out that for $\xi \in L^2$ the condition

$$\mathrm{E}(\exp(2\xi)) < \infty$$

is necessary and sufficient for the existence of a solution.

Section 4 is devoted to the general case where f is monotonic in y, has a linear growth in z, and satisfies a growth condition similar to (1.2), and $\xi \in L^p$. Here again is the monotonic condition that makes it possible to have a solution, if we think that we have a very general growth condition in y.

Let us introduce the basic notation and definitions we need for this paper. Let $(B_t)_{0 \leqslant t \leqslant T}$ be a d-dimensional standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here T is a fixed time. We also consider $(\mathcal{F}_t, 0 \leqslant t \leqslant T)$, the natural filtration of (B_t), where $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$. $\mathcal{P}$ denotes, as usual, the σ-algebra of predictable subsets of $\Omega \times [0, T]$. We write $L^p = L^p(\Omega, \mathcal{F}_T, \mathbb{P})$,

$$H^p(\mathbb{R}^d) = \left\{\varphi \in \mathcal{P}, \mathbb{R}^d\text{-valued} : \mathrm{E}\left(\int_0^T |\varphi_t|^p \mathrm{d}t\right) < \infty\right\}.$$

and

$$\mathcal{S}^p = \left\{ \varphi \in \mathcal{P},\ \mathbb{R}\text{-valued} : \mathrm{E}\left(\sup_{0 \leqslant t \leqslant T} |\varphi_t|^p \right) < +\infty \right\}.$$

Finally, we write $S = \bigcup_{p>1} S^p$.

We are given a random variable $\xi \in \mathcal{F}_T$ and a function $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, which is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. We say that $(Y_t, Z_t)_{0 \leqslant t \leqslant T}$, a pair of $\mathcal{F}_t$-progressively measurable processes, valued in $\mathbb{R}$ and $\mathbb{R}^d$ respectively, is a solution to the BSDE with terminal value ξ and generator f if the integrals $\int_t^T f(s, Y_s, Z_s)\mathrm{d}s$ and $\int_t^T Z_s \cdot \mathrm{d}B_s$ are well defined and

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \cdot \mathrm{d}B_s, \tag{1.3}$$

for all $0 \leqslant t \leqslant T$. We denote by $\mathcal{E}(\xi, f)$ the set of solutions of such a BSDE. We say that the solution is bounded whenever Y is a bounded process.

2. The general case with ξ bounded

The purpose of this section is to prove the following result:

Theorem 2.1. *Assume that ξ is bounded and the generator f satisfies the following conditions:*

- *for all (t, ω), the function $f(t, \omega, \cdot, \cdot)$ is continuous;*
- *(monotonicity in y) there exists $\mu \geqslant 0$ such that, for all $t, \omega, y, \hat{y}, z$,*

$$(f(t, \omega, y, z) - f(t, \omega, \hat{y}, z))(y - \hat{y}) \leqslant \mu(y - \hat{y})^2;$$

- *there exist $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, non-decreasing and continuous, and a constant $A \geqslant 0$ such that, for all t, ω, y, z,*

$$|f(t, \omega, y, z)| \leqslant \varphi(|y|) + A|z|^2.$$

Then there exists a bounded solution $(Y, Z) \in \mathcal{E}(\xi, f)$ with $Z \in H^2(\mathbb{R}^d)$. Moreover, there is a maximal bounded solution (Y^*, Z^*), that is, if $(\hat{Y}, \hat{Z})$ is another solution such that $\hat{Y} \in S$, $\hat{Z} \in H^2(\mathbb{R}^d)$, then $\hat{Y} \leqslant Y^*$.

