Bernoulli **11**(1), 2005, 29–36

On the quantiles of Brownian motion and their hitting times

ANGELOS DASSIOS

Department of Statistics, London School of Economics, Houghton Street, London WC2A 2AE, UK. E-mail: A.Dassios@lse.ac.uk

The distribution of the α-quantile of a Brownian motion on an interval $[0, t]$ has been obtained motivated by a problem in financial mathematics. In this paper we generalize these results by calculating an explicit expression for the joint density of the α-quantile of a standard Brownian motion, its first and last hitting times and the value of the process at time t. Our results can easily be generalized to a Brownian motion with drift. It is shown that the first and last hitting times follow a transformed arcsine law.

Keywords: arcsine law; hitting times; quantiles of Brownian motion

1. Introduction

Let $(X(s), s \geqslant 0)$ be a real-valued stochastic process on a probability space $(\Omega, \mathcal{F}, \Pr)$. For $0 < \alpha < 1$, define the α–quantile of the path of $(X(s), s \geqslant 0)$ up to a fixed time t by

$$M_X(\alpha, t) = \inf\left\{x : \int_0^t \mathbf{1}(X(s) \leqslant x)\mathrm{d}s > \alpha t\right\}. \tag{1}$$

The study of the quantiles of various stochastic processes has been undertaken as a response to a problem arising in the field of mathematical finance, the pricing of a particular path-dependent financial option; see Miura (1992), Akahori (1995) and Dassios (1995). This involves calculating quantities such as $\mathrm{E}(h(M_X(\alpha, t)))$, where $h(x) = (\mathrm{e}^x - b)^+$ or some other appropriate function, and requires obtaining the distribution of $X(t)$. In the case where $(X(s), s \geqslant 0)$ is a process with exchangeable increments the following result was obtained:

Proposition 1. *Let $X'(s) = X(\alpha t + s) - X(\alpha t)$. Then*

$$(M_X(\alpha, t), X(t)) \overset{\text{(law)}}{=} (N_X(\alpha, t), X(\alpha t) + X'((1-\alpha)t)), \tag{2}$$

where

$$N_X(\alpha, t) = \sup_{0 \leqslant s \leqslant \alpha t} X(s) + \inf_{0 \leqslant s \leqslant (1-\alpha)t} X'(s). \tag{3}$$

Note that if $(X(s), s \geqslant 0)$ is a Lévy process (having stationary and independent increments), then $X'(s)$ is an independent copy of $X(s)$.

When $(X(s),\ s \geqslant 0)$ is a Brownian motion, we can use this result and obtain an explicit formula for the joint density of $M_X(\alpha,\ t)$ and $X(t)$. This result was first proved for a Brownian motion with drift by Dassios (1995) and Embrechts *et al*. (1995), and for Lévy processes by Dassios (1996). There is also a similar result for discrete-time random walks first proved by Wendel (1960).

We now let

$$L_X(\alpha,\ t) = \inf\{s \in [0,\ t] : X(s) = M_X(\alpha,\ t)\}$$

be the first, and

$$K_X(\alpha,\ t) = \sup\{s \in [0,\ t] : X(s) = M_X(\alpha,\ t)\}$$

the last time the process hits $M_X(\alpha,\ t)$. One can now introduce a 'barrier' element to the financial application by making the option worthless if the quantile is hit too early or too late. For example, this can involve calculating quantities such as $\mathrm{E}(h(M_X(\alpha,\ t))\mathbf{1}(L_X(\alpha,\ t) > v,\ K_X(\alpha,\ t) < u))$.

The first study of these quantities can be found in Chaumont (1999). By using combinatorial arguments he derives results of the same type as Proposition 1 that are extensions of Wendel's results in discrete time. In the case where the random walk steps can only take the value +1 or -1, a representation for the analogues of $L_X(\alpha,\ t)$ and $K_X(\alpha,\ t)$ is obtained. Finally, he derives a continuous-time representation for the triple law of $M_X(\alpha,\ t)$, $L_X(\alpha,\ t)$ and $X(t)$, extending Proposition 1 when $X(t)$ is a Brownian motion. We will demonstrate that Chaumont's results point to a representation involving $K_X(\alpha,\ t)$ as well. We will use this to obtain an explicit form in Section 3. We will also derive alternative representations and prove a remarkable arcsine law.

