Bernoulli **13**(1), 2007, 54–79
DOI: 10.3150/07-BEJ5151

Inequalities for dominated martingales

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Let (M_n), (N_n) be two Hilbert-space-valued martingales adapted to some filtration $(\mathcal{F}_n)$, with corresponding difference sequences (d_n), (e_n), respectively. We assume that (N_n) weakly dominates (M_n), that is, for any convex non-decreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ and any $n = 1, 2, \ldots$ we have, almost surely, $\mathrm{E}(\phi(|d_n|)|\mathcal{F}_{n-1}) \leqslant \mathrm{E}(\phi(|e_n|)|\mathcal{F}_{n-1})$. We apply the Burkholder method to show that for a convex non-decreasing function $\Phi: \mathbb{R}_+ \to \mathbb{R}_+$ satisfying some extra conditions we have, for any $n = 1, 2, \ldots$, $\|M_n\|_\Phi \leqslant C_\Phi \|N_n\|_\Phi$, where $\|\cdot\|_\Phi$ denotes an Orlicz norm with respect to Φ and C_Φ is a constant which depends only on Φ. This approach unifies and extends the classical Burkholder inequalities for subordinated martingales and the inequalities for tangent martingales. The method leads to moment inequalities for Rosenthal-type dominated martingales and variance-dominated Gaussian martingales. All the constants obtained in the moment inequalities are of optimal order.

Keywords: martingales; Orlicz space; subordinated martingales

1. Introduction and notation

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n), \mathcal{P})$ be a probability space equipped with a discrete filtration (we will assume that $\mathcal{F}_0 = \{\varnothing, \Omega\}$). Suppose that (M_n), (N_n) are Hilbert-space-valued martingales with difference sequences (d_n), (e_n), respectively, and $M_0 = N_0 = 0$ almost surely (a.s.). Burkholder's famous result (see Burkholder 1988, 1989) states that if (M_n) is differentially subordinated by (N_n), that is, with probability 1,

$$|d_n| \leqslant |e_n|,$$

then we have the following inequalities: for all $t > 0$ and all natural n,

$$t\mathrm{P}(|M_n| \geqslant t) \leqslant 2\mathrm{E}|N_n| \tag{1}$$

(the so-called *weak-type inequality*) and, for any $1 < p < \infty$ and any natural n,

$$(\mathrm{E}|M_n|^p)^{1/p} \leqslant (p^* - 1)(\mathrm{E}|N_n|^p)^{1/p} \tag{2}$$

(the *strong-type inequality*), where $p^* = \max\{p, p/(p-1)\}$.

It is worth mentioning that these results were obtained by a method, invented by Burkholder, which reduces the problem of proving a martingale inequality to one of finding a function of two variables having special convex-type properties.

These results have many extensions; the subordination principle may be replaced by various conditions (called *dominations*). For examples of such extensions we refer to Kwapień and Woyczyński (1992) and references therein.

The aim of this paper is to give another generalization of these results. First we fix some notation. Throughout the paper, $\mathcal{H}$ is a Hilbert space, with inner product denoted by $(\cdot, \cdot)$ and norm denoted by $|\cdot|$. The symbol I_A stands for the indicator function of a set A, and the complement of any set A is denoted by A^c. (M_n), (N_n) are two $(\mathcal{F}_n)$-adapted martingales taking values in $\mathcal{H}$. Their difference sequences are denoted by (d_n), (e_n), respectively. All the random variables considered in this paper are assumed to take values in some separable Hilbert space. For any convex function $\Phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\Phi(0) = \Phi'(0) = 0$, we define an $\mathcal{H}$-Orlicz space $L_\Phi^{\mathcal{H}}$ as a set of such $\mathcal{H}$-valued random variables X, such that for some $\varepsilon > 0$,

$$\mathrm{E}\Phi(\varepsilon|X|) < \infty.$$

The set $L_\Phi^{\mathcal{H}}$ is a Banach space with a norm

$$\|X\|_\Phi = \inf\left\{c > 0 : \mathrm{E}\Phi\left(\frac{|X|}{c}\right) \leqslant 1\right\}.$$

We now introduce the following essential idea.

Definition 1. *We say that a martingale (M_n) is weakly dominated by a martingale (N_n) if, for any non-decreasing convex function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ and any $n \geqslant 1$, we have, almost surely,*

$$\mathrm{E}(\phi(|d_n|)|\mathcal{F}_{n-1}) \leqslant \mathrm{E}(\phi(|e_n|)|\mathcal{F}_{n-1}). \tag{3}$$

We will write $M \prec_C N$.

Such domination generalizes the subordination as well as tangency of martingales and leads to other interesting (and much weaker) dominations, as we will see. The weak domination was investigated by Stephen Montgomery-Smith and Shih-Chi Shen (n.d.), where the strong-type inequality in this setting was proved. We will generalize this result to the inequality between Orlicz norms of weakly dominated martingales and, in particular, obtain that the strong-type inequality holds with a constant C_p of order p as $p \to \infty$ and $1/(p-1)$ as $p \to 1^+$, which is optimal, since it is already optimal in the case of subordinated martingales. Moreover, we prove weak-type inequality and give further extensions and applications. Our paper refines the results given by Kwapień and Woyczyński (1989, 1992), as the weak domination generalizes the conditions on martingales investigated in these papers. For other related results concerning tail probabilities, see, for example, de la Peña (1993).

The paper is organized as follows. The next section is devoted to the inequality between Orlicz norms of weakly dominated martingales; Burkholder's method turns out to be very useful in this setting. In particular, we obtain the strong-type inequality and, as a by-product, the weak-type inequality. In Section 3 we show that the assumption of weak domination may be replaced by much weaker conditions.

We then present some applications: we obtain that, for $p \geqslant 2$, the strong-type inequality (2) holds for martingales (M_n), (N_n) satysfying the following condition: for any positive integer n, with probability 1,

$$\mathrm{E}(|d_n|^2|\mathcal{F}_{n-1}) \leqslant \mathrm{E}(|e_n|^2|\mathcal{F}_{n-1}), \qquad \mathrm{E}(|d_n|^p|\mathcal{F}_{n-1}) \leqslant \mathrm{E}(|e_n|^p|\mathcal{F}_{n-1}).$$

The constant in the inequality is of optimal order $O(p)$.

As a second application, we prove that for Gaussian martingales the weak and strong $(1 < p < \infty)$ type inequalities (1), (2) hold if we assume that for any positive integer $n \geqslant 1$, almost surely,

$$\mathrm{E}(|d_n|^2|\mathcal{F}_{n-1}) \leqslant \mathrm{E}(|e_n|^2|\mathcal{F}_{n-1}).$$

The constant in the strong-type inequality is of optimal order $O(p)$ as $p \to \infty$ and $O(1/(p-1))$ as $p \to 1^+$.

In the final section we present some remarks concerning best constant in the weak-type inequality and present some arguments which lead to the special functions $u_{<2}$, $u_{>2}$, defined below.

2. The inequality between Orlicz norms

2.1. Two basis functions

We start by defining two important functions. Let

$$\begin{aligned} u_{<2}(x, y) &:= \begin{cases} 9|y|^2 - 9|x|^2, & \text{if } (x, y) \in D, \\ 2|y| - 1 + 8|y|^2 I_{\{|y| \leqslant 1\}} + (16|y| - 8) I_{\{|y|>1\}}, & \text{if } (x, y) \notin D, \end{cases} \\ u_{>2}(x, y) &:= \begin{cases} 0, & \text{if } (x, y) \in E, \\ 9|y|^2 - (|x|-1)^2 - 8(|x|-1)^2 I_{\{|x| \geqslant 1\}}, & \text{if } (x, y) \notin E, \end{cases} \end{aligned} \tag{4}$$

where

$$D = \{(x, y) \in \mathcal{H}^2 : |y| + 3|x| \leqslant 1\}, \qquad E = \{(x, y) \in \mathcal{H}^2 : 3|y| + |x| \leqslant 1\}.$$

It is straightforward to verify the following remarkable identity:

$$u_{>2}(x, y) = 9(|y|^2 - |x|^2) + u_{<2}(y, x), \qquad (x, y) \in \mathcal{H}^2. \tag{5}$$

We now prove the fundamental property of these functions.

Lemma 1. *If (M_n), (N_n) are two $\mathcal{H}$-valued martingales such that $M \prec_C N$, then for any $n = 1, 2, \ldots,$*

$$\mathrm{E}u_{<2}(M_n, N_n) \geqslant 0, \qquad \mathrm{E}u_{>2}(M_n, N_n) \geqslant 0. \tag{6}$$

Proof. Let $u = u_{<2}$ or $u = u_{>2}$. We will prove that u has the following crucial property: for any $(x, y) \in \mathcal{H}^2$, there exist operators $A_{x,y}, B_{x,y} \in \mathcal{H}^*$ and a convex non-decreasing function $\phi = \phi_{x,y}\colon \mathbb{R}_+ \to \mathbb{R}$ such that, for any $h, k \in \mathcal{H}$, we have the inequality

$$u(x, y) + A_{x,y}h + B_{x,y}k + \phi(|k|) - \phi(|h|) \leqslant u(x+h, y+k). \tag{7}$$

First, we consider the case $u = u_{<2}$. Assume that $(x, y) \in D^c \cup \partial D$. Let us introduce the function $\hat{u}\colon \mathcal{H}^2 \to \mathbb{R}$ defined by

$$\hat{u}(x, y) = 2|y| - 1 + 8|y|^2 I_{\{|y|\leqslant 1\}} + (16|y| - 8)I_{\{|y|>1\}}.$$

If $y = 0$, then inequality (7) holds with $A_{x,y} = B_{x,y} = 0$, $\phi = \phi_{x,y} = 0$; indeed, we have $u(x+h, k) \geqslant -1 = u(x, 0)$. Suppose, then, that $y \neq 0$ and take

$$A_{x,y} = \frac{\partial \hat{u}}{\partial x}(x, y), \qquad B_{x,y} = \frac{\partial \hat{u}}{\partial y}(x, y), \qquad \phi = \phi_{x,y} \equiv 0.$$

The function $\hat{u}$ is obviously convex and it is easy to check that $u \geqslant \hat{u}$, with equality on the set $D^c \cup \partial D$. Therefore, we have

$$u(x, y) + A_{x,y}h + B_{x,y}k = \hat{u}(x, y) + \frac{\partial \hat{u}}{\partial x}(x, y)h + \frac{\partial \hat{u}}{\partial y}(x, y)k$$

$$\leqslant \hat{u}(x+h, y+k) \leqslant u(x+h, y+k).$$

Assume, then, that (x, y) belongs to D. We will prove that (7) holds with

$$A_{x,y} = \frac{\partial u}{\partial x}(x, y) = -18(x, \cdot), \qquad B_{x,y} = \frac{\partial u}{\partial y}(x, y) = 18(y, \cdot),$$

$$\phi(s) := \phi_{x,y}(s) = \begin{cases} 9s^2, & \text{if } s \leqslant 1 - |y|, \\ 18(1-|y|)s - 9(1-|y|)^2, & \text{if } s > 1 - |y|. \end{cases} \tag{8}$$

The function $s \mapsto 9s^2 - \phi(s)$ is non-decreasing; in particular, we have $\phi(s) \leqslant 9s^2$.