Proof. Consider $C > 0$ and a continuous function $g^C : \mathbb{R} \to [0, 1]$ such that $g^C(y) = 1$ for $-C \leqslant y \leqslant C$ and $g^C(y) = 0$ if $|y| > 2C$. Define the new generator

$$h^C(t, y, z) = g^C(y) f(t, y, z).$$

Then we have the bound $|h^C(t, y, z)| \leqslant g^C(y)(\varphi(|y|) + A|z|^2) \leqslant \varphi(2C) + A|z|^2$. From Theorem 1 in Lepeltier and San Martín (1998) there exists a maximal bounded solution $(Y^C, Z^C) \in \mathcal{E}(\xi, h^C)$; in particular,

$$Y_t^C = \xi + \int_t^T g^C(Y_s^C) f(s,\, Y_s^C,\, Z_s^C)\mathrm{d}s - \int_t^T Z_s^C \cdot \mathrm{d}B_s.$$

For all even $n \geqslant 2$, and $a \in \mathbb{R}$, we have from Itô's formula,

$$\begin{aligned}\exp(at)(Y_t^C)^n = \exp(aT)\xi^n + n\int_t^T \exp(as)(Y_t^C)^{n-1} g^C(Y_s^C) f(s,\, Y_s^C,\, Z_s^C)\mathrm{d}s \\ - n\int_t^T \exp(as)(Y_s^C)^{n-1} Z_s^C \cdot \mathrm{d}B_s - \frac{n(n-1)}{2}\int_t^T \exp(as)(Y_s^C)^{n-2}|Z_s^C|^2\mathrm{d}s \\ - a\int_t^T \exp(as)(Y_s^C)^n\mathrm{d}s.\end{aligned}$$

Since $yf(s,\, y,\, z) \leqslant yf(s,\, 0,\, z) + \mu y^2$ and $y^{n-2} \geqslant 0$, we obtain

$$y^{n-1} f(s,\, y,\, z) \leqslant y^{n-1} f(s,\, 0,\, z) + \mu y^n$$

and, moreover,

$$\begin{aligned}g^C(y) y^{n-1} f(s,\, y,\, z) &\leqslant |y|^{n-1}(\varphi(0) + A|z|^2) g^C(y) + \mu y^n \\ &\leqslant (1 + y^n)\varphi(0) + A|z|^2|y|^{n-1} g^C(y) + \mu y^n \\ &\leqslant (1 + y^n)\varphi(0) + 2CA|z|^2 y^{n-2} + \mu y^n.\end{aligned}$$

We then obtain the following bound:

$$\begin{aligned}\exp(at)(Y_t^C)^n \leqslant \exp(aT)\xi^n + n\int_t^T \exp(as)(1 + (Y_s^C)^n)\varphi(0)\mathrm{d}s \\ + 2CnA\int_t^T \exp(as)|Y_s^C|^{n-2}|Z_s|^2\mathrm{d}s + n\mu\int_t^T \exp(as)(Y_s^C)^n\mathrm{d}s \\ - \frac{n(n-1)}{2}\int_t^T \exp(as)(Y_s^C)^{n-2}|Z_s^C|^2\mathrm{d}s - a\int_t^T \exp(as)(Y_s^C)^n\mathrm{d}s \\ + M_T^n - M_t^n,\end{aligned}$$

where M^n is a martingale.

If we choose n, a such that $n - 1 \geqslant 4CA$ and $a = (\varphi(0) + \mu)n$, we obtain

$$\exp(n(\varphi(0) + \mu)t)(Y_t^C)^n \leqslant \exp(n(\varphi(0) + \mu)T)(\xi^n + 1) + M_T^n - M_t^n,$$

and taking the conditional expectation with respect to $\mathcal{F}_t$ yields

$$\begin{aligned}\exp(n(\varphi(0) + \mu)t)(Y_t^C)^n &\leqslant \exp(n(\varphi(0) + \mu)T)\mathrm{E}(\xi^n + 1|\mathcal{F}_t) \\ &\leqslant \exp(n(\varphi(0) + \mu)T)(\|\xi\|_\infty^n + 1)\end{aligned}$$

This gives the a priori bound

$$|Y_t^C| \leqslant \exp((\varphi(0)+\mu)T)(\|\xi\|_\infty^n + 1)^{1/n} \leqslant \exp((\varphi(0)+\mu)\mathrm{T})(\|\xi\|_\infty + 1).$$

If C is chosen such that $C \geqslant \exp((\varphi(0)+\mu)T)(\|\xi\|_\infty + 1)$, then $|Y_t^C| \leqslant C$ which implies $g(Y_t^C) = 1$. Therefore (Y^C, Z^C) satisfies

$$Y_t^C = \xi + \int_t^T f(s, Y_s^C, Z_s^C)\mathrm{d}s - \int_t^T Z_s^C \cdot \mathrm{d}B_s$$

and is a bounded solution of the desired BSDE. The rest of the result follows immediately. □

Remark 2.1. We have in fact used the monotonicity only for the pair y and $\hat{y} = 0$.

