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Fractional Lévy processes with an application to long memory moving average processes

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Starting from the moving average (MA) integral representation of fractional Brownian motion (FBM), the class of fractional Lévy processes (FLPs) is introduced by replacing the Brownian motion by a general Lévy process with zero mean, finite variance and no Brownian component. We present different methods of constructing FLPs and study second-order and sample path properties. FLPs have the same second-order structure as FBM and, depending on the Lévy measure, they are not always semimartingales. We consider integrals with respect to FLPs and MA processes with the long memory property. In particular, we show that the Lévy-driven MA process with fractionally integrated kernel coincides with the MA process with the corresponding (not fractionally integrated) kernel and driven by the corresponding FLP.

Keywords: CARMA process; fractional integration; fractional Lévy process; long memory; Lévy process; stochastic integration

1. Introduction

In this paper we consider fractional Lévy processes. The term 'fractional Lévy process' itself suggests that it can be regarded as a generalization of fractional Brownian motion (FBM). Let us recall that FBM is the Gaussian stochastic process $\{B_H(t)\}_{t\geqslant 0}$ satisfying $B_H(0)=0$, $\mathrm{E}[B_H(t)]=0$ for all $t\geqslant 0$, and

$$\mathrm{E}[B_H(t)B_H(s)] = \tfrac{1}{2}(|t|^{2H} - |t-s|^{2H} + |s|^{2H}) \tag{1.1}$$

for all $s,\ t\geqslant 0$, where $0<H<1$. The parameter H is also referred to as the Hurst coefficient. FBM is the only self-similar Gaussian process with stationary increments. We can define a parametric family of FBMs in terms of the stochastic Weyl integral (see Doukhan *et al.* 2003: Part A; Samorodnitsky and Taqqu 1994: Section 7.2). For any $a,\ b\in\mathbb{R}$,

$$\{B_H(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{\int_{\mathbb{R}}\left\{a\,[(t-s)_+^{H-1/2} - (-s)_+^{H-1/2}] + b[(t-s)_-^{H-1/2} - (-s)_-^{H-1/2}]\right\}\mathrm{d}B(s)\right\}_{t\in\mathbb{R}}, \tag{1.2}$$

where $u_+=\max(u,0)$, $u_-=\max(-u,0)$ and $\{B(t)\}_{t\in\mathbb{R}}$ is a standard Brownian motion. If $H=\frac{1}{2}$, it is clear that $\{B_{1/2}(t)\}_{t\in\mathbb{R}}=\{B(t)\}_{t\in\mathbb{R}}$.

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If we choose $a = \sqrt{\Gamma(2H+1)\sin(\pi H)}/\Gamma(H+\frac{1}{2})$ and $b = 0$ in (1.2) then $\{B_H(t)\}_{t\in\mathbb{R}}$ is an FBM satisfying (1.1).

In this paper we are interested in fractionally integrated processes. Therefore, we will work with the fractional integration parameter $d := H - \frac{1}{2} \in (-0.5, 0.5)$ rather than the Hurst parameter. Moreover, we restrict ourselves to $0 < d < 0.5$ as we are interested in the long memory case.

The integral representation of FBM was generalized to a fractional Lévy motion by Benassi *et al.* (2004), who started with the so-called 'well-balanced' FBM with $a = b = 1$ in (1.2). Their approach is the basis of our definition of an FLP since, like them, we replace the Brownian motion B in the moving average (MA) representation (1.2) by a two-sided Lévy process. However, we will go into further detail and also consider integrals with respect to FLPs. Furthermore, like Mandelbrot and Van Ness (1968) for FBM, we choose $a = 1/\Gamma(H+\frac{1}{2}) = 1/\Gamma(d+1)$ and $b = 0$ in (1.2). This choice will simplify calculations when we apply our results to long memory MA processes. Long memory processes are models in which the decay of the autocorrelations follows a power law:

Definition 1.1. *Long memory process. Let $X = \{X_t\}_{t\in\mathbb{R}}$ be a stationary stochastic process and $\gamma_X(h) = \mathrm{cov}(X_{t+h}, X_t)$, $h \in \mathbb{R}$, be its autocovariance function. If there exist $0 < d < 0.5$ and a constant $c_\gamma > 0$ such that*

$$\lim_{h\to\infty} \frac{\gamma_X(h)}{h^{2d-1}} = c_\gamma, \tag{1.3}$$

then X is a stationary process with long memory.

The subject of long memory has sparked considerable research interest over the last few years. A good survey of the present state of the art is Doukhan *et al.* (2003).

The rest of this paper is organized as follows. Section 2 contains the preliminaries. We review elementary properties of Lévy processes in Section 2.1 and consider Lévy-driven stochastic integrals in Section 2.2. In Section 3 we present different methods of constructing an FLP. We introduce an L^2 approach in Section 3.1, where an FLP is defined as an integral with respect to a Poisson random measure. In Section 3.2 we obtain a continuous modification of an FLP by showing that the integral is almost surely equal to an improper Riemann integral. Furthermore, in Section 3.3 we construct FLPs using series representations for Lévy processes. Section 4 is devoted to the second-order and sample path properties of FLPs. They have almost the same second-order structure as FBMs and have stationary increments which exhibit long memory. Moreover, FLPs are Hölder continuous of every order $\beta < d$ and, for a broad class of Lévy processes, cannot be semimartingales. Since any FLP has stationary increments and the long memory property, it is, like FBM, a suitable model for driving noise in various applications. Therefore one needs to define a stochastic calculus with respect to FLPs. However, since in general a FLP is not a semimartingale, we cannot use the Itô calculus. In Section 5 we define integrals with respect to FLPs and focus in Section 6 on MA processes with the long memory property. Our main result of Section 6 states that the Lévy-driven long memory MA process with fractionally integrated kernel has a moving average integral representation where the integrand is not fractionally integrated and the driving process is a FLP.

The following notation will be used throughout this paper. We denote the distribution of the random variable X by $\mathcal{L}(X)$. $\stackrel{d}{=}$ denotes equality in (all finite-dimensional) distribution(s) and $\stackrel{L^2}{\to}$ denotes L^2 convergence. Moreover, p-lim stands for the limit in probability and d-lim is the limit in distribution for all finite-dimensional margins. Furthermore, we set $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$ and write a.s. if something holds almost surely. Finally, we assume as given an underlying complete, filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, P)$ with right continuous filtration $(\mathcal{F}_t)_{t\geqslant 0}$.

2. Preliminaries

2.1. Lévy processes

We state some elementary properties of Lévy processes that will be needed below. For a more general treatment and proofs, we refer to Protter (2004) and Sato (1999).

Throughout this paper we consider a Lévy process $L = \{L(t)\}_{t\geqslant 0}$ in $\mathbb{R}$ without Brownian component. Like every Lévy process, L is determined by its characteristic function in the Lévy–Khintchine form $\mathrm{E}[\exp\{\mathrm{i}uL(t)\}] = \exp\{t\psi(u)\},\ t \geqslant 0,$ where

$$\psi(u) = \mathrm{i}\gamma u + \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}ux} - 1 - \mathrm{i}ux1_{|x|\leqslant 1})\nu(\mathrm{d}x), \qquad u \in \mathbb{R}, \tag{2.4}$$

where $\gamma \in \mathbb{R}$ and ν is a measure on $\mathbb{R}$ that satisfies

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} (|x|^2 \wedge 1)\nu(\mathrm{d}x) < \infty. \tag{2.5}$$

The measure ν is referred to as the Lévy measure of L. We always assume that ν additionally satisfies

$$\int_{|x|>1} |x|^2 \nu(\mathrm{d}x) < \infty. \tag{2.6}$$

This is a necessary and sufficient condition (Sato 1999: Example 25.12) for L to have finite mean and variance given by

$$\mathrm{var}(L(t)) = t\,\mathrm{var}(L(1)) = t\int_{\mathbb{R}} x^2 \nu(\mathrm{d}x), \qquad t \geqslant 0.$$

Furthermore, we restrict ourselves to the case where $\mathrm{E}[L(1)] = 0$. Then $\gamma = -\int_{|x|>1} x\nu(\mathrm{d}x)$ and (2.4) reduces to

$$\psi(u) = \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}ux} - 1 - \mathrm{i}ux)\nu(\mathrm{d}x), \qquad u \in \mathbb{R}. \tag{2.7}$$

It is a well-known fact that with every cadlag Lévy process L on $\mathbb{R}$ one can associate a random measure J on $\mathbb{R}_0 \times \mathbb{R}$ describing the jumps of L. For any measurable set $B \subset \mathbb{R}_0 \times \mathbb{R}$, $J(B) = \sharp\{s \in \mathbb{R} : (L(s) - L(s-), s) \in B\}$.

The jump measure J is a Poisson random measure on $\mathbb{R}_0 \times \mathbb{R}$ (see Cont and Tankov 2004: Definition 2.18) with intensity measure $n(\mathrm{d}x, \mathrm{d}s) = \nu(\mathrm{d}x)\mathrm{d}s$. Then by the Lévy–Itô decomposition we can rewrite L a.s. as

$$L(t) = \int_0^t \int_{\mathbb{R}_0} x\, \tilde{J}(\mathrm{d}x, \mathrm{d}s), \qquad t \geqslant 0. \tag{2.8}$$

Here $\tilde{J}(\mathrm{d}x, \mathrm{d}s) = J(\mathrm{d}x, \mathrm{d}s) - \nu(\mathrm{d}x)\mathrm{d}s$ is the compensated jump measure of L. Moreover, L is a martingale.

Throughout this paper we will work with a two-sided Lévy process $L = \{L(t)\}_{t\in\mathbb{R}}$ constructed by taking two independent copies $\{L_1(t)\}_{t\geqslant 0}$, $\{L_2(t)\}_{t\geqslant 0}$ of a one-sided Lévy process and setting

$$L(t) = \begin{cases} L_1(t), & \text{if } t \geqslant 0, \\ -L_2(-t-), & \text{if } t < 0. \end{cases} \tag{2.9}$$

2.2. Stochastic integrals with respect to Lévy processes

In this section we consider the stochastic process $X = \{X(t)\}_{t\in\mathbb{R}}$ in $\mathbb{R}$ given by

$$X(t) = \int_{\mathbb{R}} f(t, s) L(\mathrm{d}s), \qquad t \in \mathbb{R}, \tag{2.10}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a measurable function and $L = \{L(t)\}_{t\in\mathbb{R}}$ is a Lévy process without Brownian component. We again stress that throughout this work we assume a two-sided Lévy process L with zero mean and finite variance, that is, L can be represented as in (2.8) together with (2.9).