For the rest of the paper we assume that $(X(s),\ s \geqslant 0)$ is a standard Brownian motion, unless otherwise specified. Without loss of generality, we will restrict our attention to the case $t = 1$, taking advantage of the Brownian scaling. For simplicity we set $M_X(\alpha,\ t) = M_X(\alpha)$, $L_X(\alpha,\ t) = L_X(\alpha)$ and $K_X(\alpha,\ t) = K_X(\alpha)$.We will derive the joint density of $M_X(\alpha)$, $L_X(\alpha)$, $K_X(\alpha)$ and $X(1)$. If we denote this density by $f(y,\ x,\ u,\ v)$, our results can be generalized for a Brownian motion with drift m, using a Cameron–Martin–Girsanov transformation. The corresponding density will be

$$f(y,\ x,\ u,\ v)\exp(mx - m^2/2).$$

Before we obtain the density of $(M_X(\alpha),\ L_X(\alpha),\ K_X(\alpha),\ X(1))$, we will first show that the law of $L_X(\alpha)$ (and $K_X(\alpha)$) is a transformed arcsine law.

2. An arcsine law for $L_X(\alpha,\ t)$

Let $S_X(t) = \sup_{0 \leqslant s \leqslant t}\{X(s)\}$ and $\theta_X(t) = \sup\{s \in [0,\ t] : X(s) = S_X(t)\}$. Define also the stopping time $\tau_c = \inf\{s > 0 : X(s) = c\}$. We will first obtain the joint distribution of $(M_X(\alpha),\ L_X(\alpha))$ and of $(M_X(\alpha) - X(1),\ 1 - K_X(\alpha))$.

Theorem 1. *For* $b > 0$,

$$\Pr(M_X(\alpha) \in \mathrm{d}b,\ L_X(\alpha) \in \mathrm{d}u) = \Pr(S_X(1) \in \mathrm{d}b,\ \theta_X(1) \in \mathrm{d}u)\mathbf{1}(0 < u < \alpha), \tag{4}$$

and for, $b < 0$,

$$\Pr(M_X(\alpha) \in \mathrm{d}b,\ L_X(\alpha) \in \mathrm{d}u) = \Pr(S_X(1) \in \mathrm{d}|b|,\ \theta_X(1) \in \mathrm{d}u)\mathbf{1}(0 < u < 1-\alpha). \tag{5}$$

Furthermore, $(M_X(\alpha),\ L_X(\alpha))$ and $(M_X(\alpha) - X(1),\ 1 - K_X(\alpha))$ *have the same distribution.*

Proof. Let $b > 0$ and $u < \alpha$. We then have that

$$\begin{aligned}\Pr(M_X(\alpha) > b,\ L_X(\alpha) > u) &= \Pr(S_X(u) < M_X(\alpha),\ M_X(\alpha) > b) \\ &= \Pr(b < S_X(u) < M_X(\alpha)) + \Pr(S_X(u) < b < M_X(\alpha)). \end{aligned} \tag{6}$$

Let $\tau_b = \inf\{s > 0 : X(s) = b\}$ and $X^*(s) = X(\tau_b + s) - b$; $(X^*(s),\ s \geqslant 0)$ is a standard Brownian motion which is independent of $(X(s),\ 0 \leqslant s \leqslant \tau_b)$. We then have

$$\begin{aligned}\Pr(b < S_X(u) < M_X(\alpha)) \\ &= \Pr\left(S_X(u) > b,\ \int_0^1 \mathbf{1}(X(s) \leqslant S_X(u))\mathrm{d}s < \alpha \right) \\ &= \Pr\left(S_X(u) > b,\ \int_u^1 \mathbf{1}(X(s) - X(u) \leqslant S_X(u) - X(u))\mathrm{d}s < \alpha - u \right). \end{aligned} \tag{7}$$

We now condition on $\sigma\{X(s),\ 0 \leqslant s \leqslant u\}$. Let $X^*(s) = X(u+s) - X(u)$. $(X^*(s),\ s \geqslant 0)$ is a standard Brownian motion which is independent of $(X(s),\ 0 \leqslant s \leqslant u)$. We condition on $S_X(u) - X(u) = c$, and set $\tau_c = \inf\{s > 0 : X^*(s) = c\}$ and $X^{**}(s) = X^*(\tau_c + s) - c$. $(X^{**}(s),\ s \geqslant 0)$ is a standard Brownian motion which is independent of both $(X(s),\ 0 \leqslant s \leqslant u)$ and $(X^*(s),\ 0 \leqslant s \leqslant \tau_c)$. We have that