Suppose that $(x+h, y+k) \in D$. We must prove that

$$9|y+k|^2 - 9|x+h|^2 \geqslant 9|y|^2 - 9|x|^2 + 18(y, k) - 18(x, h) + \phi(|k|) - \phi(|h|)$$

or, equivalently,

$$9|k|^2 - \phi(|k|) \geqslant 9|h|^2 - \phi(|h|). \tag{9}$$

If $|h| \leqslant 1 - |y|$, the right-hand side is equal to 0 and the left-hand side is non-negative. For $|h| > 1 - |y|$, we have

$$|k| \geqslant |y| - |y+k| \geqslant |y| + 3|x+h| - 1 \geqslant |y| + 3|h| - 3|x| - 1$$

$$\geqslant |h| + (|h| - 1 + |y|) + (|h| - 3|x|) > |h| + 0 + (1 - |y| - 3|x|) \geqslant |h|$$

and (9) holds: the function $s \mapsto 9s^2 - \phi(s)$ is non-decreasing.

Now let $(x+h, y+k) \notin D$. If $|y+k| \geqslant 1$, then we must show that

$$18|y+k| - 9 \geqslant 9|y|^2 + 18(y, k) + \phi(|k|) - 9|x|^2 - 18(x, h) - \phi(|h|)$$

$$= 9|y+k|^2 + \phi(|k|) - 9|k|^2 - 9|x|^2 - 18(x, h) - \phi(|h|).$$

But $|k| \geqslant |y+k| - |y|$, so

$$9|y+k|^2 + \phi(|k|) - 9|k|^2 \leqslant 9|y+k|^2 + \phi(|y+k| - |y|) - 9(|y+k| - |y|)^2$$

$$= 18|y+k| - 9.$$

Hence, it suffices to show that $0 \geqslant -9|x|^2 - 18(x, h) - \phi(|h|)$ or

$$9|x + h|^2 \geqslant 9|h|^2 - \phi(|h|). \tag{10}$$

If $|h| \leqslant 1 - |y|$, then the inequality holds true. Suppose, conversely, that $|h| > 1 - |y|$. Fix $|x + h|$ and x. The right-hand side is a non-decreasing function of $|h|$; hence, it is maximal for $|h| = |x + h| + |x|$ and then

$$9|x|^2 - 18|x||h| \geqslant -18(1 - |y|)|h| + 9(1 - |y|)^2$$

or, after simplifications,

$$(1 - |x| - |y|)(2|h| - 1 - |x| + |y|) \geqslant 0.$$

This inequality holds: for $(x, y) \in D$ we have $1 - |x| - |y| \geqslant 0$ and $2|h| - 1 - |x| + |y| = 2(|h| + |y| - 1) + 1 - |x| - |y| \geqslant 0$.

The only remaining case is $(x + h, y + k) \notin D$, $|y + k| \leqslant 1$. We must prove that

$$8|y + k|^2 + 2|y + k| - 1 \geqslant 9|y|^2 + 18(y, k) + \phi(|k|) - 9|x|^2 - 18(x, h) - \phi(|h|).$$

If $|h| \leqslant 1 - |y|$, then $\phi(|h|) = 9|h|^2$, $\phi(|k|) \leqslant 9|k|^2$ and we may write

$$\begin{aligned} 8|y + k|^2 + 2|y + k| - 1 &= 9|y + k|^2 - (1 - |y + k|)^2 \geqslant 9|y + k|^2 - 9|x + h|^2 \\ &\geqslant 9|y|^2 + 18(y, k) + \phi(|k|) - 9|x|^2 - 18(x, h) - \phi(|h|). \end{aligned} \tag{11}$$

Suppose, then, that $|h| > 1 - |y|$ and consider a vector

$$h' = \frac{1 - |y|}{|h|} h.$$

We may use (11) for h', because $|h'| = 1 - |y|$:

$$\begin{aligned} &8|y + k|^2 + 2|y + k| - 1 \geqslant 9|y|^2 + 18(y, k) + \phi(|k|) - 9|x|^2 - 18(x, h') - \phi(|h'|) \\ &\quad = 9|y|^2 + 18(y, k) + \phi(|k|) - 9|x|^2 - 18(x, h) - \phi(|h|) + 18(x, h - h') + \phi(|h|) - \phi(|h'|) \end{aligned}$$

and we have

$$\begin{aligned} 18(x, h - h') + \phi(|h|) - \phi(|h'|) &\geqslant -18|x||h - h'| + 18(1 - |y|)(|h| - |h'|) \\ &= 18(|h| - 1 + |y|)(-|x| + 1 - |y|) \geqslant 0. \end{aligned}$$

Hence the proof for $u = u_{<2}$ is complete.

The case $u = u_{>2}$ follows easily from the preceding case due to formula (5). Let us write (7) for (y, x), (k, h):

$$u_{<2}(y + k, x + h) \geqslant u_{<2}(y, x) + A_{y,x}k + B_{y,x}h + \phi(|h|) - \phi(|k|).$$

Adding the equality

$$9|y + k|^2 - 9|x + h|^2 = 9|y|^2 - 9|x|^2 + 18(y, k) - 18(x, h) - 9|h|^2 + 9|k|^2$$

yields the desired result. Note that in this case the function $\phi = \phi_{x,y}$ is given by $\phi_{x,y}(s) = 9s^2$ if $(x, y) \notin E$ and

$$\phi_{x,y}(s) := \begin{cases} 0, & \text{if } s < 1 - |x|, \\ 9(s - 1 + |x|)^2, & \text{if } s \geqslant 1 - |x| \end{cases} \tag{12}$$

for $(x, y) \in E$. These functions are convex.

We now turn to inequalities (6). Fix $n \geqslant 1$, put in (7) $x = M_{n-1}$, $y = N_{n-1}$, $h = d_n$, $k = e_n$ and take conditional expectations with respect to $\mathcal{F}_{n-1}$; we obtain

$$\begin{aligned} \mathrm{E}(u(M_n, N_n)|\mathcal{F}_{n-1}) &\geqslant u(M_{n-1}, N_{n-1}) + \mathrm{E}(\phi(|e_n|)|\mathcal{F}_{n-1}) - \mathrm{E}(\phi(|d_n|)|\mathcal{F}_{n-1}) \\ &\geqslant u(M_{n-1}, N_{n-1}), \end{aligned} \tag{13}$$

hence, taking expectations, $\mathrm{E}u(M_n, N_n) \geqslant \mathrm{E}u(M_{n-1}, N_{n-1})$, which yields $\mathrm{E}u(M_n, N_n) \geqslant \mathrm{E}u(M_0, N_0) = 0$. The proof is complete. □

Remark 1. In the proof we use the condition (3) only for special functions ϕ defined by (8) or (12); therefore the inequality $\mathrm{E}u_{<2}(M_n, N_n) \geqslant 0$ (or $\mathrm{E}u_{>2}(M_n, N_n) \geqslant 0$) holds if we assume that (3) holds for the ϕs defined by (8) (or (12)). This will be taken up in Section 3.

As a corollary we will prove the weak-type inequality.

Theorem 1. *For all martingales (M_n), (N_n) taking values in the Hilbert space $\mathcal{H}$, such that $M \prec_C N$, and any $t > 0$, we have*

$$t\mathrm{P}(|M_n| \geqslant t) \leqslant 6\mathrm{E}|N_n|, \qquad n = 0, 1, 2, \ldots .$$

Proof. We will show that

$$v(x, y) := 18|y| - I_{|x| \geqslant 1/3} \geqslant u_{<2}(x, y). \tag{14}$$

If $(x, y) \in D$, then $|x| \leqslant \frac{1}{3}$, $|y| < 1$, hence

$$-9|x|^2 + I_{|x| \geqslant 1/3} \leqslant 0 \leqslant 18|y| - 9|y|^2,$$

which is (14). If $(x, y) \notin D$ and $|y| \leqslant 1$, then we must prove that

$$18|y| - I_{|x| \geqslant 1/3} \geqslant 2|y| - 1 + 8|y|^2,$$

which reduces to a trivial inequality $16|y| + 1 - I_{|x| \geqslant 1/3} \geqslant 8|y|^2$. The only remaining case is $(x, y) \notin D$ and $|y| > 1$. Then the inequality (14) takes form $18|y| - I_{|x| \geqslant 1/3} \geqslant 18|y| - 9$, which again is trivial.

Suppose now that (M_n), (N_n) are two martingales such that $M \prec_C N$, and fix $t > 0$. Then the martingale $(M_n/3t)$ is weakly dominated by the martingale $(N_n/3t)$. Therefore, by (14) and Lemma 1,

$$6\mathrm{E}|N_n| - t\mathrm{P}(|M_n| \geqslant t) = t\mathrm{E}v\left(\frac{M_n}{3t}, \frac{N_n}{3t}\right) \geqslant t\mathrm{E}u_{<2}\left(\frac{M_n}{3t}, \frac{N_n}{3t}\right) \geqslant 0,$$

which is the claim. □

2.2. The main result

In this subsection we will compare the Orlicz norms of two weakly dominated martingales.

