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Passage times for a spectrally negative Lévy process with applications to risk theory

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The distributions of the last passage time at a given level and the joint distributions of the last passage time, the first passage time and their difference for a general spectrally negative process are derived in the form of Laplace transforms. The results are applied to risk theory.

Keywords: first passage time; last passage time; spectrally negative Lévy process; risk theory

1. Introduction

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X(t), \theta_t, P_x)$ denote the canonical realization of a spectrally negative Lévy process with $P_x\{X(0) = x\} = 1$. Thus X is a Hunt process with stationary independent increments specified by

$$\mathrm{E}_0 \mathrm{e}^{\alpha X(t)} = \mathrm{e}^{t\psi(\alpha)}, \qquad \alpha \geqslant 0, \tag{1.1}$$

where E_x is the expectation with respect to P_x, and

$$\psi(\alpha) = a\alpha + \frac{1}{2}\sigma^2\alpha^2 + \int_{-\infty}^{0} \{\mathrm{e}^{\alpha x} - 1 - \alpha x \mathbf{1}(x > -1)\}\nu(\mathrm{d}x), \tag{1.2}$$

with $a \in \mathbb{R}$, $\sigma^2 \geqslant 0$, and ν is a non-negative measure supported on $(-\infty, 0)$ satisfying

$$\int_{-\infty}^{-1} \nu(\mathrm{d}x) < \infty \quad \text{and} \quad \int_{-1}^{0} x^2 \nu(\mathrm{d}x) < \infty.$$

The measure ν and the function ψ are called the Lévy measure and the Lévy exponent of X, respectively. Such a Lévy process has bounded variation if and only if $\sigma = 0$ and $\int_{-1}^{0} |x|\nu(\mathrm{d}x) < \infty$. In this case the Lévy exponent can be re-expressed as

$$\psi(\alpha) = b\alpha + \int_{-\infty}^{0} (\mathrm{e}^{\alpha x} - 1)\nu(\mathrm{d}x),$$

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where $b = a - \int_{-1}^{0} x\nu(\mathrm{d}x)$ is known as the drift coefficient. If $\sigma^2 > 0$, X is said to have a Gaussian component. As usual, we also assume that X is not a subordinator.

If Y is a spectrally positive Lévy process, then $X := -Y$ is a spectrally negative Lévy process. Thus the results for Y can be obtained from the corresponding results for X.

The following facts on ψ can be found in Bertoin (1996): ψ is strictly increasing and continuous on $[z, \infty)$, where z is the largest real zero of ψ; $\psi(0) = 0$; $\psi(\alpha) \to \infty$ as $\alpha \to \infty$; $\psi'(0+) = \mathrm{E}_0 X(1)$; and $\psi'(\alpha) > 0$ for $\alpha > 0$. The right inverse of ψ is denoted by ψ^{-1}.

For $x \in \mathbb{R}$, denote by $\tau_x = \inf\{t \geqslant 0 : X(t) > x\}$ and $T_x = \inf\{t \geqslant 0 : X(t) < x\}$ the first passage time above and below the level x, respectively; denote by $l_x = \sup\{t \geqslant 0 : X(t) < x\}$ and $T'_x = \inf\{t \geqslant T_x : X(t) > x\}$ the last passage time below the level x and the first passage time above the level x after T_x, respectively, with the usual convention that $\inf \varnothing = \infty$ and $\sup \varnothing = 0$. Since $\mathrm{E}_0 X(1) \leqslant 0$ leads to $\liminf_{t\to\infty} X(t) = -\infty$ almost surely (Zolotarev 1964), which implies $l_x = \infty$ almost surely for any fixed x, we assume that $\mathrm{E}_0 X(1) > 0$, which is equivalent to $P_0\{\lim_{t\to\infty} X(t) = +\infty\} = 1$ (Bingham 1975). For ease of presentation, let $Q(x) = 1 - \overline{Q}(x) = P_0\{\inf_{t\geqslant 0} X(t) < -x\}$ for $x \geqslant 0$, $Q(0) := \lim_{x\downarrow 0} Q(x)$ and $\mathrm{E}_0 X(1) = m$.

Lemma 1.1 *(Bertoin, 1996, p. 189, Theorem 1). For the spectrally negative Lévy process X, τ_x with $x \geqslant 0$ is a subordinator with the Laplace exponent ψ^{-1}, that is,*

$$\mathrm{E}_0 \mathrm{e}^{-\alpha \tau_x} = \mathrm{e}^{-x\psi^{-1}(\alpha)}, \qquad \alpha \geqslant 0. \tag{1.3}$$

Consequently ψ^{-1} is differentiable.