$$\Pr\left(\int_0^{1-u} \mathbf{1}(X^*(s) \leqslant c)\mathrm{d}s < \alpha - u\right) = \int_0^{\alpha-u} \Pr(\tau_c \in \mathrm{d}r)\Pr\left(\int_0^{1-u-r} \mathbf{1}(X^{**}(s) \leqslant 0)\mathrm{d}s < \alpha - u - r\right)$$

and since $\int_0^{1-u-r} \mathbf{1}(X^{**}(s) \leqslant 0)\mathrm{d}s$ has the same (arcsine) law as $\theta_{X^{**}}(1-u-r)$, this is equal to

$$\begin{aligned}\int_0^{\alpha-u} \Pr(\tau_c \in \mathrm{d}r)\Pr(\theta_{X^{**}}(1-u-r) < \alpha - u - r) \\ &= \int_0^{\alpha-u} \Pr(\tau_c \in \mathrm{d}r)\Pr\left(\sup_{0 \leqslant s \leqslant \alpha-u-r} X^{**}(s) > \sup_{\alpha-u-r \leqslant s \leqslant 1-u-r} X^{**}(s) \right) \\ &= \Pr\left(\sup_{0 \leqslant s \leqslant \alpha-u} X^*(s) > \sup_{\alpha-u \leqslant s \leqslant t-u} X^*(s),\ \sup_{0 \leqslant s \leqslant \alpha-u} X^*(s) > c \right), \end{aligned}$$

and so (7) is equal to

$$\Pr\left(\begin{array}{c}\sup_{u\leqslant s\leqslant\alpha} X(s)-X(u) > \sup_{\alpha\leqslant s\leqslant 1} X(s)-X(u),\\ \sup_{u\leqslant s\leqslant\alpha} X(s)-X(u) > \sup_{0\leqslant s\leqslant u} X(s)-X(u),\\ \sup_{0\leqslant s\leqslant u} X(s) > b\end{array}\right)$$
$$= \Pr(S_X(u) > b,\ u < \theta_X(1) < \alpha). \tag{8}$$

Furthermore,

$$\begin{aligned}\Pr(S_X(u) < b < M_X(\alpha)) &= \Pr\left(S_X(u) < b,\ \int_0^1 \mathbf{1}(X(s)\leqslant b)\mathrm{d}s < \alpha\right)\\ &= \int_u^\alpha \Pr(\tau_b\in \mathrm{d}r)\Pr\left(\int_0^{1-r}\mathbf{1}(X^*(s)\leqslant 0) < \alpha - r\right)\\ &= \int_u^\alpha \Pr(\tau_b\in \mathrm{d}r)\Pr(\theta_{X^*}(1-r) < \alpha - r)\\ &= \Pr\left(u < \theta_X(1) < \alpha,\ S_X(u) < b,\ \sup_{u\leqslant s\leqslant\alpha} X(s) > b\right).\end{aligned} \tag{9}$$

Adding (8) and (9) together, we see that (6) is equal to

$$\Pr\left(u < \theta_X(1) < \alpha,\ \sup_{u\leqslant s\leqslant\alpha} X(s) > b\right) = \Pr(u < \theta_X(1) < \alpha,\ S_X(1) > b),$$

which leads to (4).

Since $(-X(s),\ s\geqslant 0)$ is a standard Brownian motion and $M_{-X}(\alpha) = -M_X(1-\alpha)$ almost surely, we use $-X(s)$ instead of $X(s)$ and obtain that, for $b<0$,

$$\Pr(M_X(\alpha) < b,\ L_X(\alpha) > u) = \Pr(u < \theta_X(1) \leqslant (1-\alpha),\ S_X(1) > |b|),$$

which leads to (5).

To see that $(t - K_X(\alpha),\ M_X(\alpha) - X(1))$ has the same distribution as $(L_X(\alpha),\ M_X(\alpha))$, again set $\tilde{X}(s) = X(1-s) - X(1)$. Clearly $(\tilde{X}(s),\ 0\leqslant s\leqslant t)$ is a standard Brownian motion and we can easily see that $M_{\tilde{X}}(\alpha) = M_X(\alpha) - X(1)$, $M_{\tilde{X}}(\alpha) - \tilde{X}(1) = M_X(\alpha)$ and $K_{\tilde{X}}(\alpha) = 1 - L_X(\alpha)$. □