Let $\Phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a convex, twice continuously differentiable function such that $\Phi(0) = \Phi'(0) = 0$. Let $\Psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be the function given by the differential equation

$$\Psi''(s) = \frac{\Phi'(s)}{s},\ s > 0,\ \Psi(0) = \Psi_+(0) = 0.$$

Our main result is the following.

Theorem 2. *Suppose that the function Φ satisfies one of the following conditions:*

(A) $\Psi''' < 0$, $\lim_{t\to\infty}\Phi''(t) = 0$, *and there exists* $C > 0$ *such that*

$$\Psi(Cs) + 8\Psi((C-1)s) \leqslant \Psi'(Cs)s, \qquad \textit{for all } s > 0. \tag{15}$$

(B) $\Psi''' > 0$, $\Phi''(0) = 0$ *and there exists* $C > 0$ *such that*

$$\Psi(Cs) + 8\Psi((C-1)s) \geqslant \Psi'(Cs)s, \qquad \textit{for all } s > 0. \tag{16}$$

Then for all martingales (M_n), (N_n) *taking values in the Hilbert space* $\mathcal{H}$, *such that* $M \prec_C N$, *the following inequality holds true:*

$$\|M_n\|_\Phi \leqslant C_\Phi \|N_n\|_\Phi, \qquad n = 0, 1, 2, \ldots,$$

where

$$C_\Phi = \begin{cases} 1/(3(C-1)), & \textit{if } (A) \textit{ holds}, \\ 3(C-1), & \textit{if } (B) \textit{ holds}. \end{cases}$$

Remark 2. Conditions (A) and (B) appear complicated and it is difficult to verify inequalities (15), (16). The following easier conditions imply (A) and (B), respectively.

(A′) $\lim_{t\to\infty}\Phi''(t) = 0$ and there exists $C > 0$ such that

$$9(C-1)\Phi'(s) \leqslant s\Phi''(s) < \Phi'(s), \qquad \text{for all } s > 0. \tag{17}$$

(B′) $\Phi''(0) = 0$ and there exists $C > 0$ such that

$$(C-1)\Phi'(s) \geqslant s\Phi''(s) > \Phi'(s), \qquad \text{for all } s > 0. \tag{18}$$

Proof of Remark 2. Suppose that (A′) holds and C satisfies (17). We have

$$\Psi'''(s) = \frac{s\Phi''(s) - \Phi'(s)}{s^2} < 0.$$

We will prove that

$$\Psi(Cs) + 8\Psi((C-1)s) \leqslant \frac{C}{9C-8}\Psi'(Cs)s + \frac{8C-8}{9C-8}\Psi'((C-1)s)s,$$

which is stronger than (15), because Ψ' is non-decreasing. It suffices to show that for all $s > 0$ we have

$$f(s) := \Psi(s) - \frac{1}{9C-8}\Psi'(s)s \leqslant 0.$$

But

$$f'(s) = \Psi'(s) - \frac{1}{9C-8}(\Psi''(s)s - \Psi'(s)) = \frac{9C-9}{9C-8}\Psi'(s) - \frac{1}{9C-8}\Phi'(s),$$

$$f''(s) = \frac{1}{9C-8}\left[(9C-9)\frac{\Phi'(s)}{s} - \Phi''(s)\right].$$

Hence $f(0) = f'(0) = 0$ and f is concave. Therefore it is non-positive.

Assume now that (B′) holds and C satisfies (18). We have

$$\Psi'''(s) = \frac{s\Phi''(s) - \Phi'(s)}{s^2} > 0.$$

Let $f(s) = \Psi(Cs) + 8\Psi((C-1)s) - \Psi'(Cs)s$. It can be computed that

$$f''(s) = C\left[(C-1)\frac{\Phi'(Cs)}{Cs} - \Phi''(Cs)\right] + 8(C-1)^2\frac{\Phi'((C-1)s)}{(C-1)s} \geqslant 0,$$

hence f is convex. In addition, $f(0) = f'(0) = 0$, so f is non-negative and the proof is complete. □

We now are ready to define an important class of functions. If Φ satisfies (A), then we take

$$w_\Phi(x, y) := -\int_0^\infty t^2\Psi'''(t)u_{<2}\left(\frac{x}{t}, \frac{y}{t}\right) \mathrm{d}t.$$

If Φ satisfies (B), then we define

$$w_\Phi(x, y) := \int_0^\infty t^2\Psi'''(t)u_{>2}\left(\frac{x}{t}, \frac{y}{t}\right) \mathrm{d}t.$$

These functions will henceforth be called *Burkholder functions (with respect to* Φ*)*. In the lemma below we derive the formulae for w_Φ.

Lemma 2. *If* Φ *satisfies* (A)*, then*

$$w_\Phi(x, y) = -2\Psi'(3|x| + |y|)3|x| + 16\Psi(|y|) + 2\Psi(3|x| + |y|).$$

Otherwise, if (B) holds, then

$$w_\Phi(x, y) = 2\Psi'(|x| + 3|y|)3|y| - 16\Psi(|x|) - 2\Psi(|x| + 3|y|).$$

Proof. We only prove the formula for when (A) holds; the calculations for when (B) holds are analogous. By the definition of $u_{<2}$,

$$\begin{aligned} w_\Phi(x,y) &= -\int_0^\infty t^2\Psi'''(t)u_{<2}\left(\frac{x}{t},\frac{y}{t}\right)\mathrm{d}t \\ &= -\int_0^{3|x|+|y|} t^2\Psi'''(t)\frac{2|y|}{t}\,\mathrm{d}t + \int_0^{3|x|+|y|} t^2\Psi'''(t)\,\mathrm{d}t \\ &\quad - 8\int_{|y|}^{3|x|+|y|} t^2\Psi'''(t)\frac{|y|^2}{t^2}\,\mathrm{d}t - 16\int_0^{|y|} t^2\Psi'''(t)\frac{|y|}{t}\,\mathrm{d}t \\ &\quad + 8\int_0^{|y|} t^2\Psi'''(t)\,\mathrm{d}t - \int_{3|x|+|y|}^\infty t^2\Psi'''(t)\frac{9|y|^2-9|x|^2}{t^2}\,\mathrm{d}t. \end{aligned}$$

Integrating by parts, using $\Psi(0)=\Psi'(0)=\lim_{s\to\infty}\Psi''(s)=0$, we obtain

$$\begin{aligned} w_\Phi(x,y) &= -2|y|[\Psi''(3|x|+|y|)(3|x|+|y|)-\Psi'(3|x|+|y|)] \\ &\quad + \Psi''(3|x|+|y|)(3|x|+|y|)^2 - 2\Psi'(3|x|+|y|)(3|x|+|y|) + 2\Psi(3|x|+|y|) \\ &\quad + 8|y|^2[-\Psi''(3|x|+|y|)+\Psi''(|y|)] - 16|y|[\Psi''(|y|)|y|-\Psi'(|y|)] \\ &\quad + 8[\Psi''(|y|)|y|^2 - 2\Psi'(|y|)|y| + 2\Psi(|y|) + 9(|y|^2-|x|^2)\Psi''(3|x|+|y|)] \\ &= -2\Psi'(3|x|+|y|)3|x| + 16\Psi(|y|) + 2\Psi(3|x|+|y|). \end{aligned}$$

□

Proof of Theorem 2. Suppose that (A) holds and C satisfies (15). We will prove that for all $(x,y)\in\mathcal{H}^2$ we have

$$\Phi\left(\frac{C}{C-1}|y|\right) - \Phi(3C|x|) \geqslant \frac{C^2}{2}w_\Phi(x,y). \tag{19}$$

As $\mathrm{E}w_\Phi(M_n,N_n)\geqslant 0$ (which follows from the definition of the function w_Φ and Lemma 1), this will prove that

$$\mathrm{E}\Phi\left(\frac{C}{C-1}|N_n|\right) \geqslant \mathrm{E}\Phi(3C|M_n|),$$

which yields the inequality from the theorem. Fix y and let $s=3|x|$. We must show that the function f given by

$$f(s) := \Phi(Cs) - \Phi\left(\frac{C}{C-1}|y|\right) - C^2[\Psi'(s+|y|)s - 8\Psi(|y|) - \Psi(s+|y|)]$$

is non-positive. But

$$f'(s) = C\Phi'(Cs) - C^2\Psi''(s+|y|)s = C^2s[\Psi''(Cs) - \Psi''(s+|y|)].$$

The function Ψ'' is decreasing, hence f has maximum in $\bar{s}$ such that $C\bar{s} = \bar{s} + |y|$. This maximal value is equal to

$$f(\bar{s}) = C^2[-\Psi'(C\bar{s})\bar{s} + 8\Psi((C-1)\bar{s}) + \Psi(C\bar{s})],$$

which is non-positive due to (15).

We will skip the proof of the case where (B) holds as it is tedious and very similar to that of (A). We wish only to mention that in this case the 'dual' inequality to (19) is

$$\Phi(3C|y|) - \Phi\left(\frac{C}{C-1}|x|\right) \geqslant \frac{C^2}{2} w_\Phi(x, y). \tag{20}$$

□

As an application, we will use Theorem 2 to obtain the inequality between pth moments of weakly dominated martingales. Let $p \in (1, \infty)$, $p \neq 2$ and set $\Phi(t) = t^p$. Then $\Psi(t) = (p-1)^{-1} t^p$ and inequalities (15), (16) take form

$$C^p + 8(C-1)^p \leqslant pC^{p-1}, \qquad C^p + 8(C-1)^p \geqslant pC^{p-1}.$$

Hence in both cases $1 < p < 2$, $2 < p < \infty$ the best constant C we can get from our proof is defined by the equation

$$C^p + 8(C-1)^p = pC^{p-1}. \tag{21}$$

Theorem 2 yields the strong-type inequality with $C_p = (3(C-1))^{-1}$ for $1 < p < 2$ and $C_p = 3(C-1)$ for $p > 2$; before we state this as a theorem, let us study the asymptotics of C_p. Let

$$s_0 = C^{-1}, \qquad A_p = C - 1 = \frac{1}{s_0} - 1.$$

Equation (21) can be rewritten in the form

$$8(1-s_0)^p = ps_0 - 1. \tag{22}$$

Hence, $s_0 \in (1/p, 1)$.

