Saddlepoint approximations to the trimmed mean

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Saddlepoint approximations for the trimmed mean and the studentized trimmed mean are established. Some numerical evidence on the quality of our saddlepoint approximations is also included.

Keywords: saddlepoint approximations; studentizing; trimmed mean

1. Introduction

The centre of a distribution is often estimated by the sample mean or the sample median. However, it is well known that the sample mean is sensitive to outliers and thus not robust. On the other hand, the sample median is robust against outliers but is not very efficient if the underlying distribution is, for instance, normal. An estimator showing intermediate behaviour, and which includes both the sample mean and sample median, is the trimmed (sample) mean. Compared with robust M-estimates of maximum likelihood type, the trimmed mean not only has the same asymptotic variance but also is easy to compute.

The asymptotic normality of the trimmed mean is derived by Stigler (1973) under minimal conditions, while Bjerve (1974) and Helmers (1982) derive Edgeworth expansions under general conditions. Easton and Ronchetti (1986) obtained approximations to the density of trimmed means. The Edgeworth expansion for the studentized trimmed mean was established by Hall and Padmanabhan (1992), while a simple explicit form of the Edgeworth expansion was obtained in Gribkova and Helmers (2002). It is well known that Edgeworth expansions generally provide accurate approximation near the centre of the distribution, but the relative error can become unacceptably large in the far tail of the distribution. On the other hand, saddlepoint approximation will offer an approximation whose relative error is controlled both near the centre and in the far tail of the distribution. Therefore, in this paper, we derive saddlepoint approximations to the densities and tail probabilities of the trimmed mean and its studentized version. To do this, we shall exploit the special structure of the trimmed mean and employ a simple conditioning argument in the same way as Bjerve (1974) does in his derivation of an Edgeworth expansion for the trimmed mean. Conditionally given the values of the two extreme order statistics appearing

in the trimmed mean, the conditional distribution of a trimmed mean reduces to a sum of independent and identically distributed (i.i.d.) random variables, to which we can apply a saddlepoint approximation. Finally, we integrate out these two extreme order statistics by means of a Laplace approximation. Tail probabilities of Lugannani–Rice type are derived by another Laplace approximation of Temme type (see Temme 1982), as was done in Daniels and Young (1991) and Jing and Robinson (1994). For a rigorous account of saddlepoint approximations, the reader is referred to a recent monograph by Jensen (1995). A general approach dealing with saddlepoint approximations for L-estimators was presented by Easton and Ronchetti (1986).

The layout of this paper is as follows. Some basic notation will be introduced in Section 2. In Section 3, we shall derive saddlepoint approximations to the density and tail-area probabilities for the trimmed mean. In Section 4, we shall carry out the same analysis as in Section 3 for the studentized trimmed mean. Numerical results are given in Section 5. Finally, some of the technical details are given in the Appendix.

2. Some preliminaries

Let $X_1, \ldots, X_n$ be a random sample from a population with distribution function $F(\cdot)$ and density $f(\cdot)$, respectively. Let $X_{1:n} \leq \ldots \leq X_{n:n}$ be the corresponding order statistics. Define the trimmed mean by

$$\overline{X}_{\alpha\beta} = \frac{1}{m} \sum_{i=r}^{s} X_{i:n},$$

where

$$r = [n\alpha] + 1, \qquad s = n - [n\beta], \qquad m = n - [n\alpha] - [n\beta],$$

and $0 \leq \alpha < \frac{1}{2}$, $0 \leq \beta < \frac{1}{2}$; $[x]$ is the largest integer less than or equal to x. That is, we discard the smallest $[n\alpha]$ and the largest $[n\beta]$ observations and take the average of the rest of data in the middle. (In particular, if we suspect that the underlying distribution is symmetric, we can take $\alpha = \beta$.) For any $0 \leq p \leq 1$, we define

$$\xi_p = F^{-1}(p) = \inf \{x: F(x) \geq p\}.$$

It is well known that the asymptotic mean and variance of $\overline{X}_{\alpha\beta}$ are respectively given by (see Stigler 1973)

$$\mu = \frac{1}{1 - \alpha - \beta} \int_{\xi_\alpha}^{\xi_{1-\beta}} x \, dF(x),$$

$$\tau_{\alpha\beta}^2 = \frac{1}{(1 - \beta - \alpha)^2} ((1 - \beta - \alpha)\sigma^2 + \beta(1 - \beta)(\xi_{1-\beta} - \mu)^2$$

$$- 2\alpha\beta(\xi_\alpha - \mu)(\xi_{1-\beta} - \mu) + \alpha(1 - \alpha)(\xi_\alpha - \mu)^2),$$

where

$$\sigma^2 = \frac{1}{(1-\beta-\alpha)} \int_{\xi_\alpha}^{\xi_{1-\beta}} x^2 \, dF(x) - \mu^2.$$