Remark 1. The distribution of $(\theta_X(1),\ S_X(1))$ is well known (see, for example, Karatzas and Shreve 1988, p. 102). From this and Theorem 2, we can deduce the density of $(L_X(\alpha),\ M_X(\alpha))$. This is given by

$$\Pr(M_X(\alpha)\in \mathrm{d}b,\ L_X(\alpha)\in\mathrm{d}u) = \frac{|b|}{\pi\sqrt{u^3(1-u)}}\exp\left(-\frac{b^2}{2u}\right)$$
$$\cdot\ [\mathbf{1}(0<u<\alpha,\ b>0) + \mathbf{1}(0<u<1-\alpha,\ b<0)]\mathrm{d}b\,\mathrm{d}u. \tag{10}$$

Remark 2. Theorem 1 also leads to an alternative expression for the distribution of $M_X(\alpha)$; that is,

$$\Pr(M_X(\alpha) \in \mathrm{d}b) = \Pr(S_X(1) \in \mathrm{d}b,\ 0 < \theta_X(1) < \alpha),$$

for $b > 0$, and

$$\Pr(M_X(\alpha) \in \mathrm{d}b) = \Pr(S_X(1) \in \mathrm{d}|b|,\ 0 < \theta_X(1) < 1 - \alpha),$$

for $b < 0$.

From Theorem 1, we can immediately obtain the following corollary:

Corollary 1. *For* $u > 0$,

$$\Pr(L_X(\alpha) > u) = \Pr(u < \theta_X(1) \leqslant \alpha) + \Pr(u < \theta_X(1) \leqslant 1 - \alpha) \tag{11}$$

and

$$\Pr(L_X(\alpha) \in \mathrm{d}u) = \frac{\mathbf{1}(u \leqslant \alpha) + \mathbf{1}(u \leqslant 1 - \alpha)}{\pi\sqrt{u(1-u)}}\,\mathrm{d}u. \tag{12}$$

Furthermore, $K_X(\alpha)$ *has the same distribution as* $1 - L_X(\alpha)$.

3. The joint law of $(L_X(\alpha), K_X(\alpha), M_X(\alpha), X(1))$

From now on we will denote the density of τ_b by $k(\cdot, \cdot)$; that is, for $v > 0$,

$$\Pr(\tau_b \in \mathrm{d}v) = k(v, b)\mathrm{d}v = \frac{|b|}{\sqrt{2\pi v^3}} \exp\left(-\frac{b^2}{2v}\right)\mathrm{d}v. \tag{13}$$

We will also denote the joint density of $(M_X(v/t, t), X(t))$ by $g(\cdot, \cdot, \cdot, \cdot)$; that is, for $0 < v < t$,

$$\Pr\left(M_X\left(\frac{v}{t}, t\right) \in \mathrm{d}b,\ X(t) \in \mathrm{d}a\right) = g(b, a, v, t)\mathrm{d}b\,\mathrm{d}a.$$

From Proposition 1 this is also the density of

$$(N_X(\alpha, t),\ X(\alpha t) + X'((1 - \alpha t))),$$

where $N_X(\alpha, t)$ is defined by (3).

We can calculate $g(\cdot, \cdot, \cdot, \cdot)$ by using Proposition 1. Note that

$$\inf_{0 \leqslant s \leqslant (1-\alpha)t} X'(s) = -\sup_{0 \leqslant s \leqslant (1-\alpha)t} (-X'(s))$$

and that the density of $(S_X(t), X(t))$ is given by

$$\Pr(S_X(t) \in \mathrm{d}b,\ X(t) \in \mathrm{d}a) = \frac{2(2b - a)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2b-a)^2}{2t}\right)\mathbf{1}(b \geqslant 0,\ b \geqslant a)\mathrm{d}a\,\mathrm{d}b \tag{14}$$

(see Karatzas and Shreve, 1988, p. 95). We observe that since (14) is bounded, $g(\cdot, \cdot, \cdot, \cdot)$ is a bounded density. We first need to calculate $g(0, 0, v, t)$. This is the same as the value of the density of $(M_X(v/t, t), M_X(v/t, t) - X(t))$ at (0, 0). From (14) we see that

$$\Pr(S_X(t) \in \mathrm{d}y, S_X(t) - X(t) \in \mathrm{d}x) = \frac{2(y+x)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(y+x)^2}{2t}\right)\mathbf{1}(y \geqslant 0, x \geqslant 0)\mathrm{d}y\,\mathrm{d}x, \quad (15)$$

and it is a simple exercise to verify that

$$g(0, 0, v, t) = \int_0^\infty \int_0^\infty \frac{2(y+x)}{\sqrt{2\pi v^3}} \exp\left(-\frac{(y+x)^2}{2v}\right) \frac{2(y+x)}{\sqrt{2\pi(t-v)^3}} \exp\left(-\frac{(y+x)^2}{2(t-v)}\right)\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{4\sqrt{v(t-v)}}{\pi t^2}. \quad (16)$$