It has been shown by Rajput and Rosinski (1989) that (2.10) is well defined as a limit in probability of integrals of step functions approximating f under specified conditions. These conditions are formulated in terms of the kernel function f and the generating triplet (γ, σ^2, ν) of the driving Lévy process. In particular, if L can be represented by (2.8), the process X can be rewritten as

$$X(t) = \int_{\mathbb{R}\times\mathbb{R}_0} f(t, s) x \tilde{J}(\mathrm{d}x, \mathrm{d}s), \qquad t \in \mathbb{R}, \tag{2.11}$$

where $\tilde{J}(\mathrm{d}x, \mathrm{d}s) = J(\mathrm{d}x, \mathrm{d}s) - \nu(\mathrm{d}x)\,\mathrm{d}s$ is the compensated jump measure of L. Then a necessary and sufficient condition for the existence of the stochastic integral (2.11) is that

$$\int_{\mathbb{R}} \int_{\mathbb{R}_0} (|f(t, s)x|^2 \wedge |f(t, s)x|)\, \nu(\mathrm{d}x)\, \mathrm{d}s < \infty, \qquad \text{for all } t \in \mathbb{R}. \tag{2.12}$$

If (2.12) holds, the integral (2.11) may be defined as a limit in probability of elementary integrals $\int_{\mathbb{R}} \int_{\mathbb{R}_0} f_n(t, s) x\, \tilde{J}(\mathrm{d}x, \mathrm{d}s)$, where the f_n are bounded with compact support such that $|f_n| \leqslant |f|$ and $f_n \to f$. Observe that the integral is independent of the choice of approximating functions f_n (Kallenberg 1997: Theorem 10.5).

Moreover, the law of $X(t)$ is for all $t \in \mathbb{R}$ infinitely divisible with characteristic function (Rajput and Rosinski 1989)

$$\mathrm{E}[\exp\{\mathrm{i}uX(t)\}] = \exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}}(\mathrm{e}^{\mathrm{i}uf(t,s)x} - 1 - \mathrm{i}uf(t,\, s)x)\nu(\mathrm{d}x)\mathrm{d}s\right\}, \qquad u \in \mathbb{R}. \tag{2.13}$$

The following proposition shows that the integral (2.10) or (2.11) may be well defined in an L^2 sense.

Proposition 2.1. *If* $f(t,\, \cdot) \in L^2(\mathbb{R})$*, the stochastic integral* (2.11)*, and hence* (2.10)*, exists as an* $L^2(\Omega,\, P)$ *limit of approximating step functions and does not depend on the choice of the approximating sequence. Moreover,*

$$\mathrm{E}[X(t)^2] = \mathrm{E}[L(1)^2]\|f(t,\, \cdot)\|^2_{L^2(\mathbb{R})}, \qquad t \in \mathbb{R}. \tag{2.14}$$

Proof. Applying Rajput and Rosinski (1989: Theorem 3.3), it follows that (2.10) is well defined and $\mathrm{E}|\int f\,\mathrm{d}L|^2 < \infty$ if and only if

$$\int_{\mathbb{R}}\left[f(t,\, s)\gamma + \int_{\mathbb{R}} f(t,\, s)x[1_{\{|f(t,s)x|\leq 1\}} - 1_{\{|x|\leq 1\}}]\nu(\mathrm{d}x) + \int_{\mathbb{R}}(f(t,\, s)x)^2\nu(\mathrm{d}x)\right]\mathrm{d}s < \infty. \tag{2.15}$$

Since we have $\gamma = -\int_{|x|>1} x\,\nu(\mathrm{d}x)$, (2.15) is implied by

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(t,\, s)x1_{\{|f(t,s)x|>1\}}\,\nu(\mathrm{d}x)\,\mathrm{d}s + \int_{\mathbb{R}}\int_{\mathbb{R}}(f(t,\, s)x)^2\,\nu(\mathrm{d}x)\,\mathrm{d}s$$
$$\leq 2\int_{\mathbb{R}}\int_{\mathbb{R}} f^2(t,\, s)x^2\,\nu(\mathrm{d}x)\,\mathrm{d}s = 2\mathrm{E}[L(1)^2]\|f(t,\, \cdot)\|^2_{L^2(\mathbb{R})} < \infty.$$

It follows from Rajput and Rosinski (1989: Theorem 3.4) that the mapping $f \to \int_{\mathbb{R}} f\,\mathrm{d}L$ is an isomorphism from $L^2(\mathbb{R})$ to $L^2(\Omega,\, P)$. To prove (2.14) we consider, for fixed $t \in \mathbb{R}$, step functions

$$f_n(t,\, s) = \sum_{k=0}^{n-1} a_k 1_{(s_k, s_{k+1}]}(s),$$

where $a_0, \ldots a_{n-1} \in \mathbb{R}$, $n \in \mathbb{N}$ and $-\infty < s_0 < \ldots < s_n < \infty$. Then we define

$$\int_{\mathbb{R}} f_n(t,\, s)\,L(\mathrm{d}s) = \sum_{k=0}^{n-1} a_k(L(s_{k+1}) - L(s_k)).$$

It is easy to check that

$$\mathrm{E}\left[\int_{\mathbb{R}} f_n(t,\, s)\,L(\mathrm{d}s)\right]^2 = \mathrm{E}\left[\int_{\mathbb{R}} f_n^2(t,\, s)\mathrm{d}[L,\, L]_s\right] = \int_{\mathbb{R}}\int_{\mathbb{R}} f_n^2(t,\, s)x^2\,\nu(\mathrm{d}x)\mathrm{d}s$$
$$= \mathrm{E}[L(1)^2]\|f_n(t,\, \cdot)\|^2_{L^2(\mathbb{R})}.$$

This isometry property is preserved when we approximate $f(t, \cdot)$ by a sequence of step functions $(f_n(t, \cdot))$ satisfying $f_n \xrightarrow{L^2} f$ (observe that the step functions are dense in $L^2(\mathbb{R})$). □

3. Construction of fractional Lévy processes

3.1. The L^2 approach

We are now in a position to introduce a fractional Lévy process (FLP) as a natural counterpart to FBM. Based on the MA representation (1.2) of FBM we define an FLP as follows.

Definition 3.1 *Fractional Lévy process. Let* $L = \{L(t)\}_{t\in\mathbb{R}}$ *be a two-sided Lévy process on* $\mathbb{R}$ *with* $\mathrm{E}[L(1)] = 0$, $\mathrm{E}[L(1)^2] < \infty$ *and without Brownian component. For fractional integration parameter* $0 < d < 0.5$, *a stochastic process*

$$M_d(t) = \frac{1}{\Gamma(d+1)} \int_{-\infty}^{\infty} [(t-s)_+^d - (-s)_+^d] L(\mathrm{d}s), \qquad t \in \mathbb{R}, \tag{3.16}$$

is called a fractional Lévy process.

Remark 3.1. The general Lévy–Itô representation (Sato 1999: Theorem 19.2) guarantees that every Lévy process can be decomposed into a linear term, a Brownian and a jump component which is independent of the Brownian part. However, the Brownian part induces an FBM which has already been extensively studied (Doukhan *et al.* 2003; Samorodnitsky and Taqqu 1994). Therefore we have assumed a Lévy process without Brownian component.

Before we take a closer look at the integral (3.16), we summarize the following two important properties of the kernel function

$$f_t(s) := \frac{1}{\Gamma(1+d)} [(t-s)_+^d - (-s)_+^d], \qquad s \in \mathbb{R}, \tag{3.17}$$

that can be shown by simple calculus.

Proposition 3.1. *For* $0 < d < 0.5$ *and for each* $t \in \mathbb{R}$, *the kernel function* (3.17) *is bounded. Moreover,* $f_t \in L^p(\mathbb{R})$ *for* $p > (1-d)^{-1}$. *In particular,* $f_t \in L^2(\mathbb{R})$ *but* $f_t \notin L^1(\mathbb{R})$ *for* $t \neq 0$.

Proposition 3.2. *The function* $t \mapsto (t-s)_+^d - (-s)_+^d$ *is locally Hölder continuous of every order* $\beta \leqslant d$, *and for an order* $\beta > d$ *it is not Hölder continuous on any interval containing s. Furthermore, the total variation is finite on compacts.*

The following theorem makes precise the meaning of (3.16).

Theorem 3.3 *Fractional Lévy process in* L^2 *sense. Let* $L = \{L(t)\}_{t\in\mathbb{R}}$ *be a Lévy process* *without Brownian component satisfying* $\mathrm{E}[L(1)] = 0$, $\mathrm{E}[L(1)^2] < \infty$ *and* $\tilde{J}(\mathrm{d}s, \mathrm{d}u) = J(\mathrm{d}s, \mathrm{d}u) - \mathrm{d}s\nu(\mathrm{d}u)$ *be the compensated jump measure of L. For* $t \in \mathbb{R}$, *let the kernel function* f_t *be defined as in* (3.17). *Then for every* $t \in \mathbb{R}$, $M_d(t) = \int_{\mathbb{R}} f_t(s)\, L(\mathrm{d}s)$ *exists as an* $L^2(\Omega, P)$ *limit of approximating step functions in the sense that*

$$M_d(t) = \int_{\mathbb{R}_0 \times \mathbb{R}} f_t(s) x \tilde{J}(\mathrm{d}x, \mathrm{d}s), \qquad t \in \mathbb{R}. \tag{3.18}$$

Moreover, for all $t \in \mathbb{R}$, *the distribution of* $M_d(t)$ *is infinitely divisible and*

$$\mathrm{E}[M_d(t)]^2 = \|f_t\|^2_{L^2(\mathbb{R})} \mathrm{E}[L(1)^2], \qquad t \in \mathbb{R}. \tag{3.19}$$

Let $u_1, \ldots, u_m \in \mathbb{R}$, $-\infty < t_1 < \ldots < t_m < \infty$ *and* $m \in \mathbb{N}$. *Then the finite-dimensional distributions of the process* M_d *have the characteristic functions*

$$\mathrm{E}[\exp\{\mathrm{i}u_1 M_d(t_1) + \ldots + \mathrm{i}u_m M_d(t_m)\}] = \exp\left\{ \int_{\mathbb{R}} \psi\left(\sum_{j=1}^m u_j f_{t_j}(s) \right) \mathrm{d}s \right\}, \tag{3.20}$$

where ψ *is given as in* (2.7).

Proof. The assertions are direct consequences of the results of Section 2.2, since $f_t \in L^2(\mathbb{R})$. Equation (3.20) follows from (2.13) when we write

$$\sum_{j=1}^m u_j M_d(t_j) = \sum_{j=1}^m u_j \int_{\mathbb{R}} f_{t_j}(s)\, L(\mathrm{d}s) = \int_{\mathbb{R}} \sum_{j=1}^m u_j f_{t_j}(s)\, L(\mathrm{d}s).$$

□

Remark 3.2. As a consequence of (3.20) the generating triplet of $M_d(t)$ is $(\gamma_M^t, 0, \nu_M^t)$, where

$$\gamma_M^t = -\int_{\mathbb{R}} \int_{\mathbb{R}} f_t(s) x 1_{\{|f_t(s)x| > 1\}}\, \nu(\mathrm{d}x)\, \mathrm{d}s$$

and

$$\nu_M^t(B) = \int_{\mathbb{R}} \int_{\mathbb{R}} 1_B(f_t(s)x)\, \nu(\mathrm{d}x)\, \mathrm{d}s, \qquad B \in \mathcal{B}(\mathbb{R}). \tag{3.21}$$

We have seen that (3.16) can be understood as L^2 limit and we can now apply the Kolmogorov–Centsov theorem to obtain a continuous modification of $\{M_d(t)\}_{t \in \mathbb{R}}$ (see Theorem 4.3(i) below). However, we can also show that $\{M_d(t)\}_{t \in \mathbb{R}}$ has a continuous modification by proving in the following section that $M_d(t)$ is a.s. equal to an improper Riemann integral for all $t \in \mathbb{R}$.