Lemma 1.2 *(Zolotarev 1964). For the spectrally negative Lévy process X, $Q(x)$ can be determined by*

$$\alpha \int_0^\infty \mathrm{e}^{-\alpha x} Q(x)\mathrm{d}x = 1 - \frac{\alpha m}{\psi(\alpha)}. \tag{1.4}$$

It follows from (1.4) that $Q(0) = 1$ if $\sigma^2 > 0$ or $\int_{-1}^{0} |x|\nu(\mathrm{d}x) = \infty$ (see also Rogozin 1965; Prabhu 1970), and that

$$Q(0) = \frac{\displaystyle\int_{-\infty}^{0} x\nu(\mathrm{d}x)}{\displaystyle\int_{-1}^{0} x\nu(\mathrm{d}x) - a}$$

if $\sigma = 0$ and $\int_{-1}^{0} |x|\nu(\mathrm{d}x) < \infty$.

It is known that $\Lambda_t(c) = \exp\{cX(t) - \psi(c)t\}$ is a martingale under P_0 for any c such that $\psi(c)$ is finite (see Avram *et al*. 2004). Let $P_0^{(c)}$ denote the probability measure on $\mathcal{F}$ defined by

$$\left.\frac{\mathrm{d}P_0^{(c)}}{\mathrm{d}P_0}\right|_{\mathcal{F}_t} = \Lambda_t(c),$$

for all $0 \leqslant t < \infty$. We have for the stopping time T_x $(x \leqslant 0)$,

$$P_0^{(c)}(T_x < \infty) = \mathrm{E}_0\{\Lambda_{T_x}(c)\mathbf{1}(T_x < \infty)\} = \mathrm{E}_0\{\Lambda_{T_x}(c)\}. \tag{1.5}$$

Under the measure $P_0^{(c)}$, X remains within the class of spectrally negative processes and the Laplace exponent of X is given by (see Avram *et al*. 2004)

$$\psi_c(\theta) = \psi(\theta + c) - \psi(c), \qquad \theta \geqslant \min\{-c, 0\}. \tag{1.6}$$

Definition 1.1 *(Avram et al. 2004; Bertoin 1997). Consider* $(X, P_0^{(c)})$. *For* $p \geqslant 0$, *the p-scale function* $W_c^{(p)} : (-\infty, \infty) \to [0, \infty)$ *is the unique function whose restriction to* $(0, \infty)$ *is continuous and has Laplace transform*

$$\int_0^\infty \mathrm{e}^{-\theta x} W_c^{(p)}(x)\mathrm{d}x = \{\psi_c(\theta) - p\}^{-1}, \qquad \theta > \psi_c^{-1}(p),$$

and is defined to be identically zero for $x \leqslant 0$. *Further, for every* $x \geqslant 0$, *the mapping* $p \to W_c^{(p)}(x)$ *can be extended to* $p \in \mathbb{C}$ *by* $W_c^{(p)}(x) = \sum_{k=0}^\infty p^k W_c^{*(k+1)}(x)$, *where* W_c^{*k} *denotes the kth convolution power of the function* $W_c^{(0)}$. *Moreover, let*

$$Z_c^{(p)}(x) = 1 + p\int_{-\infty}^x W_c^{(p)}(y)\mathrm{d}y.$$

Lemma 1.3 *(Emery 1973). For* $\alpha \geqslant 0$ *and* $\beta \geqslant 0$, *the joint Laplace transform of* T_y *and* $X(T_y)$ *is given by*

$$\mathrm{E}_x(\mathrm{e}^{-\alpha T_y + \beta X(T_y)}) = \mathrm{e}^{\beta x}\{Z_\beta^{(p)}(x-y) - W_\beta^{(p)}(x-y)p/\psi_\beta^{-1}(p)\}, \qquad x \geqslant y, \tag{1.7}$$

where $p = \alpha - \psi(\beta)$.

Lemma 1.4 *(Doney 1991).* (i) *For any bounded variation spectrally negative Lévy process we have* $\mathrm{E}_0(\mathrm{e}^{-\theta T_0'}) = 1 - \{b(\psi^{-1})'(\theta)\}^{-1}$ *for* $\theta > 0$.

(ii) *For any spectrally negative Lévy process with paths of unbounded variation,* $T_0' = 0$ *almost surely.*

The first passage time for Lévy processes has been well studied; see Doney (1991), Emery (1973), Prabhu (1970), Rogers (1990; 2000) and Zolotarev (1964). This paper is more concerned with the last passage time and its joint distribution with the duration between the first and the last visit at x and the duration of the first period spent below the level x. More precisely, the main purpose of this paper is to develop results related to the random variables $l_x - \tau_x$, $l_x - T_x$, l_x and $T_x' - T_x$ for spectrally negative Lévy processes. Explicit solutions of the Laplace transforms of the distributions are obtained.

The outline of the paper is as follows. Section 2 gives some primary results for a Lévy process with bounded variation. Section 3 considers the general case, which of course includes Lévy processes with bounded variation as a special case. We present the bounded variation case separately to demonstrate a different methodology. Applications in risk theory are discussed in Section 4.

2. Special case

This section considers the spectrally negative Lévy process with bounded variation, that is, the case of X with $\sigma = 0$ and $\int_{-1}^{0} |x|\nu(\mathrm{d}x) < \infty$. Under these conditions, X is essentially a positive drift minus a subordinator (see Bingham 1975, Proposition 6). To avoid ambiguity, we use a different notation for the Lévy component in this case:

$$\phi(\alpha) = b\alpha + \int_{-\infty}^{0} (\mathrm{e}^{\alpha x} - 1)\nu(\mathrm{d}x),$$

where $b = a - \int_{-1}^{0} x\nu(\mathrm{d}x)$.