We will also use the following lemma

Lemma 1. *Let* $(X(s), s \geqslant 0)$ *be a standard Brownian motion,* $\tau_x = \inf\{s > 0 : X(s) = x\}$ *and* $\underline{\tau}_y = \sup\{s \leqslant t : X(s) = y\}$. *Then, for* $0 < x < z$ *and* $w < y < z$,

$$\Pr(\tau_x \in \mathrm{d}u, \underline{\tau}_y \in \mathrm{d}v, S_X(t) \in \mathrm{d}z, X(t) \in \mathrm{d}w)$$

$$= k(u, x)k(t-v, y-w)\Pr(S_X(v-u) \in \mathrm{d}(z-y), X(v-u) \in \mathrm{d}(x-y)). \quad (17)$$

Proof. Using the strong Markov property as in the previous section, we see that the right-hand side of (17) is equal to

$$\Pr(\tau_x \in \mathrm{d}u)\Pr(\underline{\tau}_{y-x} \in \mathrm{d}(v-u), S_X(t-u) \in \mathrm{d}(z-x), X(t-u) \in \mathrm{d}(w-x))$$

and, replacing $X(s)$ by the standard Brownian motion $X(t-u-s) - X(t-u)$, this is equal to

$$\Pr(\tau_x \in \mathrm{d}u)\Pr(\tau_{y-w} \in \mathrm{d}(1-v), S_X(t-u) \in \mathrm{d}(z-w), X(t-u) \in \mathrm{d}(x-w)),$$

which leads to (17). □

The following extension to Proposition 1 can be derived as a direct consequence of the results of Chaumont (1999) (see Theorem 7 and the remark after Theorem 4 in his paper):

Proposition 2. *Let* $(X(s), s \geqslant 0)$ *be a continuous process with exchangeable increments and* $X'(s) = X(\alpha + s) - X(\alpha)$. *Then,*

$$(L_X(\alpha), K_X(\alpha), M_X(\alpha), X(1)) \overset{\text{(law)}}{=} (T_X(\alpha), U_X(\alpha), N_X(\alpha), X(\alpha) + X'(1-\alpha)), \quad (18)$$

where

$$T_X(\alpha) = \inf\{s \geqslant 0 : X(s) = N_X(\alpha)\}\mathbf{1}(N_X(\alpha) \geqslant 0) + \inf\{s \geqslant 0 : X'(s) = N_X(\alpha)\}\mathbf{1}(N_X(\alpha) \leqslant 0)$$

and

$$U_X(\alpha) = (1 - \alpha + \sup\{s \leqslant \alpha : X(s) = N_X(\alpha) - X'(1-\alpha)\})\mathbf{1}(N_X(\alpha) \geqslant X(\alpha) + X'(1-\alpha))$$

$$+ (\alpha + \sup\{s \leqslant 1 - \alpha : X'(s) = N_X(\alpha) - X(\alpha)\})\mathbf{1}(N_X(\alpha) \leqslant X(\alpha) + X'(1-\alpha)).$$

Note that the expression for $U_X(\alpha)$ is a slight modification of the one in Chaumont's paper that better serves our purpose. We now deduce the law of $(L_X(\alpha), K_X(\alpha), M_X(\alpha), X(1))$.

Theorem 2. *For the standard Brownian motion* $(X(s), s \geqslant 0)$,

$$\Pr(L_X(\alpha) \in \mathrm{d}u, K_X(\alpha) \in \mathrm{d}v, M_X(\alpha) \in \mathrm{d}b, X(1) \in \mathrm{d}a)$$

$$= \frac{2|b||b-a|\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}b\,\mathrm{d}a}{\pi^2(v-u)^2\sqrt{u^3(1-v)^3}} \exp\left(-\frac{b^2}{2u} - \frac{(b-a)^2}{2(1-v)}\right)$$

$$\times \begin{cases} \sqrt{(v-u-(1-\alpha))(1-\alpha)}\mathbf{1}(u > 0, u + (1-\alpha) < v < 1), & b > 0, b > a, \\ \sqrt{(\alpha - u)(v - \alpha)}\mathbf{1}(0 < u < \alpha < v < 1), & b > 0, b < a, \\ \sqrt{(v - u - \alpha)\alpha}\mathbf{1}(u > 0, u + \alpha < v < 1), & b < 0, b > a, \\ \sqrt{(1 - \alpha - u)(v - (1-\alpha))}\mathbf{1}(0 < u < 1 - \alpha < v < 1), & b < 0, b < a. \end{cases} \tag{19}$$