Lemma 3. *Let K be a positive real number such that*

$$(K-1)e^K = 8 \tag{23}$$

(i.e. $K \approx 2.04$). The asymptotics of A_p is as follows: $A_p/(p-1)$, considered as a function of variable p, is increasing, and

$$\lim_{p \to 1^+} \frac{A_p}{p-1} = \frac{1}{9}, \qquad \lim_{p \to \infty} \frac{A_p}{p-1} = \frac{1}{K}.$$

Proof. If we take the logarithm of both sides of (22), we get

$$\log 8 + p \log(1 - s_0) = \log(ps_0 - 1).$$

Now differentiate over p; after some simplifications we obtain

$$s_0' = \frac{1}{p(p-1)}\left(\frac{(1-s_0)\log(1-s_0)}{s_0}\cdot(ps_0-1)-(1-s_0)\right). \tag{24}$$

Let us investigate the function

$$f(p) := \frac{A_p}{p-1} = \frac{1}{(p-1)s_0} - \frac{1}{p-1}, \qquad p > 1.$$

We have

$$f'(p) = \frac{1}{(p-1)^2 s_0^2}(-s_0-(p-1)s_0'+s_0^2).$$

But f' is non-negative:

$$\begin{aligned} s_0+(p-1)s_0' &= \frac{ps_0-1}{p}\left(\frac{(1-s_0)\log(1-s_0)}{s_0}+\frac{ps_0-1+s_0}{ps_0-1}\right) \\ &= \left(s_0-\frac{1}{p}\right)\left(\frac{(1-s_0)\log(1-s_0)}{s_0}+1\right)+\frac{s_0}{p}-s_0^2+s_0^2 \\ &= s_0^2+\left(s_0-\frac{1}{p}\right)\left(\frac{(1-s_0)\log(1-s_0)}{s_0}+1-s_0\right) \\ &= s_0^2+\left(s_0-\frac{1}{p}\right)\cdot\frac{1-s_0}{s_0}\cdot(\log(1-s_0)+s_0) \leqslant s_0^2. \end{aligned} \tag{25}$$

Hence f is non-decreasing; it is also non-negative, therefore there exists a finite limit

$$g = \lim_{p\to 1^+} f(p) = \lim_{p\to 1^+}\frac{1-s_0}{(p-1)s_0} = \lim_{p\to 1^+}\frac{1-s_0}{p-1},$$

because $s_0 \to 1$ when $p \to 1^+$. Dividing both sides of (22) by $(p-1)^{p-1}(1-s_0)$ gives

$$8\left(\frac{1-s_0}{p-1}\right)^{p-1} = \frac{(p-1)s_0}{(p-1)^{p-1}(1-s_0)} - \frac{1}{(p-1)^{p-1}},$$

and if $p \to 1^+$, we see that g must satisfy the equation $8 = 1/g - 1$, hence $g = \frac{1}{9}$ and $f \geqslant \frac{1}{9}$.

On the other hand, $s_0 > 1/p$, hence f is bounded from above by 1 and there exists the finite limit $\bar{g} = \lim_{p\to\infty} f(p)$. Moreover, $s_0 \to 0$ as $p \to \infty$. Therefore,

$$\lim_{p\to\infty}(p-1)s_0 = \lim_{p\to\infty} ps_0 = \frac{1}{\bar{g}}$$

and letting $p \to \infty$ in (22) implies that $K = 1/\bar{g}$. Hence $f \leqslant 1/K$. □

Thus we have proved the following:

Theorem 3. *Let (M_n), (N_n) be two martingales as in Theorem 2. Then for any $n \geqslant 1$, we have*

$$(\mathrm{E}|M_n|^p)^{1/p} \leqslant C_p(\mathrm{E}|N_n|^p)^{1/p},$$

where

$$C_p = \begin{cases} \dfrac{1}{3A_p} < \dfrac{3}{p-1}, & \textit{for } 1 < p < 2, \\ 3A_p < \dfrac{3}{K}(p-1) \approx 1.48(p-1), & \textit{for } 2 < p < \infty. \end{cases}$$

3. Weakening the assumptions in Theorems 1–3

The crucial part of the proofs of all the theorems is Lemma 1. Following Remark 1, Theorem 2 holds true if for any $n \geqslant 1$, $t > 0$, $(x, y) \in \mathcal{H}^2$, we have

$$\mathrm{E}\left(\phi_{x,y}\left(\frac{|d_n|}{t}\right)|\mathcal{F}_{n-1}\right) \leqslant \mathrm{E}\left(\phi_{x,y}\left(\frac{|e_n|}{t}\right)|\mathcal{F}_{n-1}\right) \text{ a.s.,}$$

where $\phi_{x,y}$ is defined by (8) or (12), depending on whether (A) or (B) holds. Moreover, this assumption may be further relaxed; in the proof of the Theorem 2 we integrate the functions $u_{<2}$, $u_{>2}$ with the kernel $s \mapsto |s^2\Psi'''(s)|$; hence, we also integrate the inequalities (13) to obtain

$$\begin{aligned} \mathrm{E}(w_\Phi(M_n, N_n)|\mathcal{F}_{n-1}) - w_\Phi(M_{n-1}, N_{n-1}) \geqslant{}& \mathrm{E}(\Upsilon_{\Phi,M_{n-1},N_{n-1}}(|e_n|)|\mathcal{F}_{n-1}) \\ &- \mathrm{E}(\Upsilon_{\Phi,M_{n-1},N_{n-1}}(|d_n|)|\mathcal{F}_{n-1}), \end{aligned} \tag{26}$$

where

$$\Upsilon(s) = \Upsilon_{\Phi,x,y}(s) := \int_0^\infty |s^2\Psi'''(s)|\phi_{x/t,y/t}\left(\frac{s}{t}\right) \mathrm{d}t. \tag{27}$$

Therefore, the following inequality for 'integrated' ϕ is sufficient: for all x, y,

$$\mathrm{E}(\Upsilon_{\Phi,x,y}(|d_n|)|\mathcal{F}_{n-1}) \leqslant \mathrm{E}(\Upsilon_{\Phi,x,y}(|e_n|)|\mathcal{F}_{n-1}) \text{ a.s.,} \qquad n = 1,\ 2\ \ldots. \tag{28}$$

We now turn to Theorem 3, where we may derive the formulae for the function Υ. For $\Phi(t) = t^p$ we write $\Upsilon_{p,x,y} := \Upsilon_{\Phi,x,y}$. Note that $t^2\Psi'''(t) = p(p-2)t^{p-1}$.

Lemma 4. *If* $1 < p < 2$, *then*

$$\Upsilon_{p,x,y}(s) = \begin{cases} 9p(3|x| + |y|)^{p-2}s^2, & \textit{if } s \leqslant 3|x|, \\ 9p(3|x| + |y|)^{p-2}s^2 + \dfrac{18}{p-1}\Theta_p(s - 3|x|), & \textit{if } s \geqslant 3|x|, \end{cases} \tag{29}$$

where

$$\Theta_p(s) = \Theta_{p,x,y}(s) := (s + 3|x| + |y|)^p - (3|x| + |y|)^p$$
$$- p(3|x| + |y|)^{p-1}s - \frac{p(p-1)}{2}(3|x| + |y|)^{p-2}s^2.$$

If $p > 2$, then

$$\Upsilon_{p,x,y}(s) = \begin{cases} 9p(|x| + 3|y|)^{p-2}s^2, & \text{if } s \leqslant 3|y|, \\ 9p(|x| + 3|y|)^{p-2}s^2 + \frac{18}{p-1}\Theta_p(s - 3|y|), & \text{if } s \leqslant 3|y|, \end{cases} \tag{30}$$

where

$$\Theta_p(s) = \Theta_{p,x,y}(s) = (s + |x| + 3|y|)^p - (|x| + 3|y|)^p$$
$$- p(|x| + 3|y|)^{p-1}s - \frac{p(p-1)}{2}(|x| + 3|y|)^{p-2}s^2.$$

Proof. The following equation can easily be verified: for $a, b > 0$, $p > 1$, we have

$$\int_a^{a+b} t^{p-3}(a + b - t)^2 \,\mathrm{d}t$$
$$= \frac{2}{p(p-1)(p-2)}\left[(a+b)^p - a^p - pa^{p-1}b - \frac{p(p-1)}{2}a^{p-2}b^2\right]. \tag{31}$$

Let $1 < p < 2$. If $t < 3|x| + |y|$, then $(x/t, y/t) \notin D$ and $\phi_{x/t,y/t} \equiv 0$, so

$$\Upsilon_{p,x,y}(s) = p(2-p)\int_{3|x|+|y|}^{\infty} t^{p-1}\phi_{x/t,y/t}\left(\frac{s}{t}\right) \mathrm{d}t.$$

If $t \geqslant 3|x| + |y|$ (or $(x/t, y/t) \in D$), then we have

$$\phi_{x/t,y/t}\left(\frac{s}{t}\right) = \frac{9s^2}{t^2} - 9\left(\frac{s}{t} - 1 + \frac{|y|}{t}\right)_+^2.$$

If $s \leqslant 3|x|$, then for $t \geqslant 3|x| + |y|$ we have

$$\frac{s}{t} - 1 + \frac{|y|}{t} \leqslant \frac{3|x| + |y|}{t} - 1 \leqslant 0$$

and

$$\Upsilon_{p,x,y}(s) = 9p(2-p)\int_{3|x|+|y|}^{\infty} t^{p-1}\frac{s^2}{t^2}\,\mathrm{d}t = 9p(3|x| + |y|)^{p-2}s^2.$$

If, conversely, $s > 3|x|$, then for $t < s + |y|$ we have

$$\frac{s}{t} - 1 + \frac{|y|}{t} > 0$$

and

$$\Upsilon_{p,x,y}(s) = 9p(2-p)\left[\int_{3|x|+|y|}^{\infty} t^{p-1}\frac{s^2}{t^2}\,\mathrm{d}t - \int_{3|x|+|y|}^{s+|y|} t^{p-1}\frac{(s-t+|y|)^2}{t^2}\,\mathrm{d}t\right],$$

which, by (31), used with $a = 3|x| + |y|$ and $b = s - 3|x|$, is equal to

$$9p(3|x|+|y|)^{p-2}s^2 + \frac{18}{p-1}\left[(s-3|x|+3|x|+|y|)^p - (3|x|+|y|)^p\right.$$

$$\left. - p(3|x|+|y|)^{p-1}(s-3|x|) - \frac{p(p-1)}{2}(3|x|+|y|)^{p-2}(s-3|x|)^2\right],$$

as required. Again we skip the remaining part (for $p > 2$), which can be proved in the same manner. □