We first give the Laplace transform of $l_x - \tau_x$, the duration between the first and the last visit at x, then deduce the Laplace transform of l_x, and finally find the joint Laplace transform of T_x, $l_x - T_x$ and $T'_x - T_x$.

Theorem 2.1. *For any $y \leqslant x$ and $\alpha > 0$, we have*

$$\mathrm{E}_y \mathrm{e}^{-\alpha(l_x - \tau_x)} = m(\phi^{-1})'(\alpha). \tag{2.1}$$

Proof. We first consider $y = 0$. For $x \geqslant 0$, let $S_n(x)$ denote the time elapsed between the nth and the $(n+1)$th visit at x. Since x is both non-polar and non-regular, it is clear that $l_x - \tau_x = S_1(x) + \ldots + S_N(x)$ on $\{\omega \in \Omega : S_i(x)(\omega) < \infty\}$, where $N = \max\{k \geqslant 1 : S_k(x) < \infty\}$. The Markov property of the process implies that the random variables $S_i(x)$ are independent and identically distributed. Moreover, they are independent of N, which has a geometric distribution

$$P_0(N = k) = \overline{Q}(0)Q^k(0), \qquad k = 0, 1, 2, \ldots.$$

Note that $P_0(l_x - \tau_x < \infty) = 1$ and $P_0\{S_1(x) \geqslant T_x\} = 1$. Thus, for $\alpha > 0$, we have

$$\begin{aligned} \mathrm{E}_0 \mathrm{e}^{-\alpha(l_x - \tau_x)} &= \mathrm{E}_0(\mathrm{e}^{-\alpha(l_x - \tau_x)} | l_x - \tau_x < \infty) \\ &= \sum_{k=0}^{\infty} \{\mathrm{E}_0(\mathrm{e}^{-\alpha S_1(x)} | S_1(x) < \infty)\}^k P_0(N = k) \\ &= \frac{\overline{Q}(0)}{1 - Q(0)\mathrm{E}_0(\mathrm{e}^{-\alpha S_1(x)} | S_1(x) < \infty)}. \end{aligned} \tag{2.2}$$

The distribution of $S_1(x)$ does not depend on x, since the process X has stationary independent increments. It follows from Lemma 1.4 that

$$\mathrm{E}_0\{\mathrm{e}^{-\alpha S_1(0)} \mathbf{1}(T_0 < \infty)\} = \mathrm{E}_0 \mathrm{e}^{-\alpha S_1(0)} = 1 - \{b(\phi^{-1})'(\alpha)\}^{-1},$$

which implies that

$$\mathrm{E}_0(\mathrm{e}^{-\alpha S_1(0)} | S_1(0) < \infty) = \mathrm{E}_0(\mathrm{e}^{-\alpha S_1(0)} | T_0 < \infty) = [1 - \{b(\phi^{-1})'(\alpha)\}^{-1}]/Q(0).$$

Substituting this into (2.2) and noting that $\overline{Q}(0)b = m$, we obtain

$$\mathrm{E}_0 \mathrm{e}^{-\alpha(l_x - \tau_x)} = m(\phi^{-1})'(\alpha). \tag{2.3}$$

Since X has stationary independent increments, for $x,\ y \in \mathbb{R}$ and $y \leqslant x$,

$$\mathrm{E}_y \mathrm{e}^{-\alpha(l_x - \tau_x)} = \mathrm{E}_0 \mathrm{e}^{-\alpha(l_{x-y} - \tau_{x-y})},$$

and the result follows.

Theorem 2.2. *For $\alpha > 0$ and $x,\ y \in \mathbb{R}$,*

$$\mathrm{E}_y\{\mathrm{e}^{-\alpha l_x}\mathbf{1}(l_x > 0)\} = \begin{cases} m(\phi^{-1})'(\alpha)\mathrm{e}^{-(x-y)\phi^{-1}(\alpha)}, & y < x, \quad (2.4) \\ m(\phi^{-1})'(\alpha) - \dfrac{m}{b}, & y = x, \quad (2.5) \\ \dfrac{m(\phi^{-1})'(\alpha)P_0^{(\phi^{-1}(\alpha))}(T_{x-y} < \infty)}{\mathrm{e}^{(x-y)\phi^{-1}(\alpha)}}, & y > x. \quad (2.6) \end{cases}$$

Proof. Since $P_0\{X(\tau_x) = x\} = 1$, the strong Markov property of X at time τ_x implies that, for $x,\ s,\ t > 0$,

$$\begin{aligned} P_0(l_x - \tau_x < t,\, \tau_x < s) &= \mathrm{E}_0\{P_{X(\tau_x)}(l_x < t)\mathbf{1}(\tau_x < s)\} \\ &= P_x(l_x < t)P_0(\tau_x < s). \end{aligned} \tag{2.7}$$