3.2. The improper Riemann integral

We give here a pathwise construction of an FLP as an improper Riemann integral.

Theorem 3.4. *Let $L = \{L(t)\}_{t\in\mathbb{R}}$ be a Lévy process without Brownian component satisfying* $\mathrm{E}[L(1)] = 0$ *and* $\mathrm{E}[L(1)^2] < \infty$. *For $t \in \mathbb{R}$, define the kernel function f_t as in* (3.17). *Then for all $t \in \mathbb{R}$, $M_d(t) = \int_{\mathbb{R}} f_t(s)\,L(\mathrm{d}s)$ has a modification which is equal to the improper Riemann integral*

$$M_d(t) = \frac{1}{\Gamma(d)} \int_{\mathbb{R}} [(t-s)_+^{d-1} - (-s)_+^{d-1}]\,L(s)\,\mathrm{d}s, \qquad t \in \mathbb{R}. \tag{3.22}$$

Moreover, (3.22) *is continuous in t.*

Proof. We assume $t > 0$; for $t \leqslant 0$ the proof is analogous. For a Lévy process L on $\mathbb{R}$ that satisfies $\mathrm{E}[L(1)] = 0$ and $\mathrm{E}[L(1)^2] < \infty$ we have a generalization of the law of the iterated logarithm of random walks (Sato 1999: Proposition 48.9), that is,

$$\limsup_{t\to\infty} \frac{|L(t)|}{(2t\log\log t)^{1/2}} = (\mathrm{E}[L(1)^2])^{1/2} \text{ a.s.}$$

Moreover, $(t-s)^d - (-s)^d \sim td(-s)^{d-1}$ as $s \to -\infty$, and therefore

$$\lim_{s\to-\infty} L(s)[(t-s)^d - (-s)^d] = 0 \quad \text{a.s.}$$

If g is a continuously differentiable function on $[a, b] \subset \mathbb{R}$ it is always possible to use the integration by parts formula to define $\int_a^b g(s)\,L(\mathrm{d}s)$ as a Riemann integral by

$$\int_{[a,b]} g(s)\,L(\mathrm{d}s) = g(b)L(b) - g(a)L(a) - \int_{[a,b]} L(s)\mathrm{d}g(s) \tag{3.23}$$

(see Eberlein and Raible 1999: Lemma 2.1). Since we have

$$M_d(t) = \frac{1}{\Gamma(d+1)} \lim_{a\to-\infty} \int_{[a,0]} [(t-s)^d - (-s)^d]\,L(\mathrm{d}s) + \frac{1}{\Gamma(d+1)} \int_{[0,t]} (t-s)^d\,L(\mathrm{d}s),$$

it follows by (3.23) that

$$\begin{aligned} M_d(t) = {} & \frac{1}{\Gamma(d)} \int_{[0,t]} (t-s)^{d-1}\,L(s)\,\mathrm{d}s - \frac{1}{\Gamma(d+1)} \lim_{a\to-\infty} \{L(a)[(t-a)^d - (-a)^d]\} \\ & + \frac{1}{\Gamma(d+1)} \lim_{a\to-\infty} \left\{ d\int_{[a,0]} [(t-s)^{d-1} - (-s)^{d-1}]\,L(s)\,\mathrm{d}s \right\} \\ = {} & \frac{1}{\Gamma(d)} \int_{\mathbb{R}} [(t-s)_+^{d-1} - (-s)_+^{d-1}]\,L(s)\,\mathrm{d}s, \qquad t \in \mathbb{R}. \end{aligned}$$

To show that (3.22) is continuous in t we define, for $t > 0$, $g_t(s) = (t-s)^{d-1} L(s) 1_{[0,t]}(s)$, $s \in \mathbb{R}$. Then, for all $T > 0$, the family $\{g_t\}_{t\in[0,T]}$ is uniformly integrable with respect to the Lebesgue measure and the continuity of $\int_0^t (t-s)^{d-1} L(s)\,\mathrm{d}s$ follows from Theorem 5, Section II.6 of Shiryaev (1996). Furthermore, by Lebesgue's dominated convergence theorem, $\int_{-\infty}^0 [(t-s)^{d-1} - (-s)^{d-1}]\,L(s)\,\mathrm{d}s$ is continuous in t. □

3.3. Series representations of fractional Lévy processes

The results in this section are based on the series representation of Lévy processes summarized in Rosinski (2001).

Theorem 3.5. *Let $L = \{L(t)\}_{t\in\mathbb{R}}$ be a Lévy process without Brownian component satisfying* $\mathrm{E}[L(1)] = 0$ *and* $\mathrm{E}[L(1)^2] < \infty$, *and, for $t \in \mathbb{R}$, define the kernel function f_t as in* (3.17). *Suppose the Lévy measure ν of L is symmetric. Set $\nu^{\leftarrow}(s) = \inf\{x > 0 : \nu((x, \infty)) \leqslant s\}$, $s > 0$, the right continuous inverse of $x \mapsto \nu((x, \infty))$. Let Λ be an arbitrary probability measure on $\mathbb{R}$ with nowhere vanishing density ρ. Moreover, let $\{T_i\}_{i=1,2,\ldots}$ and $\{U_i\}_{i=1,2,\ldots}$ be independent sequences of random variables, such that $\{T_i\}_{i=1,2,\ldots}$ is a sequence of independent indentically distributed (i.i.d.) standard exponential random variables and $\{U_i\}_{i=1,2\ldots}$ is a sequence of i.i.d. random variables with distribution Λ. Put $\tau_0 = 0$ and $\tau_i = \sum_{j=1}^{i} T_j$, $i = 1, 2, \ldots$. Furthermore, let $\{\varepsilon_i\}_{i=1,2\ldots}$ be an i.i.d. sequence of random variables with $P(\varepsilon_i = -1) = P(\varepsilon_i = 1) = \frac{1}{2}$. Then, for every $t \in \mathbb{R}$, the series*

$$X(t) = \sum_{i=1}^{\infty} \varepsilon_i \nu^{\leftarrow}(\tau_i \rho(U_i)) f_t(U_i) \tag{3.24}$$

converges a.s. and

$$\{M_d(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \{X(t)\}_{t\in\mathbb{R}}. \tag{3.25}$$

Proof. As ν is symmetric, we have

$$E[\mathrm{e}^{\mathrm{i}uM_d(t)}] = \exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}} \left[\mathrm{e}^{\mathrm{i}uxf_t(s)} - 1 - \mathrm{i}uxf_t(s)\right] \nu(\mathrm{d}x)\,\mathrm{d}s\right\}$$
$$= \exp\left\{2\int_{\mathbb{R}}\int_{0}^{\infty} [\cos(uxf_t(s)) - 1]\,\nu(\mathrm{d}x)\,\mathrm{d}s\right\}.$$

Therefore, the assertion is an immediate consequence of Rosinski (1989: Proposition 2). □

If ν is not symmetric we obtain a similar result by taking into account the left continuous inverse of ν.

Theorem 3.6. *Let $L = \{L(t)\}_{t\in\mathbb{R}}$ be a Lévy process without Brownian component satisfying* $\mathrm{E}[L(1)] = 0$ *and* $\mathrm{E}[L(1)^2] < \infty$. *Set $\nu^{\leftarrow}(s) = \inf\{x > 0 : \nu((x, \infty)) \leqslant s\}$, $s > 0$, and $\nu^{\rightarrow}(s) = \sup\{x < 0 : \nu((-\infty, x)) \leqslant s\}$, $s > 0$, the right and left continuous inverse of ν, respectively. Define Λ and the sequences $\{T_i\}$, $\{U_i\}$ and $\{\tau_i\}$ as in Theorem 3.5. Then for every $t \in \mathbb{R}$ the series*

$$X(t) = \sum_{i=1}^{\infty} \{[\nu^{\leftarrow}(\tau_i\rho(U_i)) + \nu^{\rightarrow}(\tau_i\rho(U_i))] f_t(U_i) - C_t(\tau_i)\} \tag{3.26}$$

converges a.s., where

$$C_t(\tau_i) = \int_{\mathbb{R}} \int_{\tau_{i-1}}^{\tau_i} [\nu^{\leftarrow}(\tau\rho(u)) + \nu^{\rightarrow}(\tau\rho(u))] f_t(u) \mathrm{d}\tau\, \rho(u)\, \mathrm{d}u.$$

Moreover, $\{M_d(t)\}_{t\in\mathbb{R}} \overset{d}{=} \{X(t)\}_{t\in\mathbb{R}}$.

Proof. $X(t)$ in (3.26) is a generalized shot noise series which converges a.s. if we show that, for $B \in \mathcal{B}(\mathbb{R})$,

$$G^t(B) := \int_{\mathbb{R}} \int_0^\infty 1_{\{B\setminus\{0\}\}}(H_t(\tau, u)) \mathrm{d}\tau\, \Lambda(\mathrm{d}u) = \int_{\mathbb{R}} \int_0^\infty 1_{\{B\setminus\{0\}\}}(H_t(\tau, u))\, \mathrm{d}\tau\, \rho(u)\, \mathrm{d}u$$

defines a Lévy measure, where

$$H_t(\tau, u) = [\nu^{\leftarrow}(\tau\rho(u)) + \nu^{\rightarrow}(\tau\rho(u))] f_t(u), \qquad \tau > 0,\ t,\ u \in \mathbb{R}$$

(see Rosinski (1990: Theorem 2.4). Observe that, for every $x \geqslant 0$, $u \in \mathbb{R}$,

$$\mathrm{Leb}(\{\tau > 0 : \nu^{\leftarrow}(\tau\rho(u)) > x\}) = \mathrm{Leb}(\{\tau > 0 : \nu^{\leftarrow}(\tau) > x\})/\rho(u) = \nu((x, \infty))/\rho(u)$$

and thus

$$\int_{\mathbb{R}} \int_0^\infty 1_{\{B\setminus\{0\}\}}(\nu^{\leftarrow}(\tau\rho(u)) f_t(u))\, \mathrm{d}\tau\, \rho(u)\, \mathrm{d}u = \int_{\mathbb{R}} \int_0^\infty 1_{\{B\setminus\{0\}\}}(x f_t(u))\, \nu(\mathrm{d}x)\, \mathrm{d}u.$$

Analogously, for every $x \leqslant 0$ and $u \in \mathbb{R}$,

$$\mathrm{Leb}(\{\tau > 0 : \nu^{\rightarrow}(\tau\rho(u)) < x\}) = \nu((-\infty, x))/\rho(u),$$

which yields

$$\int_{\mathbb{R}} \int_0^\infty 1_{\{B\setminus\{0\}\}}(\nu^{\rightarrow}(\tau\rho(u)) f_t(u))\, \mathrm{d}\tau\, \rho(u)\, \mathrm{d}u = \int_{\mathbb{R}} \int_{-\infty}^0 1_{\{B\setminus\{0\}\}}(x f_t(u))\, \nu(\mathrm{d}x)\, \mathrm{d}u.$$

Therefore,

$$G^t(B) = \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{\{B\setminus\{0\}\}}(x f_t(u))\, \nu(\mathrm{d}x)\, \mathrm{d}u.$$

From (3.21) it follows that $G^t = \nu_M^t$ is the Lévy measure of an infinitely divisible random variable. Furthermore, it follows from Theorem 3.1(iii) in Rosinski (1990) and its proof that $X(t)$ has characteristic function given by

$$\mathrm{E}[\mathrm{e}^{\mathrm{i}uX(t)}] = \exp\left\{ \int_{\mathbb{R}} \int_{\mathbb{R}} [\mathrm{e}^{\mathrm{i}uf_t(s)x} - 1 - \mathrm{i}uf_t(s)x]\, \nu(\mathrm{d}x)\, \mathrm{d}s \right\},$$

that is, $X(t) \overset{d}{=} M_d(t)$. Finally, repeating the same arguments for $\sum_{j=1}^m w_j H_{t_j}(\tau, u)$, where $m \in \mathbb{N}$, $t_1, \ldots, t_m \in \mathbb{R}$ and $w_1, \ldots, w_m \in \mathbb{R}$, we obtain that the finite-dimensional distributions of X are identical to those of M_d. □

The series representation (3.25) can be used for simulations of FLPs. Of course, for practical simulations the series must be truncated. However, simulation from it is not so easy since the inverse of the tail mass of the Lévy measure is rarely known in closed form.