We will formulate the weakened conditions in terms of the following functions:

$$\psi_{<2,r}(s) := \begin{cases} s^2, & \text{if } s \leqslant r, \\ 2rs - r^2, & \text{if } s > r, \end{cases} \tag{32}$$

$$\psi_{>2,r}(s) := \begin{cases} 0, & \text{if } s \leqslant r, \\ (s-r)^2, & \text{if } s > r. \end{cases} \tag{33}$$

Moreover, for $1 < p < \infty$, $\alpha \in [0, 1]$, let

$$\Pi_{p,\alpha}(s) := \frac{p(p-1)}{2}s^2 + \left[(1+(s-\alpha)_+)^p - 1 - p(s-\alpha)_+ - \frac{p(p-1)}{2}(s-\alpha)_+^2\right]. \tag{34}$$

The function $\psi_{\cdot,r}$ has the following property: for $\lambda > 0$, $s > 0$, we have

$$\lambda^2\psi_{\cdot,r}(s = \psi_{\cdot,r}(\lambda s). \tag{35}$$

Theorem 4. *In Theorem 2 it suffices to assume that, for all $n \geqslant 1$ and any $r \in \mathbb{R}_+$ we have*

$$\mathrm{E}(\psi_{<2,r}(|d_n|)|\mathcal{F}_{n-1}) \leqslant \mathrm{E}(\psi_{<2,r}(|e_n|)|\mathcal{F}_{n-1}) \text{ a.s.,} \qquad \text{if (A) holds,}$$

$$\mathrm{E}(\psi_{>2,r}(|d_n|)|\mathcal{F}_{n-1}) \leqslant \mathrm{E}(\psi_{>2,r}(|e_n|)|\mathcal{F}_{n-1}) \text{ a.s.,} \qquad \text{if (B) holds.}$$

Moreover, Theorem 3 holds if, for any $n \geqslant 1$, $\alpha \in [0, 1]$ and $\beta \in \mathbb{R}_+$, we have

$$\mathrm{E}(\Pi_{p,\alpha}(\beta|d_n|)|\mathcal{F}_{n-1}) \leqslant \mathrm{E}(\Pi_{p,\alpha}(\beta|e_n|)|\mathcal{F}_{n-1}) \text{ a.s.} \tag{36}$$

Theorem 1 holds if, for all $n \geqslant 1$ and any $r \in \mathbb{R}_+$, we have

$$\mathrm{E}(\psi_{<2,r}(|d_n|)|\mathcal{F}_{n-1}) \leqslant \mathrm{E}(\psi_{<2,r}(|e_n|)|\mathcal{F}_{n-1}) \text{ a.s.}$$

Proof. Suppose Φ satisfies (A). It is enough to show that, for any $(x, y) \in D$, $n \geqslant 1$ and $t > 0$, we have

$$\mathrm{E}\left(\phi_{x/t,y/t}\left(\frac{|d_n|}{t}\right)|\mathcal{F}_{n-1}\right) \leqslant \mathrm{E}\left(\phi_{x/t,y/t}\left(\frac{|e_n|}{t}\right)|\mathcal{F}_{n-1}\right) \text{ a.s.} \tag{37}$$

If $(x/t,\ y/t) \notin D$, then $\phi_{x/t,y/t} \equiv 0$ and the inequality holds. If $(x/t,\ y/t) \in D$, then $t \geqslant |y|$ and, as one can easily verify, we have the identity

$$\phi_{x/t,y/t}\left(\frac{s}{t}\right) = \frac{9}{t^2}\psi_{<2,t-|y|}(s),$$

which yields (37) and also implies the weak-type inequality. When (B) holds, the proof is similar.

We will now deal with the second statement. It suffices to show that, for any $(x, y) \in \mathcal{H}^2$ and $n \geqslant 1$, we have

$$\mathrm{E}(\Upsilon_{p,x,y}(|d_n|)|\mathcal{F}_{n-1}) \leqslant \mathrm{E}(\Upsilon_{p,x,y}(|e_n|)|\mathcal{F}_{n-1}) \text{ a.s.} \tag{38}$$

For $1 < p < 2$ it is an immediate consequence of the equation

$$\Upsilon_{p,x,y}(s) = \frac{18}{p-1}(3|x| + |y|)^p \Pi_{p,\alpha}(\beta s),$$

where

$$\alpha = \frac{3|x|}{3|x| + |y|} \in [0,\ 1], \qquad \beta = \frac{1}{3|x| + |y|} > 0.$$

For $p > 2$ we use the fact

$$\Upsilon_{p,x,y}(s) = \frac{18}{p-1}(|x| + 3|y|)^p \Pi_{p,\alpha}(\beta s), \tag{39}$$

for

$$\alpha = \frac{3|y|}{|x| + 3|y|} \in [0,\ 1], \qquad \beta = \frac{1}{|x| + 3|y|} > 0. \tag{40}$$

□

3.1. Burkholder–Rosenthal-type inequality

Burkholder (1973) proved the following inequality: for $p \geqslant 2$, any positive integer n and any real-valued martingale (M_n) with difference sequence (d_n),

$$c_p^{-1}\left\{\left[\mathrm{E}\left(\sum_{k=1}^{n} \mathrm{E}(|d_k|^2|\mathcal{F}_{k-1})\right)^{p/2}\right]^{1/p} + \left(\mathrm{E}\sum_{k=1}^{n} |d_k|^p\right)^{1/p}\right\} \leqslant (\mathrm{E}|M_n|^p)^{1/p}$$

$$\leqslant C_p\left\{\left[\mathrm{E}\left(\sum_{k=1}^{n} \mathrm{E}(|d_k|^2|\mathcal{F}_{k-1})\right)^{p/2}\right]^{1/p} + \left(\mathrm{E}\sum_{k=1}^{n} |d_k|^p\right)^{1/p}\right\}. \tag{41}$$

The best order of the constants c_p and C_p as $p \to \infty$ is $O(p)$ and $O(p/\ln p)$ (see Hitczenko 1990). For extensions of this inequality see de la Peña *et al.* (2003) and references therein.

Therefore, in order to compare the pth moments of two martingales, $p \geqslant 2$, it suffices to compare the second and pth conditional moments of corresponding difference sequences; this motivates a domination introduced in the following theorem.

Theorem 5. *Let* $p \geqslant 2$. *Assume that* (M_n), (N_n) *are* $\mathcal{H}$*-valued martingales, such that, for* $k = 0, 1, 2, \ldots$, *we have, with probability* 1,

$$\mathrm{E}(|d_k|^2|\mathcal{F}_{k-1}) \leqslant \mathrm{E}(|e_k|^2|\mathcal{F}_{k-1}), \qquad \mathrm{E}(|d_k|^p|\mathcal{F}_{k-1}) \leqslant \mathrm{E}(|e_k|^p|\mathcal{F}_{k-1}).$$

Then

$$(\mathrm{E}|M_n|^p)^{1/p} \leqslant K_p(\mathrm{E}|N_n|^p)^{1/p}, \qquad n = 0, 1, 2, \ldots,$$

where

$$K_p = 2\left(C_p^p + \frac{9(C-1)^p(p-1)^{p-3}}{C^{p-2}}\right)^{1/p},$$

C *is defined by equation* (21) *and* C_p *comes from Theorem 3.*

Remark 3. It is easy to check that K_p is of order $O(p)$ as $p \to \infty$; in fact, $K_p < 3p$.

Remark 4. As noted by Zinn (1985), inequality (41), used twice, immediately yields the theorem, however, with the constant K_p of order $O(p^2/\ln p)$ as $p \to \infty$.

First we prove an additional fact.

Lemma 5. *Let* $\mathcal{H}$ *be a Hilbert space. For any* $x, y \in \mathcal{H}$, *we have*

$$|x+y|^p - |y|^p - p|y|^{p-2}(y, x) \geqslant \frac{|x|^2|y|^{p-2}}{2} I_{\{|x| \leqslant 2|y|\}} + (|x| - |y|)^p I_{\{|x| > 2|y|\}}.$$

Proof. We have

$$|x+y|^p - |y|^p - p|y|^{p-2}(y, x) = |x+y|^p - \frac{p}{2}|x+y|^2|y|^{p-2} - |y|^p + \frac{p}{2}|y|^p + \frac{p}{2}|y|^{p-2}|x|^2.$$

Fix $|y|$ and $|x|$; we will minimize the expression above. Consider a function $g\colon \mathbb{R}_+ \to \mathbb{R}$ given by $g(t) = t^p - \frac{1}{2}p|y|^{p-2}t^2$. Then $g'(t) = pt(t^{p-2} - |y|^{p-2})$, so g has global minimum in $|y|$ and for $t > |y|$ it is increasing.