The process X has only negative jumps and $P_0\{\lim_{t\to\infty} X(t) = +\infty\} = 1$, thus $P_0(\tau_x < \infty) = 1$ for $x > 0$. Consequently, letting $s \to \infty$ in (2.7) yields $P_0(l_x - \tau_x < t) = P_x(l_x < t)$, and hence (2.7) shows that $l_x - \tau_x$ and τ_x are independent with respect to P_0. As a result, by (2.3) and Lemma 1.1, we have, for $x > 0$,

$$\mathrm{E}_0\mathrm{e}^{-\alpha l_x} = \mathrm{E}_0\mathrm{e}^{-\alpha(l_x - \tau_x)}\mathrm{E}_0\mathrm{e}^{-\alpha\tau_x} = m(\phi^{-1})'(\alpha)\mathrm{e}^{-x\phi^{-1}(\alpha)}.$$

Letting $x \downarrow 0$ yields

$$\mathrm{E}_0\mathrm{e}^{-\alpha l_0} = m(\phi^{-1})'(\alpha),$$

and due to the stationarity and independence of increments, for any $x \in \mathbb{R}$, we have

$$\mathrm{E}_x\mathrm{e}^{-\alpha l_x} = \mathrm{E}_0\mathrm{e}^{-\alpha l_0} = m(\phi^{-1})'(\alpha), \tag{2.8}$$

which is equivalent to (2.5).

For any $y < x$, $P_y(l_x > 0) = P_y(\tau_x < \infty) = 1$ and $P_y(X(\tau_x) = x) = 1$. By the strong Markov property of X at time τ_x,

$$\begin{aligned} \mathrm{E}_y\{\mathrm{e}^{-\alpha l_x}\mathbf{1}(l_x > 0)\} &= \mathrm{E}_y[\mathrm{e}^{-\alpha\tau_x}\mathrm{E}_{X(\tau_x)}\{\mathrm{e}^{-\alpha l_x}\mathbf{1}(l_x \geqslant 0)\}\mathbf{1}(\tau_x < \infty)] \\ &= \mathrm{E}_0\mathrm{e}^{-\alpha\tau_{x-y}}\mathrm{E}_x\mathrm{e}^{-\alpha l_x}. \end{aligned} \tag{2.9}$$

Equation (2.4) follows from (2.8), (2.9) and Lemma 1.1.

Since X has stationary independent increments and is strong Markov, for any $y > x$, we have

$$\begin{aligned}\mathrm{E}_y\{\mathrm{e}^{-\alpha l_x}\mathbf{1}(l_x>0)\} &= \mathrm{E}_0\{\mathrm{e}^{-\alpha l_{x-y}}\mathbf{1}(l_{x-y}>0)\}\\ &= \mathrm{E}_0[\mathrm{e}^{-\alpha T_{x-y}}\mathrm{E}_{X(T_{x-y})}\{\mathrm{e}^{-\alpha l_{x-y}}\mathbf{1}(l_{x-y}\geqslant 0)\}\mathbf{1}(T_{x-y}<\infty)]\\ &= \int_0^\infty\int_{-\infty}^{x-y}\mathrm{e}^{-\alpha t}\mathrm{E}_z(\mathrm{e}^{-\alpha l_{x-y}})P_0\{T_{x-y}\in \mathrm{d}t,\, X(T_{x-y})\in \mathrm{d}z,\, T_{x-y}<\infty\},\end{aligned}$$

so that equation (2.6) follows from (2.4) and (1.5). □

Theorem 2.3. *Suppose that* $\alpha, \beta, \delta > 0$ and $x, y \in \mathbb{R}$. *If* $y > x$, *then*

$$\begin{aligned}&\mathrm{E}_y\{\mathrm{e}^{-\alpha T_x-\beta(l_x-T_x)-\delta(T'_x-T_x)}\mathbf{1}(T_x<\infty)\}\\ &\quad= m(\phi^{-1})'(\beta)\mathrm{e}^{-x\phi^{-1}(\beta+\delta)}\Delta_1(\alpha,\beta,\delta,x,y),\end{aligned}\tag{2.10}$$

where

$$\Delta_1(\alpha,\beta,\delta,x,y)=\mathrm{e}^{\phi^{-1}(\beta+\delta)y}\left\{Z^{(p_1)}_{\phi^{-1}(\beta+\delta)}(y-x)-\frac{W^{(p_1)}_{\phi^{-1}(\beta+\delta)}(y-x)p_1}{\phi^{-1}_{\phi^{-1}(\beta+\delta)}(p_1)}\right\},$$

in which $p_1=\alpha-\beta-\delta$.

