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Infinite divisibility and generalized subexponentiality

TAKAAKI SHIMURA[1] and TOSHIRO WATANABE[2]

[1]*The Institute of Statistical Mathematics, 4-6-7 Minami-Azabu Minato-ku, Tokyo 106-8569, Japan. E-mail: shimura@ism.ac.jp*
[2]*Center for Mathematical Sciences, The University of Aizu, Aizu-Wakamatsu, Fukushima 965-8580, Japan. E-mail: t-watanabe@u-aizu.ac.jp*

We introduce a new class of distributions by generalizing the subexponential class to investigate the asymptotic relation between the tails of an infinitely divisible distribution and its Lévy measure. We call a one-sided distribution μ O-subexponential if it has positive tail satisfying $\limsup_{x\to\infty}\mu * \mu(x, \infty)/\mu(x, \infty) < \infty$. Necessary and sufficient conditions for an infinitely divisible distribution to be O-subexponential are given in a similar way to the subexponential case in work by Embrechts *et al*. It is of critical importance that the O-subexponential is not closed under convolution roots. This property leads to the difference between our result and that corresponding to the subexponential class. Moreover, under the assumption that an infinitely divisible distribution has exponential tail, it is shown that an infinitely divisible distribution is convolution equivalent if and only if the ratio of its tail and its Lévy measure goes to a positive constant as x goes to infinity. Additionally, the upper and lower limits of the ratio of the tails of a semi-stable distribution and its Lévy measure are given.

Keywords: convolution equivalent class; infinitely divisible distribution; O-subexponential distribution; subexponential distribution

1. Introduction and main results

In the study of infinitely divisible distributions, one of the most important problems to investigate is the relation between an infinitely divisible distribution and its Lévy measure. In this paper, we consider the asymptotic relation between the tails of an infinitely divisible distribution on $[0, \infty)$ and its Lévy measure by generalizing the subexponential class. In this section, we mention some known results and give our results without proofs.

Hereafter, $\mathbb{N}$ and $\mathbb{R}$ denote the set of positive integers and the set of real numbers, respectively. The function $\bar{\mu}(x)$ denotes the tail of a measure μ, that is, $\bar{\mu}(x) = \mu(x, \infty)$, and $\mathcal{P}_+$ denotes the class of all distributions μ on $[0, \infty)$ satisfying $\bar{\mu}(x) > 0$ for every $x > 0$.

The class $\mathcal{ID}_+$ denotes the class of all infinitely divisible distributions μ on $[0, \infty)$ with Laplace transform

$$\hat{\mu}(s) = \exp\left\{\int_0^\infty (\mathrm{e}^{-st} - 1)\nu(\mathrm{d}t)\right\},$$

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where the Lévy measure ν satisfies $\int_0^\infty (1 \wedge t)\nu(\mathrm{d}t) < \infty$ and $\bar{\nu}(t) > 0$ for every $t > 0$. Define the normalized Lévy measure ν_1 as $\nu_1 = 1_{\{x>1\}}\nu/\nu(1, \infty)$.

Our concern is with the asymptotics of the ratio $\bar{\mu}(x)/\bar{\nu}(x)$. We define two relations for positive functions $f(x)$ and $g(x)$ on $[A, \infty)$ for some $A \geqslant 0$. The relation $f(x) \sim g(x)$ is defined by $\lim_{x\to\infty} f(x)/g(x) = 1$ and the relation $f(x) \asymp g(x)$ by $\liminf_{x\to\infty} f(x)/g(x) > 0$ and $\limsup_{x\to\infty} f(x)/g(x) < \infty$. Denote by $\mu * \rho$ the convolution of distributions μ and ρ on $[0, \infty)$ and by μ^{n*} the nth convolution power of μ. The relation '$\asymp$' is closed under convolution (Proposition 2.7), but the relation '$\sim$' is not (Remark 2.2). Several distribution classes play important roles. The first one is the subexponential class $\mathcal{S}$ introduced by Chistyakov (1964).

A distribution μ on $[0, \infty)$ is said to be *subexponential* ($\mu \in \mathcal{S}$) if

$$\lim_{x\to\infty} \overline{\mu * \mu}(x)/\bar{\mu}(x) = 2.$$

If a distribution on $[0, \infty)$ has regularly varying tail, then it belongs to $\mathcal{S}$. But there are distributions in $\mathcal{S}$ which do not have regularly varying tail. One-sided stable distributions and the Pareto distribution are examples of the former case, and the Weilbull distributions with parameter less than 1 and the lognormal distribution are examples of the latter case. Furhermore, these are in $\mathcal{ID}_+$. See Embrechts (1984), Embrechts *et al*. (1997) and Goldie and Klüppelberg (1998).

The following result shows that the subexponentiality is very important in the study of the tail behaviours of an infinitely divisible distribution and its Lévy measure.

Theorem A (Embrechts *et al.* 1979). *Let μ be a distribution in $\mathcal{ID}_+$ with Lévy measure ν. The following are equivalent:*

(i) $\mu \in \mathcal{S}$;
(ii) $\nu_1 \in \mathcal{S}$;
(iii) $\bar{\mu}(x) \sim \bar{\nu}(x)$.

Motivated by the above theorem, we introduce some distribution classes. Define the class $\mathcal{IS}$ as $\mathcal{IS} = \mathcal{S} \cap \mathcal{ID}_+$. Let $\mathcal{WS}$ be the class of distributions μ in $\mathcal{P}_+$ such that

$$\lim_{x\to\infty} \overline{\mu * \mu}(x)/\bar{\mu}(x) = c \qquad \text{for some } c > 0.$$

Let $\mathcal{WIS}$ be the class of distributions μ in $\mathcal{ID}_+$ such that

$$\lim_{x\to\infty} \bar{\mu}(x)/\bar{\nu}(x) = c \qquad \text{for some } c > 0.$$

We are interested in the relationships among these distribution classes. For instance, as a generalization of Theorem A, it is an interesting problem whether $\mathcal{WIS} = \mathcal{WS} \cap \mathcal{ID}_+$. Indeed, Chover *et al*. (1973a; 1973b) define the convolution equivalent class and Sgibnev (1990) gives a theorem corresponding to this class, which is an extension of Theorem A.

Define the class $\mathcal{L}(\gamma)$ with $\gamma \geqslant 0$ as the class of μ such that $\lim_{x\to\infty} \bar{\mu}(x+k)/\bar{\mu}(x) = \mathrm{e}^{-\gamma k}$ for every $k \in \mathbb{R}$. A distribution μ on $[0, \infty)$ is said to be in *the convolution equivalent class* $\mathcal{S}(\gamma)$ with $\gamma \geqslant 0$ if μ satisfies

(i) $\mu \in \mathcal{L}(\gamma)$,
(ii) $\lim_{x\to\infty}\overline{\mu * \mu}(x)/\bar{\mu}(x) = 2\hat{\mu}(-\gamma)$, where $\hat{\mu}(-\gamma) = \int_0^\infty e^{\gamma t}\mu(dt) < \infty$.

It is obvious that $\mathcal{S}(0) = \mathcal{S}$. A distribution μ in $\mathcal{L}(\gamma)$ is said to have exponential tail with rate γ. Let $\mathcal{L} = \mathcal{L}(0)$. Some generalized inverse Gaussian distributions belong to $\mathcal{S}(\gamma)$ for some $\gamma \geqslant 0$ (Embrechts 1983).

Theorem B (Sgibnev 1990). *Let $\gamma > 0$ and μ be a distribution in $\mathcal{ID}_+$. The following are equivalent:*

(i) $\mu \in \mathcal{S}(\gamma)$;
(ii) $\nu_1 \in \mathcal{S}(\gamma)$;
(iii) $\mu \in L(\gamma)$, $(\hat{\mu} - \gamma) < \infty$ *and* $\overline{\mu}(x) \sim \hat{\mu}(-\gamma)\overline{\nu}(x)$.

We deal with the more general asymptotic relation

$$\bar{\mu}(x) \asymp \bar{\nu}(x).$$

To begin with, we refer to the first result on this generalization. A distribution $\mu \in \mathcal{P}_+$ is said to have a *dominatedly varying tail* if $\limsup_{x\to\infty}\bar{\mu}(x)/\bar{\mu}(2x) < \infty$. Let $\mathcal{D}$ be the class of all distributions $\mu \in \mathcal{P}_+$ with dominatedly varying tail. A one-sided semi-stable distribution belongs to $\mathcal{D}$ (Theorem 1.3). The Peter and Paul distribution defined by $\mu(\{2^k\}) = 2^{-k}$ for $k \in \mathbb{N}$ is in $\mathcal{D}$. However, it is not in $\mathcal{S}$ (Goldie 1978).

Theorem C (Watanabe 1996). *Let μ be a distribution in $\mathcal{ID}_+$.*

(i) *The following are equivalent:*
   (a) $\mu \in \mathcal{D}$;
   (b) $\nu_1 \in \mathcal{D}$.
(ii) *If $\nu_1 \in D$, then $\overline{\mu}(x) \asymp \overline{\nu}(x)$.*

Remark 1.1. The opposite direction in (ii) does not hold. For example, the lognormal distribution is in $\mathcal{S}$ but not in $\mathcal{D}$.

We now define a new class of interest. We say that a distribution μ on $[0, \infty)$ is *O-subexponential* if

$$\limsup_{x\to\infty}\overline{\mu * \mu}(x)/\bar{\mu}(x) < \infty.$$

The class of all O-subexponential distributions is denoted by $\mathcal{OS}$. The class $\mathcal{OS}$ includes all of the classes $\mathcal{S}$, $\mathcal{S}(\gamma)$, $\mathcal{D}$ and $\mathcal{WS}$.

Remark 1.2. Klüppelberg (1990) calls an O-subexponential distribution weak idempotent.

In relation to the class $\mathcal{OS}$, we show the same type of theorem as Theorems A, B and C. The situation is different here, because the class $\mathcal{OS}$ is not closed under convolution roots.
The following proposition clarifies the difference of the property on convolution between

the class $\mathcal{OS}$ and the other classes. We say that a class $\mathcal{C}$ is closed under convolution roots if $\mu^{n*} \in \mathcal{C}$ for some $n \in \mathbb{N}$ implies $\mu \in \mathcal{C}$. Moreover, for a class $\mathcal{C}_1$, we say that $\mathcal{C}$ is closed under convolution roots in the class $\mathcal{C}_1$ if $\mu^{n*} \in \mathcal{C}$ for some $n \in \mathbb{N}$ and $\mu \in \mathcal{C}_1$ imply $\mu \in \mathcal{C}$.

Proposition 1.1.