Therefore, if $|x| \leqslant 2|y|$, then $g \geqslant g(|y|)$:

$$|x+y|^p - |y|^p - p|y|^{p-2}(y, x) \geqslant |y|^p - \frac{p}{2}|y|^2|y|^{p-2} - |y|^p + \frac{p}{2}|y|^p + \frac{p}{2}|y|^{p-2}|x|^2$$
$$= \frac{p}{2}|x|^2|y|^{p-2}.$$

If $|x| > 2|y|$, then $|x+y| \geqslant |x| - |y| > |y|$ and $g \geqslant g(|x| - |y|)$:

$$|x+y|^p - |y|^p - p|y|^{p-2}(y,x) \geqslant (|x|-|y|)^p - \frac{p}{2}(|x|-|y|)^2|y|^{p-2} - |y|^p + \frac{p}{2}|y|^p + \frac{p}{2}|y|^{p-2}|x|^2$$
$$= (|x|-|y|)^p - |y|^p + p|y|^{p-1}|x| \geqslant (|x|-|y|)^p.$$

□

Lemma 6. *Let* $p \geqslant 2$. *Then, for* $s \geqslant 0$ *and any* $\gamma \in [0, 1]$, *we have*

$$(1+s)^p - 1 - ps \leqslant \frac{e}{2}(p-1)^2 s^2 + (p-1)^{p-2} s^p$$
$$\leqslant \frac{p(p-1)}{2}(2s)^2 + (p-1)^{p-2}\left[\frac{\gamma^{p-2}}{2}(2s)^2 I_{\{2s\leqslant 2\gamma\}} + (2s-\gamma)^p I_{\{2s>2\gamma\}}\right].$$

Proof. To prove the first inequality, consider the function

$$f(s) := \frac{e}{2}(p-1)^2 s^2 + (p-1)^{p-2} s^p - (1+s)^p + 1 + ps.$$

Since $f(0) = f'(0) = 0$, it suffices to prove that f is convex. We have

$$f''(s) = e(p-1)^2 + p(p-1)[(p-1)s]^{p-2} - p(p-1)(1+s)^{p-2}.$$

If $s \leqslant 1/(p-1)$, then

$$f''(s) \geqslant e(p-1)^2 - p(p-1)\left(1+\frac{1}{p-1}\right)^{p-2} = (p-1)^2\left[e - \left(1+\frac{1}{p-1}\right)^{p-1}\right] > 0.$$

If $1/(p-1) < s < 1/(p-2)$, then

$$f''(s) > e(p-1)^2 + p(p-1) - p(p-1)\left(1+\frac{1}{p-2}\right)^{p-2}. \tag{42}$$

If $p \leqslant 3$ the right-hand side is greater than

$$e(p-1)^2 + p(p-1) - 2p(p-1) = (p-1)[p(e-1) - e] > 0.$$

If $p > 3$, then the right-hand side of (42) exceeds

$$e(p-1)^2 + p(p-1) - p(p-1)e = (p-1)(p-e) > 0.$$

The only remaining case is $s \geqslant 1/(p-2)$. Then we have $1 + s \leqslant (p-1)s$ and we immediately obtain $f''(s) > 0$.

Let us deal with the second inequality. We have

$$\frac{e}{2}(p-1)^2 s^2 = \frac{p(p-1)}{2}s^2 \cdot \frac{p-1}{p}e \leqslant \frac{p(p-1)}{2}(2s)^2.$$

Moreover, if $s \leqslant \gamma$, then

$$(p-1)^{p-2}s^p \leqslant (p-1)^{p-2}\gamma^{p-2}s^2 \leqslant (p-1)^{p-2}\frac{\gamma^{p-2}}{2}(2s)^2,$$

as required; if $s > \gamma$, then $s < 2s - \gamma$ and

$$(p-1)^{p-2}s^p = (p-1)^{p-2}\left(\frac{s}{2s-\gamma}\right)^p (2s-\gamma)^p \leqslant (p-1)^{p-2}(2s-\gamma)^p.$$

□

Proof of Theorem 5. Let w_p be the Burkholder function with respect to $\Phi(t) = t^p$. Unfortunately, inequalities (28) do not hold under the assumptions of Theorem 5. We will modify the function w_p; consider a function $\overline{w}\colon \mathcal{H}^2 \to \mathbb{R}$ given by

$$\overline{w}(x, y) := w_p(x, y) + 18(p-1)^{p-3}|y|^p.$$

By (20), we have

$$w_p(x, y) \leqslant \frac{2}{C^2}\left[(3C|y|)^p - \left(\frac{C}{C-1}|x|\right)^p\right] = \frac{2C^{p-2}}{(C-1)^p}[C_p^p|y|^p - |x|^p].$$

Therefore $\overline{w} \leqslant v$, where $v\colon \mathcal{H}^2 \to \mathbb{R}$ is given by

$$v(x, y) = \frac{2C^{p-2}}{(C-1)^p}\left[\left(C_p^p + \frac{9(C-1)^p(p-1)^{p-3}}{C^{p-2}}\right)|y|^p - |x|^p\right].$$

Therefore it suffices to prove the inequality $\mathrm{E}\overline{w}(M_n, 2N_n) \geqslant 0$. We have, by (26),

$$\begin{aligned}
&\mathrm{E}(\overline{w}(M_n, 2N_n)|\mathcal{F}_{n-1}) - \overline{w}(M_{n-1}, 2N_{n-1})\\
&\quad\geqslant \mathrm{E}(\Upsilon_{p,M_{n-1},2N_{n-1}}(2|e_n|)|\mathcal{F}_{n-1}) - \mathrm{E}(\Upsilon_{p,M_{n-1},2N_{n-1}}(|d_n|)|\mathcal{F}_{n-1}) \qquad (43)\\
&\qquad + 18(p-1)^{p-3}\mathrm{E}\big((|2N_n|^p - |2N_{n-1}|^p - p|2N_{n-1}|^{p-2}(2N_{n-1}, 2e_n)|\mathcal{F}_{n-1}\big).
\end{aligned}$$

We will show, that for any $x, y \in \mathcal{H}$ and any centred bounded variables X, Y, taking values in Hilbert-space $\mathcal{H}$, such that

$$\mathrm{E}|X|^2 \leqslant \mathrm{E}|Y|^2, \qquad \mathrm{E}|X|^p \leqslant \mathrm{E}|Y|^p,$$

we have

$$\mathrm{E}\Upsilon_{p,x,2y}(|X|) \leqslant \mathrm{E}\Upsilon_{p,x,2y}(|2Y|) + 18(p-1)^{p-3}\mathrm{E}\big(|2y+2Y|^p - |2y|^p - p|2y|^{p-2}(2y, 2Y)\big).$$

This will immediately imply that the right-hand side of (43) is non-negative almost surely. Divide both sides of the inequality above by $18(|x| + 6|y|)^p/(p-1)$ and substitute

$$\alpha = \frac{6|y|}{|x|+6|y|}, \qquad X := \frac{X}{|x|+6|y|},\ Y := \frac{Y}{|x|+6|y|}, \qquad y := \frac{2y}{|x|+6|y|}.$$

Then, by (39) and (40), we obtain the following inequality to prove:

$$\mathrm{E}\Pi_{p,\alpha}(|X|) \leqslant \mathrm{E}\Pi_{p,\alpha}(|Y|) + (p-1)^{p-2}\mathrm{E}(|y+2Y|^p - |y|^p - p|y|^{p-2}(y, 2Y)).$$

But $\mathrm{E}\Pi_{p,\alpha}(|X|) \leqslant \mathrm{E}\Pi_{p,0}(|X|)$, as for any $s \geqslant 0$ the function $\alpha \mapsto \Pi_{p,\alpha}(s)$ is non-increasing. Therefore, by Lemma 6 we have

$$\mathrm{E}\Pi_{p,0}(|X|) = \mathrm{E}[(1+|X|)^p - 1 - p|X|] \leqslant \frac{\mathrm{e}}{2}(p-1)^2\mathrm{E}|X|^2 + (p-1)^{p-2}\mathrm{E}|X|^p$$
$$\leqslant \frac{\mathrm{e}}{2}(p-1)^2\mathrm{E}|Y|^2 + (p-1)^{p-2}\mathrm{E}|Y|^p.$$

Now we apply the second inequality from Lemma 6 with $\gamma = |y|$; we may bound the expression from above by

$$\frac{p(p-1)}{2}\mathrm{E}(2|Y|)^2 + (p-1)^{p-2}\mathrm{E}\left(\frac{|y|^{p-2}}{2}(2|Y|)^2 I_{\{2|Y|\leqslant 2|y|\}} + (2|Y| - |y|)^p I_{\{2|Y|>2|y|\}}\right),$$

which, due to Lemma 6, does not exceed

$$\frac{p(p-1)}{2}\mathrm{E}(2|Y|)^2 + (p-1)^{p-2}\mathrm{E}\big(|y+2Y|^p - |y|^p - p|y|^{p-2}(y, 2Y)\big)$$
$$\leqslant \mathrm{E}\Pi_{p,\alpha}(|2Y|) + (p-1)^{p-2}\mathrm{E}\big(|y+2Y|^p - |y|^p - p|y|^{p-2}(y, 2Y)\big).$$

□

3.2. Gaussian martingales

The purpose of this subsection is to prove the strong- and weak-type inequalities for Gaussian martingales. We start from the definition.

Definition 2. *A martingale $(M_n, \mathcal{F}_n)$ is Gaussian if for any $n \geqslant 1$ the conditional distribution of d_n with respect to the σ-algebra $\mathcal{F}_{n-1}$ is Gaussian almost surely.*

We now introduce a new domination.

Definition 3. *Let (M_n), (N_n) be $(\mathcal{F}_n)$-martingales taking values in Hilbert space $\mathcal{H}$. We say that (M_n) is variance-dominated by (N_n) if, for any $n \geqslant 1$, we have*

$$\mathrm{E}(|d_n|^2|\mathcal{F}_{n-1}) \leqslant \mathrm{E}(|e_n|^2|\mathcal{F}_{n-1}) \text{ a.s.}$$

First we prove a lemma which enables us to relate the variance domination with the 'weakened' domination from Theorem 4.

Lemma 7. *Let X, Y be two centred Gaussian random variables with values in the Hilbert space $\mathcal{H}$, such that $\mathrm{E}|X|^2 \leqslant \mathrm{E}|Y|^2$. Then, for any $r > 0$, we have*

$$\mathrm{E}\psi_r(|X|) \leqslant \mathrm{E}\psi_r\left(\sqrt{\frac{\pi}{2}}|Y|\right), \tag{44}$$

where $\psi_r = \psi_{<2,r}$ is given by (32).