An alternative generalized shot noise representation for fractional fields has recently been developed by Cohen *et al.* (2005).

4. Second-order and sample path properties

As the isometry property (3.19) of an FLP is the same as that of an FBM, it is obvious that up to a constant FLPs have the same second-order structure as for FBM. Therefore, we omit the proofs of the following two theorems.

Theorem 4.1 Autocovariance function. *For $s, t \in \mathbb{R}$, the autocovariance function of an FLP $M_d = \{M_d(t)\}_{t\in\mathbb{R}}$ is given by*

$$\operatorname{cov}(M_d(t), M_d(s)) = \frac{\mathrm{E}[L(1)^2]}{2\Gamma(2d+2)\sin(\pi[d+\frac{1}{2}])}\left[|t|^{2d+1} - |t-s|^{2d+1} + |s|^{2d+1}\right]. \tag{4.27}$$

Theorem 4.2 Covariance between two increments. *Let $h > 0$ and the FLP M_d be given as in* (3.18). *The covariance between two increments $M_d(t+h) - M_d(t)$ and $M_d(s+h) - M_d(s)$, where $s + h \leqslant t$ and $t - s = nh$, is*

$$\begin{aligned}\delta_d(n) &= \frac{\mathrm{E}[L(1)^2]}{2\Gamma(2d+2)\sin(\pi[d+\frac{1}{2}])}h^{2d+1}\left[(n+1)^{2d+1} + (n-1)^{2d+1} - 2n^{2d+1}\right] \\ &= \frac{\mathrm{E}[L(1)^2]d(2d+1)}{\Gamma(2d+2)\sin(\pi[d+\frac{1}{2}])}h^{2d+1}n^{2d-1} + O(n^{2d-2}), \qquad n \to \infty. \end{aligned} \tag{4.28}$$

Remark 4.1. As a consequence of (4.28), the increments of an FLP exhibit long memory in the sense of Definition 1.1. It is this long memory property that allows us in Section 5 to construct long memory MA processes without a fractional integration of the kernel.

We also note that for a martingale X with zero expectation the covariance function must be identically zero, since

$$\begin{aligned}&\operatorname{cov}(X(h) - X(h-1), X(h+n) - X(h+n-1)) \\ &\quad = \mathrm{E}[(X(h) - X(h-1))\mathrm{E}[X(h+n) - X(h+n-1) \,|\, \mathcal{F}_{h+n-1}]] = 0.\end{aligned}$$

This shows that M_d cannot be a martingale. We will prove later that, for a fairly large class of Lévy processes, M_d is not a semimartingale either.

We now consider sample path properties of FLPs.

Theorem 4.3 Sample path properties. *Let $M_d = \{M_d(t)\}_{t\in\mathbb{R}}$ be a FLP.*

(i) Hölder continuity. *For every $\beta < d$ there exists a continuous modification of M_d and there exist an a.s. positive random variable H_ϵ and a constant $\delta > 0$ such that*

$$P\left[\omega \in \Omega : \sup_{0<h<H_\epsilon(\omega)} \left(\frac{M_d(t+h,\omega) - M_d(t,\omega)}{h^\beta}\right) \leqslant \delta\right] = 1.$$

This means that the sample paths of FLPs are a.s. locally Hölder continuous of any order $\beta < d$. Moreover, for every modificaton of M_d and for every $\beta > d$, $P(\{\omega \in \Omega : M_d(\cdot,\omega) \notin C^\beta[a,b]\}) > 0$, where $C^\beta[a,b]$ is the space of Hölder continuous functions on $[a,b]$. Furthermore, if $\nu(\mathbb{R}) = \infty$ then $P(\{\omega \in \Omega : M_d(\cdot,\omega) \notin C^\beta[a,b]\}) = 1$.

(ii) Stationary increments. *M_d is a process with stationary increments.*

(iii) Symmetry. $\{M_d(-t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \{-M_d(t)\}_{t\in\mathbb{R}}$.

Proof. (i) The first assertion follows directly from (4.27) and an application of the Kolmogorov–Centsov theorem (Loève 1960: 519). Furthermore, from Proposition 3.2 we know that $t \mapsto (t-s)_+^d - (-s)_+^d \notin C^\beta[a,b]$ for every $\beta > d$. Therefore, the proof of the second part is analogous to the proof of Proposition 3.3 in Benassi *et al.* (2004).

(ii) For any $s, t \in \mathbb{R}$, $s < t$, we have

$$M_d(t) - M_d(s) = \frac{1}{\Gamma(d+1)} \int_{\mathbb{R}} [(t-u)_+^d - (s-u)_+^d]\, L(\mathrm{d}u)$$

$$\stackrel{d}{=} \frac{1}{\Gamma(d+1)} \int_{\mathbb{R}} [(t-s-v)_+^d - (-v)_+^d]\, L(\mathrm{d}v) = M_d(t-s),$$

where equality in distribution follows from the stationarity of the increments of L.

(iii) $M_d(-t) = M_d(-t) - M_d(0) \stackrel{d}{=} M_d(0) - M_d(t) = -M_d(t)$. □

Theorem 4.4 Self-similarity. *An FLP M_d cannot be self-similar.*

Proof. Assume that M_d is self-similar with index $H \in [0.5, \infty)$. Then we have, for all $c > 0$,

$$\{M_d(ct)\}_{t\in\mathbb{R}} \stackrel{d}{=} c^H \{M_d(t)\}_{t\in\mathbb{R}}. \tag{4.29}$$

The generating triplet of $M_d(t)$ is $(\gamma_M^t, 0, \nu_M^t)$ (see (3.21)). Define, for $r > 0$, the transformation T_r of measures ν on $\mathbb{R}$ by $(T_r\nu)(B) = \nu(r^{-1}B)$, $B \in \mathcal{B}(\mathbb{R})$. Then the Lévy measure of $c^{-H}M_d(ct)$ is given by $c(T_b\nu_M^t)$ with $b = c^{d-H}$. Therefore, if M_d is self-similar, by the uniqueness of the generating triplet $\nu_M^t = b^{-1/(H-d)}(T_b\nu_M^t)$, for all $b > 0$. Then by Sato (1999: Theorem 14.3(ii)) and its proof it follows that $1/(H-d) < 2$ and that ν_M^t is the Lévy measure of an α-stable process with $\alpha = 1/(H-d)$. Hence, $\mathrm{E}[M_d(t)^2] = \infty$, contradicting (4.27). □

Remark 4.2. For a fractional stable process to be well defined one has to choose a different kernel function. If L is α-stable a possible choice is $f_t(s) = |t-s|^{H-1/\alpha} - |s|^{H-1/\alpha}$, where H is the Hurst parameter and α denotes the index of stability (see Samorodnitsky and Taqqu 1994: Section 7.4).

Theorem 4.5. *For* $1 < \alpha < 2$ *define the parameter* $\tilde{H}$ *by* $\tilde{H} = d + 1/\alpha$ *such that* $0 < \tilde{H} < 1$. *Assume that* $\nu(\mathrm{d}x) = g(x)\,\mathrm{d}x$, *where* $g : \mathbb{R} \to \mathbb{R}_+$ *is measurable and satisfies*

$$\begin{aligned} g(x) &\sim |x|^{-1-\alpha}, \qquad x \to 0, \\ g(x) &\leqslant C|x|^{-1-\alpha}, \qquad \text{for all } x \in \mathbb{R}, \end{aligned} \tag{4.30}$$

with a constant $C > 0$. *Then* M_d *is locally self-similar with parameter* $\tilde{H}$, *that is, for every fixed* $t \in \mathbb{R}$,

$$d\text{-}\lim_{\epsilon \downarrow 0} \left\{ \frac{M_d(t + \epsilon x) - M_d(t)}{\epsilon^{\tilde{H}}} \right\}_{x \in \mathbb{R}} \stackrel{d}{=} \{Y_{\tilde{H}}(x)\}_{x \in \mathbb{R}}. \tag{4.31}$$

Here $Y_{\tilde{H}}$ *is a linear fractional stable motion with representation*

$$Y_{\tilde{H}}(t) = \frac{1}{\Gamma(d)} \int_{\mathbb{R}} [(t-s)_+^{\tilde{H}-1/\alpha} - (-s)_+^{\tilde{H}-1/\alpha}]\, L_\alpha(\mathrm{d}s),$$

where L_α *is a symmetric* α*-stable Lévy process (Samorodnitsky and Taqqu 1994).*

Proof. Since M_d has stationary increments it is enough to show the convergence for $t = 0$. For $u_1, \ldots, u_n \in \mathbb{R}$, $-\infty < t_1 < \ldots < t_n < \infty$ and $n \in \mathbb{N}$, we have by (3.20),

$$\begin{aligned} &\log \mathrm{E}\left[\exp\left\{\mathrm{i} \sum_{k=1}^n u_k \frac{M_d(\epsilon t_k)}{\epsilon^{\tilde{H}}}\right\}\right] \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\exp\left\{\mathrm{i}x \sum_{k=1}^n u_k \frac{f_{\epsilon t_k}(s)}{\epsilon^{\tilde{H}}}\right\} - 1 - \mathrm{i}x \sum_{k=1}^n u_k \frac{f_{\epsilon t_k}(s)}{\epsilon^{\tilde{H}}}\right] \nu(\mathrm{d}x)\,\mathrm{d}s \\ &\stackrel{\epsilon v = s}{=} \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\exp\left\{\mathrm{i}x\epsilon^{d-\tilde{H}} \sum_{k=1}^n u_k f_{t_k}(v)\right\} - 1 - \mathrm{i}x\epsilon^{d-\tilde{H}} \sum_{k=1}^n u_k f_{t_k}(v)\right] \epsilon\, \nu(\mathrm{d}x)\,\mathrm{d}v \\ &\stackrel{\epsilon^{d-\tilde{H}}x = y}{=} \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\exp\left\{\mathrm{i}y \sum_{k=1}^n u_k f_{t_k}(v)\right\} - 1 - \mathrm{i}y \sum_{k=1}^n u_k f_{t_k}(v)\right] \epsilon\, \nu(\epsilon^{\tilde{H}-d}\mathrm{d}y)\,\mathrm{d}v. \end{aligned}$$