(i) *$\mathcal{S}$ is closed under convolution roots* (*Embrechts et al. 1979.*)
(ii) *$\mathcal{S}(\gamma)$ is closed under convolution roots in the class $\mathcal{L}(\gamma)$* (*Embrechts and Goldie 1982.*)
(iii) *$\mathcal{D}$ is closed under convolution roots.*
(iv) *$\mathcal{OS}$ is not closed under convolution roots. Let ρ be the distribution defined as*

$$\rho(\mathrm{d}x) = c \sum_{k \in T} \frac{\mathrm{e}^{-k}}{(k+1)^2} \delta_k(\mathrm{d}x),$$

where $T = \{k \in \mathbb{N} : k = \sum_{i=1}^{\infty} j_i 3^{i-1},\ j_i = 0$ or $1\}$, δ_k is the probability measure concentrated at k and c is the normalizing constant. Then ρ does not belong to $\mathcal{OS}$, but the nth convolution power ρ^{n*} belongs to $\mathcal{OS}$ for all $n \geqslant 2$.

Theorem 1.1. *Let μ be a distribution in $\mathcal{ID}_+$.*

(i) *The following are equivalent*:
 (a) $\nu_1 \in \mathcal{OS}$;
 (b) $\bar{\mu}(x) \asymp \bar{\nu}(x)$.
(ii) *The following are equivalent*:
 (a) $\mu \in \mathcal{OS}$;
 (b) $\nu_1^{n*} \in \mathcal{OS}$ *for some* $n \geqslant 1$;
 (c) $\bar{\mu}(x) \asymp \bar{\nu}_1^{n*}(\mathrm{x})$ *for some* $n \geqslant 1$.
 If (b) holds, then the set of $n \in \mathbb{N}$ *such that $\nu_1^{n*} \in \mathcal{OS}$ is identical with the set of $n \in \mathbb{N}$ such that $\bar{\mu}(x) \asymp \overline{\nu_1^{n*}}(\mathrm{x})$ and is equal to $\{m \in \mathbb{N} : m \geqslant n_0\}$ with some $n_0 \in \mathbb{N}$.*
(iii) *If ν_1 is in $\mathcal{OS}$, then μ is in $\mathcal{OS}$. However, the converse does not hold.*

Remark 1.3. There exists $\mu \in \mathcal{ID}_+ \cap \mathcal{OS}$ with $\nu_1 \notin \mathcal{OS}$. In other words, in (ii)(c) n_0 is not always equal to 1. An example is given by taking ρ from Proposition 1.1 as the Lévy measure ν. Then we see that $\nu_1 \notin \mathcal{OS}$ but $\nu_1^{2*} \in \mathcal{OS}$. An early result of Grübel (1983) has some faults in this respect.

Remark 1.4. Theorem C is obtained by using Theorem 1.1 and Proposition 1.1(iii). The original proof in Watanabe (1996) is more complicated by virtue of employing a Tauberian theorem.

We give two corollaries of Theorem 1.1.

Corollary 1.1. *Let μ be a distribution in $\mathcal{ID}_+$ with $\nu_1 \in \mathcal{OS}$.*

(i) *For a non-negative locally integrable function* $f(t)$ *on* $[0, \infty)$,

$$\int_1^x f(t)\bar{\mu}(t)\mathrm{d}t \asymp \int_1^x f(t)\bar{\nu}(t)\mathrm{d}t.$$

(ii) *For a non-negative non-decreasing function* $g(t)$ *on* $[1, \infty)$, *the integral* $\int_1^\infty g(t)\mu(\mathrm{d}t)$ *is finite if and only if the integral* $\int_1^\infty g(t)\nu(\mathrm{d}t)$ *is finite.*

Corollary 1.2. *If* $\mu^{n*} \in \mathcal{OS} \cap \mathcal{ID}_+$ *for some integer* $n \geqslant 1$, *then* $\mu \in \mathcal{OS} \cap \mathcal{ID}_+$. *Thus* $\mathcal{OS}$ *is closed under convolution roots in the class* $\mathcal{ID}_+$.

We consider $\mathcal{WS}$ and $\mathcal{WIS}$ again. While it is obvious that $\cup_{\gamma\geqslant 0}\mathcal{S}(\gamma) \subset \mathcal{WS}$, we do not know whether $\cup_{\gamma\geqslant 0}\mathcal{S}(\gamma) = \mathcal{WS}$. However, it is known that they are identical in the class $\mathcal{L}(\gamma)$.

Theorem D (Chover *et al.* 1973a; 1973b; Cline 1987; Rogozin 2000). *For* $\gamma \geqslant 0$,

$$\mathcal{WS} \cap \mathcal{L}(\gamma) = \mathcal{S}(\gamma).$$

On the other hand, Theorem B implies that $\cup_{\gamma\geqslant 0}\mathcal{S}(\gamma) \cap \mathcal{ID}_+ \subset \mathcal{WIS}$. We conjecture the following three identities:

(i) $\mathcal{WS} = \cup_{\gamma\geqslant 0}\mathcal{S}(\gamma)$;
(ii) $\mathcal{WIS} = \cup_{\gamma\geqslant 0}\mathcal{S}(\gamma) \cap \mathcal{ID}_+$;

and thus,

(iii) $\mathcal{WIS} = \mathcal{WS} \cap \mathcal{ID}_+$.

Combined with Theorem D, the following theorem shows that the above conjecture is true in the class $\mathcal{L}(\gamma)$.

Theorem 1.2. *For* $\gamma \geqslant 0$,

$$\mathcal{WIS} \cap \mathcal{L}(\gamma) = \mathcal{S}(\gamma) \cap \mathcal{ID}_+.$$

Corollary 1.3. *Let* μ *be a distribution in* $\mathcal{ID}_+$. *If, for every* $a \in \mathbb{R}$,

$$c(a) = \lim_{x\to\infty} \frac{\overline{\mu}(x+a)}{\overline{\nu}(x)}$$

exists and is positive and finite, then there exists $\gamma \geqslant 0$ *such that*

$$\mu \in \mathcal{S}(\gamma) \cap \mathcal{ID}_+ \quad \textit{and} \quad c(a) = \mathrm{e}^{-\gamma a}\hat{\mu}(-\gamma).$$

Remark 1.5. The condition can be weakened to the existence of $c(a)$ for every a in a set with positive Lebesgue measure including 0. See Theorem 1.4.1 in Bingham *et al.* (1987).

Although it is not easy to give the upper or lower limit of $\bar{\mu}(x)/\bar{\nu}(x)$ for a distribution μ

in $\mathcal{OS} \cap \mathcal{ID}_+$, we can show them for semi-stable distributions on $[0, \infty)$. An infinitely divisible distribution $\mu \in \mathcal{P}_+$ is called semi-stable if, for some $a > 1$, there exist $b > 1$ and $c \geqslant 0$ satisfying

$$\hat{\mu}(s)^a = \hat{\mu}(bs)\mathrm{e}^{-cs} \qquad \text{for } s \geqslant 0.$$

For a semi-stable distribution the span b and the index $\alpha = \log a / \log b$ $(0 < \alpha < 1)$ are important. A span is not unique, but the index is independent of the choice of span. See Sato (1999) for semi-stable distributions on $\mathbb{R}^d$.

Theorem 1.3. *Let $\mu \in \mathcal{ID}_+$ be a semi-stable distribution on $[0, \infty)$ with Lévy measure ν, index α and span b.*

(i) *The distribution μ is in $\mathcal{D}$.*

(ii) *We have*

$$\liminf_{x\to\infty} \frac{\bar{\mu}(x)}{\bar{\nu}(x)} = 1 \quad \text{and} \quad \limsup_{x\to\infty} \frac{\bar{\mu}(x)}{\bar{\nu}(x)} = \sup_{1 \leqslant x < b} \frac{\bar{\nu}(x-)}{\bar{\nu}(x)}.$$

Thus the distribution μ is in $\mathcal{S}$ if and only if $\bar{\nu}(x)$ is continuous on $(0, \infty)$.

(iii) *The upper limit in (ii) is bounded by b^α. This bound is attained if and only if $\nu(\mathrm{d}x) = c\sum_{k\in\mathbb{Z}} b^{-k\alpha}\delta_{b^k x_0}(\mathrm{d}x)$ with some $c, x_0 > 0$.*

We show some fundamental properties of distributions in $\mathcal{OS}$ in Section 2. Proofs of all the statements in this section are given in Section 3.

2. Properties of the class $\mathcal{OS}$

In this section, we state the properties of $\mathcal{OS}$, comparing them with those of $\mathcal{S}$.

We start with definitions and notation. Let $\mathcal{OL}$ be the class of all distributions such that $\limsup_{x\to\infty} \bar{\mu}(x+k)/\bar{\mu}(x) < \infty$ for every $k \in \mathbb{R}$. Let $\int_a^b f(t)\mu(\mathrm{d}t) = \int_{(a,b]} f(t)\mu(\mathrm{d}t)$ for $0 \leqslant a \leqslant b < \infty$. Let $l^*(\mu) = \limsup_{x\to\infty} \overline{\mu * \mu}(x)/\bar{\mu}(x)$ for $\mu \in \mathcal{P}_+$.

Proposition 2.1.

(i) $\mathcal{S} \subset \mathcal{L}$ (*Chistyakov 1964.*)

(ii) $\mathcal{OS} \subset \mathcal{OL}$.

Proposition 2.2.

(i) *If $\mu \in \mathcal{L}$, then $\lim_{x\to\infty} \mathrm{e}^{\epsilon x}\bar{\mu}(x) = \infty$ for every $\epsilon > 0$.*

(ii) *If $\mu \in \mathcal{OL}$, then $\lim_{x\to\infty} \mathrm{e}^{\epsilon x}\bar{\mu}(x) = \infty$ for some $\epsilon > 0$.*

Proof of Propositions 2.1 and 2.2. We only prove (ii) of each proposition. Assume that $\mu \in \mathcal{OS}$ and $0 < k \leqslant x$. Then we have

$$\begin{aligned}\overline{\mu^{2*}}(x) &= \int_{0-}^{\infty} \bar{\mu}(x-t)\mu(\mathrm{d}t) \\ &= \bar{\mu}(x) + \int_{0-}^{k} \bar{\mu}(x-t)\mu(\mathrm{d}t) + \int_{k}^{x} \bar{\mu}(x-t)\mu(\mathrm{d}t) \\ &\geqslant \bar{\mu}(x) + \bar{\mu}(x)\mu[0,\, k] + \bar{\mu}(x-k)\mu(k,\, x].\end{aligned}$$

Therefore,

$$\frac{\bar{\mu}(x-k)}{\bar{\mu}(x)} \leqslant \frac{\overline{\mu * \mu}(x)/\bar{\mu}(x) - 1 - \mu[0,\, k]}{\mu(k,\, x]}.$$

Letting $x \to \infty$, we obtain

$$\limsup_{x\to\infty} \frac{\bar{\mu}(x-k)}{\bar{\mu}(x)} \leqslant \bar{\mu}(k)^{-1}(l^*(\mu) - 1 - \mu[0,\, k]).$$

It follows from the assumption that the right-hand side is finite. The proof of Proposition 2.1(ii) is complete.

Note that $\mu \in \mathcal{OL}$ if and only if $\bar{\mu}(\log x)$ is O-regularly varying. It follows from Theorem 2.2.7 in Bingham *et al.* (1987, p. 74) that $\lim_{x\to\infty} x^{\epsilon}\bar{\mu}(\log x) = \infty$ for some $\epsilon > 0$. □

Both $\mathcal{S}$ and $\mathcal{OS}$ are characterized by their tail behaviours as follows.