Proof. By the strong Markov property of X and (2.4),

$$\begin{aligned}&\mathrm{E}_y\{\mathrm{e}^{-\alpha T_x-\beta(l_x-T_x)-\delta(T'_x-T_x)}\mathbf{1}(T_x<\infty)\}\\ &= \mathrm{E}_y\{\mathrm{e}^{-\alpha T_x}\mathrm{E}_{X(T_x)}\mathrm{e}^{-\beta l_x-\delta T'_x}\mathbf{1}(T_x<\infty)\}\\ &= \mathrm{E}_y\{\mathrm{e}^{-\alpha T_x}\mathrm{E}_{X(T_x)}\{\mathrm{e}^{-(\beta+\delta)T'_x}\cdot \mathrm{E}_x\mathrm{e}^{-\beta l_x}\}\mathbf{1}(T_x<\infty)\}\\ &= m(\phi^{-1})'(\beta)\mathrm{E}_y\{\mathrm{e}^{-\alpha T_x-\phi^{-1}(\beta+\delta)\{x-X(T_x)\}}\mathbf{1}(T_x<\infty)\}\\ &= m(\phi^{-1})'(\beta)\mathrm{e}^{-x\phi^{-1}(\beta+\delta)}\mathrm{E}_y\{\mathrm{e}^{-\alpha T_x+\phi^{-1}(\beta+\delta)X(T_x)}\},\end{aligned}$$

and the result follows from Lemma 1.3. □

The following corollary generalizes the result in Doney (1991, equation (2.19)).

Corollary 2.1. *For any* $\alpha, \beta > 0$, *we have*

$$\begin{aligned}&\mathrm{E}_0\{\mathrm{e}^{-\alpha T_0-\beta(l_0-T_0)-\delta(T'_0-T_0)}\mathbf{1}(T_0<\infty)\}\\ &\quad= m(\phi^{-1})'(\beta)\left[1-\frac{\alpha-\beta-\delta}{b\{\phi^{-1}(\alpha)-\phi^{-1}(\beta+\delta)\}}\right].\end{aligned}$$

Proof. From the proof of Theorem 2.3 one finds

$$E_0\{e^{-\alpha T_0-\beta(l_0-T_0)-\delta(T_0'-T_0)}\mathbf{1}(T_0<\infty)\}$$
$$=m(\phi^{-1})'(\beta)E_0\{e^{-\alpha T_0+\phi^{-1}(\beta+\delta)X(T_0)}\}.$$

Since

$$E_0 e^{-\alpha T_0+\phi^{-1}(\beta)X(T_0)}=1-\frac{\alpha-\beta}{b\{\phi^{-1}(\alpha)-\phi^{-1}(\beta)\}}$$

(Prabhu 1970), the result follows. □

3. General case

In the previous section, we computed the Laplace transforms of l_x, $l_x-\tau_x$ and the joint Laplace transform of T_x, l_x-T_x and $T_x'-T_x$ for a spectrally negative Lévy process X with bounded variation. In this section, we consider the general Lévy process defined by (1.1) and (1.2), which includes the bounded variation case in Section 2. The argument used in Section 2, however, cannot be applied to the case where $\sigma\neq 0$ or $\int_{-1}^0|x|\nu(\mathrm{d}x)=\infty$, because then X has unbounded variation and hence $S_i(x)=0$ almost surely (see Lemma 1.4(ii)).

Theorem 3.1. *For $\alpha>0$ and $x,\ y\in\mathbb{R}$, we have*

$$E_y\{e^{-\alpha l_x}\mathbf{1}(l_x>0)\}=\begin{cases} m(\psi^{-1})'(\alpha)e^{-(x-y)\psi^{-1}(\alpha)}, & y<x, \quad (3.1)\\ m(\psi^{-1})'(\alpha)-\overline{Q}(0), & y=x, \quad (3.2)\\ \dfrac{m(\psi^{-1})'(\alpha)P_0^{(\psi^{-1}(\alpha))}(T_{x-y}<\infty)}{e^{(x-y)\psi^{-1}(\alpha)}}, & y>x, \quad (3.3)\end{cases}$$

where $\overline{Q}(0)=0$ if $\sigma^2>0$ or $\int_{-1}^0|x|\nu(\mathrm{d}x)=\infty$, and $\overline{Q}(0)=m/b$ if $\sigma^2=0$ and $\int_{-1}^0|x|\nu(\mathrm{d}x)<\infty$.

Proof. Since $\{l_0<t\}=\{X(t)>0,\ \inf_{s\geq t}X(s)>0\}$, using the Markov property and right continuity, we have

$$E_0 e^{-\alpha l_0}=\alpha\int_0^\infty e^{-\alpha t}P_0(l_0<t)\mathrm{d}t$$
$$=\alpha\int_0^\infty\int_0^\infty e^{-\alpha t}\overline{Q}(x)P_0\{X(t)\in\mathrm{d}x\}\mathrm{d}t.$$