We shall need the joint distribution of two order statistics. Define $q_{r,s:n}(x, y)$ to be the joint density function of order statistics $(X_{r:n}, X_{s:n})$. From David (1981),

$$q_{r,s:n}(x, y) = D_{n\alpha\beta}[F(x)]^{r-1}[F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s}f(x)f(y)I\{x < y\},$$

where $D_{n\alpha\beta} = n!/\{(r-1)!(s-r-1)!(n-s)!\}$ and $I\{\cdot\}$ is the indicator function.

Finally, for fixed values x and y with $x < y$, let $F_{x,y}(t)$ be the truncation of F to the left of x and to the right of y. That is,

$$F_{x,y}(t) = \begin{cases} 0, & t \leq x, \\ \dfrac{F(t) - F(x)}{F(y) - F(x)}, & x \leq t \leq y, \\ 1, & t \geq y. \end{cases}$$

Also, let $Y_1, \ldots, Y_m$ be a random sample from a distribution $F_{x,y}(t)$. Let $Y_{1:m} \leq \ldots \leq Y_{m:m}$ be the order statistics of $Y_1, \ldots, Y_m$. Write $\bar{Y} = m^{-1}\sum_{i=1}^{m} Y_i$, and denote its density and distribution functions by $f_{\bar{Y}}(\cdot)$ and $F_{\bar{Y}}(\cdot)$, respectively.

3. Saddlepoint approximation to the trimmed mean

In this section, we shall derive the saddlepoint approximation to the distribution and density function of $\bar{X}_{\alpha\beta}$ defined by

$$G(t) = P(\bar{X}_{\alpha\beta} \leq t), \qquad g(t) = G'(t).$$

For any t, denote $\Omega = \{(x, y) \in \mathbb{R}^2 : x < y\}$ and $\Omega(t) = \{(x, y) \in \mathbb{R}^2 : x < t < y\}$. First, we note that

$$G(t) = P(\bar{X}_{\alpha\beta} \leq t)$$

$$= \iint_\Omega P\left(m^{-1} \sum_{i=r}^{s} X_{i:n} \leq t \, \middle| \, X_{r-1:n} = x, X_{s+1:n} = y \right) q_{r-1,s+1:n}(x, y) dx \, dy$$

$$= \iint_\Omega P\left(m^{-1} \sum_{i=1}^{m} Y_{i:m} \leq t \right) q_{r-1,s+1:n}(x, y) dx \, dy$$

$$= \iint_\Omega P(\bar{Y} \leq t) q_{r-1,s+1:n}(x, y) dx \, dy$$

$$= \iint_\Omega F_{\bar{Y}}(t) q_{r-1,s+1:n}(x, y) dx \, dy,$$

and

$$g(t) = \iint_{\Omega(t)} f_{\overline{Y}}(t) q_{r-1,s+1:n}(x, y) \mathrm{d}x \, \mathrm{d}y. \tag{3.1}$$

The first step in obtaining a saddlepoint approximation to $g(t)$ is to replace $f_{\overline{Y}}(t)$ in (3.1) by its saddlepoint approximation. To do this, define the cumulant generating function of Y_1 by

$$K_{Y_1}(\lambda) = \log \mathrm{E} \exp\{\lambda Y_1\} = \log\left(\frac{\int_x^y \mathrm{e}^{\lambda z} \, \mathrm{d}F(z)}{F(y) - F(x)}\right).$$