For any $y \neq 0$, the asymptotic behaviour of g yields

$$\epsilon\nu(\epsilon^{\tilde{H}-d}\,\mathrm{d}y) = \epsilon g(\epsilon^{\tilde{H}-d}y)\epsilon^{\tilde{H}-d}\,\mathrm{d}y \sim \epsilon^{\tilde{H}-d+1}|\epsilon^{\tilde{H}-d}y|^{-1-\alpha}\,\mathrm{d}y = |y|^{-1-\alpha}\,\mathrm{d}y, \qquad \epsilon \to 0,$$

which is the Lévy measure of a symmetric α-stable Lévy process. By (4.30) we have $|G_\epsilon| \leqslant F$ for all $\epsilon > 0$, where

$$G_\epsilon(y, v) = \left[\exp\left\{\mathrm{i}y \sum_{k=1}^n u_k f_{t_k}(v)\right\} - 1 - \mathrm{i}y \sum_{k=1}^n u_k f_{t_k}(v)\right] \epsilon g(\epsilon^{\tilde{H}-d}y)\epsilon^{\tilde{H}-d}$$

and

$$F(y, v) = \left| \exp\left\{ \mathrm{i}y \sum_{k=1}^{n} u_k f_{t_k}(v) \right\} - 1 - \mathrm{i}y \sum_{k=1}^{n} u_k f_{t_k}(v) \right| C|y|^{-1-\alpha}.$$

It can be shown that $F \in L^1(\mathbb{R}^2)$. Hence, it follows, by dominated convergence,

$$\begin{aligned}
&\lim_{\epsilon \downarrow 0} \log \mathrm{E}\left[\exp\left\{ \mathrm{i} \sum_{k=1}^{n} u_k \frac{M_d(\epsilon t_k)}{\epsilon^{\tilde{H}}} \right\} \right] \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\exp\left\{ \mathrm{i}y \sum_{k=1}^{n} u_k f_{t_k}(v) \right\} - 1 - \mathrm{i}y \sum_{k=1}^{n} u_k f_{t_k}(v) \right] |y|^{-1-\alpha} \,\mathrm{d}y \,\mathrm{d}v \\
&= \int_{\mathbb{R}} \int_0^{\infty} \left[2\cos\left(y \sum_{k=1}^{n} u_k f_{t_k}(v) \right) - 2 \right] |y|^{-1-\alpha} \,\mathrm{d}y \,\mathrm{d}v \\
&= \int_{\mathbb{R}} \int_0^{\infty} [2\cos(x) - 2] \left| \sum_{k=1}^{n} u_k f_{t_k}(v) \right|^{\alpha} \frac{\mathrm{d}x}{x^{1+\alpha}} \,\mathrm{d}v = C(\alpha) \int_{\mathbb{R}} \left| \sum_{k=1}^{n} u_k f_{t_k}(v) \right|^{\alpha} \mathrm{d}v,
\end{aligned}$$

where

$$C(\alpha) = 2\int_0^{\infty} [\cos(x) - 1] \frac{\mathrm{d}x}{x^{1+\alpha}}.$$

Since

$$\log \mathrm{E}\left[\exp\left\{ \mathrm{i} \sum_{k=1}^{n} u_k Y_{\tilde{H}}(\epsilon t_k) \right\} \right] = C(\alpha) \int_{\mathbb{R}} \left| \sum_{k=1}^{n} u_k f_{t_k}(v) \right|^{\alpha} \mathrm{d}v$$

(see Samorodnitsky and Taqqu 1994: 114), the proof is complete. □

In the following let $\mathrm{Var}_{[a,b]}(M_d)$ denote the total variation of the sample paths of M_d on the interval $[a, b] \subset \mathbb{R}$.

Theorem 4.6 Total variation. *If ν is given as in Theorem 4.5, the sample paths of M_d are a.s. of infinite total variation on compacts, that is* $\mathrm{Var}_{[a,b]}(M_d) = \infty$ *a.s. If $\nu(\mathbb{R}) < \infty$, they are of finite total variation.*

Proof. We know from (4.31) that

$$d\text{-}\lim_{h \downarrow 0} \frac{M_d(t \pm h) - M_d(t)}{h^{\tilde{H}}} \stackrel{d}{=} Y_{\tilde{H}}(\pm 1).$$

Thus,

$$d\text{-}\lim_{h \downarrow 0} \frac{|M_d(t \pm h) - M_d(t)|}{|h|^{\tilde{H}}} \stackrel{d}{=} |Y_{\tilde{H}}(\pm 1)| > 0 \text{ a.s.} \tag{4.32}$$

As $|Y_{\tilde{H}}(\pm 1)| > 0$ a.s., it follows for all $\Omega' \subset \Omega$ with $P(\Omega') > 0$ that

$$\lim_{h\downarrow 0} \mathrm{E}\left[1_{\Omega'} \frac{|M_d(t \pm h) - M_d(t)|}{|h|^{\tilde{H}}}\right] > 0. \tag{4.33}$$

In fact, let $\Omega' \subset \Omega$ with $P(\Omega') > 0$. Then $\lim_{\delta\downarrow 0} P(|Y_{\tilde{H}}(\pm 1)| \leqslant \delta) \to 0$. Choose $\delta > 0$ small enough such that δ is a continuity point of the distribution function of $|Y_{\tilde{H}}(\pm 1)|$ and $P(|Y_{\tilde{H}}(\pm 1)| \leqslant \delta) \leqslant P(\Omega')/4$, which implies by (4.32) that

$$\lim_{h\downarrow 0} P\left(\frac{|M_d(t \pm h) - M_d(t)|}{|h|^{\tilde{H}}} \leqslant \delta\right) = P(|Y_{\tilde{H}}(\pm 1)| \leqslant \delta) \leqslant \frac{P(\Omega')}{4}.$$

Hence, there exists $\epsilon_t > 0$ such that

$$P\left(\frac{|M_d(t + h) - M_d(t)|}{|h|^{\tilde{H}}} \leqslant \delta\right) \leqslant \frac{P(\Omega')}{2} \qquad \text{for all } h \neq 0,\ |h| \leqslant \epsilon_t.$$

This yields

$$P\left(\Omega' \cap \left\{\frac{|M_d(t + h) - M_d(t)|}{|h|^{\tilde{H}}} \leqslant \delta\right\}\right) \leqslant \frac{P(\Omega')}{2} \qquad \text{for all } h \neq 0,\ |h| \leqslant \epsilon_t,$$

and hence

$$P\left(\Omega' \cap \left\{\frac{|M_d(t + h) - M_d(t)|}{|h|^{\tilde{H}}} > \delta\right\}\right) \geqslant \frac{P(\Omega')}{2} \qquad \text{for all } h \neq 0,\ |h| \leqslant \epsilon_t.$$

Therefore,

$$\begin{aligned}
&\mathrm{E}\left[1_{\Omega'} \frac{|M_d(t + h) - M_d(t)|}{|h|^{\tilde{H}}}\right] \\
&\quad = \mathrm{E}\left[1_{\Omega' \cap \{|M_d(t+h) - M_d(t)|/|h|^{\tilde{H}} \leqslant \delta\}} \frac{|M_d(t + h) - M_d(t)|}{|h|^{\tilde{H}}}\right] \\
&\qquad + \mathrm{E}\left[1_{\Omega' \cap \{|M_d(t+h) - M_d(t)|/|h|^{\tilde{H}} > \delta\}} \frac{|M_d(t + h) - M_d(t)|}{|h|^{\tilde{H}}}\right] \\
&\quad \geqslant 0 + \mathrm{E}\left[1_{\Omega' \cap \{|M_d(t+h) - M_d(t)|/|h|^{\tilde{H}} > \delta\}} \delta\right] = \delta\, P\left(\Omega' \cap \left\{\frac{|M_d(t + h) - M_d(t)|}{|h|^{\tilde{H}}} > \delta\right\}\right) \\
&\quad \geqslant \frac{P(\Omega')}{2}\delta, \qquad \text{for all } h \neq 0,\ |h| \leqslant \epsilon_t.
\end{aligned}$$

This shows (4.33).

Now assume that $P(\mathrm{Var}_{[a,b]}(M_d) < \infty) > 0$. Then there exist $\Omega' \subset \Omega$, $P(\Omega') > 0$ and $K > 0$ such that $\mathrm{Var}_{[a,b]}(M_d) < K$ on Ω'. Hence,

$$\mathrm{E}\left[1_{\Omega'} \mathrm{Var}_{[a,b]}(M_d)\right] \leqslant K. \tag{4.34}$$

We obtain a contradiction as follows. For any sequence $a \leqslant t_0 < t_1 < \ldots < b$, we have

$$\mathrm{E}\left[1_{\Omega'}\mathrm{Var}_{[a,b]}(M_d)\right] \geqslant \mathrm{E}\left[1_{\Omega'}\sum_{i=0}^{\infty}|M_d(t_{i+1}) - M_d(t_i)|\right] = \sum_{i=0}^{\infty}\mathrm{E}[1_{\Omega'}|M_d(t_{i+1}) - M_d(t_i)|]. \tag{4.35}$$

Fix $[a, b'] \subset [a, b]$, $a < b' < b$. We construct a sequence $a \leqslant t_0 < t_1 < \ldots < t_n \leqslant b' < t_{n+1} < b$ for some n with

$$\mathrm{E}[1_{\Omega'}|M_d(t_{i+1}) - M_d(t_i)|] \geqslant (t_{i+1} - t_i)\frac{2K}{b' - a'}, \qquad 0 \leqslant i \leqslant n. \tag{4.36}$$

Since $\tilde{H} < 1$, (4.33) yields

$$\lim_{h\downarrow 0}\mathrm{E}\left[1_{\Omega'}\frac{|M_d(t \pm h) - M_d(t)|}{h}\right] = \lim_{h\downarrow 0}h^{\tilde{H}-1}E\left[1_{\Omega'}\frac{|M_d(t \pm h) - M_d(t)|}{h^{\tilde{H}}}\right] = \infty. \tag{4.37}$$

Thus, for any $t \in [a, b']$, we find $0 < \epsilon_t < b - b'$ with

$$E[1_{\Omega'}|M_d(t + h) - M_d(t)|] \geqslant |h|\frac{2K}{b' - a'}, \qquad \text{for all } h, |h| \leqslant \epsilon_t. \tag{4.38}$$

Now $(]t - \epsilon_t, t + \epsilon_t[)$ is an open covering of $[a, b']$ and thus we find a finite covering $(]t_{2i} - \epsilon_{t_{2i}}, t_{2i} + \epsilon_{t_{2i}}[)$, $t_0 < t_2 < \ldots < t_{2m}$, $t_{2m} + \epsilon_{t_{2m}} = t_{2m+1} > b'$. We choose $t_{2i+1} \in]t_{2i}, t_{2i} + \epsilon_{t_{2i}}[\cap]t_{2i+2} - \epsilon_{t_{2i+2}}, t_{2i+2}[$. Then by (4.38) in fact (4.36) holds for all i, $0 \leqslant i \leqslant 2m =: n$. Now summation of (4.36) gives, together with (4.35),

$$\begin{aligned}\mathrm{E}[1_{\Omega'}\,\mathrm{Var}_{[a,b]}(M_d)] &\geqslant \sum_{i=0}^{n}\mathrm{E}[1_{\Omega'}\,|M_d(t_{i+1}) - M_d(t_i)|] \geqslant \sum_{i=0}^{n}|t_{i+1} - t_i|\frac{2K}{b' - a'}\\ &= (t_{n+1} - t_0)\frac{2K}{b' - a'} \geqslant 2K.\end{aligned}$$

This contradicts (4.34). Consequently, $\mathrm{Var}_{[a,b]}(M_d) = \infty$ a.s.