Proof. The function $s \mapsto \psi_r(\sqrt{s})$ is concave, so by the Jensen inequality we may write

$$\mathrm{E}\psi_r(|X|) = \mathrm{E}\psi_r(\sqrt{|X|^2}) \leqslant \psi_r(\sqrt{\mathrm{E}|X|^2}).$$

With no loss of generality we may assume that $\mathcal{H}$ is finite-dimensional. Let $k_1, k_2, \ldots, k_m$ be its orthonormal basis. The random variables

$$W_i = \sqrt{\frac{\mathrm{E}|Y|^2}{\mathrm{E}|(Y, k_i)|^2}}(Y, k_i)$$

are real, centred, Gaussian and have variance $\mathrm{E}|Y|^2$. Again by the Jensen inequality,

$$\begin{aligned}
\mathrm{E}\psi_r\left(\sqrt{\frac{\pi}{2}}|Y|\right) &= \mathrm{E}\psi_r\left(\sqrt{\frac{\pi}{2}|Y|^2}\right) \\
&= \mathrm{E}\psi_r\left(\sqrt{\frac{\pi}{2}\sum_{i=1}^m \frac{\mathrm{E}|(Y, k_i)|^2}{\mathrm{E}|Y|^2}\left[\frac{\mathrm{E}|Y|^2}{\mathrm{E}|(Y, k_i)|^2}|(Y, k_i)|^2\right]}\right) \\
&\geqslant \sum_{i=1}^m \frac{\mathrm{E}|(Y, k_i)|^2}{\mathrm{E}|Y|^2}\mathrm{E}\psi_r\left(\sqrt{\frac{\pi}{2}|W_i|^2}\right) = \mathrm{E}\psi_r\left(\sqrt{\frac{\pi}{2}}|W_1|\right).
\end{aligned}$$

Hence, it is enough to show that

$$\mathrm{E}\psi_r\left(\sqrt{\frac{\pi}{2}}|W_1|\right) \geqslant \psi_r(\sqrt{\mathrm{E}|W_1|^2}).$$

By (35), this inequality is equivalent to

$$\frac{\pi}{2}\mathrm{E}\psi_r(|W_1|) \geqslant \psi_{\sqrt{(\pi/2)}r}(\sqrt{\mathrm{E}|W_1|^2}). \tag{45}$$

We may assume that $\mathrm{E}|W_1|^2 = 1$. Suppose that $r \leqslant \sqrt{(2/\pi)}$. Then $\psi_{\sqrt{(\pi/2)}r}(1) = \sqrt{2\pi}r - \frac{\pi}{2}r^2$ and (45) takes the form

$$\frac{\pi}{2}\left(\sqrt{\frac{2}{\pi}}\int_0^r s^2 \exp\left(-\frac{s^2}{2}\right)\mathrm{d}s + \sqrt{\frac{2}{\pi}}\int_r^\infty (2rs - r^2)\exp\left(-\frac{s^2}{2}\right)\mathrm{d}s\right) \geqslant \sqrt{2\pi}r - \frac{\pi}{2}r^2. \tag{46}$$

Integrating by parts, we obtain

$$\sqrt{\frac{\pi}{2}}\left(r\exp\left(-\frac{r^2}{2}\right) + (1 + r^2)\int_0^r \exp\left(-\frac{s^2}{2}\right)\mathrm{d}s\right) \geqslant \sqrt{2\pi}r,$$

hence we must show that

$$f(r) := \int_0^r \exp\left(-\frac{s^2}{2}\right)\mathrm{d}s + \frac{r}{r^2+1}\exp\left(-\frac{r^2}{2}\right) - \frac{2r}{1+r^2} \geqslant 0.$$

But we have

$$f'(r) = \frac{2}{(r^2+1)^2}\left[\exp\left(-\frac{r^2}{2}\right) - 1 + r^2\right] \geqslant 0,$$

so inequality (46) holds.

On the other hand, if $r > \sqrt{2/\pi}$, then $\psi_{\sqrt{(\pi/2)}r}(1) = 1$ and the left-hand side of (45) is increasing (as a function of r). But the inequality is true for $r = \sqrt{2/\pi}$, so the proof is complete. □

Inequality (44) enables us to prove the weak-type inequality and the strong-type inequality for $1 < p < 2$; unfortunately, it fails to hold for $\psi_r = \psi_{>2,r}$ and in this case we need a different argument.

Lemma 9. *Let $p > 2$. Then, for any $\sigma > 0$, we have*

$$\sqrt{\frac{2}{\pi}}\int_0^\infty [(1+\sigma s)^p - 1 - ps\sigma]\exp\left(-\frac{s^2}{2}\right)\mathrm{d}s \leqslant \frac{p(p-1)}{2}(2\sigma)^2 + 2(2p\sigma)^p.$$

Proof. Consider the function $f(x) = [(1+x)^p - 1 - px]/x^2$. It is increasing for $x \geqslant 0$, as we have $f(x) = p(p-1)\int_0^1 (1-t)(1+tx)^{p-2}\,\mathrm{d}t$. Suppose that $\sigma \leqslant 1/p$. Then

$$\begin{aligned}
&\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{(1+\sigma s)^p - 1 - ps\sigma}{\sigma^2}\exp\left(-\frac{s^2}{2}\right)\mathrm{d}s\\
&\leqslant \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{(1+(1/p)s)^p - 1 - ps(1/p)}{1/p^2}\exp\left(-\frac{s^2}{2}\right)\mathrm{d}s\\
&\leqslant \sqrt{\frac{2}{\pi}}p^2\int_0^\infty [\exp(s) - 1 - s]\exp\left(-\frac{s^2}{2}\right)\mathrm{d}s\\
&= p^2\left(\frac{\sqrt{2e}}{\sqrt{\pi}}\int_{-\infty}^1 \exp\left(-\frac{s^2}{2}\right)\mathrm{d}s - 1 - \sqrt{\frac{2}{\pi}}\right) \leqslant p^2 \leqslant \frac{p(p-1)}{2}\cdot 4.
\end{aligned}$$

Suppose now that $\sigma > 1/p$. Then

$$\begin{aligned}
&\sqrt{\frac{2}{\pi}}\int_0^\infty [(1+\sigma s)^p - 1 - ps\sigma]\exp\left(-\frac{s^2}{2}\right)\mathrm{d}s\\
&\leqslant \sqrt{\frac{2}{\pi}}\int_0^{1/\sigma} (1+\sigma s)^p\exp\left(-\frac{s^2}{2}\right)\mathrm{d}s + \sqrt{\frac{2}{\pi}}\int_{1/\sigma}^\infty (1+\sigma s)^p\exp\left(-\frac{s^2}{2}\right)\mathrm{d}s\\
&\leqslant 2^p + \sqrt{\frac{2}{\pi}}\int_{1/\sigma}^\infty (2s\sigma)^p\exp\left(-\frac{s^2}{2}\right)\mathrm{d}s\\
&\leqslant (2p\sigma)^p + (2\sigma)^p\sqrt{\frac{2}{\pi}}\int_0^\infty s^p\exp\left(-\frac{s^2}{2}\right)\mathrm{d}s \leqslant 2\cdot(2p\sigma)^p.
\end{aligned}$$

□

We are ready to extend the previous results to the Gaussian setting.

Theorem 5. *Let (M_n), (N_n) be two Gaussian martingales taking values in the Hilbert space $\mathcal{H}$, such that N variance-dominates M. Then*

(a) *For $t > 0$, we have*

$$\mathrm{P}(|M_n| \geqslant t) \leqslant 3\sqrt{2\pi}\mathrm{E}|N_n|.$$

(b) *For $1 < p < \infty$ and $n \geqslant 0$, we have*

$$(\mathrm{E}|M_n|^p)^{1/p} \leqslant K_p(\mathrm{E}|N_n|^p)^{1/p},$$

where

$$K_p = \begin{cases} \sqrt{\frac{\pi}{2}}C_p & \text{if } 1 < p < 2, \\ 1 & \text{if } p = 2, \\ 2\left(C_p^p + \frac{18(C-1)^p p^p}{C^{p-2}(p-1)}\right)^{1/p}, & \text{if } p > 2, \end{cases}$$

C is defined by equation (21) *and C_p is the constant from Theorem 3.*

Remark 5. One can verify that for $1 < p < 2$ we have $K_p < 3.76/(p-1)$ and for $p > 2$ we have $K_p < 4p$.

Proof of Theorem 6. (a) This is an immediate consequence of the Theorem 4 and the Lemma 7.

(b) For $1 < p < 2$ we again apply Theorem 4 and Lemma 7. For $p = 2$ the proof is trivial. Suppose, then, that $p > 2$. The proof is similar to that of Theorem 5: we consider the function $\bar{w}\colon \mathcal{H}^2 \to \mathbb{R}$ given by

$$\bar{w}(x, y) = w_p(x, y) + \frac{36p^p}{p-1}|y|^p,$$

where w_p is the Burkholder function with respect to $\Phi(t) = t^p$. Then $\bar{w} \leqslant v$, where

$$v(x, y) = \frac{2C^{p-2}}{(C-1)^p}\left[(C_p^p + \frac{18(C-1)^p p^p}{C^{p-2}(p-1)})|y|^p - |x|^p\right].$$

To prove the theorem, it suffices to show that for any $x, y \in \mathcal{H}$ and any centred Gaussian random variables X, Y taking values in $\mathcal{H}$, such that $\mathrm{E}|X|^2 \leqslant \mathrm{E}|Y|^2$, we have

$$\mathrm{E}\Upsilon(|X|) \leqslant \mathrm{E}\Upsilon(2|Y|) + \frac{36p^p}{p-1}(\mathrm{E}|2y + 2Y|^p - |2y|^p),$$

where $\Upsilon = \Upsilon_{p,x,2y}$ is given by (30). The variable Y is symmetric, hence

$$\begin{aligned}\mathrm{E}|2y+2Y|^p &= \frac{1}{2}\mathrm{E}[|2y+2Y|^p + |2y-2Y|^p] \\ &= \frac{1}{2}\mathrm{E}\Big[(|2y|^2+|2Y|^2+2(2y,\,2Y))^{p/2} + (|2y|^2+|2Y|^2-2(2y,\,2Y))^{p/2}\Big] \\ &\geqslant \mathrm{E}(|2y|^2+|2Y|^2)^{p/2} \geqslant |2y|^p + \ \mathrm{E}|2Y|^p.\end{aligned}$$

Therefore it suffices to show that

$$\mathrm{E}\Upsilon(|X|) \leqslant \mathrm{E}\Upsilon(|2Y|) + \frac{36p^p}{p-1}\mathrm{E}|2Y|^p.$$