It follows that

$$K'_{Y_1}(\lambda) = \frac{\mathrm{d}K_{Y_1}(\lambda)}{\mathrm{d}\lambda} = \frac{\int_x^y z \, \mathrm{e}^{\lambda z} \, \mathrm{d}F(z)}{\int_x^y \mathrm{e}^{\lambda z} \, \mathrm{d}F(z)},$$

$$K''_{Y_1}(\lambda) = \frac{\mathrm{d}^2 K_{Y_1}(\lambda)}{\mathrm{d}\lambda^2} = \frac{\int_x^y z^2 \, \mathrm{e}^{\lambda z} \, \mathrm{d}F(z)}{\int_x^y \mathrm{e}^{\lambda z} \, \mathrm{d}F(z)} - (K'_{Y_1}(\lambda))^2.$$

Therefore, the saddlepoint approximation to $f_{\overline{Y}}(t)$ is (see Daniels 1954, for instance)

$$f_{\overline{Y}}(t) = \sqrt{\frac{m}{2\pi K''_{Y_1}(\tilde{\lambda})}} \exp\{-m[\tilde{\lambda}t - K_{Y_1}(\tilde{\lambda})]\}\{1 + m^{-1} r_m(x, y, t)\}, \tag{3.2}$$

where $\tilde{\lambda} = \tilde{\lambda}(t)$ satisfies the saddlepoint equation

$$K'_{Y_1}(\tilde{\lambda}(t)) = t, \tag{3.3}$$

and $r_m(x, y, t)$ is an error term.

3.1. Saddlepoint approximation to the density of the trimmed mean

We shall now derive a saddlepoint approximation to the density of the trimmed mean. Substituting (3.2) into (3.1), we obtain

$$g(t) = \iint_{\Omega(t)} \sqrt{\frac{m}{2\pi K''_{Y_1}(\tilde{\lambda})}} \exp\{-m[\tilde{\lambda}t - K_{Y_1}(\tilde{\lambda})]\} q_{r-1,s-1:n}(x, y)$$

$$\times \{1 + m^{-1} r_m(x, y, t)\} dx\, dy$$

$$= \iint_{\Omega(t)} \sqrt{\frac{m}{2\pi K''_{Y_1}(\tilde{\lambda})}} f(x) f(y) \exp[-m\Lambda(x, y, t)]$$

$$\times \{1 + m^{-1} r_m(x, y, t)\} dx\, dy, \tag{3.4}$$

where $\tilde{\lambda}$ is the solution to $K'_{Y_1}(\tilde{\lambda}) = t$ and

$$\Lambda(x, y, t) = \Lambda_1(x, y, t) + \Lambda_2(x, y),$$

with

$$\Lambda_1(x, y, t) = \tilde{\lambda}t - K_{Y_1}(\tilde{\lambda}),$$

$$\Lambda_2(x, y) = -m^{-1} \log(C_{n\alpha\beta}[F(x)]^{r-2}[F(y) - F(x)]^m[1 - F(y)]^{n-s-1}),$$

and $C_{n\alpha\beta} = n!/\{(r-2)!(s-r+1)!(n-s-1)!\}$. Define

$$\Delta(x, y, t) = (f(x)f(y))^{-1} \begin{pmatrix} \dfrac{\partial^2 \Lambda(x, y, t)}{\partial x^2} & \dfrac{\partial^2 \Lambda(x, y, t)}{\partial x \partial y} \\[2mm] \dfrac{\partial^2 \Lambda(x, y, t)}{\partial y \partial x} & \dfrac{\partial^2 \Lambda(x, y, t)}{\partial x^2} \end{pmatrix}.$$

For each t, let $x_0 = x_0(t)$, $y_0 = y_0(t)$, $\tilde{\lambda}_0 = \tilde{\lambda}_0(t)$ be the solution to

$$\frac{\partial \Lambda(x_0, y_0, t)}{\partial x} = 0,$$

$$\frac{\partial \Lambda(x_0, y_0, t)}{\partial y} = 0, \tag{3.5}$$

$$K'_{Y_1}(\tilde{\lambda}_0) = t.$$

If the density function is non-zero in the support of X, then (3.5) can be reduced to

$$\frac{m \exp\{\tilde{\lambda}_0 x_0\}}{\displaystyle\int_{x_0}^{y_0} \exp\{\tilde{\lambda}_0 z\} dF(z)} = \frac{r-2}{F(x_0)},$$

$$\frac{m \exp\{\tilde{\lambda}_0 y_0\}}{\displaystyle\int_{x_0}^{y_0} \exp\{\tilde{\lambda}_0 z\} dF(z)} = \frac{n-s-1}{1 - F(y_0)}, \tag{3.6}$$

$$K'_{Y_1}(\tilde{\lambda}_0) = t.$$

That is, $(x_0(t), y_0(t))$ is a stationary point of $\Lambda(x, y, t)$ for fixed t. For simplicity, we write $\Lambda_0(t) = \Lambda(x_0, y_0, t)$.