Proposition 2.3.

(i) *Let* $\mu_1 \in \mathcal{S}$ *and* $\mu_2 \in \mathcal{P}_+$. *If* $\bar{\mu}_1(x) \sim c\bar{\mu}_2(x)$ *for some* $c > 0$, *then* $\mu_2 \in \mathcal{S}$ (*Teugels 1975*.)
(ii) *Let* $\mu_1 \in \mathcal{OS}$ *and* $\mu_2 \in \mathcal{P}_+$. *If* $\bar{\mu}_1(x) \asymp \bar{\mu}_2(x)$, *then* $\mu_2 \in \mathcal{OS}$ (*Klüppelberg 1990*.)

The following is a generalization of Lemma 7 in Athreya and Ney (1972, p. 149).

Proposition 2.4. *If* $\mu \in \mathcal{OS}$, *then, for arbitrary* $\epsilon > 0$, *there exists* $c_1 > 0$ *such that, for* $n \geqslant 1$ *and* $x \geqslant 0$,

$$\frac{\overline{\mu^{n*}}(x)}{\bar{\mu}(x)} \leqslant c_1(l^*(\mu) + \epsilon - 1)^n.$$

Proof. Let $a_n = \sup_{x\geqslant 0}\overline{\mu^{n*}}(x)/\bar{\mu}(x)$ for $n \geqslant 2$. For any given $\epsilon > 0$, take y such that $\sup_{x\geqslant y}\overline{\mu * \mu}(x)/\bar{\mu}(x) \leqslant l^*(\mu) + \epsilon$. Since

$$\overline{\mu^{(n+1)*}}(x) = \bar{\mu}(x) + \int_{0-}^{x} \overline{\mu^{n*}}(x-t)\mu(\mathrm{d}t),$$

we have

$$
\begin{aligned}
a_{n+1} &\leqslant 1 + \sup_{0\leqslant x<y}\int_{0-}^{x} \frac{\overline{\mu^{n*}}(x-t)}{\bar{\mu}(x)}\mu(\mathrm{d}t) + \sup_{x\geqslant y}\int_{0-}^{x} \frac{\overline{\mu^{n*}}(x-t)}{\bar{\mu}(x)}\mu(\mathrm{d}t) \\
&\leqslant 1 + \frac{1}{\bar{\mu}(y)} + a_n \sup_{x\geqslant y} \frac{\overline{\mu*\mu(x)} - \bar{\mu}(x)}{\bar{\mu}(x)} \\
&\leqslant c_0 + a_n r,
\end{aligned}
$$

where $c_0 = 1 + 1/\bar{\mu}(y)$ and $r = l^*(\mu) + \epsilon - 1$. Note that $l^*(\mu) \geqslant 2$ by the argument in the proof of Proposition 2.1. It follows by induction that

$$a_n \leqslant c_0 \sum_{k=0}^{n-1} r^k + r^n \leqslant c_1 r^n,$$

where $c_1 = 1 + c_0(r-1)^{-1}$. □

Remark 2.1 (Chistyakov 1964). For $\mu \in \mathcal{P}_+$, $\liminf_{x\to\infty}\overline{\mu^{n*}}(x)/\bar{\mu}(x) \geqslant n$.

Lastly, we give two propositions on the convolution properties, a part of which corresponds to the converse of Proposition 1.1.

Proposition 2.5.

(i) *$\mathcal{S}$ is closed under convolution power, but is not closed under convolution* (*Leslie 1989.*)
(ii) *$\mathcal{S}(\gamma)$ is closed under convolution power, but is not closed under convolution* (*Klüppelberg and Villasenor 1991.*)
(iii) *$\mathcal{D}$ is closed under convolution.*
(iv) *$\mathcal{OS}$ is closed under convolution* (*Klüppelberg 1990.*)

Proof. For (iii), observe that $\bar{\mu}_1(x) \vee \bar{\mu}_2(x) \leqslant \overline{\mu_1 * \mu_2}(x) \leqslant \bar{\mu}_1(x/2) + \bar{\mu}_2(x/2)$. {□}

Proposition 2.6. *If $\mu^{n*} \in \mathcal{OS}$ for some $n \geqslant 1$, then $\mu^{k*} \in \mathcal{OS}$ for every $k \geqslant n$.*

Proof. It is easily shown by induction, on account of Proposition 2.5(iv) and the observation, that

$$\frac{\overline{\mu^{2(n+1)*}}(x)}{\overline{\mu^{(n+1)*}}(x)} \leqslant \frac{\overline{\mu^{4n*}}(x)}{\overline{\mu^{n*}}(x)} = \frac{\overline{\mu^{4n*}}(x)}{\overline{\mu^{2n*}}(x)}\frac{\overline{\mu^{2n*}}(x)}{\overline{\mu^{n*}}(x)}.$$

□

The following shows that the relation '$\asymp$' is closed under convolution.

Proposition 2.7. *Let μ_i $(i = 1, 2, 3, 4)$ be in $\mathcal{P}_+$. If $\bar{\mu}_1(x) \asymp \bar{\mu}_2(x)$ and $\bar{\mu}_3(x) \asymp \bar{\mu}_4(x)$, then $\overline{\mu_1 * \mu_3}(x) \asymp \overline{\mu_2 * \mu_4}(x)$. Therefore, if $\bar{\mu}_1(x) \asymp \bar{\mu}_2(x)$, then $\bar{\mu}_1^{n*}(x) \asymp \bar{\mu}_2^{n*}(x)$ for $n \geqslant 1$.*

Proof.

$$\begin{aligned}\overline{\mu_1 * \mu_3(x)} &= \int_0^\infty \bar{\mu}_1(x-t)\mu_3(\mathrm{d}t) \asymp \int_0^\infty \bar{\mu}_2(x-t)\mu_3(\mathrm{d}t) \\ &= \int_0^\infty \bar{\mu}_3(x-t)\mu_2(\mathrm{d}t) \asymp \int_0^\infty \bar{\mu}_4(x-t)\mu_2(\mathrm{d}t) = \overline{\mu_2 * \mu_4}(x).\end{aligned}$$

□

Remark 2.2. The relation '$\sim$' is not closed under convolution. For example, let $\mu_1 = \frac{1}{2}(\delta_0(\mathrm{d}x) + \mu)$, $\mu_2 = \frac{1}{2}(\delta_1(\mathrm{d}x) + \mu)$, where μ is the Peter and Paul distribution. Then the relation $\overline{\mu_1^{2*}}(x) \sim \overline{\mu_2^{2*}}(x)$ does not hold, although $\bar{\mu}_1(x) \sim \bar{\mu}_2(x)$.

3. Proofs of the results

In this section, we prove the results presented in Section 1. The proof of Proposition 1.1 is given last because of its difficulty. We will use Proposition 1.1 in the proof of Theorem 1.1(iii), but Proposition 1.1 will be shown independently of other results.

3.1. Proof of Theorem 1.1

We will prove Theorem 1.1 step by step, beginning with the following proposition.

Proposition 3.1. *Let $\lambda_k \geqslant 0$ with $k \in \mathbb{N}$ be a sequence satisfying $\sum_{k=0}^\infty \lambda_k = 1$, $\lambda_0 + \lambda_1 < 1$ and $\sup\{x \geqslant 1 : \sum_{k=0}^\infty \lambda_k x^k < \infty\} = \infty$. Put $\eta = \sum_{k=0}^\infty \lambda_k \rho^{k*}$ for $\rho \in \mathcal{P}_+$.*

(i) *The following are equivalent:*
 (a) *$\rho \in \mathcal{OS}$;*
 (b) *$\overline{\eta}(x) \asymp \overline{\rho}(x)$.*
(ii) *If $\sup\{k : \lambda_k > 0\} = \infty$, then the following are equivalent.*
 (a) *$\eta \in \mathcal{OS}$;*
 (b) *$\rho^{n*} \in \mathcal{OS}$ for some $n \geqslant 1$;*
 (c) *$\overline{\eta}(x) \asymp \overline{\rho^{n*}}(x)$ for some $n \geqslant 1$.*

If $\eta \in \mathcal{OS}$, then the set $\{n \in \mathbb{N} : \rho^{n*} \in \mathcal{OS}\}$ is identical with the set $\{n \in \mathbb{N} : \overline{\eta}(x) \asymp \overline{\rho^{n*}}(x)\}$ and equal to $\{n \in \mathbb{N} : n \geqslant n_0\}$ for some n_0.

Proof. Let us set $x > 0$. We omit the proof of (i) since this is proved in the same way as (ii) by using $\lambda_0+\lambda_1<1$ instead of $\sup\{k:\lambda_k>0\}=\infty$. To prove (ii), we first prove the equivalence of (b) and (c).

(b) $\Rightarrow$ (c). If $\rho^{n*}\in\mathcal{OS}$, then, by Proposition 2.4, for arbitrary $\epsilon>0$, there exists $c_1>0$ such that $\overline{\rho^{kn*}}(x)\leqslant c_1(l^*(\rho^{n*})+\epsilon-1)^k\overline{\rho^{n*}}(x)$ for every $k\geqslant 1$ and $x>0$. Therefore, we see that

$$\bar{\eta}(x)=\sum_{k=1}^{\infty}\lambda_k\overline{\rho^{k*}}(x)\leqslant\sum_{k=1}^{\infty}\lambda_k\overline{\rho^{kn*}}(x)\leqslant c_1\sum_{k=1}^{\infty}\lambda_k(l^*(\rho^{n*})+\epsilon-1)^k\overline{\rho^{n*}}(x).$$

On the other hand, for $\lambda_m>0$ $(m\geqslant n\geqslant 1)$,

$$\bar{\eta}(x)\geqslant\lambda_m\overline{\rho^{m*}}(x)\geqslant\lambda_m\overline{\rho^{n*}}(x).$$

(c) $\Rightarrow$ (b). If we take $m\geqslant 2n$ such that $\lambda_m>0$, then, it follows from (c) that

$$\overline{\rho^{2n*}}(x)\leqslant\overline{\rho^{m*}}(x)\leqslant(\lambda_m)^{-1}\bar{\eta}(x)\asymp\overline{\rho^{n*}}(x),$$

which implies (b). Thus we see that (b) and (c) are equivalent.

(b) and (c) $\Rightarrow$ (a). Since we can take the same n in (b) and (c), it is easily obtained by Proposition 2.3.