Proof. We start with the case $b > 0, b > a$; we use Proposition 2 and Lemma 1 with $z = b - \inf_{0 \leqslant s \leqslant (1-\alpha)t} X'(s)$, $w = a - X'(1-\alpha)$, $x = b$ and $y = b - X'(1-\alpha)$. This leads to

$$k(b, u)k(b-a, 1-v)g(0, 0, v-u-(1-\alpha), v-u)$$
$$\cdot\, \mathbf{1}(u > 0, u + (1-\alpha)t < v < t)\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}b\,\mathrm{d}a. \tag{20}$$

Substituting (13) and (15) into (20), we obtain the first part of the right-hand side of (19). For the case $b > 0, b < a$, note that we can rewrite $U_X(\alpha)$ in Proposition 2 as

$$U_X(\alpha) = 1 - \inf\{s \geqslant 0 : X''(s) = \sup_{0 \leqslant s \leqslant \alpha} X(s) + \inf_{0 \leqslant s \leqslant 1-\alpha} X''(s) - X(\alpha)\},$$

where $X''(s) = X'(1-\alpha-s) - X'(1-\alpha)$. The left-hand side of (19) is then the density of

$$\left(\begin{array}{c} \inf\{s \geqslant 0 : X(s) = b\}, \inf\{s \geqslant 0 : X''(s) = b - a\}, \\ \sup_{0 \leqslant s \leqslant \alpha} X(s) + \inf_{0 \leqslant s \leqslant 1-\alpha} X''(s) - X''(1-\alpha), X(\alpha) - X''(1-\alpha) \end{array}\right)$$

at $(u, 1-v, b, a)$. This in turn is equal to $k(b, u)k(|b-a|, 1-v)$ multiplied by the density of

$$\left(\sup_{0 \leqslant s \leqslant (\alpha - u)} X(s) + \inf_{0 \leqslant s \leqslant (v-\alpha)} X''(s) - X''(v-\alpha), X(\alpha-u) - X''(v-\alpha)\right),$$

which leads to the second part of the right-hand side of (19).

Considering the process $(-X(s), 0 \leqslant s \leqslant 1)$ and observing that $M_{-X}(\alpha) = -M_X(1-\alpha)$, $L_{-X}(\alpha) = L_X(1-\alpha)$ and $K_{-X}(\alpha) = K_X(1-\alpha)$ yields the rest of (19). $\square$

Remark 3. One could derive Theorem 1 from Theorem 2 by integrating out two variables. However, it is difficult to obtain the result without knowing it in advance.

Acknowledgement

The author wishes to thank an anonymous referee for many useful comments.

References

Akahori, J. (1995) Some formulae for a new type of path-dependent option. *Ann. Appl. Probab.*, **5**, 383–388.

Chaumont, L. (1999) A path transformation and its applications to fluctuation theory. *J. London Math. Soc. (2)*, **59**, 729–741.

Dassios, A. (1995) The distribution of the quantiles of a Brownian motion with drift and the pricing of related path-dependent options. *Ann. Appl. Probab.*, **5**, 389–398.

Dassios, A. (1996) Sample quantiles of stochastic processes with stationary and independent increments, *Ann. Appl. Probab.*, **6**, 1041–1043.

Embrechts, P., Rogers, L.C.G. and Yor, M. (1995) A proof of Dassios' representation of the α-quantile of Brownian motion with drift. *Ann. Appl. Probab.*, **5**, 757–767.

Karatzas, I. and Shreve, S.E. (1988) *Brownian Motion and Stochastic Calculus*. New York: Springer-Verlag.

Miura, R. (1992) A note on a look-back option based on order statistics. *Hitosubashi J. Commerce Management*, **27**, 15–28.

Wendel, J.G. (1960) Order statistics of partial sums. *Ann. Math. Statist.*, **31**, 1034–1044.

Received June 2003 and revised April 2004