Proposition 3.1. *Let t belong to the support of X. Then, for any n satisfying $[n\alpha] \geqslant 2$, $[n\beta] \geqslant 2$ and $n - [n\alpha] - [n\beta] \geqslant 1$, $\Lambda(x, y, t)$ attains its global minimum at some finite point (x_0, y_0) such that not only both x_0 and y_0 satisfy (3.5), but also both $F(x_0)$ and $F(y_0)$ are unique.*

Remark 3.1. Proposition 3.1 guarantees that the saddlepoint equation (3.5) always has a solution under condition (i) of Theorem 3.1 below; and the solution is unique in the sense of distribution functions. In fact, (x_0, y_0) is unique except in one case where $\tilde{\lambda}_0 = 0$, which causes no trouble because we can regard x_0 and y_0 as $F^{-1}(F(x_0))$ and $F^{-1}(F(y_0))$. The conditions $[n\alpha] \geqslant 2$, $[n\beta] \geqslant 2$ and $n - [n\alpha] - [n\beta] \geqslant 1$ are imposed to guarantee that the exponents in $q_{r-1,s+1:n}(x, y)$ are greater than 0.

The following theorem gives the saddlepoint approximation to the density of the trimmed mean.

Theorem 3.1. *Let t belong to the support of X. Suppose that:*

  (i) *$f(x) = F'(x)$ and $f''(x)$ exists;*
 (ii) *for any n satisfying $[n\alpha] \geqslant 2$, $[n\beta] \geqslant 2$ and $n - [n\alpha] - [n\beta] \geqslant 1$, $\Delta(x_0, y_0, t)$ is positive definite;*
(iii) *$|Ee^{i\eta X}| \in L^v(\mathrm{R})$ for some $v > 0$.*

Then we have

$$g(t) = g_{\mathrm{sp}}(t)(1 + m^{-1} R_n(t)), \tag{3.7}$$

where

$$g_{\mathrm{sp}}(t) = \sqrt{\frac{2\pi}{m}} \frac{\exp(-m\Lambda(x_0(t), y_0(t), t))}{\sqrt{K''_{Y_1}(\tilde{\lambda}_0(t))|_{x=x_0(t), y=y_0(t)}|\Delta_0(t)|}}, \tag{3.8}$$

the error term $R_n(t)$ in (3.7) is bounded when t is in some compact set and $x_0(t)$, $y_0(t)$ and $\tilde{\lambda}_0(t)$ are solutions to (3.5).

Remark 3.2. Condition (i) is a natural smoothness condition, which we need to validate Laplace approximation. Also note that (3.8) involves $f'(x)$. Since $\Lambda(x, y, t)$ attains its minimum at (x_0, y_0), $\Delta(x_0, y_0, t)$ is non-negative definite. The purpose of condition (ii) is to simplify the proof. Conditon (iii) ensures smoothness so that we may apply the Fourier inversion theorem.

The proof of Proposition 3.1 and Theorem 3.1 is postponed to the Appendix.

3.2. Saddlepoint approximation to the tail probability of trimmed mean

One way to obtain an approximation to the tail probability $1 - G(t) = P(\overline{X}_{\alpha\beta} \geq t)$ is to integrate the saddlepoint approximation $g_{sp}(t)$ numerically. Since $\int_{-\infty}^{\infty} g_{sp}(t)\mathrm{d}t$ may not be one in general, renormalization will usually improve the accuracy of the resulting saddlepoint approximation. The resulting approximation to $1 - G(t)$ will be denoted by $1 - G_{ss}(t)$. However, it would be more convenient to have an explicit approximation formula for the tail probability. Theorem 3.2 below will give a saddlepoint approximation to the tail probability $1 - G(t)$ of the trimmed mean. For ease of notation, let

$$
\begin{aligned}
a(t) &= (2\pi/m)^{1/2}(K''_{Y_1}(\tilde{\lambda}_0(t))\Big|_{x=x_0(t), y=y_0(t)} \cdot |\Delta_0(t)|)^{-1/2}, \\
h(t) &= \Lambda(x_0(t),\ y_0(t),\ t).
\end{aligned}
$$