(a) $\Rightarrow$ (c). Let $\eta\in\mathcal{OS}$. Using $\bar{\rho}(x)\leqslant\overline{\rho^{k*}}(x)$ for $k\geqslant 1$, we have

$$\sum_{k=1}^{\infty}\lambda_k\bar{\rho}(x)\leqslant\sum_{k=1}^{\infty}\lambda_k\overline{\rho^{k*}}(x)=\bar{\eta}(x).$$

That is to say,

$$\bar{\rho}(x)\leqslant(1-\lambda_0)^{-1}\bar{\eta}(x).$$

Moreover, it follows by induction that

$$\overline{\rho^{k*}}(x)\leqslant(1-\lambda_0)^{-k}\overline{\eta^{k*}}(x). \tag{3.1}$$

By Proposition 2.4,

$$\overline{\eta^{k*}}(x)\leqslant c_1(l^*(\eta)+\epsilon-1)^k\bar{\eta}(x). \tag{3.2}$$

We obtain from (3.1) and (3.2) that

$$\overline{\rho^{k*}}(x)\leqslant c_1(l^*(\eta)+\epsilon-1)^k(1-\lambda_0)^{-k}\bar{\eta}(x).$$

Therefore, we see that

$$1=\sum_{k=1}^{\infty}\lambda_k\frac{\overline{\rho^{k*}}(x)}{\bar{\eta}(x)}\leqslant\sum_{k=1}^{n}\lambda_k\frac{\overline{\rho^{k*}}(x)}{\bar{\eta}(x)}+c_1\sum_{k=n+1}^{\infty}\lambda_k(l^*(\eta)+\epsilon-1)^k(1-\lambda_0)^{-k}. \tag{3.3}$$

The assumption allows us to choose $n\geqslant 2$ such that $\lambda_n>0$ and the second term of the right-hand side is less than 1. Let us prove that

$$\inf_{x\geqslant 0}\frac{\overline{\rho^{n*}}(x)}{\bar{\eta}(x)}>0.$$

Assume that $\{x_j : j \in \mathbb{N}\}$ is an increasing sequence satisfying $\lim_{j\to\infty} x_j = \infty$ and $\lim_{j\to\infty} \overline{\rho^{n*}}(x_j)/\bar{\eta}(x_j) = 0$. Since

$$\limsup_{j\to\infty} \frac{\overline{\rho^{k*}}(x_j)}{\bar{\eta}(x_j)} \leqslant \lim_{j\to\infty} \frac{\overline{\rho^{n*}}(x_j)}{\bar{\eta}(x_j)} = 0 \qquad \text{for } 1 \leqslant k \leqslant n,$$

the first term of the right-hand side of (3.3) goes to 0 as $j \to \infty$. Then the second term should be not less than 1. This is a contradiction.

Conversely, noticing $\lambda_n > 0$, we obtain $\overline{\rho^{n*}}(x) \leqslant (\lambda_n)^{-1}\bar{\eta}(x)$. Thus we obtain (c).

Since we can take the same n in (b) and (c), one half of the last assertion is true. The other half follows from Proposition 2.6. □

Applying this proposition to the compound Poisson distribution gives the following, which corresponds to the special case of Theorem 1.1 with finite Lévy measure.

Proposition 3.2. *Let $\mu \in \mathcal{ID}_+$ with finite Lévy measure ν and put*

$$\nu_0 = \frac{\nu}{\nu(0, \infty)}.$$

(i) *The following are equivalent:*
  (a) $\nu_0 \in \mathcal{OS}$, *equivalently*, $\nu_1 \in \mathcal{OS}$;
  (b) $\bar{\mu}(x) \asymp \bar{\nu}(x)$.

(ii) *The following are equivalent:*
  (a) $\mu \in \mathcal{OS}$;
  (b) $\nu_0^{n*} \in \mathcal{OS}$ *for some* $n \geqslant 1$;
  (c) $\bar{\mu}(x) \asymp \overline{\nu_0^{n*}}(x)$ *for some* $n \geqslant 1$.

If $\mu \in \mathcal{OS}$, *then the set* $\{n \in \mathbb{N} : \nu_0^{n*} \in \mathcal{OS}\}$ *is identical to the set* $\{n \in \mathbb{N} : \bar{\mu}(x) \asymp \overline{\nu_0^{n*}}(x)\}$ *and equal to* $\{n \in \mathbb{N} : n \geqslant n_0\}$ *for some* n_0.

We can now prove Theorem 1.1.

Proof of Theorem 1.1. Let $x > 0$. First we decompose μ into the convolution of two distributions as follows. Let $c = \nu(1, \infty)$ and define the distribution μ_1 in $\mathcal{ID}_+$ with Lévy measure $c\nu_1$ and the distribution μ_2 in $\mathcal{ID}_+$ as $\mu = \mu_1 * \mu_2$. Comparing the tails of these distributions with $\mathrm{e}^{-\epsilon x \log x}$ $(0 < \epsilon < 1)$ by Theorem 26.1 of Sato (1999), we see $\bar{\mu}_2(x) = o(\bar{\mu}_1(x))$. Let $c_1 > 0$ such that $\bar{\mu}_2(x) \leqslant c_1\bar{\mu}_1(x)$. Since

$$\bar{\mu}(x) = \int_{0-}^{\infty} \bar{\mu}_2(x - t)\mu_1(\mathrm{d}t) \leqslant c_1 \int_{0-}^{\infty} \bar{\mu}_1(x - t)\mu_1(\mathrm{d}t) = c_1\overline{\mu_1^{2*}}(x),$$

we obtain

$$\bar{\mu}_1(x) \leqslant \bar{\mu}(x) \leqslant c_1\overline{\mu_1^{2*}}(x) \leqslant c_1\overline{\mu^{2*}}(x). \tag{3.4}$$

It follows from (3.4) that $\mu_1 \in \mathcal{OS}$ implies $\mu \in \mathcal{OS}$ and $\mu \in \mathcal{OS}$ implies $\mu_1^{2*} \in \mathcal{OS}$. Since μ_1 is a compound Poisson distribution and μ_1 and μ_1^{2*} have the same normalized Lévy measure, we see that $\mu_1 \in \mathcal{OS}$ and $\mu_1^{2*} \in \mathcal{OS}$ are equivalent by Proposition 3.2(ii). Thus, $\mu_1 \in \mathcal{OS}$ and $\mu \in \mathcal{OS}$ are equivalent. Moreover, in this case, $\bar{\mu}(x) \asymp \bar{\mu}_1(x)$.

(i) (a) $\Rightarrow$ (b). If ν_1 is in $\mathcal{OS}$, by Proposition 3.2, $\mu_1 \in \mathcal{OS}$ and $\bar{\mu}_1(x) \asymp \bar{\nu}_1(x)$. Moreover, since $\bar{\mu}_1(x) \leqslant \bar{\mu}(x) \leqslant c_1\overline{\mu_1^{2*}}(x) \asymp \bar{\mu}_1(x)$, we have $\bar{\mu}(x) \asymp \bar{\mu}_1(x)$. Thus, we conclude that $\bar{\mu}(x) \asymp \bar{\mu}_1(x) \asymp \bar{\nu}_1(x) \asymp \bar{\nu}(x)$.

(b) $\Rightarrow$ (a). Conversely, suppose that $\bar{\mu}(x) \asymp \bar{\nu}(x)$. Then we see that

$$\mathrm{e}^{-c} c\bar{\nu}_1(x) \leqslant \bar{\mu}_1(x) \leqslant \bar{\mu}(x) \asymp \bar{\nu}(x) \asymp \bar{\nu}_1(x).$$

Thus we obtain $\bar{\mu}_1(x) \asymp \bar{\nu}_1(x)$, and Proposition 3.2 thereby implies that $\nu_1 \in \mathcal{OS}$.

(ii) Applying Proposition 3.2 to μ_1 and by using the fact mentioned after (3.4), we can easily see that (a) and (b) are equivalent. Additionally, (c) is obtained from (a) and (b) by Proposition 2.3. Hence, it is enough to show (b) from (c).

(c) $\Rightarrow$ (b). From the assumption, we have

$$\overline{\nu_1^{n*}}(x) \asymp \bar{\mu}(x) \geqslant \bar{\mu}_1(x) \geqslant \mathrm{e}^{-c}\frac{c^n}{n!}\overline{\nu_1^{n*}}(x).$$

Hence $\bar{\mu}_1(x) \asymp \overline{\nu_1^{n*}}(x)$, which implies $\nu_1^{n*} \in \mathcal{OS}$ by Proposition 3.2. The last assertion in (ii) follows from that of Proposition 3.2.

(iii) This is obvious from (i), (ii) and Remark 1.3. □

Proof of Corollary 1.1. Assertion (i) is easily shown from Theorem 1.1(i).

Noticing that $\mathcal{OS} \subset \mathcal{OL}$, assertion (ii) follows from the equivalence of the following: $\int_1^\infty g(t)\mu(\mathrm{d}t) < \infty$, $\sum_{k=1}^\infty (g(k+1) - g(k))\bar{\mu}(k) < \infty$ and $\sum_{k=1}^\infty (g(k+1) - g(k))\bar{\mu}(k-1) < \infty$. □

Proof of Corollary 1.2. Observe that μ^{n*} and μ have the same normalized Lévy measure. □

3.2. Proof of Theorem 1.2

Before proving Theorem 1.2 we give a number of useful lemmas and propositions. We start with an interesting equality which is a generalized version of that in Lemma 3 of Rogozin (2000). The proof is similar and omitted.

Lemma 3.1. *Let ρ be a distribution on $[0, \infty)$. If $\int_{0-}^\infty t\rho(\mathrm{d}t) < \infty$, then, for $n \geqslant 2$ and $x > 0$,*

$$\int_x^\infty \overline{\rho^{n*}}(t)\mathrm{d}t = \sum_{k=1}^{n-1}\int_0^x \overline{\rho^{k*}}(x-t)\bar{\rho}(t)\mathrm{d}t + n\int_x^\infty \bar{\rho}(t)\mathrm{d}t.$$

The following lemmas, due to Rudin (1973), will be used in the proof of Proposition 3.3.

Lemma 3.2. *Let $\eta = \sum_{k=0}^\infty \lambda_k\rho^{k*}$ for $\rho \in \mathcal{P}_+$ with $\lambda_k \geqslant 0$ and $\sum_{k=0}^\infty \lambda_k = 1$. If there exists $n \in \mathbb{N}$ such that $\int_0^\infty t^n\rho(\mathrm{d}t) = \infty$ and $\sum_{k=0}^\infty k^n\lambda_k < \infty$, then*

$$\liminf_{x\to\infty} \bar{\eta}(x)/\bar{\rho}(x) = \sum_{k=0}^{\infty} k\lambda_k.$$

Lemma 3.3. *Let* $f(x) = \sum_{k=0}^{\infty} f_k x^k$ *with* $f_k > 0$, $F(x) = \sum_{k=0}^{\infty} \lambda_k x^k$ *with* $\lambda_k \geqslant 0$. *Suppose that* $f(1) < \infty$, $f(x) = \infty$ *for every* $x > 1$, $F(f(1)) < \infty$ *and there exists* $x > f(1)$ *such that* $F(x) < \infty$. *Then*

$$\liminf_{n\to\infty} \frac{[F(f(x))]_n}{f_n} \leqslant \sum_{k=1}^{\infty} k\lambda_k f(1)^{k-1},$$

where $[\cdot]_n$ *denotes the Taylor expansion coefficient of* x^n.