Applying the identity $tP_0\{\tau_x\in\mathrm{d}t\}\mathrm{d}x=xP_0\{X(t)\in\mathrm{d}x\}\mathrm{d}t$ (Bertoin 1996, p. 190, Corollary 3) to this leads to

$$
\begin{aligned}
\mathrm{E}_0\mathrm{e}^{-\alpha l_0} &= \alpha\int_0^\infty\left\{x^{-1}\overline{Q}(x)\int_0^\infty t\mathrm{e}^{-\alpha t}P_0(\tau_x\in \mathrm{d}t)\right\}\mathrm{d}x \\
&= -\alpha\int_0^\infty\left[x^{-1}\overline{Q}(x)\frac{\mathrm{d}}{\mathrm{d}\alpha}\left\{\int_0^\infty \mathrm{e}^{-\alpha t}P_0(\tau_x\in \mathrm{d}t)\right\}\right]\mathrm{d}x \\
&= -\alpha\int_0^\infty\left[x^{-1}\overline{Q}(x)\frac{\mathrm{d}}{\mathrm{d}\alpha}\{\mathrm{e}^{-x\psi^{-1}(\alpha)}\}\right]\mathrm{d}x \\
&= \alpha(\psi^{-1})'(\alpha)\int_0^\infty \overline{Q}(x)\mathrm{e}^{-x\psi^{-1}(\alpha)}\mathrm{d}x \\
&= m(\psi^{-1})'(\alpha),
\end{aligned}
$$

where (1.3) and (1.4) have been used in the third and last step, respectively. The process X has stationary independent increments, and thus, for any $x\in\mathbb{R}$, we have

$$\mathrm{E}_x\mathrm{e}^{-\alpha l_x} = \mathrm{E}_0\mathrm{e}^{-\alpha l_0} = m(\psi^{-1})'(\alpha), \tag{3.4}$$

and (3.2) follows.

Since (2.9) remains true for processes of unbounded variation, it, together with (1.3) and (3.4), yields (3.1).

For any $y > x$, using (3.4) and an argument similar to the proof of (2.6), we obtain

$$
\begin{aligned}
&\mathrm{E}_y\{\mathrm{e}^{-\alpha l_x}\mathbf{1}(l_x>0)\} \\
&= \int_0^\infty\int_{-\infty}^{x-y}\frac{m(\psi^{-1})'(\alpha)}{\mathrm{e}^{\alpha t}}\mathrm{e}^{-(x-y-z)\psi^{-1}(\alpha)}P_0\{T_{x-y}\in \mathrm{d}t,\, X(T_{x-y})\in \mathrm{d}z,\, T_{x-y}<\infty\},
\end{aligned}
$$

and (3.3) follows. □

Remark 3.1. By using Lemma 1.2 or Kyprianou and Palmowski (2005, Theorem 4), we have

$$P_0^{(\psi^{-1}(\alpha))}(T_{x-y}<\infty) = 1-\psi'_{\psi^{-1}(\alpha)}(0+)W^{(0)}_{\psi^{-1}(\alpha)}(y-x),$$

and replacing ψ^{-1} by ϕ^{-1} yields an expression for the probability $P_0^{(\phi^{-1}(\alpha))}(T_{x-y}<\infty)$ in the special case presented in Section 2.

Theorem 3.2. *Suppose that* α, β, $\delta>0$ and x, $y\in\mathbb{R}$. *If* $y<x$, *then*

$$\mathrm{E}_y\{\mathrm{e}^{-\alpha\tau_x-\beta(l_x-\tau_x)}\mathbf{1}(\tau_x<\infty)\} = m(\psi^{-1})'(\beta)\mathrm{e}^{-(x-y)\psi^{-1}(\alpha)}, \tag{3.5}$$

and if $y>x$, *then*

$$\mathrm{E}_y\{\mathrm{e}^{-\alpha T_x-\beta(l_x-T_x)+\delta X(T_x)}\mathbf{1}(T_x<\infty)\} = m(\psi^{-1})'(\beta)\mathrm{e}^{-x\psi^{-1}(\beta)}\Delta_2(\alpha,\beta,\delta,x,y), \tag{3.6}$$

where

$$\Delta_2(\alpha, \beta, \delta, x, y) = \mathrm{e}^{\kappa y}\left\{ Z_\kappa^{(p_2)}(y-x) - \frac{W_\kappa^{(p_2)}(y-x)p_2}{\psi_\kappa^{-1}(p_2)}\right\},$$

with $p_2 = \alpha - \psi(\delta + \psi^{-1}(\beta))$ *and* $\kappa = \delta + \psi^{-1}(\beta)$.

Proof. For any $y, x \in \mathbb{R}$ such that $y < x$, $P_y(\tau_x < \infty) = 1$ and $P_y(X(\tau_x) = x) = 1$. The strong Markov property of X at time τ_x, the stationarity and independence of increments and equation (3.2) imply that, for $\alpha, \beta > 0$,

$$\begin{aligned} \mathrm{E}_y\{\mathrm{e}^{-\alpha\tau_x - \beta(l_x - \tau_x)}\mathbf{1}(\tau_x < \infty)\} &= \mathrm{E}_y[\mathrm{e}^{-\alpha\tau_x}\mathrm{E}_{X(\tau_x)}\{\mathrm{e}^{-\beta l_x}\mathbf{1}(l_x \geqslant 0)\}\mathbf{1}(\tau_x < \infty)] \\ &= \mathrm{E}_y(\mathrm{e}^{-\alpha\tau_x})\mathrm{E}_x\{\mathrm{e}^{-\beta l_x}\mathbf{1}(l_x \geqslant 0)\} \\ &= \mathrm{E}_0(\mathrm{e}^{-\alpha\tau_{x-y}})\mathrm{E}_x(\mathrm{e}^{-\beta l_x}). \end{aligned} \tag{3.7}$$

Equation (3.5) follows from (1.3), (3.4) and (3.7).