It remains to show that $\mathrm{Var}_{[a,b]}(M_d) < \infty$ if $\nu(\mathbb{R}) < \infty$. The proof is elementary and based on the series representation of FLPs, and we skip the details. For simplicity, assume that the Lévy measure ν of the driving Lévy process L is symmetric. Now, consider the series representation (3.24). Since $\nu(\mathbb{R}) < \infty$, there is only a finite number $n \in \mathbb{N}$ of jumps τ_i on every interval $[a, b]$. We divide the interval $[a, b]$ into subintervals $]\tau_{i-1}, \tau_i[$, $i = 1, \ldots, n - 1$. Since the total variation of the function $t \mapsto (t - s)_+^d - (-s)_+^d$ is finite on every interval $[\tau_{i-1}, \tau_i]$ and since there are only finitely many τ_i, we can conclude (by an interchange of sumation) that the sample paths of M_d have finite variation on compacts.

If ν is not symmetric the proof uses the series representation (3.26) and the same arguments. □

Remark 4.3. Observe that as a consequence of Theorem 4.6, the FLP M_d is a semimartingale if $\nu(\mathbb{R}) < \infty$.

Theorem 4.7 Semimartingale. *If the Lévy measure ν is given as in Theorem 4.5, then the corresponding fractional Lévy process M_d is not a semimartingale.*

Proof. Let $0 = t_0^n < \ldots < t_n^n = t$, $n \in \mathbb{N}$, be a partition of $[0, t]$ such that $\max_{0 \leqslant i \leqslant n} |t_{i+1}^n - t_i^n| \to 0$ as $n \to \infty$. Assume that M_d is a semimartingale. Then its quadratic variation

$$[M_d, M_d]_t = p\text{-}\lim_{n\to\infty} \sum_{i=0}^{n-1} |M_d(t_{i+1}^n) - M_d(t_i^n)|^2$$

exists for all $t \in [0, T]$, $T > 0$. Hence, there exists a refining subsequence $\{t_i^{n_k}\}$ such that

$$\sum_{i=0}^{n_k-1} |M_d(t_{i+1}^{n_k}) - M_d(t_i^{n_k})|^2 \to [M_d, M_d]_t \text{ a.s.} \qquad \text{as } k \to \infty.$$

Therefore we can apply Fatou's lemma and obtain, together with Theorem 4.1,

$$\begin{aligned} \mathrm{E}[M_d, M_d]_t &= \mathrm{E}\left[\lim_{k\to\infty} \sum_{i=0}^{n_k-1} [M_d(t_{i+1}^{n_k}) - M_d(t_i^{n_k})]^2\right] \qquad (4.39) \\ &\leqslant \liminf_{k\to\infty} \mathrm{E}\left[\sum_{i=0}^{n_k-1} [M_d(t_{i+1}^{n_k}) - M_d(t_i^{n_k})]^2\right] \\ &= \liminf_{k\to\infty} \sum_{i=0}^{n_k-1} \mathrm{E}[M_d(t_{i+1}^{n_k}) - M_d(t_i^{n_k})]^2 \\ &= \frac{\operatorname{var}(L(1))}{\Gamma(2d+2)\sin(\pi[d+\frac{1}{2}])} \liminf_{k\to\infty} \sum_{i=0}^{n_k-1} |t_{i+1}^{n_k} - t_i^{n_k}|^{2d+1} = 0. \end{aligned}$$

It follows from $M_d(0) = 0$ a.s., (4.39) and Protter (2004: Theorem II.22(ii)) that $[M_d, M_d]_t = 0$ a.s. for all $t \in [0, T]$, $T > 0$. If $[M_d, M_d]_t$ is identically zero, the semimartingale M_d with continuous sample paths is known to be of finite variation (Protter 2004: Theorem II.27)). However, by Theorem 4.8, M_d is not of finite variation if ν is of the form given in Theorem 4.5, leading to a contradiction. □

5. Integrals with respect to fractional Lévy processes

In this section we define integrals with respect to FLPs. As pointed out in Theorem 4, an FLP is not always a semimartingale. Therefore, classical Itô integration theory cannot be applied. Recently, integration with respect to FBMs has been studied extensively and various approaches have been used to define a stochastic integral with respect to FBM (for a survey, see Nualart 2003). For instance, Zähle (1998) introduced a pathwise stochastic integral using fractional integrals and derivatives. If the integrand is β-Hölder continuous with $\beta > 1 - H$, then the integral with respect to FBM can be interpreted as a Riemann–Stieltjes integral. Other approaches use the Gaussianity and define a Wiener integral, or they apply Malliavin calculus to obtain Skorohod-like integrals with respect to FBM (see

Decreusefond and Üstünel 1999, and the references therein). Malliavin calculus was also used by Decreusefond and Savy (2004) to construct a stochastic calculus for filtered Poisson processes. A new integral of Itô type with zero mean defined by means of the Wick product was introduced in Duncan *et al.* (2000) who give some Itô formulae (see also Bender 2003).

In this section we consider the special case of a deterministic integrand which is sufficient for our present purposes and turns out to be easy to handle. We give a general definition of integrals with respect to FLPs which is closely related to the integral with respect to FBM defined in Pipiras and Taqqu (2000). First we introduce the Riemann–Liouville fractional integrals and derivatives. For details see Samko *et al.* (1993).

For $0 < \alpha < 1$, the Riemann–Liouville fractional integrals $I^{\alpha}_{\pm}$ are defined by

$$(I^{\alpha}_{-} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} f(t)(t-x)^{\alpha-1}\,\mathrm{d}t, \tag{5.40}$$

$$(I^{\alpha}_{+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} f(t)(x-t)^{\alpha-1}\,\mathrm{d}t, \tag{5.41}$$

if the integrals exist for almost all $x \in \mathbb{R}$. In fact, fractional integrals $I^{\alpha}_{\pm}$ are defined for functions $f \in L^p(\mathbb{R})$ if $0 < \alpha < 1$ and $1 \leqslant p < 1/\alpha$ (Samko *et al.* 1993: 94). We refer to the integrals I^{α}_{-} and I^{α}_{+} as right-sided and left-sided, respectively. Fractional differentiation was introduced as the inverse operation. Let $0 < \alpha < 1$, $1 \leqslant p < 1/\alpha$ and denote by $I^{\alpha}_{\pm}(L^p)$ the class of functions $\phi \in L^p(\mathbb{R})$ which may be represented as an $I^{\alpha}_{\pm}$-integral of some function $f \in L^p(\mathbb{R})$. If $\phi \in I^{\alpha}_{\pm}(L^p)$, there exists a unique function $f \in L^p(\mathbb{R})$ such that $\phi = I^{\alpha}_{\pm} f$ and f agrees with the Riemann–Liouville derivative $\mathcal{D}^{\alpha}_{\pm}$ of ϕ of order α defined by

$$(\mathcal{D}^{\alpha}_{-}\phi)(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_x^{\infty} \phi(t)(t-x)^{-\alpha}\,\mathrm{d}t,$$

$$(\mathcal{D}^{\alpha}_{+}\phi)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_{-\infty}^{x} \phi(t)(x-t)^{-\alpha}\,\mathrm{d}t,$$

where the convergence of the integrals at the singularity $t = x$ holds pointwise for almost all x if $p = 1$ and in the L^p sense if $p > 1$.

Observe that we can rewrite

$$M_d(t) = \int_{\mathbb{R}} (I^d_{-} I_{(0,t)})(s)\, L(\mathrm{d}s).$$

For $g \in L^1(\mathbb{R})$ consider the right-sided Riemann–Liouville fractional integral $I^d_{-} g$ of order d and denote by $\tilde{H}$ the set of functions $g : \mathbb{R} \to \mathbb{R}$, $g \in L^1(\mathbb{R})$ such that

$$\int_{-\infty}^{\infty} (I^d_{-} g)^2(u)\mathrm{d}u < \infty. \tag{5.42}$$

Proposition 5.1. *If* $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, *then* $g \in \tilde{H}$.

Proof. Starting from the fact that $(I^d_{-} g) \in L^2(\mathbb{R})$ if and only if $\int_{\mathbb{R}} |h(u)(I^d_{-} g)(u)|\,\mathrm{d}u \leqslant C\|h\|_{L^2}$ for all $h \in L^2(\mathbb{R})$, it is sufficient to show that for all $h \in L^2(\mathbb{R})$,

$$\int_{\mathbb{R}} \int_0^{\infty} |h(u) s^{d-1} g(s+u)| \, ds \, du \leqslant C \|h\|_{L^2}. \tag{5.43}$$

Now (5.43) holds if

$$I_1 = \int_{\mathbb{R}} \int_1^{\infty} |h(u) s^{d-1} g(s+u)| \, ds \, du \leqslant C \|h\|_{L^2}$$

and

$$I_2 = \int_{\mathbb{R}} \int_0^{1} |h(u) s^{d-1} g(s+u)| \, ds \, du \leqslant C \|h\|_{L^2}.$$

Applying Fubini's theorem and the Hölder inequality, we obtain that

$$I_2 = \int_0^1 s^{d-1} \int_{\mathbb{R}} |h(u) g(s+u)| \, du \, ds \leqslant \int_0^1 s^{d-1} \|h\|_{L^2} \|g\|_{L^2} \, ds = d^{-1} \|g\|_{L^2} \|h\|_{L^2}.$$

Furthermore, setting $t = s + u$ and using Fubini's theorem and Hölder's inequality,

$$\begin{aligned} I_1 &= \int_{\mathbb{R}} |g(t)| \int_1^{\infty} |h(t-s)| s^{d-1} \, ds \, dt \leqslant \int_{\mathbb{R}} \|h\|_{L^2} \left(\int_1^{\infty} s^{2(d-1)} \, ds \right)^{1/2} |g(t)| \, dt \\ &= \int_{\mathbb{R}} \|h\|_{L^2} \frac{1}{\sqrt{1-2d}} |g(t)| \, dt \leqslant (1-2d)^{-1/2} \|g\|_{L^1} \|h\|_{L^2}. \end{aligned}$$

□

We define the space H as the completion of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with respect to the norm

$$\|g\|_H := \left(\mathrm{E}[L(1)^2] \int_{\mathbb{R}} (I_-^d g)^2(u) \, du \right)^{1/2}.$$

If follows from Pipiras and Taqqu (2000: Theorem 3.2) that $\|\cdot\|_H$ defines a norm. Then from the proof of Proposition 5.1 we know that for $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

$$\|g\|_H \leqslant C [\|g\|_{L^1} + \|g\|_{L^2}]. \tag{5.44}$$

To construct the integral $I_{M_d}(g) := \int_{\mathbb{R}} g(s) \, M_d(ds)$ for $g \in H$ we proceed as follows. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a simple function,

$$\phi(s) = \sum_{i=1}^{n-1} a_i 1_{(s_i, s_{i+1}]}(s),$$

where $a_i \in \mathbb{R}$, $i = 1, \ldots, n$ and $-\infty < s_1 < s_2 < \ldots < s_n < \infty$. Notice that $\phi \in H$. Define

$$I_{M_d}(\phi) = \int_{\mathbb{R}} \phi(s) \, M_d(ds) = \sum_{i=1}^{n-1} a_i [M_d(s_{i+1}) - M_d(s_i)].$$

Obviously, I_{M_d} is linear in the simple functions.