Proposition 3.3. *Suppose that* $\lambda_k \geqslant 0$ *for* $k \geqslant 0$, $\sum_{k=0}^{\infty} \lambda_k = 1$ *and* $\sup\{x \geqslant 1 : \sum_{k=0}^{\infty} \lambda_k x^k < \infty\} = \infty$. *Put* $\eta = \sum_{k=0}^{\infty} \lambda_k \rho^{k*}$ *for* $\rho \in \mathcal{P}_+$. *If* $\rho \in \mathcal{OS} \cap \mathcal{L}$, *then*

$$\liminf_{x\to\infty} \frac{\bar{\eta}(x)}{\bar{\rho}(x)} = \sum_{k=1}^{\infty} k\lambda_k. \tag{3.5}$$

Proof. Put $c = \sum_{k=1}^{\infty} k\lambda_k$. If $\int_{0-}^{\infty} t\rho(\mathrm{d}t) = \infty$, we obtain (3.5) from Lemma 3.2 directly. If $\int_{0-}^{\infty} t\rho(\mathrm{d}t) < \infty$, we consider the quantity

$$\liminf_{x\to\infty} \frac{1}{\bar{\rho}(x)} \int_x^{\infty} (\bar{\eta}(t) - c\bar{\rho}(t))\mathrm{d}t.$$

It follows from Proposition 2.4 that $\sum_{k=0}^{\infty} \lambda_k \int_x^{\infty} \overline{\rho^{k*}}(t)\mathrm{d}t < \infty$. Thus, in the following estimation, exchanging the order of sum and integral is permitted. Using Lemma 3.1 and Proposition 2.4, we obtain

$$\begin{aligned}
\liminf_{x\to\infty} \frac{1}{\bar{\rho}(x)} \int_x^{\infty} (\bar{\eta}(t) - c\bar{\rho}(t))\mathrm{d}t &= \liminf_{x\to\infty} \frac{1}{\bar{\rho}(x)} \sum_{k=2}^{\infty} \int_x^{\infty} \lambda_k(\overline{\rho^{k*}}(t) - k\bar{\rho}(t))\mathrm{d}t \\
&= \liminf_{x\to\infty} \frac{1}{\bar{\rho}(x)} \sum_{k=2}^{\infty} \lambda_k \sum_{j=1}^{k-1} \int_0^x \overline{\rho^{j*}}(x-t)\bar{\rho}(t)\mathrm{d}t \\
&\leqslant c_2 \liminf_{x\to\infty} \frac{1}{\bar{\rho}(x)} \int_0^x \bar{\rho}(x-t)\bar{\rho}(t)\mathrm{d}t,
\end{aligned} \tag{3.6}$$

where the constant c_2 defined by $c_2 = c_1 \sum_{k=2}^{\infty} \lambda_k (l^*(\rho) + \epsilon - 1)^{k-1}(k-1)$ is finite. Moreover, we have

$$\liminf_{x\to\infty}\frac{1}{\bar{\rho}(x)}\int_0^x\bar{\rho}(x-t)\bar{\rho}(t)\mathrm{d}t \leqslant \liminf_{n\to\infty}\frac{1}{\bar{\rho}(n)}\sum_{k=0}^{n-1}\bar{\rho}(n-k-1)\bar{\rho}(k)$$

$$\leqslant \sup_{k\geqslant 0}\frac{\bar{\rho}(k)}{\bar{\rho}(k+1)}\liminf_{n\to\infty}\frac{1}{\bar{\rho}(n)}\sum_{k=0}^{n}\bar{\rho}(n-k)\bar{\rho}(k). \tag{3.7}$$

Note that $\sup_{k\geqslant 0}\bar{\rho}(k)/\bar{\rho}(k+1)<\infty$ because $\rho\in\mathcal{OS}\subset\mathcal{OL}$. Let $f_k=\bar{\rho}(k)$ and $F(x)=x^2$ in Lemma 3.3. Then, $f(1)=\sum_{k=0}^{\infty}\bar{\rho}(k)\leqslant\int_0^{\infty}(t+1)\rho(\mathrm{d}t)<\infty$. We see from $\rho\in\mathcal{L}$ that $f(x)=\sum_{k=0}^{\infty}\bar{\rho}(k)x^k\geqslant x^{-1}\int_0^{\infty}\mathrm{e}^{\epsilon t}\bar{\rho}(t)\mathrm{d}t=\infty$ for $x=\mathrm{e}^{\epsilon}$ with every $\epsilon>0$. Hence, using Lemma 3.3,

$$\liminf_{n\to\infty}\frac{1}{\bar{\rho}(n)}\sum_{k=0}^{n}\bar{\rho}(n-k)\bar{\rho}(k)\leqslant\sum_{k=1}^{\infty}k\lambda_k f(1)^{k-1}=2\sum_{k=0}^{\infty}\bar{\rho}(k)<\infty. \tag{3.8}$$

We obtain from (3.6), (3.7) and (3.8) that

$$\liminf_{x\to\infty}\frac{1}{\bar{\rho}(x)}\int_x^{\infty}(\bar{\eta}(t)-c\bar{\rho}(t))\mathrm{d}t<\infty. \tag{3.9}$$

On the other hand, assume that $\liminf_{x\to\infty}\bar{\eta}(x)/\bar{\rho}(x)>c$. Then there exists $\epsilon>0$ such that $\bar{\eta}(x)/\bar{\rho}(x)>c+\epsilon$ for all x large enough. Thus

$$\liminf_{x\to\infty}\frac{1}{\bar{\rho}(x)}\int_x^{\infty}(\bar{\eta}(t)-c\bar{\rho}(t))\mathrm{d}t\geqslant\liminf_{x\to\infty}\frac{\epsilon}{\bar{\rho}(x)}\int_x^{\infty}\bar{\rho}(t)\mathrm{d}t.$$

However, since $\rho\in\mathcal{L}$, the right-hand side goes to infinity by Feller (1971, Theorem 1, p. 281). Since this contradicts (3.9), we must have $\liminf_{x\to\infty}\bar{\eta}(x)/\bar{\rho}(x)\leqslant c$. It follows from Fatou's lemma and Remark 2.1 that $\liminf_{x\to\infty}\bar{\eta}(x)/\bar{\rho}(x)\geqslant c$. Therefore, we conclude that $\liminf_{x\to\infty}\bar{\eta}(x)/\bar{\rho}(x)=c$. □

The following proposition is the key to proving Theorems 1.2 and 1.3. Part (i) is due to Embrechts and Goldie (1981, Proposition 2).

Proposition 3.4. (i) *For $\mu\in\mathcal{ID}_+$,*

$$\liminf_{x\to\infty}\frac{\bar{\mu}(x)}{\bar{\nu}(x)}\geqslant 1.$$

(ii) *If $\mu\in\mathcal{ID}_+$ and $\nu_1\in\mathcal{OS}\cap\mathcal{L}$, then* $\liminf_{x\to\infty}\bar{\mu}(x)/\bar{\nu}(x)=1$.

Proof. (ii) Let $x>0$. Let μ_1 and μ_2 be the same as in the proof of Theorem 1.1. Write $\bar{\mu}_1(x)$ as the sum of two terms as follows:

$$\bar{\mu}_1(x)=\sum_{k=1}^{\infty}\frac{\mathrm{e}^{-c}c^k}{k!}\overline{\nu_1^{k*}}(x)=\sum_{k=1}^{n}+\sum_{k=n+1}^{\infty}=J_{1,n}(x)+J_{2,n}(x).$$

From Proposition 2.4, $\lim_{n\to\infty}\limsup_{x\to\infty}J_{2,n}(x)/J_{1,n}(x)=0$. Since $\mathcal{L}$ is closed under convolution (Embrechts and Goldie 1980), $\lim_{x\to\infty}J_{1,n}(x+k)/J_{1,n}(x)=1$ for every $k\in\mathbb{R}$.

Hence we have $\lim_{x\to\infty}\bar{\mu}_1(x+k)/\bar{\mu}_1(x) = 1$ for every $k \in \mathbb{R}$, which means that $\mu_1 \in \mathcal{L}$. Combining with Theorem 1.1, we obtain $\mu_1 \in \mathcal{OS} \cap \mathcal{L}$. We have, for $x > y > 0$,

$$\bar{\mu}(x) = \overline{\mu_1 * \mu_2}(x) = \int_{0-}^{y} \bar{\mu}_1(x-t)\mu_2(\mathrm{d}t) + \int_{0-}^{x-y} \bar{\mu}_2(x-t)\mu_1(\mathrm{d}t) + \bar{\mu}_1(x-y)\bar{\mu}_2(y)$$

$$= I_1(x) + I_2(x) + I_3(x).$$

Since $\mu_1 \in \mathcal{L}$, we easily see that $\lim_{y\to\infty}\lim_{x\to\infty}I_1(x)/\bar{\mu}_1(x) = 1$ and $\lim_{y\to\infty}\lim_{x\to\infty}I_3(x)/\bar{\mu}_1(x) = 0$. Furthermore, since $\bar{\mu}_2(x) = o(\bar{\mu}_1(x))$, for arbitrary $\epsilon > 0$, we can take y so large that, for $x > y$,

$$\frac{I_2(x)}{\bar{\mu}_1(x)} = \int_{0-}^{x-y} \frac{\bar{\mu}_2(x-t)}{\bar{\mu}_1(x-t)}\frac{\bar{\mu}_1(x-t)}{\bar{\mu}_1(x)}\mu_1(\mathrm{d}t)$$

$$< \epsilon\int_{0-}^{x-y} \frac{\bar{\mu}_1(x-t)}{\bar{\mu}_1(x)}\mu_1(\mathrm{d}t) \leqslant \epsilon \sup_{x>0}\frac{\overline{\mu_1 * \mu_1}(x)}{\overline{\mu_1}(x)}.$$

Thus we obtain by $\mu_1 \in \mathcal{OS}$ that $\lim_{y\to\infty}\limsup_{x\to\infty}I_2(x)/\bar{\mu}_1(x) = 0$. Therefore, we have $\bar{\mu}(x) \sim \bar{\mu}_1(x)$. It follows from Proposition 3.3 that

$$\liminf_{x\to\infty} \bar{\mu}(x)/\bar{\nu}(x) = \liminf_{x\to\infty} \bar{\mu}_1(x)/c\bar{\nu}_1(x) = \sum_{k=1}^{\infty} k\lambda_k/c = 1.$$

□

Proof of Theorem 1.2. First we prove the case of $\gamma = 0$. By Theorem A, we easily see that $\mathcal{S} \cap \mathcal{ID}_+ \subset \mathcal{WIS} \cap \mathcal{L}$. Conversely, suppose that $\mu \in \mathcal{WIS} \cap \mathcal{L}$. We can take $c > 0$ such that $\bar{\mu}(x) \sim c\bar{\nu}(x)$. By using this and Theorem 1.1(i) we see that $\nu_1 \in \mathcal{OS} \cap \mathcal{L}$. Thus we see from Proposition 3.4 that

$$c = \lim_{x\to\infty} \frac{\bar{\mu}(x)}{\bar{\nu}(x)} = \liminf_{x\to\infty} \frac{\bar{\mu}(x)}{\bar{\nu}(x)} = 1.$$

This implies $\mu \in \mathcal{S} \cap \mathcal{ID}_+$ by Theorem A.