Then (3.8) can be rewritten as $g_{sp}(t) = a(t)\exp\{-mh(t)\}$. Let $\hat{t}$ be the solution to $h'(\hat{t}) = 0$. Then we define

$$
w = \sqrt{2(h(t) - h(\hat{t}))}\mathrm{sgn}(t - \hat{t}), \tag{3.9}
$$

$$
\psi(w) = (2\pi/m)^{1/2}a(t(w))\exp\{-mh(\hat{t})\}\left|\frac{\mathrm{d}t}{dw}\right|. \tag{3.10}
$$

Then we have:

Theorem 3.2. *Under the conditions of Theorem 3.1, we have*

$$
P(\overline{X}_{\alpha\beta} \geq t) = 1 - \Phi(w\sqrt{m}) - \frac{\phi(w\sqrt{m})}{\sqrt{m}}\left(\frac{\psi(0) - \psi(w)}{w\psi(0)} + O(m^{-1})\right),
$$

where w and $\psi(w)$ are given in (3.9) *and* (3.10).

The proof of the theorem is similar to but simpler than that of Theorem 4.2 below, hence omitted here.

4. Saddlepoint approximation to studentized trimmed mean

4.1. Introduction

In Section 3 we derived saddlepoint approximations to the density and tail probabilities of the trimmed mean. In this section, we shall carry out the same derivations for the studentized trimmed mean. This will have greater practical relevance if we are interested in constructing confidence intervals or hypothesis testing concerning the centre of the distribution.

To studentize the trimmed mean, we employ the plug-in estimate of the variance, which is given by

$$\hat{\tau}_{\alpha\beta}^2 = \frac{1}{(1-\beta-\alpha)^2}((1-\beta-\alpha)\hat{\sigma}_{\alpha\beta}^2 + \beta(1-\beta)(\hat{\xi}_{1-\beta} - \overline{X}_{\alpha\beta})^2$$

$$- 2\alpha\beta(\hat{\xi}_\alpha - \overline{X}_{\alpha\beta})(\hat{\xi}_{1-\beta} - \overline{X}_{\alpha\beta}) + \alpha(1-\alpha)(\hat{\xi}_\alpha - \overline{X}_{\alpha\beta})^2),$$

where $\hat{\xi}_p = \inf\{x: \hat{F}(x) \geqslant p\}$ for any $0 < p < 1$ and $\hat{F}(x)$ is the empirical distribution, and

$$\hat{\sigma}_{\alpha\beta}^2 = \frac{1}{(1-\beta-\alpha)} \int_{\hat{\xi}_\alpha}^{\hat{\xi}_{1-\beta}} x^2 \, d\hat{F}(x) - \overline{X}_{\alpha\beta}^2,$$

(see Hall and Padmanabhan 1992). Because of its complicated form, we shall restrict our attention from now on to the special case where we assume that:

(C1) $f(x)$ is symmetric around the origin, i.e., $f(x) = f(-x)$;
(C2) the trimming proportions are the same, i.e., $\alpha = \beta$.

Clearly, (C1) and (C2) imply that $\mu = 0$. The case for the non-zero mean can be dealt with by a simple mean shift.

Under assumptions (C1) and (C2), $\hat{\tau}_{\alpha\beta}^2$ above reduces to

$$\hat{\tau}_{\alpha\alpha}^2 = \frac{1}{n(1-2\alpha)^2} \left(\sum_{i=r+1}^{s-1} (X_{i:n} - \overline{X}_{\alpha\alpha})^2 + r[(X_{r:n} - \overline{X}_{\alpha\alpha})^2 + (X_{s:n} - \overline{X}_{\alpha\alpha})^2] \right).$$

Therefore, we can define the studentized trimmed mean by

$$T = \frac{\overline{X}_{\alpha\alpha}}{\hat{\tau}_{\alpha\alpha}}.$$

The purpose of this section is to derive saddlepoint approximations to the density and tail probability for the studentized trimmed mean T, denoted by

$$\tilde{G}(t) = P(T \leqslant t), \qquad \tilde{g}(t) = \tilde{G}'(t).$$

As in Section 2, let $Y_1, \ldots, Y_{m-2}$ be a random sample from the truncated distribution $F_{x,y}(t)$. We further define $Z_i = Y_i^2$ and

$$\overline{Y} = \frac{1}{m-2} \sum_{i=1}^{m-2} Y_i, \qquad \overline{Z} = \frac{1}{m-2} \sum_{i=1}^{m-2} Y_i^2.$$