3. The case $f(t, y, z) = |z|^2$

We consider the BSDE given by

$$Y_t = \xi + \int_t^T |Z_s|^2 \mathrm{d}s - \int_t^T Z_s \cdot \mathrm{d}B_s. \tag{3.1}$$

Theorem 3.1. *Assume* $\xi \in L^2$. *Then, there exists a solution* $(Y, Z) \in H^2(\mathbb{R}^{d+1})$ *to* (3.1) *if and only if* $\mathrm{E}(\exp(2\xi)) < \infty$.

Proof. Let (Y, Z) be a solution of (3.1). By Itô's formula we have

$$\exp(2Y_t) = \exp(2Y_0) + 2\int_0^t \exp(2Y_s) Z_s \cdot \mathrm{d}B_s = \exp(2Y_0) + M_t,$$

where M is a local martingale.

Consider $\tau_n = \inf\{t > 0 : Y_t \geqslant n\}$. We have that $\tau_n \uparrow T$, when $n \to \infty$. Then $M_{t\wedge\tau_n}$ is a bounded martingale and we have

$$\mathrm{E}(\exp(2Y_{\tau_n})) = \mathrm{E}(\exp(2Y_0)).$$

Finally, from Fatou's lemma we obtain

$$\mathrm{E}\Big(\liminf_{n\to\infty} \exp(2Y_{\tau_n})\Big) = \mathrm{E}(\exp(2\xi)) \leqslant \mathrm{E}(\exp(2Y_0)) < \infty.$$

We now assume that $\mathrm{E}(\exp(2\xi)) < \infty$. Let

$$M_t =: \mathrm{E}(\exp(2\xi)|\mathcal{F}_t) = M_0 + \int_0^t Z_s \cdot \mathrm{d}B_s.$$

By Itô's formula we have

$$\frac{1}{2}\log M_t = \frac{1}{2}\log M_0 + \frac{1}{2}\int_0^t \frac{Z_s}{M_s} \cdot \mathrm{d}B_s - \frac{1}{4}\int_0^t \left|\frac{Z_s}{M_s}\right|^2 \mathrm{d}s \tag{3.2}$$

and $\frac{1}{2}\log M_T = \xi$.

In order to prove that $(\frac{1}{2}\log M_t, \frac{1}{2}(Z_t/M_t))$ is solution of the BSDE, it is enough to prove that

$$\frac{Z_t}{M_t} \in H^2(\mathbb{R}^d), \qquad \frac{1}{2}\log M_t \in \mathcal{S}^2.$$

Since log is concave we have

$$\frac{1}{2}\log M_t = \frac{1}{2}\log \mathrm{E}(\exp(2\xi)|\mathcal{F}_t) \geqslant \mathrm{E}(\xi|\mathcal{F}_t) \geqslant -\mathrm{E}(\xi^-|\mathcal{F}_t) =: N_t.$$

Consider for $a > 0$ the stopping time

$$T_a = \inf\left\{ t > 0 : |N_t| > a, \int_0^t \left|\frac{Z_s}{M_s}\right|^2 \mathrm{d}s > a, \left|\int_0^t Z_s \cdot \mathrm{d}B_s\right| > a \right\} \wedge T.$$

Using (3.2) and the fact $(a+b+c)^2 \leqslant 3(a^2+b^2+c^2)$, we have

$$\int_0^{T_a} \left|\frac{Z_s}{M_s}\right|^2 \mathrm{d}s \leqslant 2\log M_0 - 4N_{T_a} + 2\int_0^{T_a} \frac{Z_s}{M_s} \cdot \mathrm{d}B_s$$

and

$$\left(\int_0^{T_a} \left|\frac{Z_s}{M_s}\right|^2 \mathrm{d}s\right)^2 \leqslant 12(\log M_0)^2 + 48(N_{T_a})^2 + 12\left(\int_0^{T_a} \frac{Z_s}{M_s} \cdot \mathrm{d}B_s\right)^2.$$