For any $y, x \in \mathbb{R}$ such that $y > x$ and $\alpha, \beta, \delta > 0$, from the strong Markov property of X and (3.1),

$$\begin{aligned} &\mathrm{E}_y\{\mathrm{e}^{-\alpha T_x - \beta(l_x - T_x) + \delta X(T_x)}\mathbf{1}(T_x < \infty)\} \\ &= \mathrm{E}_y[\mathrm{e}^{-\alpha T_x + \delta X(T_x)}\mathrm{E}_{X(T_x)}\{\mathrm{e}^{-\beta l_x}\mathbf{1}(l_x \geqslant 0)\}\mathbf{1}(T_x < \infty)] \\ &= \int_0^\infty \int_{-\infty}^x \mathrm{e}^{-\alpha t + \delta z}\mathrm{E}_z\{\mathrm{e}^{-\beta l_x}\mathbf{1}(l_x \geqslant 0)\}P_y\{T_x \in \mathrm{d}t, X(T_x) \in \mathrm{d}z, T_x < \infty\} \\ &= m(\psi^{-1})'(\beta)\int_0^\infty \int_{-\infty}^x \mathrm{e}^{-\alpha t + \delta z}\mathrm{e}^{-(x-z)\psi^{-1}(\beta)}P_y\{T_x \in \mathrm{d}t, X(T_x) \in \mathrm{d}z, T_x < \infty\}, \end{aligned}$$

and the result follows. □

4. Applications to risk theory

Consider the surplus $R(t)$ that is the classical risk process perturbed by diffusion:

$$R(t) = u + ct + \sigma B_t - \sum_{i=1}^{N_t} Z_i, \qquad t \geqslant 0, \tag{4.1}$$

where $u \geqslant 0$ is the initial surplus, σ a non-negative constant, c the positive constant premium income rate, $\{B_t, t \geqslant 0\}$ the standard Brownian motion, $\{N_t, t \geqslant 0\}$ a Poisson process with intensity $\lambda > 0$, and $\{Z_k, k \geqslant 1\}$ a sequence of non-negative independent and identically distributed claim amounts, such that $\{B_t, t \geqslant 0\}$, $\{N_t, t \geqslant 0\}$ and $\{Z_k, k \geqslant 1\}$ are independent. Denote by P and μ the distribution function and the mean, respectively, of the claim sizes Z_k, with the condition that $P(0) = 0$. Assume that the safety loading $c - \lambda\mu$

is positive, so that $\lim_{t\to\infty} R(t) = \infty$ almost surely if the process continues even when the surplus is negative. For simplicity, P is assumed to possess a density p. The diffusion term in (4.1) contributes an additional uncertainty of the premium income or the aggregate claims to the surplus. This model was first introduced and studied by Gerber (1970). When $\sigma = 0$, (4.1) is called the classical risk process. We refer to Embrechts *et al*. (1997), Rolski *et al*. (1999) and Asmussen (2000) for a complete presentation of risk theory.

Denote the Laplace transform of p by $\hat{p}(\alpha) = \int_0^\infty e^{-\alpha x} p(x)dx$. Then $E_0 e^{\alpha\{R(t)-u\}} = e^{t\xi(\alpha)}$, where $\xi(\alpha) = c\alpha + \frac{1}{2}\sigma^2\alpha^2 + \lambda\{\hat{p}(\alpha) - 1\}$. Obviously $\{R(t) - u,\ t \geqslant 0\}$ is a spectrally negative Lévy process with $E_0\{R(1) - u\} = c - \lambda\mu > 0$ and initial value zero.

Let T denote the time of ruin and $\Psi(u) = P_u(T < \infty)$ the ultimate ruin probability. It is well known that $\Psi(0)$ is 1 if $\sigma \neq 0$, and is $\lambda\mu/c$ otherwise. If we use the notation in Section 1, we have $T = T_0$ and $\Psi(u) = Q(u)$.

One central topic of risk theory is to find the probability of ruin. Recently there has been growing interest in the distributions of some other random variables related to the surplus process. These random variables include the time when the surplus reaches some level for the first time (Gerber 1990; Picard and Lefèvre 1994; Zhang and Wu 2002), the time when the surplus crosses some level for the last time (Gerber 1990), the duration of negative surplus (Egídio dos Reis 1993; Zhang and Wu 2002) and the recovery time from negative surplus (Dickson and Egídio dos Reis 1997; Egídio dos Reis 2000; Yang and Zhang 2001). A scenario where these random variables make sense is as follows. Suppose that the portfolio under consideration is one of many belonging to a company so that it has other funds available to support negative surpluses for a while, in the hope that the portfolio will recover in the future. In such a situation perhaps a more interesting question is to find the distributions of the first and last passage times and of their difference for the surplus process at a given level. Gerber (1990) considered this question for the classical risk model and obtained explicit results for their Laplace transforms and their moments. However, when we consider the time of last passage at level x and the duration between successive visits at x for the classical risk process perturbed by diffusion, the method of Gerber (1990) is not applicable because the infinite oscillation of $R(t)$ due to the Brownian motion B_t yields an arbitrarily small duration. Nevertheless, the general results of spectrally negative Lévy processes in Sections 2 and 3 can be used directly.