Now, for fixed x and y, define

$$s(\overline{Y}, \overline{Z}) = n^{-1/2}(1-2\alpha)^{-1} \left[(m-2)\overline{Z} - (m-2r+2)\left(\frac{m-2}{m}\overline{Y} + \frac{x+y}{m} \right)^2 \right.$$

$$\left. + r(x+y)^2 - 2(r-1)(x+y)\left(\frac{m-2}{m}\overline{Y} + \frac{x+y}{m} \right) \right]^{1/2}$$

and

$$b \equiv b(\overline{Y}, \overline{Z}) = \left(\frac{m-2}{m} \overline{Y} + \frac{x+y}{m} \right),$$

$$a \equiv a(\overline{Y}, \overline{Z}) = \frac{b(\overline{Y}, \overline{Z})}{s(\overline{Y}, \overline{Z})}.$$

(4.1)

Then, conditional on $X_{r:n} = x$ and $X_{s:n} = y$, we can show, after some simple algebra, that

$$\overline{X}_{\alpha\alpha} = b(\overline{Y}, \overline{Z}),$$

$$\hat{\tau}_{\alpha\alpha} = s\{\overline{Y}, \overline{Z}\}.$$

Therefore, we have

$$\tilde{G}(t) \equiv P(T \leqslant t)$$

$$= \iint_{\Omega} P(T \leqslant t | X_{r:n} = x, X_{s:n} = y) q_{r,s:n}(x, y) \mathrm{d}x \, \mathrm{d}y$$

$$= \iint_{\Omega} P(a(\overline{Y}, \overline{Z}) \leqslant t) q_{r,s:n}(x, y) \mathrm{d}x \, \mathrm{d}y$$

and

$$\tilde{g}(t) \equiv \tilde{G}'(t) = \iint_{\Omega(t)} f_{a(\overline{Y}, \overline{Z})}(t) q_{r,s:n}(x, y) \mathrm{d}x \, \mathrm{d}y,$$

(4.2)

where $a(\cdot, \cdot)$ is defined in (4.1).

Similarly to Section 3, we first obtain a saddlepoint approximation to the density of $a(\overline{Y}, \overline{Z})$ and then substitute this into the above to obtain saddlepoint approximations to $\tilde{g}(t)$ and $\tilde{G}(t)$. For this purpose, we shall need the joint cumulant generating function of $(Y_i, Z_i) = (Y_i, Y_i^2)$,

$$K(d, u) = \log E\{\exp(dY + uY^2)\} = \log \frac{\int_x^y \exp(dz + uz^2) \mathrm{d}F(z)}{F(y) - F(x)}.$$

Note that $K(d, u)$ is also a function of x and y, and its derivatives with respect to x and y are given by

$$\frac{\partial K(d, u)}{\partial x} = (1 - \exp(dx + ux^2 - K(d, u)))\frac{f(x)}{F(y) - F(x)},$$

$$\frac{\partial K(d, u)}{\partial y} = (\exp(dy + uy^2 - K(d, u)) - 1)\frac{f(y)}{F(y) - F(x)},$$

$$\frac{\partial^2 K(d, u)}{\partial^2 x} = -\frac{(d + 2ux)f(x)}{F(y) - F(x)} + \frac{\partial K(d, u)}{\partial x} \times \left(d + 2ux + \frac{f'(x)}{f(x)} + \frac{2f(x)}{F(y) - F(x)}\right) - \left(\frac{\partial K(d, u)}{\partial x}\right)^2,$$

$$\frac{\partial^2 K(d, u)}{\partial^2 y} = \frac{(d + 2uy)f(y)}{F(y) - F(x)} + \frac{\partial K(d, u)}{\partial y} \times \left(d + 2uy + \frac{f'(y)}{f(y)} - \frac{2f(y)}{F(y) - F(x)}\right) - \left(\frac{\partial K(d, u)}{\partial y}\right)^2,$$

$$\frac{\partial^2 K(d, u)}{\partial x \partial y} = \frac{f(x)\partial K(d, u)/\partial y - f(y)\partial K(d, u)/\partial x}{F(y) - F(x)} - \frac{\partial K(d, u)}{\partial x}\frac{\partial K(d, u)}{\partial y}.$$