Taking the expectation and using the fact that N_t^2 is a submartingale, we obtain

$$\mathrm{E}\left(\int_0^{T_a} \left|\frac{Z_s}{M_s}\right|^2 \mathrm{d}s\right)^2 \leqslant 12(\log M_0)^2 + 48\mathrm{E}(\xi^-)^2 + 12\mathrm{E}\left(\int_0^{T_a} \left|\frac{Z_s}{M_s}\right|^2 \mathrm{d}s\right),$$

and finally

$$\mathrm{E}\left(\int_0^{T_a} \left|\frac{Z_s}{M_s}\right|^2 \mathrm{d}s\right)^2 \leqslant C_1\big((\log M_0)^2 + \mathrm{E}(\xi^-)^2 + 1\big) \leqslant C_2,$$

for some $C_2 > 0$. To obtain the last inequality we have used the fact $12x \leqslant x^2/2 + 72$. Since $T_a \nearrow T$ as $a \to \infty$ we obtain from the monotone convergence theorem

$$\mathrm{E}\left(\int_0^{T} \left|\frac{Z_s}{M_s}\right|^2 \mathrm{d}s\right) \leqslant \sqrt{C_2}, \tag{3.3}$$

that is, $Z/M \in H^2(\mathbb{R}^d)$. Now from (3.2) we have

$$(\log M_t)^2 \leqslant 3\left((\log M_0)^2 + \left(\int_0^t \frac{Z_s}{M_s} \cdot \mathrm{d}B_s\right)^2 + \frac{1}{4}\left(\int_0^t \left|\frac{Z_s}{M_s}\right|^2 \mathrm{d}s\right)^2\right),$$

which, together with the Burkholder–Davis–Gundy inequality, finally gives

$$\mathrm{E}\left(\sup_{0\leqslant t\leqslant T}\log M_t\right)^2 < \infty.$$

Remark 3.1. We notice that from the results in Briand and Hu (2005), when the generator is $f(t, y, z) = |z|^p$ for $p \in [1, 2)$, a sufficient condition for having a solution is $\mathrm{E}(\mathrm{e}^{\varepsilon|\xi|}) < \infty$, for some positive ε. This follows from the fact that for any $\delta > 0$ there exists a positive constant $A(\delta)$ such that, for all z,

$$|z|^p \leqslant A(\delta) + \delta|z|^2.$$

Remark 3.2. Consider a bounded ξ and the BSDE

$$Y_t = \xi + \int_t^T (\alpha_s + |Z_s|^2)\mathrm{d}s - \int_t^T Z_s \cdot \mathrm{d}B_s \tag{3.4}$$

where $\alpha \in H^2(\mathbb{R})$ is such that

$$\mathrm{E}\left(\exp\left(2\int_0^T \alpha_s \mathrm{d}s\right)\right) = \infty \tag{3.5}$$

It is easy to see that (3.4) has a solution (Y, Z) if and only if $\widetilde{Y}_t = Y_t + \int_0^t \alpha_s \mathrm{d}s$ and Z is a solution of

$$\widetilde{Y}_t = \xi + \int_0^T \alpha_s \mathrm{d}s + \int_t^T |Z_s|^2 \mathrm{d}s - \int_t^T Z_s \cdot \mathrm{d}B_s.$$

This BSDE has no solution (in H^2) because of (3.5). This shows with a simple example that ξ bounded is not enough to obtain a solution when the generator f has quadratic growth in z. This is mainly due to the fact that, from (3.5), the process α is not bounded.

4. The linear increasing case in z

In this section we state and prove a general existence result for a BSDE where the generator is monotonic in y, has a general growth condition in this variable and has linear growth in z. We also assume that the terminal value has a finite p-moment. The following result is the main theorem of this section.