Using the same notation τ_x, l_x, T_x and T'_x as before and applying the results in previous sections to the special Lévy process defined by equation (4.1), we obtain the following results.

Theorem 4.1. *For any* $y \leqslant x$ *and* $\alpha > 0$, *we have*

$$E_y e^{-\alpha(l_x - \tau_x)} = \frac{c - \lambda\mu}{c + \lambda\hat{p}'(\xi^{-1}(\alpha))}.$$

Theorem 4.2. *For* $\alpha > 0$ *and* $x,\ y \in \mathbb{R}$,

$$
\mathrm{E}_y\{\mathrm{e}^{-\alpha l_x}\mathbf{1}(l_x > 0)\} = \begin{cases} \dfrac{c-\lambda\mu}{c+\xi^{-1}(\alpha)\sigma^2+\lambda\hat{p}'(\xi^{-1}(\alpha))}\mathrm{e}^{-(x-y)\xi^{-1}(\alpha)}, & y < x, \\[2ex] -\dfrac{\lambda(c-\lambda\mu)\hat{p}'(\xi^{-1}(\alpha))}{c\{c+\lambda\hat{p}'(\xi^{-1}(\alpha))\}}, & y = x,\ \sigma = 0, \\[2ex] \dfrac{c-\lambda\mu}{c+\xi^{-1}(\alpha)\sigma^2+\lambda\hat{p}'(\xi^{-1}(\alpha))}, & y = x,\ \sigma \neq 0, \\[2ex] \dfrac{(c-\lambda\mu)P_0^{(\xi^{-1}(\alpha))}(T_{x-y} < \infty)}{c+\xi^{-1}(\alpha)\sigma^2+\lambda\hat{p}'(\xi^{-1}(\alpha))}\mathrm{e}^{-(x-y)\xi^{-1}(\alpha)}, & y > x. \end{cases}
$$

Theorem 4.3. *For any α, β, $\delta > 0$ and $x,\ y \in \mathbb{R}$, if $y < x$, then*

$$
\mathrm{E}_y\{\mathrm{e}^{-\alpha\tau_x-\beta(l_x-\tau_x)}\mathbf{1}(\tau_x < \infty)\} = \frac{(c-\lambda\mu)\mathrm{e}^{-(x-y)\xi^{-1}(\alpha)}}{c+\xi^{-1}(\beta)\sigma^2+\lambda\hat{p}'(\xi^{-1}(\beta))},
$$

and if $y > x$, then

$$
\mathrm{E}_y\{\mathrm{e}^{-\alpha T_x-\beta(l_x-T_x)+\delta R(T_x)}\mathbf{1}(T_x < \infty)\} = \frac{(c-\lambda\mu)\mathrm{E}_0\mathrm{e}^{-\alpha T_{x-y}+\{\delta+\xi^{-1}(\beta)\}R(T_{x-y})}}{\{c+\xi^{-1}(\beta)\sigma^2+\lambda\hat{p}'(\xi^{-1}(\beta))\}\mathrm{e}^{x\xi^{-1}(\beta)}},
$$

where $\mathrm{E}_0\mathrm{e}^{-\alpha T_{x-y}+\{\delta+\xi^{-1}(\beta)\}R(T_{x-y})}$ can be computed by using Lemma 1.3.

Theorem 4.4. *Suppose that α, β, $\delta > 0$ and $\sigma = 0$. For any $x,\ y \in \mathbb{R}$, if $y > x$, then*

$$
\mathrm{E}_y\{^{-\alpha T_x-\beta(l_x-T_x)-\delta(T'_x-T_x)}\mathbf{1}(T_x < \infty)\}
$$

$$
= \frac{c-\lambda\mu}{c+\lambda\hat{p}'(\xi_1^{-1}(\beta))}\mathrm{e}^{-x\xi_1^{-1}(\beta+\delta)}\mathrm{E}_0\{\mathrm{e}^{-\alpha T_{x-y}+\xi_1^{-1}(\beta+\delta)R(T_{x-y})}\},
$$

where $\xi_1(\alpha) = c\alpha + \lambda\{\hat{p}(\alpha) - 1\}$ and $\mathrm{E}_0\mathrm{e}^{-\alpha T_{x-y}+\xi_1^{-1}(\beta+\delta)R(T_{x-y})}$ can be computed by using Lemma 1.3.

Remark 4.1. Taking $c = 1$, $y = 0$ and $\alpha = -s$ in the case of $\sigma = 0$ in Theorem 4.2, we get Gerber (1990, p. 118, equation (43)).

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