4.2. Saddlepoint approximation to the density of the studentized trimmed mean

Note that the inverse transformation of (4.1) is

$$\bar{Y} \equiv \bar{Y}(a, b) = (m - 2)^{-1}(mb - x - y),$$

$$\bar{Z} \equiv \bar{Z}(a, b) = (m - 2)^{-1}\left(\frac{n(1 - 2\alpha)^2 b^2}{a^2} + (m - 2r + 2)b^2 - r(x + y)^2 + 2(r - 1)(x + y)b\right),$$

whose Jacobian is given by

$$J \equiv J(a, b) = \begin{vmatrix} \dfrac{\partial \bar{Y}}{\partial a} & \dfrac{\partial \bar{Y}}{\partial b} \\ \dfrac{\partial \bar{Z}}{\partial a} & \dfrac{\partial \bar{Z}}{\partial b} \end{vmatrix} = \frac{2n(1 - 2\alpha)^2 mb^2}{(m - 2)^2 a^3}.$$

Define

$$\Lambda_s(a, b) = d\bar{Y} + u\bar{Z} - K(d, u),$$

$$\Delta_s(a, b) = \begin{vmatrix} \dfrac{\partial^2 K(d, u)}{\partial^2 d} & \dfrac{\partial^2 K(d, u)}{\partial d \partial u} \\ \dfrac{\partial^2 K(d, u)}{\partial u \partial d} & \dfrac{\partial^2 K(d, u)}{\partial^2 u} \end{vmatrix},$$

$$G(a, b) = |\Delta_s(a, b)|\frac{\partial^2 \Lambda_s(a, b)}{\partial b^2}.$$

Similarly to Daniels and Young (1991) and Jing and Robinson (1994), a saddlepoint approximation to the density of $a(\bar{Y}, \bar{Z})$ is given by

$$f_{a(\overline{Y}, \overline{Z})}(t)$$

$$= \sqrt{\frac{m-2}{2\pi}} J(t, b_0(t)) G^{-1/2}(t, b_0(t)) \exp[-(m-2)\Lambda_s(t, b_0(t))] \times \{1 + m^{-1}\tilde{r}_m(x, y, t)\},$$

where $\tilde{r}_m(x, y, t)$ is an error term which will not be given here explicitly. Substituting this into (4.2), we obtain

$$\tilde{g}(t) = \iint_{\Omega(t)} \sqrt{\frac{m-2}{2\pi}} J(t, b_0(t)) G^{-1/2}(t, b_0(t)) \exp[-(m-2)\Lambda_s(t, b_0(t))]$$

$$\times q_{r,s:n}(x, y)\{1 + m^{-1}\tilde{r}_m(x, y, t)\}\mathrm{d}x\,\mathrm{d}y$$

$$= \iint_{\Omega(t)} \sqrt{\frac{m-2}{2\pi}} \frac{J(t, b_0(t))}{G^{1/2}(t, b_0(t))} \exp[-(m-2)\tilde{\Lambda}(x, y, t)]$$

$$\times f(x)f(y)\{1 + m^{-1}\tilde{r}_m(x, y, t)\}\mathrm{d}x\,\mathrm{d}y, \tag{4.3}$$

where $d_0 = d_0(t)$, $u_0 = u_0(t)$ and $b_0 = b_0(t)$ are solutions to the following three equations,

$$\frac{\partial \Lambda_s(a, b_0(t))}{\partial b}\Big|_{d=d_0(t), u=u_0(t)} = 0,$$

$$\frac{\partial K(d_0(t), u_0(t))}{\partial d} = \overline{Y}(t, b_0(t)),$$

$$\frac{\partial K(d_0(t), u_0(t))}{\partial u} = \overline{Z}(t, b_0(t))$$

and, further,

$$\tilde{\Lambda}(x, y, t) = \tilde{\Lambda}_1(x, y, t) + \tilde{\Lambda}_2(x, y, t),$$

$$\tilde{\Lambda}_1(x, y, t) = d_0(t)\overline{Y}(t, b_0(t)) + u_0(t)\overline{Z}(t, b_0(t)) - K(d_0(t), u_0(t)),$$

$$\tilde{\Lambda}_2(x, y, t) = -(m-2)^{-1}\log(D_{n\alpha\beta}[F(x)]^{r-1}[F(y) - F(x)]^{m-2}[1 - F(y)]^{n-s})$$

with $D_{n\alpha\beta} = n!/\{(r-1)!(m-2)!(n-s)!\}$.