Theorem 4.1. *Assume that the following conditions hold:*

- *for all (t, ω), $f(t, \omega, \cdot, \cdot)$ is continuous;*
- *(monotonicity in y) there exists $\mu \geqslant 0$ such that, for all $t, \omega, y, \hat{y}, z$,*

$$(f(t, \omega, y, z) - f(t, \omega, \hat{y}, z))(y - \hat{y}) \leqslant \mu(y - \hat{y})^2;$$

- *there exist a non-decreasing and continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$\varphi(0)=0$, a constant $A \geqslant 0$ and a nonnegative continuous adapted process $\{g_t\}_{t\in[0,T]}$ such that $\mathbb{P}$-almost surely

$$|f(t, \omega, y, z)| \leqslant g_t(\omega) + \varphi(|y|) + A|z|;$$

- *for some $p > 1$,*

$$\mathrm{E}\left[|\xi|^p + \int_0^T g_s^p\,\mathrm{d}s\right] < \infty.$$

Then the BSDE (1.3) has a minimal solution in S which belongs to $S^p \times H^p$. Moreover, we have, if $c = \mu + A^2/(1 \wedge (p-1))$,

$$\mathrm{E}\left[\sup_{t\in[0,T]} \mathrm{e}^{cpt}|Y_t|^p + \left(\int_0^T \mathrm{e}^{2cs}|Z_s|^2\,\mathrm{d}s\right)^{p/2}\right] \leqslant C_p\,\mathrm{E}\left[\mathrm{e}^{cpT}|\xi|^p + \left(\int_0^T \mathrm{e}^{cs} g_s\,\mathrm{d}s\right)^p\right],$$

where the constant C_p depends only on p and also, for $a = |\mu| + (1-1/p) + A^2/2[1 \wedge (p-1)]$,

$$\forall t \in [0, T], \qquad |Y_t|^p \leqslant \mathrm{e}^{apT}\,\mathrm{E}\left(|\xi|^p + \int_0^T g_s^p\,\mathrm{d}s \middle| \mathcal{F}_t\right).$$

Remark 4.1. We can also construct a maximal solution in S with the same properties.

Proof. Let us assume for the moment that $\mu = 0$. Thus, for each (t, z), $y \mapsto f(t, y, z)$ is non-increasing.

For $n \geqslant A$, let us introduce the function

$$f_n(t, y, z) = \inf\{f(t, y, q) + n|z-q| : q \in \mathbb{Q}^d\}.$$

Then it is easy to check that, for each $n \geqslant A$, $\mathbb{P}$-almost surely,

- for all (t, z), $y \mapsto f_n(t, y, z)$ is non-increasing;
- for all (t, y), $z \mapsto f_n(t, y, z)$ is n–Lipschitz;
- for all (t, y, z), $|f_n(t, y, z)| \leqslant g_t + \varphi(|y|) + A|z|$.

It follows from Briand *et al.* (2003: Theorem 4.2) that, for each $n \geqslant A$, the BSDE

$$Y_t^n = \xi + \int_t^T f_n(s, Y_s^n, Z_s^n)\,\mathrm{d}s - \int_t^T Z_s^n \cdot \mathrm{d}B_s$$

has a unique solution $(Y^n, Z^n) \in S^p \times H^p$.

On the other hand, we have, for each $n \geqslant A$,

$$y f_n(t, y, z) \leqslant g_t|y| + A|y|\,|z|.$$

Thus, from Briand *et al.* (2003: Proposition 3.2), we have the following inequality, with $c = A^2/(1 \wedge (p-1))$:

$$\mathrm{E}\left[\sup_{t\in[0,T]} \mathrm{e}^{cpt}|Y_t^n|^p + \left(\int_0^T \mathrm{e}^{2cs}|Z_s^n|^2\,\mathrm{d}s\right)^{\frac{p}{2}}\right] \leqslant C_p\,\mathrm{E}\left[\mathrm{e}^{cpT}|\xi|^p + \left(\int_0^T \mathrm{e}^{cs}g_s\,\mathrm{d}s\right)^p\right], \qquad (4.1)$$

where the constant C_p depends only on p. In particular, the sequence $((Y^n, Z^n))_{n\geqslant A}$ is bounded in $S^p \times H^p$.

We now turn to the key estimate. Let us fix $n \geqslant A$. Let a be a real number to be chosen later. We set $U_t = \mathrm{e}^{at}Y_t^n$ and $V_t = \mathrm{e}^{at}Z_t^n$. Then (U, V) solves the BSDE associated with $\zeta = \mathrm{e}^{aT}\xi$ and $F(s, y, z) = \mathrm{e}^{at} f_n(t, \mathrm{e}^{-at}y, \mathrm{e}^{-at}z) - ay$.