Secondly, we deal with the case of $\gamma > 0$. By Theorem B, $\mathcal{S}(\gamma) \cap \mathcal{ID}_+ \subset \mathcal{WIS} \cap \mathcal{L}(\gamma)$. Conversely, assume that $\mu \in \mathcal{WIS} \cap \mathcal{L}(\gamma)$. Then there exists $c > 0$ satisfying $\bar{\mu}(x) \sim c\bar{\nu}(x)$. Using this and Theorem 1.1(i), we have $\nu_1 \in \mathcal{OS} \cap \mathcal{L}(\gamma)$. This implies that $\bar{\nu}_1(-\gamma) = \int_0^\infty \mathrm{e}^{\gamma t}\nu_1(\mathrm{d}t) < \infty$. In fact, thanks to Fatou's lemma, we find that

$$\int_0^\infty \mathrm{e}^{\gamma t}\nu_1(\mathrm{d}t) = \int_0^\infty \lim_{x\to\infty} \frac{\bar{\nu}_1(x-t)}{\bar{\nu}_1(x)}\nu_1(\mathrm{d}t) \leqslant \liminf_{x\to\infty}\frac{\overline{\nu_1 * \nu_1}(x)}{\bar{\nu}_1(x)} < \infty.$$

Similarly, we have $\hat{\mu}(-\gamma) < \infty$. Now we can define a measure ν^γ and a probability measure μ^γ by

$$\overline{\nu^\gamma}(x) = \int_x^\infty \mathrm{e}^{\gamma t}\nu(\mathrm{d}t),$$

$$\overline{\mu^\gamma}(x) = c_1\int_x^\infty \mathrm{e}^{\gamma t}\mu(\mathrm{d}t), \qquad \text{where } c_1 = (\hat{\mu}(-\gamma))^{-1}.$$

This is one of the density transformations called Esscher or exponential transformations. Then, μ^γ is in $\mathcal{ID}_+$ with Lévy measure ν^γ. See Sato (1999; 2000). Integral by parts leads, for $y > x > 0$, to

$$\int_x^y \mathrm{e}^{\gamma t}\mu(\mathrm{d}t) = \bar{\mu}(x)\mathrm{e}^{\gamma x} - \bar{\mu}(y)\mathrm{e}^{\gamma y} + \gamma\int_x^y \mathrm{e}^{\gamma t}\bar{\mu}(t)\mathrm{d}t.$$

We have $\lim_{y\to\infty}\bar{\mu}(y)\mathrm{e}^{\gamma y}/\gamma\int_x^y \mathrm{e}^{\gamma t}\bar{\mu}(t)\mathrm{d}t = 0$ by the slow variation of $x^\gamma\overline{\mu}(\log x)$ (Feller 1971, Theorem 1, p.281). Further, since $\int_0^\infty \mathrm{e}^{\gamma t}\mu(dt) < \infty$, we see that $\int_0^\infty \mathrm{e}^{\gamma t}\overline{\mu}(t)\mathrm{d}t < \infty$. Then, again by Feller (1971),

$$\lim_{x\to\infty} \bar{\mu}(x)\mathrm{e}^{\gamma x}/\gamma\int_x^\infty \mathrm{e}^{\gamma t}\bar{\mu}(t)\mathrm{d}t = 0.$$

Thus, we obtain

$$\lim_{x\to\infty} \overline{\mu^\gamma}(x)\Big/\int_x^\infty \mathrm{e}^{\gamma t}\bar{\mu}(t)\mathrm{d}t = \gamma c_1.$$

Similarly,

$$\lim_{x\to\infty} \overline{\nu^\gamma}(x)\Big/\int_x^\infty \mathrm{e}^{\gamma t}\bar{\nu}(t)\mathrm{d}t = \gamma.$$

Since $\lim_{x\to\infty}\bar{\mu}(x)/\bar{\nu}(x) = c$, we obtain that

$$\lim_{x\to\infty} \overline{\mu^\gamma}(x)/\bar{\nu}^\gamma(x) = \lim_{x\to\infty} c_1\int_x^\infty \mathrm{e}^{\gamma t}\bar{\mu}(t)\mathrm{d}t\bigg/\int_x^\infty \mathrm{e}^{\gamma t}\bar{\nu}(t)\mathrm{d}t = c_1 c.$$

This means that $\mu^\gamma \in \mathcal{WIS}$. On the other hand, we see that $\mu^\gamma \in \mathcal{L}$ from $\mu \in \mathcal{L}(\gamma)$. Thus $\mu^\gamma \in \mathcal{WIS}\cap\mathcal{L} = \mathcal{S}\cap\mathcal{ID}_+$. Therefore, $c_1 c = 1$ by Theorem A, that is, $c = \hat{\mu}(-\gamma)$.

Drawing everything together, we have $\mu \in \mathcal{L}(\gamma)$, $\hat{\mu}(-\gamma) < \infty$ and $\hat{\mu}(x) \sim \bar{\mu}(-\gamma)\bar{\nu}(x)$. Finally, by Theorem B, we conclude that $\mu \in \mathcal{S}(\gamma)\cap\mathcal{ID}_+$. □

Proof of Corollary 1.3. For $a \in \mathbb{R}$, put

$$g(a) = \lim_{x\to\infty}\frac{\bar{\mu}(x+a)}{\bar{\mu}(x)} = \frac{c(a)}{c(0)}.$$

Then, $g(a)$ is monotone and satisfies $g(a+b) = g(a)g(b)$. This means that $g(a) = \mathrm{e}^{-\gamma a}$ for some $\gamma \geqslant 0$. Therefore, $\mu \in \mathcal{L}(\gamma)$ and $\lim_{x\to\infty}\bar{\mu}(x)/\bar{\nu}(x) = c(0)$. It follows from the proof of Theorem 1.2 that $c(0) = \hat{\mu}(-\gamma)$ and $\mu \in \mathcal{S}(\gamma)\cap\mathcal{ID}_+$. Finally, we have $c(a) = c(0)\mathrm{e}^{-\gamma a} = \hat{\mu}(-\gamma)\mathrm{e}^{-\gamma a}$. □

3.3. Proof of Theorem 1.3

Lemma 3.4 (Yakymiv 1987). *Let* $\mu \in \mathcal{ID}_+$ *with Lévy measure* ν. *If* $\nu_1 \in \mathcal{D}$, *then, for arbitrary* ϵ, $\delta \in (0, 1)$, *there exists* x_0 *such that*

$$(1-\epsilon)\bar{\nu}((1+\delta)x) \leqslant \bar{\mu}(x) \leqslant (1+\epsilon)\bar{\nu}((1-\delta)x) \text{ for } x \geqslant x_0.$$

Proof of Theorem 1.3. Note that the Lévy measure ν has a certain periodicity, that is, $\nu(bE) = b^{-\alpha}\nu(E)$ for a Borel set E in $(0, \infty)$.

(i) It is obvious from Theorem C and $\nu_1 \in \mathcal{D}$ that $\mu \in \mathcal{D}$.

(ii) By Proposition 3.4, we have

$$1 \leqslant \liminf_{x\to\infty} \frac{\bar{\mu}(x)}{\bar{\nu}(x)}. \tag{3.10}$$

Since $\bar{\mu}(x)$ is continuous by Example 28.2 of Sato (1999),

$$\liminf_{x\to\infty} \frac{\bar{\mu}(x)}{\bar{\nu}(x)} = \liminf_{x\to\infty} \lim_{\epsilon\to 0} \frac{\bar{\mu}(x-\epsilon)}{\bar{\nu}(x-\epsilon)} = \liminf_{x\to\infty} \frac{\bar{\mu}(x)}{\bar{\nu}(x-)}. \tag{3.11}$$

By Lemma 3.4 and the periodicity of ν, we obtain

$$\limsup_{x\sim\infty} \frac{\bar{\mu}(x)}{\bar{\nu}(x)} \leqslant \inf_{r<1} \limsup_{x\to\infty} \frac{\bar{\nu}(rx)}{\bar{\nu}(x)} = \inf_{r<1} \sup_{1\leqslant x<b} \frac{\bar{\nu}(rx)}{\bar{\nu}(x)}, \tag{3.12}$$

$$\liminf_{x\to\infty} \frac{\bar{\nu}(x)}{\bar{\nu}(x-)} = \left(\limsup_{x\to\infty} \frac{\bar{\nu}(x-)}{\bar{\nu}(x)}\right)^{-1} = \left(\sup_{1\leqslant x<b} \frac{\bar{\nu}(x-)}{\bar{\nu}(x)}\right)^{-1}. \tag{3.13}$$

Choose sequences $\{x_k : k \in \mathbb{N}\}$ and $\{r_k : k \in \mathbb{N}\}$ such that $1 \leqslant x_k < b$, $x_k \to x_0$, $1 \leqslant x_0 \leqslant b$, $r_k \uparrow 1$ and $\lim_{k\to\infty}\bar{\nu}(r_k x_k)/\bar{\nu}(x_k) = \inf_{r<1}\sup_{1\leqslant x<b}\bar{\nu}(rx)/\bar{\nu}(x)$. Then, we see that

$$\begin{aligned}\inf_{r<1} \sup_{1\leqslant x<b} \frac{\bar{\nu}(rx)}{\bar{\nu}(x)} &= \lim_{k\to\infty} \frac{\bar{\nu}(r_k x_k)}{\bar{\nu}(x_k)} \leqslant \frac{\limsup_{k\to\infty} \bar{\nu}(r_k x_k)}{\liminf_{k\to\infty} \bar{\nu}(x_k)} \\ &\leqslant \frac{\bar{\nu}(x_0-)}{\bar{\nu}(x_0)} \leqslant \sup_{1\leqslant x<b} \frac{\bar{\nu}(x-)}{\bar{\nu}(x)}.\end{aligned} \tag{3.14}$$

We obtain from (3.10)–(3.14) that

$$
\begin{aligned}
1 &\leqslant \liminf_{x\to\infty} \frac{\bar{\mu}(x)}{\bar{\nu}(x)} = \liminf_{x\to\infty} \frac{\bar{\mu}(x)}{\bar{\nu}(x-)} \\
&= \liminf_{x\to\infty} \frac{\bar{\mu}(x)}{\bar{\nu}(x)} \frac{\bar{\nu}(x)}{\bar{\nu}(x-)} \\
&\leqslant \limsup_{x\to\infty} \frac{\bar{\mu}(x)}{\bar{\nu}(x)} \liminf_{x\to\infty} \frac{\bar{\nu}(x)}{\bar{\nu}(x-)} \\
&\leqslant \inf_{r<1} \sup_{1\leqslant x<b} \frac{\bar{\nu}(rx)}{\bar{\nu}(x)} \left(\sup_{1\leqslant x<b} \frac{\bar{\nu}(x-)}{\bar{\nu}(x)} \right)^{-1} = 1.
\end{aligned}
$$

Thus, we obtain $\liminf_{x\to\infty}\bar{\mu}(x)/\bar{\nu}(x) = 1$ and $\limsup_{x\to\infty}\bar{\mu}(x)/\bar{\nu}(x) = \sup_{1\leqslant x<b}\bar{\nu}(x-)/\bar{\nu}(x)$.