Proposition 5.2. *Let* $\phi\colon \mathbb{R} \to \mathbb{R}$ *be a simple function. Then*

$$\int_{\mathbb{R}} \phi(s)\, M_d(\mathrm{d}s) = \int_{\mathbb{R}} (I_-^d \phi)(u)\, L(\mathrm{d}u) \tag{5.45}$$

and $\phi \mapsto I_{M_d}(\phi) = \int_{\mathbb{R}} \phi(s)\, M_d(\mathrm{d}s)$ *is an isometry between* H *and* $L^2(\Omega, P)$.

Proof. It is sufficient to show (5.45) for indicator functions $\phi(s) = 1_{[0,t]}(s),\ t > 0$. In fact,

$$\int_{\mathbb{R}} \phi(s)\, M_d(\mathrm{d}s) = \int_{\mathbb{R}} 1_{[0,t]}(s)\, M_d(\mathrm{d}s) = M_d(t)$$

and for the right-hand side of (5.45) we obtain

$$\begin{aligned}\int_{\mathbb{R}} (I_-^d \phi)(u)\, L(\mathrm{d}u) &= \frac{1}{\Gamma(d)} \int_{\mathbb{R}} \int_u^{\infty} (s-u)^{d-1} 1_{[0,t]}(s) \mathrm{d}s\, L(\mathrm{d}u)\\ &= \frac{1}{\Gamma(d+1)} \int_{\mathbb{R}} \left[(t-u)_+^d - (-u)_+^d\right] L(\mathrm{d}u) = M_d(t).\end{aligned}$$

Moreover, for all simple functions ϕ it follows from (2.14) that

$$\|I_{M_d}(\phi)\|_{L^2(\Omega,P)}^2 = \mathrm{E}\left[\int_{\mathbb{R}} (I_-^d \phi)(u)\, L(\mathrm{d}u)\right]^2 = \mathrm{E}[L(1)^2] \int_{\mathbb{R}} (I_-^d \phi)^2(u)\, \mathrm{d}u = \|\phi\|_H^2. \tag{5.46}$$

□

Theorem 5.3. *Let* $M_d = \{M_d(t)\}_{t\in\mathbb{R}}$ *be an FLP and let the function* $g \in H$. *Then there are simple functions* $\phi_k : \mathbb{R} \to \mathbb{R},\ k \in \mathbb{N}$, *satisfying* $\|\phi_k - g\|_H \to 0$ *as* $k \to \infty$ *such that* $I_{M_d}(\phi_k)$ *converges in* $L^2(\Omega, P)$ *towards a limit denoted by* $I_{M_d}(g) = \int_{\mathbb{R}} g(s)\, M_d(\mathrm{d}s)$ *and* $I_{M_d}(g)$ *is independent of the approximating sequence* ϕ_k. *Moreover,*

$$\|I_{M_d}(g)\|_{L^2(\Omega,P)}^2 = \|g\|_H^2. \tag{5.47}$$

Proof. The simple functions are dense in H. This follows from the fact that the simple functions are dense in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and that $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in H by construction and (5.44). Hence, there exists a sequence (ϕ_k) of simple functions such that $\|\phi_k - g\|_H \to 0$ as $k \to \infty$. It follows from the isometry property (5.46) that $\int_{\mathbb{R}} \phi_k(s)\, M_d(\mathrm{d}s)$ converges in $L^2(\Omega, P)$ towards a limit denoted by $\int_{\mathbb{R}} g(s)\, M_d(\mathrm{d}s)$ and the isometry property is preserved in this procedure. Last, but not least, (5.47) implies that the integral $\int_{\mathbb{R}} g(s)\, M_d(\mathrm{d}s)$ is the same for all sequences of simple functions converging to g.

□

Corollary 5.4. *If* M_d *is a semimartingale, then* $\int_{\mathbb{R}} g(s)\, M_d(\mathrm{d}s)$ *is well defined as a limit in probability of elementary integrals. Observe that, since the limit in probability is unique, this limit is then equal to the limit* $I_{M_d}(g)$ *of Theorem 5.3.*

Using (5.45) and Theorem 5.3 the next proposition is obvious.

Proposition 5.5. *Let* $g \in H$. *Then*

$$\int_{\mathbb{R}} g(s)\, M_d(\mathrm{d}s) = \int_{\mathbb{R}} (I_-^d g)(u)\, L(\mathrm{d}u), \tag{5.48}$$

where the equality holds in the L^2 *sense.*

Remark 5.1. Notice that our conditions on the integrand g differ from those imposed by Zähle (1998). In particular, we do not require the function g to be Hölder continuous of order greater than $1 - d$. Furthermore, if the function g is Hölder continuous and g is defined on a compact interval, then $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Hence, $g \in H$.

The second-order properties of integrals which are driven by FLPs follow by direct calculation. As $\mathrm{E}[L(1)] = 0$, first note that we have, for $g \in H$,

$$\mathrm{E}\left[\int_{\mathbb{R}} g(t)\, M_d(\mathrm{d}t)\right] = \mathrm{E}\left[\frac{1}{\Gamma(d)}\int_{\mathbb{R}}\int_u^\infty (s-u)^{d-1} g(s)\,\mathrm{d}s\, L(\mathrm{d}u)\right] = 0.$$

Proposition 5.6. *Let* $|f|, |g| \in H$. *Then*

$$\mathrm{E}\left[\int_{\mathbb{R}} f(t)\, M_d(\mathrm{d}t)\int_{\mathbb{R}} g(u)\, M_d(\mathrm{d}u)\right] = \frac{\Gamma(1-2d)\mathrm{E}[L(1)^2]}{\Gamma(d)\Gamma(1-d)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)g(u)|t-u|^{2d-1}\,\mathrm{d}t\,\mathrm{d}u. \tag{5.49}$$

Proof. It is a well-known fact that

$$\int_{-\infty}^{\min(u,t)} (t-s)^{d-1}(u-s)^{d-1}\mathrm{d}s = |t-u|^{2d-1}\frac{\Gamma(d)\Gamma(1-2d)}{\Gamma(1-d)}, \qquad u,\ t \in \mathbb{R}.$$

(Gripenberg and Norros 1996: 405). Hence, by the isometry (5.47),

$$\begin{aligned}
&\mathrm{E}\left[\int_{\mathbb{R}} f(t)\, M_d(\mathrm{d}t)\int_{\mathbb{R}} g(u)\, M_d(\mathrm{d}u)\right]\\
&= \frac{\mathrm{E}[L(1)^2]}{\Gamma^2(d)}\int_{-\infty}^{\infty}\int_s^\infty\int_s^\infty f(t)g(u)(t-s)^{d-1}(u-s)^{d-1}\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}s\\
&= \frac{\mathrm{E}[L(1)^2]}{\Gamma^2(d)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)g(u)\int_{-\infty}^{\min(u,t)} (t-s)^{d-1}(u-s)^{d-1}\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}u\\
&= \frac{\Gamma(1-2d)\mathrm{E}[L(1)^2]}{\Gamma(d)\Gamma(1-d)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)g(u)|t-u|^{2d-1}\,\mathrm{d}t\,\mathrm{d}u,
\end{aligned}$$

where we have used Fubini's theorem. □

6. Long memory moving average processes

In discrete time, MA processes are very popular in classical time series analysis and are widely used in applications in engineering, physics and metrology.

We consider the continuous-time version of an MA process. Continuous-time MA processes play an important role since they are very flexible models; for example, MA processes can capture volatility jumps or exhibit long memory properties. Typical examples are the stochastic volatility models of Barndorff-Nielsen and Shephard (2001) which are based on Ornstein–Uhlenbeck processes, the CARMA processes (Brockwell 2001), the FICARMA processes (Brockwell 2004) and the stable MA processes (Samorodnitsky and Taqqu 1994). Extremes of Lévy-driven MA processes were recently studied by Fasen (2004).

We construct a special class of MA processes, the long memory MA processes. Throughout we assume as always that L is a Lévy process without Brownian component satisfying $\mathrm{E}[L(1)] = 0$ and $\mathrm{E}[L(1)^2] < \infty$.

6.1. Lévy-driven long memory moving average processes

Definition 6.1 *Stationary MA process. A stationary continuous-time MA process is a process of the form*

$$Y(t) = \int_{-\infty}^{\infty} g(t-u)\, L(\mathrm{d}u), \qquad t \in \mathbb{R}, \tag{6.50}$$

where the kernel function $g : \mathbb{R} \to \mathbb{R}$ is measurable and the driving process $L = \{L(t)\}_{t\in\mathbb{R}}$ is a Lévy process on $\mathbb{R}$.

Every MA process is well defined if the kernel g and the generating triplet $(\gamma_L, \sigma_L^2, \nu_L)$ of the driving Lévy process L satisfy (2.12).

We first consider *short memory* causal MA processes. Therefore we assume that the kernel g satisfies the following two conditions:

(M1) $g(t) = 0$ for all $t < 0$ (causality).
(M2) $|g(t)| \leqslant C\mathrm{e}^{-ct}$ for some constants $C > 0$ and $c > 0$ (short memory).

From now on, if not stated otherwise, an MA process means a short memory causal MA process, that is g satisfies (M1) and (M2), which imply $g \in L^1(\mathbb{R})$.

Remark 6.1. Substituting (M2) in (2.12), we see that a short memory MA process is well defined if

$$\int_{|x|>1} \log|x|\, \nu_L(\mathrm{d}x) < \infty. \tag{6.51}$$

Now we can use a short memory MA process to construct a long memory MA process. For this purpose we calculate the left-sided Riemann–Liouville fractional integral of the kernel g in (6.50), where we only consider functions $g \in H$. Then we obtain for $0 < d < 0.5$ the fractionally integrated kernel

$$g_d(t) := (I_+^d g)(t) = \int_0^t g(t-s) \frac{s^{d-1}}{\Gamma(d)} \, \mathrm{d}s, \qquad t \in \mathbb{R}. \tag{6.52}$$

From (M1) it follows that $g_d(t) = 0$ for $t \leqslant 0$. Furthermore, $g_d \in L^2(\mathbb{R})$ as $g \in H$. We can now define a fractionally integrated MA process by replacing the kernel g by the kernel g_d.

Definition 6.2 *FIMA process. Let $0 < d < 0.5$. Then the fractionally integrated moving average (FIMA) process $Y_d = \{Y_d(t)\}_{t \in \mathbb{R}}$ driven by the Lévy process L* with $\mathrm{E}[L(1)] = 0$ *and* $\mathrm{E}[L(1)^2] < \infty$ *is defined by*

$$Y_d(t) = \int_{-\infty}^t g_d(t-u) \, L(\mathrm{d}u), \qquad t \in \mathbb{R}, \tag{6.53}$$

where the fractionally integrated kernel g_d is given in (6.52).