It follows from Briand *et al.* (2003: Corollary 2.3) that, if we let $c(p) = p[(p-1)\wedge 1]/2$,

$$\begin{aligned}&|U_t|^p + c(p)\int_t^T |U_s|^{p-2}\mathbf{1}_{U_s\neq 0}\,|V_s|^2\,\mathrm{d}s\\ &\leqslant |\zeta|^p + p\int_t^T |U_s|^{p-1}\hat{U}_s F(s, U_s, V_s)\,\mathrm{d}s - p\int_t^T |U_s|^{p-1}\hat{U}_s V_s\cdot\mathrm{d}B_s,\end{aligned}$$

where $\hat{u} = |u|^{-1}u\mathbf{1}_{|u|\neq 0}$. But we have

$$\hat{u}F(t, u, v) \leqslant G_t - a|u| + A|v|,$$

where $G_t = \mathrm{e}^{at}g_t$.

The previous inequality shows that, with probability one,

$$\int_0^T |U_s|^{p-2}\mathbf{1}_{U_s\neq 0}\,|V_s|^2\,\mathrm{d}s < +\infty.$$

Moreover, we have

$$p|u|^{p-1}\hat{u}F(t, u, v) \leqslant p|u|^{p-1}G_t - ap|u|^p + A|u|^{p-1}|v|$$

and, using Young's inequality for the first term and the fact that $ab \leqslant a^2/2\varepsilon + \varepsilon b^2/2$ for the third, we deduce that

$$p|u|^{p-1}\hat{u}F(t, u, v) \leqslant (p-1)|u|^p + G_t^p - ap|u|^p + \frac{pA^2}{2[(p-1)\wedge 1]}|u|^p + c(p)|u|^{p-2}\mathbf{1}_{|u|>0}|v|^2.$$

Choosing $a = (1 - 1/p) + A^2/2[(p-1)\wedge 1]$, we obtain

$$|U_t|^p \leqslant \zeta^p + \int_0^T G_s^p\,\mathrm{d}s - p\int_t^T |U_s|^{p-1}\hat{U}_s V_s\cdot\mathrm{d}B_s.$$

But since $(U, V) \in S^p \times H^p$, it follows from the Burkholder–Davis–Gundy inequality that $\{\int_0^t |U_s|^{p-1}\hat{U}_s V_s\cdot\mathrm{d}B_s\}_{t\in[0,t]}$ is a uniformly integrable martingale. Thus, taking the conditional expectation of the previous estimate, we obtain

$$|U_t|^p \leqslant \mathrm{E}\left(\zeta^p + \int_0^T G_s^p\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right)$$

from which we deduce, coming back to the definition of U, ζ and $(G_t)_{t\in[0,T]}$, that, with the notation $a = (1 - 1/p) + A^2/2[(p-1)\wedge 1]$,

$$\forall t \in [0, T], \forall n \geqslant A, \qquad |Y_t^n|^p \leqslant \mathrm{e}^{apT}\,\mathrm{E}\bigg(|\xi|^p + \int_0^T g_s^p\,\mathrm{d}s\bigg|\mathcal{F}_t\bigg). \tag{4.2}$$

For the rest of the proof, we set

$$M_t = \mathrm{e}^{apT}\,\mathrm{E}\bigg(|\xi|^p + \int_0^T g_s^p\,\mathrm{d}s\bigg|\mathcal{F}_t\bigg).$$

With inequality (4.2) to hand, let us construct the minimal solution to our BSDE.

For this purpose let us observe that the sequence $(f_n)_{n\geqslant A}$ is non-decreasing. Thus, it follows from the comparison theorem (see Briand and Hu 2005: Proposition 5) that

$$\forall n \geqslant A, \forall t \in [0, T], \qquad Y_t^n \leqslant Y_t^{n+1},$$

so that we set $Y_t = sup_{n\geqslant A} Y_t^n$.

Now, we use the same localization procedure as in Briand and Hu (2005). For $k \geqslant 1$, let τ_k be the following stopping time:

$$\tau_k = \inf\{t \in [0, T] : M_t + g_t \geqslant k\} \wedge T,$$

and we introduce the stopped process $Y_k^n(t) = Y_{t\wedge\tau_k}^n$ together with $Z_k^n(t) = Z_t^n \mathbf{1}_{t\leqslant\tau_k}$. (Y_k^n, Z_k^n) solves the BSDE

$$Y_k^n(t) = \xi_k^n + \int_t^T \mathbf{1}_{s\leqslant\tau_k} f_n(s, Y_k^n(s), Z_k^n(s)) - \int_t^T Z_k^n(s)\cdot \mathrm{d}B_s, \qquad 0 \leqslant t \leqslant T,$$

where $\xi_k^n = Y_{\tau_k}^n$.