Let $\sigma^2 = \mathrm{E}|X|^2$. It is an easy matter to check that the function $s \mapsto \Upsilon(\sqrt{s})$ is convex (because $\Upsilon = \int_0^\infty t^{p-1}\phi(\cdot/t)\,\mathrm{d}t$ and all integrated functions have this property). Hence, by the Jensen inequality,

$$\begin{aligned}\mathrm{E}\Upsilon(|2Y|) + \frac{36p^p}{p-1}\mathrm{E}|2Y|^p &= \mathrm{E}\Upsilon(\sqrt{|2Y|^2}) + \frac{36p^p}{p-1}\mathrm{E}(|2Y|^2)^{p/2} \\ &\geqslant \Upsilon(2\sigma) + \frac{36}{p-1}(2p\sigma)^p \geqslant 9p(|x|+3\cdot 2|y|)^{p-2}(2\sigma)^2 + \frac{36}{p-1}(2p\sigma)^p.\end{aligned}$$

With no loss of generality we may assume that $\mathcal{H}$ is finite-dimensional; let $h_1, h_2, \ldots, h_m$ be its orthonormal basis. We have

$$X = \sum_{l=1}^m (h_l,\, X)h_l = \sum_{l=1}^m \sqrt{\frac{\mathrm{E}(h_l,\, X)^2}{\mathrm{E}|X|^2}}\left(\sqrt{\frac{\mathrm{E}|X|^2}{\mathrm{E}(h_l,\, X)^2}}(h_l,\, X)h_l\right).$$

The variables

$$W_l = \sqrt{\frac{\mathrm{E}|X|^2}{\mathrm{E}(h_l,\, X)^2}}(h_l,\, X), \qquad l = 1, 2, \ldots, m,$$

are real, Gaussian and have variance σ^2. Moreover,

$$\sum_{l=1}^m \frac{\mathrm{E}(h_l,\, X)^2}{\mathrm{E}|X|^2} = 1,$$

hence, by the Jensen inequality,

$$\begin{aligned}\mathrm{E}\Upsilon(|X|) = \mathrm{E}\Upsilon(\sqrt{|X|^2}) &= \mathrm{E}\Upsilon\left(\sqrt{\sum_{l=1}^m \frac{\mathrm{E}(h_l,\, X)^2}{\mathrm{E}|X|^2}|W_l|^2}\right) \\ &\leqslant \sum_{l=1}^m \frac{\mathrm{E}(h_l,\, X)^2}{\mathrm{E}|X|^2}\mathrm{E}\Upsilon(|W_l|) = \mathrm{E}\Upsilon(|W_1|),\end{aligned}$$

therefore we may assume that $\dim \mathcal{H} = 1$. Hence it suffices to show that, for $X \sim N(0, \sigma)$, we have

$$\mathrm{E}\Upsilon(|X|) \leqslant 9p(|x| + 6|y|)^{p-2}(2\sigma)^2 + \frac{36}{p-1}(2p\sigma)^p.$$

If we divide both sides by $18(|x| + 6|y|)^p/(p-1)$ and substitute

$$\sigma := \frac{\sigma}{|x| + 6|y|}, \qquad \alpha = \frac{6|y|}{|x| + 6|y|}, \qquad X := \frac{X}{|x| + 6|y|},$$

we obtain the following inequality to prove:

$$\mathrm{E}\Pi_{p,\alpha}(|X|) \leqslant \frac{p(p-1)(2\sigma)^2}{2} + 2(2p\sigma)^p.$$

But we have

$$\mathrm{E}\Pi_{p,\alpha}(|X|) \leqslant \mathrm{E}\Pi_{p,0}(|X|) = \sqrt{\frac{2}{\pi}}\int_0^\infty [(1+\sigma s)^p - 1 - ps\sigma]\exp(-s^2/2)\,\mathrm{d}s$$

and the result follows from Lemma 9.

4. Concluding remarks

Remark 6. As we have seen, the function $u_{<2}$ gives a constant 6 in the weak-type inequality. The function

$$u(x, y) = \begin{cases} |y|^2 - |x|^2, & \textit{if } |y| \leqslant \sqrt{2} - \sqrt{|x|^2 + 1}, \\ 2\sqrt{2}|y| - 1, & \text{otherwise} \end{cases}$$

yields the weak type inequality with the constant $2\sqrt{2}$. The best constant cannot be less than $1 + \sqrt{2}$ (an easy example). The details will appear elsewhere.

Remark 7. We will now explain how the functions $u_{<2}$, $u_{>2}$ were constructed. Let us first note, that the 'integration method', which leads from simple functions to the more complicated ones, has its origin in the case of differentially subordinated martingales; indeed, if we define

$$v_{<2}(x, y) = \begin{cases} |y|^2 - |x|^2, & \text{if } |x| + |y| \leqslant 1, \\ \\ 2|y| - 1 & \text{otherwise,} \end{cases}$$

$$v_{>2}(x, y) = (|y|^2 - (|x| - 1)^2)I_{|x|+|y|>1},$$

then it is easy to check that if a martingale (M_n) is subordinated by (N_n), then

$$\mathrm{E}v_{<2}(M_n, N_n) \geqslant 0, \qquad \mathrm{E}v_{>2}(M_n, N_n) \geqslant 0.$$

Hence, if we define, for $p < 2$,

$$w_p(x, y) = \int_0^\infty t^{p-1} v_{<2}\Big(\frac{x}{t}, \frac{y}{t}\Big)\,\mathrm{d}t = c_1\Big(\frac{1}{p-1}|y| - |x|\Big)(|x| + |y|)^{p-1}$$

and for $p > 2$,

$$w_p(x, y) = \int_0^\infty t^{p-1} v_{>2}\Big(\frac{x}{t}, \frac{y}{t}\Big)\,\mathrm{d}t = c_2((p-1)|y| - |x|)(|x| + |y|)^{p-1},$$

then we have $\mathrm{E}w_p(M_n, N_n) \geqslant 0$. The functions w_p are exactly those used by Burkholder to prove the strong-stype inequalities.

In this paper we modify the functions $v_{<2}$, $v_{>2}$ so that condition (7) is satisfied. Let us deal with $u_{<2}$; the other function can be treated in the same manner. The idea is the following: first we set $\mathcal{H} = \mathbb{R}$ and expect the constructed function to be easily extended to the Hilbert space setting. Then, in order to make the verification of condition (7) simpler, the function is sought in the class of the continuous functions $u : \mathbb{R}^2 \to \mathbb{R}$ of the form

$$u(x, y) = \begin{cases} |y|^2 - |x|^2, & (x, y) \in D, \\ f(|y|), & (x, y) \notin D. \end{cases}$$

Here D is a certain symmetric subset of $\mathbb{R}^2$ and f is a certain function, $f : \mathbb{R}_+ \to \mathbb{R}$, which satisfies the following conditions:

(a) it is quadratic for small arguments and linear for large ones;
(b) for $(x, y) \in D$ we have $|y|^2 - |x|^2 \geqslant f(y)$;
(c) for fixed x, the function $y \mapsto u(x, y)$, $y \in \mathbb{R}$, is convex.

In practice, we take a set $D = \{(x, y) : a|x| + b|y| = c\}$ (we need ∂D to be simple, as we want to integrate $u_{<2}$) and the function f is determined on a certain interval due to the continuity of $u_{<2}$. Then we extend it using conditions (a) and (c), trying to keep the slope of f (i.e. $f'(t)$ for large t) as small as possible (as it is proportional to the constants in the theorems).

Now we impose (7); then for $(x, y) \notin D$, by (b) and (c), it is automatically satisfied with $\phi_{x,y} \equiv 0$ and $A_{x,y}$, $B_{x,y}$ equal to partial derivatives of the function u. The only case which needs some calculations is $(x, y) \in D$, where again we set $A_{x,y}$, $B_{x,y}$ equal to the partial derivatives of u (i.e. $A_{x,y} = -2x$ and $B_{x,y} = 2y$). The function $\phi_{x,y}$ is sought in the set of the convex differentiable functions of the form $t \mapsto t^2 I_{\{t \leqslant T\}} + (at + b) I_{\{t > T\}}$.

It turns out that, up to the constant 9, the optimal function (i.e. such that f has the smallest slope for large arguments) is the one defined by (4).

Acknowledgement

The research for this paper was supported by KBN Grant GR-1712. It is a pleasure to thank Professor S. Kwapień for valuable remarks and encouragement.

References

Burkholder, D.L. (1973) Distribution function inequalities for martigales. *Ann. Probab.*, **1** (1973), 19–42.

Burkholder, D.L. (1988) Sharp inequalities for martingales and stochastic integrals. *Les Processus Stochastiques*, Colloque Paul Lévy, Palaiseau, 1987, *Astérisque*, **157–158**, 75–94.

Burkholder, D.L. (1989) Differential subordination of harmonic functions and martingales. In J. García-Cuerva (ed.), *Harmonic Analysis and Partial DIfferential Equations*, Lecture Notes in Math. 1384, pp. 1–23. Berlin: Springer-Verlag.

de la Peña, V.H. (1993) Inequalities for the tails of adapted processes with applications to Wald's lemma. *J. Theoret. Probab.*, **6**, 285–302.

de la Peña, G.H., Ibragimov, R. and Sharakhmetov, S. (2003) On extremal distributions and sharp L_p-bounds for sums of multilinear forms. *Ann. Probab.*, **31**, 630–675.

Hitczenko, P. (1990) Best constants in martingale version of Rosenthal's inequality. *Ann. Probab.*, **18**, 1656–1668.

Kwapień, S. Woyczyński, W.A. (1989) Tangent sequences of random variables: basic inequalities and their applications. In G.A. Edgar and L. Sucheston (eds), *Almost Everywhere Convergence*, pp. 237–265. Boston: Academic Press.

Kwapień, S. Woyczyński, W.A. (1992) *Random Series and Stochastic Integrals: Single and Multiple*. Boston: Birkhäuser.

Montgomery-Smith, S. and Shen, S.-C. (n.d.) An extension to the strong dominated martingale inequality. Unpublished paper. http://www.math.missouri.edu/_stephen/preprints/shen5.pdf.

Zinn, J. (1985) Comparison of martingale difference sequences. In A. Beck, R. Dudley, M. Hahn, J. Kuelbs and M. Marcus (eds), *Probability in Banach Spaces V*, Lecture Notes in Math. 1153, pp. 453–457. Berlin: Springer-Verlag.

Received October 2005 and revised July 2006