(iii) First, we show that $\sup_{1\leqslant x<b}\bar{\nu}(x-)/\bar{\nu}(x) \leqslant b^{\alpha}$. Let $x \in I_k = [b^k x_0, b^{k+1}x_0)$ for $k \in \mathbb{N}$ with $x_0 > 0$. Since $\bar{\nu}(x-) \leqslant \sum_{j=k}^{\infty}\nu(I_j) = (1-b^{-\alpha})^{-1}\nu(I_k)$ and $\bar{\nu}(x) \geqslant \sum_{j=k+1}^{\infty}\nu(I_j) = (1-b^{-\alpha})^{-1}\nu(I_{k+1})$, we have $\bar{\nu}(x-)/\bar{\nu}(x) \leqslant \nu(I_k)/\nu(I_{k+1}) = b^{\alpha}$. It is obvious that $\sup_{1\leqslant x<b}\bar{\nu}(x-)/\bar{\nu}(x) = b^{\alpha}$ if $\nu(\mathrm{d}x) = c\sum_{k\in\mathbb{Z}} b^{-k\alpha}\delta_{b^k x_0}(\mathrm{d}x)$. Conversely, if $\sup_{1\leqslant x<b}\bar{\nu}(x-)/\bar{\nu}(x) = b^{\alpha}$, then there exists a sequence $\{x_k : k \in \mathbb{N}\}$ such that $1 \leqslant x_k < b$, $x_k \to x_0$ and

$$
\lim_{k\to\infty} \frac{\bar{\nu}(x_k-)}{\bar{\nu}(x_k)} = b^{\alpha}.
$$

Since $\bar{\nu}(x_k) = b^{-\alpha}\bar{\nu}(b^{-1}x_k)$, this implies that $\lim_{k\to\infty}\nu(b^{-1}x_k, x_k) = 0$. Since

$$
\nu(b^{-1}x_0, x_0) \leqslant \nu(\liminf_{k\to\infty}(b^{-1}x_k, x_k)) \leqslant \liminf_{k\to\infty} \nu(b^{-1}x_k, x_k) = 0,
$$

we see that the support of ν is concentrated in $\cup_{k\in\mathbb{Z}}\{b^k x_0\}$. That is, $\nu(\mathrm{d}x) = c\sum_{k\in\mathbb{Z}} b^{-k\alpha}\delta_{b^k x_0}(\mathrm{d}x)$ with some $c > 0$. $\square$

3.4. Proof of Proposition 1.1

(iii) Use $\bar{\mu}(x) \leqslant \overline{\mu^{n*}}(x) \leqslant n\bar{\mu}(x/n)$.

(iv) Instead of ρ, we study, adding a mass on $\{0\}$, the distribution η defined by

$$
\eta(\mathrm{d}x) = c_1 \sum_{n\in T_0} \frac{\mathrm{e}^{-n}}{(n+1)^2}\delta_n(\mathrm{d}x),
$$

where $T_0 = \{n \in \mathbb{N}\cup\{0\} : n = \sum_{i=1}^{\infty} j_i 3^{i-1},\ j_i = 0 \text{ or } 1\}$ and c_1 is the normalized constant. It is enough to show that $\eta \notin \mathcal{OS}$ and $\eta^{n*} \in \mathcal{OS}$ for $n \geqslant 2$, because it is shown by Proposition 2.7 that $\overline{\eta^{n*}}(x) \asymp \overline{\rho^{n*}}(x)$ for $n \geqslant 2$.

Throughout this proof, let every integer be expressed in the ternary system and $f(n, i)$ denote the ith digit of an integer n (counting from the right). Let us prove $\eta \notin \mathcal{OL}$, which leads to $\eta \notin \mathcal{OS}$. Let n_k be the largest number consisting of k digits in T_0, that is, $f(n_k, i) = 1$ for $1 \leqslant i \leqslant k$, and m_k be the smallest number consisting of $k+1$ digits in T_0, that is, $f(m_k, k+1) = 1$ and $f(m_k, i) = 0$ for $1 \leqslant i \leqslant k$. Note that m_k is next to n_k in T_0, and

$$n_k = \sum_{i=0}^{k-1} 3^i = \frac{1}{2}(3^k - 1), \qquad m_k = 3^k.$$

From

$$\bar{\eta}(n_k - 1) > \eta(\{n_k\}) = c_1 \frac{e^{-n_k}}{(n_k + 1)^2}$$

and

$$\bar{\eta}(n_k) = \eta[m_k, \infty) < c_1 \sum_{n=m_k}^{\infty} e^{-n}(n+1)^{-2} < \frac{c_1 e}{e - 1} \frac{e^{-m_k}}{(m_k + 1)^2},$$

we see that

$$\frac{\bar{\eta}(n_k - 1)}{\bar{\eta}(n_k)} > \frac{e - 1}{e} \frac{(m_k + 1)^2}{(n_k + 1)^2} \frac{e^{m_k}}{e^{n_k}} = 4(1 - e^{-1}) \exp\left(\frac{1}{2}(3^k + 1)\right).$$

Hence,

$$\limsup_{x \to \infty} \frac{\bar{\eta}(x - 1)}{\bar{\eta}(x)} = \infty.$$

This means that $\eta \notin \mathcal{OL}$.

We have to show that $\eta^{n*} \in \mathcal{OS}$ for every $n \geqslant 2$. By Proposition 2.6, it is enough to show that $\eta^{2*} \in \mathcal{OS}$. We have

$$\eta^{2*}(\{n\}) \asymp e^{-n} \sum_{(C_0)} \frac{1}{(n_1 + 1)^2 (n_2 + 1)^2},$$

where (C_0) means the condition $n_1, n_2 \in T_0$, $n_1 \geqslant n_2$ and $n = n_1 + n_2$. It is easy to see that the support of η^{2*} is $\mathbb{N} \cup \{0\}$.

The representation of $n \in \mathbb{N}$ in the ternary system is denoted by $\langle \cdot \rangle$. For example, $3 = \langle 10 \rangle$. We now prove the following:

$$\eta^{2*}(\{n\}) \asymp \frac{e^{-n}}{n^2} \frac{2^{q(n)}}{3^{2p(n)}}, \tag{3.15}$$

where

$$p = p(n) = \sup\left\{ i : j_i = 2,\ n = \sum_{i=1}^{\infty} j_i 3^{i-1} \right\},$$

$$q = q(n) = \#\{ i : j_i = 1,\ 1 \leqslant i \leqslant p(n) - 1 \},$$

with the understanding that $p = 0$ for $n \in T_0$ and $q = 0$ for $p = 0$ or 1. For example, $p = 4$ and $q = 2$ for $n = \langle 102101 \rangle$.

Put $I(n) = \sum_{(C_0)}(n_2+1)^{-2}$. Since

$$\eta^{2*}(\{n\}) \asymp \mathrm{e}^{-n} \sum_{(C_0)} \frac{1}{(n_1+1)^2(n_2+1)^2} \asymp \frac{\mathrm{e}^{-n}}{n^2} I(n),$$

(3.15) is equivalent to the relation

$$I(n) \asymp \frac{2^{q(n)}}{3^{2p(n)}}. \tag{3.16}$$

We estimate the number of combinations of n_1 and n_2 satisfying (C_0) for given n. If $f(n, i) = 2$ (0) for some $i \geqslant 1$, then it necessarily means that $(f(n_1, i), f(n_2, i)) = (1, 1)$ ((0,0)). However, if $f(n, i) = 1$, then there may be two patterns: $(f(n_1, i), f(n_2, i)) = (1, 0)$ or $(0, 1)$. For instance, the set of (n_1, n_2) corresponding to $n = \langle 211 \rangle$ is $\{(\langle 111 \rangle, \langle 100 \rangle), (\langle 110 \rangle, \langle 101 \rangle)\}$. Hence, only the digit 1 in the ternary representation of n influences the number of combinations of n_1 and n_2 corresponding to n. Keeping this in mind, we show (3.16) in the case $p(n) = 0$ and in the case $p(n) \geqslant 1$ separately. If $p(n) = 0$, then, seeing that $n_1 = n \in T_0$ and $n_2 = 0$, we immediately obtain that $I(n) \geqslant 1$. Also, we easily obtain

$$I(n) = \sum_{(C_0)} \frac{1}{(n_2+1)^2} \leqslant \sum_{k \in \mathbb{N} \cup \{0\}} \frac{1}{(k+1)^2} < \infty.$$

Let $p(n) \geqslant 1$ and consider the lower bound of $I(n)$. Now we can find an n_2 which has $p(n)$ digits. Such an n_2 does not exceed $\frac{1}{2}(3^{p(n)} - 1)$, which is the largest number consisting of p digits in T_0. On the other hand, if $M = \#\{n_2\colon n_2$ is a number consisting of p digits appearing in (n_1, n_2) and satisfying $(C_0)\}$, then $M = 2^q$ or 2^{q-1}. For example, if $n = \langle 211 \rangle$, then $p = 3$, $q = 2$, and $M = 2^{q-1} = 2$; if $n = \langle 1211 \rangle$, then $p = 3$, $q = 2$, and $M = 2^q = 4$. Thus, we see that

$$I(n) \geqslant \frac{2^{q(n)+1}}{(3^{p(n)}+1)^2} > \frac{2^{q(n)}}{3^{2p(n)}}.$$

Next we consider the upper bound of $I(n)$, assuming $p(n) \geqslant 1$. Since $f(n_2, p) = 1$,

$$n_2 = \sum_{i=1}^{\infty} j_i 3^{i-1} \geqslant \sum_{i=p+1}^{\infty} j_i 3^{i-1} + 3^{p-1}.$$

Denote $g(m) = \sup\{i : j_i = 1, m = \sum_{i=1}^{\infty} j_i 3^{i-1}\}$ for $m \in T_0 \backslash \{0\}$ and $g(0) = 0$. Then $g(n_2) \geqslant p(n)$. We write $I(n)$ as the sum of $I_1(n)$ and $I_2(n)$, where

$$I_1(n) = \sum_{(C_0), g(n_2) \geqslant p+1} \frac{1}{(n_2+1)^2}, \qquad I_2(n) = \sum_{(C_0), g(n_2) = p} \frac{1}{(n_2+1)^2}.$$

Suppose that $g(n_2) \geqslant p(n) + 1$. Then

$$n_2 \geqslant 3^{g(n_2)-1} + 3^{p-1}. \tag{3.17}$$

Meanwhile, for $j \geqslant p(n) + 1$, $\#\{n_2 : g(n_2) = j\}$ is estimated by partitioning the digits of n_2 into three parts and considering their combinations respectively as follows. The number of possible combinations of digits of n_2 from the first to the $(p-1)$th is 2^q, $f(n_2, p)$ is necessarily equal to 1, and the number of possible combinations of digits of n_2 from the $(p+1)$th to the $(j-1)$th is not more than 2^{j-p-1}. Multiplying these together, we have

$$\#\{n_2 : g(n_2) = j\} \leqslant 2^{j-p+q-1}. \tag{3.18}$$

From (3.17) and (3.18), we obtain

$$I_1(n) \leqslant \sum_{j \geqslant p+1} \frac{2^{j-p+q-1}}{(3^{j-1} + 3^{p-1})^2} \leqslant \frac{9}{7}\frac{2^q}{3^{2p}}. \tag{3.19}$$

On the other hand, suppose that $g(n_2) = p(n)$. Since $n_2 \geqslant 3^{p-1}$ and $\#\{n_2 : g(n_2) = p\} = 2^q$ or 2^{q-1}, we see that

$$I_2(n) \leqslant \frac{2^q}{(3^{p-1} + 1)^2} \leqslant 9 \times \frac{2^q}{3^{2p}}. \tag{3.20}$$

From (3.19) and (3.20), we see in the case where $p(n) \geqslant 1$ that

$$I(n) = I_1(n) + I_2(n) \leqslant \frac{72}{7}\frac{2^q}{3^{2p}}.$$

We have finished the case where $p(n) \geqslant 1$. Thus we have (3.16), which implies (3.15).