It is very important to observe that, by construction $(Y_k^n)_{n\geqslant A}$ is non-decreasing in n and that from inequality (4.2),

$$\sup_{n\geqslant A}\sup_{t\in[0,T]} \|Y_k^n(t)\|_\infty \leqslant k.$$

Thus, if $\rho(y) = yk/max(|y|, k)$, we have

$$Y_k^n(t) = \xi_k^n + \int_t^T \mathbf{1}_{s\leqslant\tau_k} f_n\big(s, \rho(Y_k^n(s)), Z_k^n(s)\big) - \int_t^T Z_k^n(s)\cdot \mathrm{d}B_s$$

and

$$|\mathbf{1}_{s\leqslant\tau_k} f_n(s, \rho(y), z)| \leqslant k + \varphi(k) + A|z|.$$

Moreover, it follows from Dini's theorem that, $\mathbb{P}$-almost surely, for all $s \in [0, T]$, $\mathbf{1}_{s\leqslant\tau_k} f_n(s, \rho(y), z)$ converges to $\mathbf{1}_{s\leqslant\tau_k} f(s, \rho(y), z)$ uniformly on compact sets of $\mathbb{R}\times\mathbb{R}^d$.

Arguing as in Lepeltier and San Martín (1996), we can take the limit in n (k being fixed) in the previous equation in the space $S^2 \times H^2(\mathbb{R}^d)$. In particular, setting $Y_k = sup_{n\geqslant A} Y_k^n$, we know that Y_k is continuous and that there exists a process $Z_k \in H^2(\mathbb{R}^d)$ such that $\lim_{n\to\infty} Z_k^n = Z_k$ in $H^2(\mathbb{R}^d)$ and (Y_k, Z_k) solves the BSDE

$$Y_k(t) = \xi_k + \int_t^T \mathbf{1}_{s\leqslant\tau_k} f(s,\, \rho(Y_k(s)),\, Z_k(s)) - \int_t^T Z_k(s)\cdot \mathrm{d}B_s,$$

where $\xi_k = \sup_{n\geqslant A} Y^n_{\tau_k}$. Since $|Y_k(t)| \leqslant k$, this last equation can be rewritten as

$$Y_k(t) = \xi_k + \int_t^T \mathbf{1}_{s\leqslant\tau_k} f(s,\, Y_k(s),\, Z_k(s)) - \int_t^T Z_k(s)\cdot \mathrm{d}B_s. \tag{4.3}$$

But $\tau_k \leqslant \tau_{k+1}$, and thus we obtain, coming back to the definition of Y_k, Z_k and Y,

$$Y_{t\wedge\tau_k} = Y_{k+1}(t\wedge\tau_k) = Y_k(t), \qquad Z_{k+1}(t)\,\mathbf{1}_{t\leqslant\tau_k} = Z_k(t).$$

The Y_k are continuous processes and, moreover, $\mathbb{P}$-almost surely $\tau_k = T$ for k large enough so that Y is continuous on $[0, T]$.

Then we define Z on $[0, T]$ by setting

$$Z_t = Z_1(t)\,\mathbf{1}_{t\leqslant\tau_1} + \sum_{k\geqslant 2} Z_k(t)\,\mathbf{1}_{]\tau_{k-1},\tau_k]}(t),$$

so that $Z_t\,\mathbf{1}_{t\leqslant\tau_k} = Z_k(t)\,\mathbf{1}_{t\leqslant\tau_k} = Z_k(t)$ and (4.3) can be rewritten as

$$Y_{t\wedge\tau_k} = Y_{\tau_k} + \int_{t\wedge\tau_k}^{\tau_k} f(s,\, Y_s,\, Z_s)\mathrm{d}s - \int_{t\wedge\tau_k}^{\tau_k} Z_s\cdot \mathrm{d}B_s. \tag{4.4}$$