Theorem 6.1 Stationarity, infinite divisibility. *The FIMA process* (6.53) *is well defined and stationary. For all $t \in \mathbb{R}$, the distribution of $Y_d(t)$ is infinitely divisible with characteristic triplet $(\gamma_Y^t, 0, \nu_Y^t)$, where*

$$\gamma_Y^t = -\int_{-\infty}^t \int_{\mathbb{R}} x g_d(t-s) 1_{\{|g_d(t-s)x| > 1\}} \, \nu_L(\mathrm{d}x) \, \mathrm{d}s \tag{6.54}$$

and

$$\nu_Y^t(B) = \int_{-\infty}^t \int_{\mathbb{R}} 1_B(g_d(t-s)x) \, \nu_L(\mathrm{d}x) \, \mathrm{d}s, \qquad B \in \mathcal{B}(\mathbb{R}). \tag{6.55}$$

Here $(\gamma_L, 0, \nu_L)$ denotes the characteristic triplet of L.

Proof. Since $g_d \in L^2(\mathbb{R})$ we can apply Proposition 2.1 to $Y_d(0)$ and obtain that Y_d is well defined. Now let $u_1, \ldots, u_n \in \mathbb{R}$ and $-\infty < t_1 < \ldots < t_n < \infty$, $n \in \mathbb{N}$. Then by the stationarity of the increments of L,

$$\begin{aligned} u_1 Y_d(t_1 + h) + \ldots + u_n Y_d(t_n + h) &= \sum_{k=1}^n u_k \int_{-\infty}^{t_k + h} g_d(t_k + h - s) \, L(\mathrm{d}s) \\ &\stackrel{d}{=} \sum_{k=1}^n u_k \int_{-\infty}^{t_k} g_d(t_k - s) \, L(\mathrm{d}s) = u_1 Y_d(t_1) + \ldots + u_n Y_d(t_n). \end{aligned} \tag{6.56}$$

The characteristic functions of the left- and the right-hand side of (6.56) coincide. Hence, by the Cramér–Wold device, Y_d is stationary. □

So far we have constructed an FIMA process by fractional integration of the corresponding short memory kernel g. The next theorem states that we can also construct an FIMA process by replacing the driving Lévy process in the short memory MA process (6.50) by the corresponding fractional Lévy process. The resulting process coincides in L^2 with the process (6.53).

Theorem 6.2. *Suppose $Y_d = \{Y_d(t)\}_{t\in\mathbb{R}}$ to be the FIMA process $Y_d(t) = \int_{-\infty}^{t} g_d(t-s)\,L(\mathrm{d}s)$, $t \in \mathbb{R}$, with $g_d \in L^2(\mathbb{R})$ such that $g_d \in I_{+}^{d}(L^2)$. Then Y_d can be represented as*

$$Y_d(t) = \int_{-\infty}^{t} g(t-s)\,M_d(\mathrm{d}s), \qquad t \in \mathbb{R}, \tag{6.57}$$

with

$$g(x) = \frac{1}{\Gamma(1-d)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x g_d(s)(x-s)^{-d}\mathrm{d}s, \qquad x \in \mathbb{R},$$

that is, g is the Riemann–Liouville derivative $\mathcal{D}_{+}^{d} g_d$ of the kernel g_d.

On the other hand, if Y_d is given by (6.57) with $g \in H$, then Y_d can be rewritten as $Y_d(t) = \int_{-\infty}^{t} g_d(t-s)\,L(\mathrm{d}s)$, $t \in \mathbb{R}$, where $g_d(x) = (I_{+}^{d} g)(x)$.

Proof. For every $t \in \mathbb{R}$, we have that a.s.

$$\begin{aligned}Y_d(t) &= \int_{-\infty}^{t} g(t-s)\,M_d(\mathrm{d}s) = \frac{1}{\Gamma(d)}\int_{-\infty}^{t}\left(\int_u^\infty (s-u)^{d-1} g(t-s)\,\mathrm{d}s\right) L(\mathrm{d}u)\\ &= \frac{1}{\Gamma(d)}\int_{-\infty}^{t}\left(\int_0^\infty s^{d-1} g(t-u-s)\mathrm{d}s\right) L(\mathrm{d}u) = \int_{-\infty}^{t} g_d(t-u)\,L(\mathrm{d}u).\end{aligned}$$

□

Using representation (6.57) of an FIMA process it is easy to show that this class of processes has long memory properties.

Theorem 6.3 Long memory. *An FIMA process $Y_d = \{Y_d(t)\}_{t\in\mathbb{R}}$ is a long memory MA process.*

Proof. Since Y_d can be expressed as (6.57), we have from Proposition 5.6, for $h > 0$, that

$$\begin{aligned}\gamma_{Y_d}(h) &= \operatorname{cov}(Y_d(t+h), Y_d(t))\\ &= \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)}\mathrm{E}[L(1)^2]\int_{\mathbb{R}}\int_{\mathbb{R}} g(t+h-u)g(t-v)|u-v|^{2d-1}\,\mathrm{d}u\,\mathrm{d}v\\ &= \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)}\mathrm{E}[L(1)^2]\int_{\mathbb{R}}\int_{\mathbb{R}} g(s)g(\tilde{s})|h-s+\tilde{s}|^{2d-1}\,\mathrm{d}s\,\mathrm{d}\tilde{s}.\end{aligned}$$

It follows that

$$\gamma_{Y_d}(h) \sim \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \mathrm{E}[L(1)^2]\left(\int_{\mathbb{R}} g(u)\,\mathrm{d}u\right)^2 |h|^{2d-1}, \qquad \text{as } h \to \infty.$$

Hence, γ_{Y_d} satisfies condition (1.3) and Y_d is a long memory process. □

6.2. Second-order and sample path properties of FIMA processes

Theorem 6.4 Autocovariance function. *Let $0 < d < 0.5$. The autocovariance function γ_d of an FIMA process Y_d is*

$$\gamma_d(h) = \mathrm{E}[L(1)^2]\int_{\mathbb{R}} g_d(u+|h|)g_d(u)\,\mathrm{d}u, \qquad h \in \mathbb{R}, \tag{6.58}$$

where g_d is the fractionally integrated kernel given in (6.52).

Proof. Let $h \geqslant 0$. Then, from representation (6.53),

$$\gamma_d(h) = \mathrm{cov}(Y_d(t+h), Y_d(t)) = \mathrm{var}(L(1))\int_{-\infty}^{t} g_d(t+h-s)g_d(t-s)\,\mathrm{d}s$$

$$= \mathrm{E}[L(1)^2]\int_0^{\infty} g_d(u+h)g_d(u)\,\mathrm{d}u = \mathrm{E}[L(1)^2]\int_{\mathbb{R}} g_d(u+h)g_d(u)\,\mathrm{d}u,$$

since $g_d(t) = 0$ for $t \leqslant 0$. □

Theorem 6.5 Spectral density. *The spectral density f_d of an FIMA process Y_d equals*

$$f_d(\lambda) = \frac{\mathrm{E}[L(1)^2]}{2\pi}\left|G_d(\lambda)\right|^2, \qquad \lambda \in \mathbb{R}, \tag{6.59}$$

where $G_d(\lambda) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}u\lambda} g_d(u)\,\mathrm{d}u$, $\lambda \in \mathbb{R}$, is the Fourier transform of the kernel function g_d given in (6.52).

Proof. The assertion follows from (6.58), since the spectral density of a stationary process is the inverse Fourier transform of the autocovariance function. □

To obtain some insight into the behaviour of the sample paths of an FIMA process we exclude path properties that do *not* hold. In fact, Rosinski (1989) provides immediately verifiable necessary conditions for interesting sample path properties.

Proposition 6.6 p-Variation. *Let $p \geqslant 0$. If the kernel $t \mapsto g_d(t-s)$ is of unbounded p-variation then $P(\{\omega \in \Omega : Y_d(\cdot, \omega) \notin C_p[a, b]\}) > 0$, where $C_p[a, b]$ is the space of functions of bounded p-variation on $[a, b]$.*

Proof. The assertion follows by an application of Theorem 4 of Rosinski (1989), where we use the symmetrization argument of Section 5 in Rosinski (1989), if ν_L is not already symmetric. □

We noted in Theorem 6.1 that an FIMA process Y_d has infinitely divisible margins. Moreover, since $\mathrm{E}[L(1)] = 0$, $\mathrm{E}[L(1)^2] < \infty$ and the Lévy–Itô representation (2.8) of L is given by $L(t_1) - L(t_2) = \int_{\mathbb{R}_0 \times (t_1, t_2]} x\tilde{J}(\mathrm{d}x, \mathrm{d}s)$, we can write

$$Y_d(t) = \int_{-\infty}^{t} \int_{\mathbb{R}_0} x g_d(t-s)\, \tilde{J}(\mathrm{d}x, \mathrm{d}s).$$

Therefore we can apply the results of Marcus and Rosinski (2005) to determine the continuity of Y_d.

Proposition 6.7 Continuity. *Let* $g_d \in C_b^1(\mathbb{R})$. *Then the FIMA process* Y_d *has a continuous version on every bounded interval* I *of* $\mathbb{R}$.

Proof. Applying Theorem 2.5 of Marcus and Rosinski (2005), we obtain that Y_d has a continuous version on $I \subset \mathbb{R}$ if $g_d(0) = 0$ and if, for some $\epsilon > 0$,

$$\sup_{u,v \in I} \left(\log \frac{1}{|u-v|} \right)^{1/2+\epsilon} |g_d(u) - g_d(v)| < \infty.$$

We have

$$|g_d(u) - g_d(v)| \leqslant |g_d{}'(\xi)(u-v)| \leqslant C|u-v|, \qquad u \leqslant \xi \leqslant v,\ \xi \in I.$$

Therefore,

$$\sup_{u,v \in I} \left(\log \frac{1}{|u-v|} \right)^{1/2+\epsilon} |g_d(u) - g_d(v)| \leqslant \sup_{t \in I'} C|t|(-\log|t|)^{1/2+\epsilon} = \sup_{t \in I'} m(t),$$

where $m(t) = C|t|(-\log|t|)^{1/2+\epsilon} \leqslant C|t|(-\log|t|) \to 0$ as $t \to 0^+$. Moreover m is continuous and assumes its maximum on any compact interval. Hence, $\sup_{t \in I'} m(t) < \infty$. □

Remark 6.2. If the process L has paths of bounded variation then

$$Y_d(t) = \int_{-\infty}^{t} g_d(t-s)\, L(\mathrm{d}s) = (g_d * L)(t), \qquad t \in \mathbb{R},$$

is the convolution of the kernel g_d with the jumps of L, taken pathwise. In this case, as g_d is continuous, it is obvious that Y_d is continuous.

Remark 6.3. Finally, we remark that, like an FLP, an FIMA process has a generalized shot noise representation (3.25) with the kernel function $f_t(\cdot)$ replaced by the kernel $g_d(t - \cdot)$ given in (6.52).

The results of this section can be applied to CARMA and FICARMA processes, which are the continuous-time analogues of the well-known autoregressive moving average (ARMA) and fractionally integrated ARMA processes, respectively. Details on CARMA and FICARMA processes can be found in Brockwell (2001, 2004) and Brockwell and Marquardt (2005). Due to the slow decay of the fractionally integrated kernel g_d, simulation algorithms for FICARMA processes have been very slow and expensive. The rapid decay of the kernel g in the new representation (6.57) allows much more efficient simulation of these processes.

The results of a simulation of FICARMA processes will be available at http://www-m4.ma.tum.de/pers/marquardt/.

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