Now we shall show that

$$\eta^{3*}(\{n\}) \asymp \eta^{2*}(\{n\}). \tag{3.21}$$

Considering the case where $n_3 = 0$, we easily obtain the lower bound of $\eta^{3*}(\{n\})$, namely, $\eta^{3*}(\{n\}) \geqslant \eta(\{0\})\eta^{2*}(\{n\})$.

Let us show that $\eta^{3^*}(\{n\})/\eta^{2^*}(\{n\})$ is bounded above. For a given n, let (C_1) denote the condition that $n_1, n_2, n_3 \in T_0$, $n_1 \geqslant n_2 \geqslant n_3$ and $n = n_1 + n_2 + n_3$. Then

$$\eta^{3^*}(\{n\}) \asymp \mathrm{e}^{-n} \sum_{(C_1)} \frac{1}{(n_1+1)^2(n_2+1)^2(n_3+1)^2}.$$

This time we estimate the number of combinations when n is the sum of three elements in T_0. In this case, we have to take into account the contribution from the lower places. Let $J(n) = \sum_{(C_1)}(n_2+1)^{-2}(n_3+1)^{-2}$. Recalling (3.15) and noticing that

$$\sum_{(C_1)} \frac{1}{(n_1+1)^2(n_2+1)^2(n_3+1)^2} \leqslant \frac{9}{(n+1)^2} J(n),$$

we see that, in order to prove that $\eta^{3*}(\{n\})/\eta^{2*}(\{n\})$ is bounded above, it is sufficient to show that

$$J(n) \leqslant c_0 \frac{2^q}{3^{2p}}, \tag{3.22}$$

where c_0 is a positive constant that does not depend on n.

If $p(n) = 0$ or 1, then (3.22) is obtained from

$$\sum_{(C_1)} \frac{1}{(n_2+1)^2(n_3+1)^2} < \sum_{n_2, n_3 \in \mathbb{N} \cup \{0\}} \frac{1}{(n_2+1)^2(n_3+1)^2} < \infty.$$

Next, we consider the case where $p \geqslant 2$. Then note that $g(n_2) \geqslant p-1 \geqslant 1$. We have

$$\begin{aligned} J(n) &= \sum_{j \geqslant p-1} \sum_{(C_1), g(n_2)=j} \frac{1}{(n_2+1)^2(n_3+1)^2} \\ &\leqslant \sum_{j \geqslant p-1} \frac{1}{(3^{j-1}+1)^2} \sum_{(C_1), g(n_2)=j} \frac{1}{(n_3+1)^2} \\ &\leqslant \sum_{j \geqslant p-1} \frac{1}{(3^{j-1}+1)^2} \sum_{0 \leqslant k \leqslant j} \frac{N(C_1, j, k)}{(3^{k-1}+1)^2}, \end{aligned} \tag{3.23}$$

where $N(C_1, j, k) = \#\{(n_1, n_2, n_3) : (C_1), g(n_2) = j, g(n_3) = k\}$. We shall show that

$$N(C_1, j, k) \leqslant 2^{j-p+q+1} 8^k. \tag{3.24}$$

Classifying $(g(n_2), g(n_3)) = (j, k)$ into seven cases in the following way, we can prove (3.24) in each case: (a) $j \geqslant p+1$, $k \leqslant p-1$; (b) $j \geqslant p+1$, $k = p$; (c) $j \geqslant p+1$, $k \geqslant p+1$; (d) $j = p$, $k \leqslant p-1$; (e) $j = k = p$; (f) $j = p-1$, $k \leqslant p-2$; (g) $j = k = p-1$. Since every case can be handled in a similar way, we give the proof only for case (a). In this case, $k < p < j$. If $k = 0$, then $N(C_1, j, 0) \leqslant 2^{j-p+q}$. Let $k \geqslant 1$. When we count the number of combinations, we partition the digits of n into several parts and consider the combinations in each part. The parts in case (a) are as follows: from the first digit to the kth; from the $(k+1)$th to the pth; from the $(p+1)$th to the $(j-1)$th; and from the jth to the last. The last of these does not increase the number of combinations. The number of combinations from the first to the kth digit is not more than 8^k, since each digit in n_1, n_2 and n_3 is 0 or 1. The combinations in this first part are of two kinds: whether the contribution to the $(k+1)$th place from the kth place does not exist (call this the first kind), or does exist (call this the second kind). In the case of the first kind, we can deal separately with the first to the kth digits and with the $(k+1)$th digit onwards. From the $(k+1)$th to the pth digits, the number of combinations is determined by the number of 1s from the $(k+1)$th to the $(p-1)$th digits of n, which is not more than q. Hence, the number of combinations in this part is bounded by 2^q. From the $(p+1)$th digit to the $(j-1)$th, the number of combinations is determined by the number of 1s in this sequence, which is not more than $j-p-1$. Thus, the number of combinations in this part is bounded by 2^{j-p-1}. Combining the combinations of each part, we see that the number of combinations of the first kind from the $(k+1)$th digit does not exceed $2^{j-p+q-1}$. For combinations of the second kind, we pay attention to the $(k+1)$th digit of n. If $f(n, k+1) = 2$, then $(f(n_1, k+1), f(n_2, k+1)) = (1, 0)$ or $(0, 1)$ and there is no contribution to the $(k+2)$th place from the $(k+1)$th place. If $f(n, k+1) = 1$, then necessarily $(f(n_1, k+1), f(n_2, k+1)) = (0, 0)$. Lastly, if $f(n, k+1) = 0$, then $(f(n_1, k+1), f(n_2, k+1)) = (1, 1)$ and there is a contribution to the $(k+2)$th place from the

$(k+1)$th place. Then we consider the same thing for the $(k+2)$th digit and repeat the same procedure inductively. Noticing that contribution to the next-left place stops before the pth place, we see that the number of combinations of the second kind from the $(k+1)$th digit to the last does not exceed twice the number of the first kind, that is, 2^{j-p+q}. Combining these with the combinations in the part from the first digit to the kth, we have finished the proof of (3.24) in case (a) with $k \geqslant 1$. Thus, we obtain

$$\sum_{0 \leqslant k \leqslant j} \frac{N(C_1, j, k)}{(3^{k-1}+1)^2} \leqslant \sum_{0 \leqslant k \leqslant j} \frac{2^{j-p+q+1} 8^k}{(3^{k-1}+1)^2} < 81 \times 2^{j-p+q+1}. \tag{3.25}$$

From (3.23) and (3.25), we have (3.22). This completes the proof of (3.21).

Finally, we prove $\eta^{2*} \in \mathcal{OS}$. We have

$$\frac{\overline{\eta^{4*}}(x)}{\overline{\eta^{2*}}(x)} = \frac{\overline{\eta^{4*}}(x)}{\overline{\eta^{3*}}(x)} \frac{\overline{\eta^{3*}}(x)}{\overline{\eta^{2*}}(x)}.$$

Since it follows from (3.21) that, for positive constants c_1 and c_2 which do not depend on n,

$$0 < c_1 < \frac{\eta^{3*}(\{n\})}{\eta^{2*}(\{n\})} < c_2 < \infty,$$

we see that $c_1 < \overline{\eta^{3*}}(x)/\overline{\eta^{2*}}(x) < c_2$. Furthermore, since

$$\frac{\overline{\eta^{4*}}(x)}{\overline{\eta^{3*}}(x)} = \frac{\int_0^\infty \overline{\eta^{3*}}(x-t)\eta(\mathrm{d}t)}{\int_0^\infty \overline{\eta^{2*}}(x-t)\eta(\mathrm{d}t)},$$

we obtain $c_1 < \overline{\eta^{4*}}(x)/\overline{\eta^{3*}}(x) < c_2$. Thus, we conclude that $\eta^{2*} \in \mathcal{OS}$.

Remark 3.1. We conjecture that, for every $p \in \mathbb{N}$, it is possible to construct a distribution η on $[0, \infty)$ such that $\eta^{k*} \notin \mathcal{OS}$ for $1 \leqslant k \leqslant p$ and $\eta^{k*} \in \mathcal{OS}$ for $k \geqslant p+1$ by generalizing the above method.

4. Additional remarks

Remark 4.1. Rogozin (2000) states that the proof of Theorem D by Chover *et al*. (1973a; 1973b) is not complete for $\gamma = 0$ and that there is a logical gap in the equality on line 12 on p. 356 of Cline (1987). See also Rogozin and Sgibnev (1999). We agree with him, but found that the logical gap in Cline's proof is not critical and thus Cline's proof of Theorem D is correct.

Remark 4.2. Lemma 2.1(iv) of Cline (1987) is not correct. A counterexample is as follows. For $n \in \mathbb{N}$, define $A_i(x)$ $(i = 1, 2, 3, 4)$ as $A_1(x) = 1/n^2$, $A_2(x) = 1$, $A_3(x) = 1/n$, $A_4(x) = 1$ for $2n \leqslant x < 2n+1$, and $A_1(x) = 1$, $A_2(x) = 1/n^2$, $A_3(x) = 1$, $A_4(x) = 1/n$ for $2n+1 \leqslant x < 2(n+1)$.

This lemma was used in Cline's (1987) Lemma 3.1(iii), Corollary 3.2(i), Theorem 2.13(iii), Corollary 2.14, and Theorem 3.4. Moreover, Corollary 3.3, Theorem 4.1, and Corollary 4.2 of Klüppelberg (1989) depend on Cline's Corollary 2.14. Lemmas 1 and 2 and Theorem 1 of Klüppelberg and Villasenor (1991) depend on Cline's Theorem 3.4. Finally, Theorem 5.2 of Goldie and Klüppelberg (1998) depends on Cline's Theorem 2.13(iii). Thus the all above results should be reconsidered.

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