Finally, we have

$$\mathbb{P}\left(\int_0^T |Z_s|^2\,\mathrm{d}s = \infty\right) = \mathbb{P}\left(\int_0^T |Z_s|^2\,\mathrm{d}s = \infty,\, \tau_k = T\right) + \mathbb{P}\left(\int_0^T |Z_s|^2\,\mathrm{d}s = \infty,\, \tau_k < T\right)$$

$$\leqslant \mathbb{P}\left(\int_0^{\tau_k} |Z_k(s)|^2\,\mathrm{d}s = \infty\right) + \mathbb{P}(\tau_k < T),$$

and we deduce, since $\tau_k\uparrow T$, that $\mathbb{P}$-almost surely

$$\int_0^T |Z_s|^2\,\mathrm{d}s < \infty.$$

By sending k to infinity in (4.4), we deduce that (Y, Z) is a solution of (1.3). Since $Y^n\uparrow Y$, (4.2) holds true for Y and thus Y belongs to S^p. It follows from Lemma 3.1 and Proposition 3.2 in Briand *et al.* (2003) that $Z\in H^p$ and also that (4.1) is true for (Y, Z).

We now prove that (Y, Z) is minimal. Let (Y', Z') be a solution to the BSDE associated with ξ' and f', where $\xi\leqslant\xi'$ and $f\leqslant f'$. Since f_n is Lipschitz-continuous in z and decreasing in y, we obtain from the comparison theorem (see Briand and Hu 2005: Proposition 5), for all $t\in[0, T]$ and all $n\geqslant A$, that $Y^n_t\leqslant Y'_t$. Since $Y_t = sup_{n\geqslant A} Y^n_t$, we finally obtain $Y_t\leqslant Y'_t$.

For the general case, namely $\mu\neq 0$, we only have to make the change of variables $\widetilde{Y}_t = \mathrm{e}^{\mu t} Y_t$. □

Remark 4.2. Arguing exactly as in the proof of Proposition 3.2 in Briand *et al.* (2003), we can prove that, for all $t\in[0, T]$ and for all $n\geqslant A$,

$$|Y_t^n|^p \leqslant \mathrm{E}\left(\sup_{u\in[t,T]}|Y_u^n|^p \middle| \mathcal{F}_t\right) \leqslant C_p\,\mathrm{e}^{A^2T/(1\wedge(p-1))}\,\mathrm{E}\left(|\xi|^p + \left(\int_t^T g_s\,\mathrm{d}s\right)^p \middle| \mathcal{F}_t\right).$$

Thus we can weaken the last assumption of Theorem 4.1 slightly to

$$\exists p > 1, \qquad \mathrm{E}\left[|\xi|^p + \left(\int_0^T g_s\,\mathrm{d}s\right)^p\right] < +\infty.$$

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References

Briand, Ph., Delyon, B., Hu, Y., Pardoux, E. and Stoica, L. (2003) L^p solutions of backward stochastic differential equations. *Stochastic Process. Appl.*, **108**, 109–129.

Briand, Ph. and Hu, Y. (2005) BSDE with quadratic growth and unbounded terminal value. Preprint 05-07, IRMAR, Université Rennes 1.

Kobylanski, M. (2000) Backward stochastic differential equations and partial differential equations with quadratic growth. *Ann. Probab.*, **28**, 558–602.

Lepeltier, J.-P. and San Martín, J. (1996) Backward SDE's with continuous coefficient. *Statist. Probab. Lett.*, **32**, 425–430.

Lepeltier, J.-P. and San Martín, J. (1998) Existence for BSDE with superlinear-quadratic coefficient. *Stochastics Stochastics Rep.*, **63**, 227–240.

Lepeltier, J.-P. and San Martín, J. (2002) On the existence or non-existence of solutions for certain backward stochastic differential equations. *Bernoulli*, **8**, 123–137.

Pardoux, E. (1999) BSDE's, weak convergence and homogenization of semilinear PDE's in nonlinear analysis. In F.M. Clarke and J. Stern (eds), *Nonlinear Analysis, Differential Equations and Control*, pp. 503–549. Dordrecht: Kluwer Academic.

Pardoux, E. and Peng S. (1990) Adapted solutions of a backward stochastic differential equation. *Systems Control Lett.*, **14**, 55–61.

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