Note that $\overline{Y}(t, b_0(t))$, $\overline{Z}(t, b_0(t))$ and $K(d_0(t), u_0(t))$ are also functions of x and y. We can therefore find their partial derivatives with respect to x and y. Some simple algebra yields

$$\frac{\partial \tilde{\Lambda}(x, y, t)}{\partial x} = (m-2)^{-1}(-d_0 - 2ru_0(x+y) + 2(r-1)u_0 b_0)$$

$$+ \left(\frac{\exp(d_0 x + u_0 x^2 - K(d_0, u_0))}{F(y) - F(x)} - \frac{r-1}{(m-2)F(x)} \right) f(x),$$

$$\frac{\partial \tilde{\Lambda}(x, y, t)}{\partial y} = (m-2)^{-1}(-d_0 - 2ru_0(x+y) + 2(r-1)u_0 b_0)$$

$$- \left(\frac{\exp(d_0 y + u_0 y^2 - K(d_0, u_0))}{F(y) - F(x)} - \frac{n-s}{(m-2)(1-F(y))} \right) f(y),$$

$$\frac{\partial^2 \tilde{\Lambda}(x, y, t)}{\partial x^2} = \left(\frac{1}{(F(y)-F(x))^2} + \frac{r-1}{(m-2)F^2(x)} \right) f^2(x)$$

$$+ \left(\frac{1}{F(y)-F(x)} - \frac{r-1}{(m-2)F(x)} \right) f'(x)$$

$$- \left(\frac{2ru_0}{m-2} + \frac{\partial^2 K(d_0, u_0)}{\partial^2 x} \right),$$

$$\frac{\partial^2 \tilde{\Lambda}(x, y, t)}{\partial y^2} = \left(\frac{1}{(F(y)-F(x))^2} + \frac{n-s}{(m-2)(1-F(y))^2} \right) f^2(y)$$

$$+ \left(-\frac{1}{F(y)-F(x)} + \frac{n-s}{(m-2)(1-F(y))} \right) f'(y)$$

$$- \left(\frac{2ru_0}{m-2} + \frac{\partial^2 K(d_0, u_0)}{\partial^2 y} \right),$$

$$\frac{\partial^2 \tilde{\Lambda}(x, y, t)}{\partial x \partial y} = -\frac{f(x)f(y)}{[F(y)-F(x)]^2} - \left(\frac{2ru_0}{m-2} + \frac{\partial^2 K(d_0, u_0)}{\partial x \partial y} \right).$$

Define

$$\tilde{\Delta}(t) \equiv \tilde{\Delta}(x, y, t) = (f(x)f(y))^{-1} \begin{pmatrix} \dfrac{\partial^2 \tilde{\Lambda}(x, y, t)}{\partial x^2} & \dfrac{\partial^2 \tilde{\Lambda}(x, y, t)}{\partial x \partial y} \\ \dfrac{\partial^2 \tilde{\Lambda}(x, y, t)}{\partial y \partial x} & \dfrac{\partial^2 \tilde{\Lambda}(x, y, t)}{\partial y^2} \end{pmatrix}.$$

For each t, let $\tilde{x}_0 = \tilde{x}_0(t)$, $\tilde{y}_0 = \tilde{y}_0(t)$, $\tilde{d}_0 = \tilde{d}_0(t)$, $\tilde{u}_0 = \tilde{u}_0(t)$ and $\tilde{b}_0 = \tilde{b}_0(t)$ be the solutions to

$$\frac{\partial \Lambda_s(a, \tilde{b}_0)}{\partial b}\Big|_{d=\tilde{d}_0, u=\tilde{u}_0} = 0,$$

$$\frac{\partial K(\tilde{d}_0, \tilde{u}_0)}{\partial d} = \bar{Y}(t, \tilde{b}_0),$$

$$\frac{\partial K(\tilde{d}_0, \tilde{u}_0)}{\partial u} = \bar{Z}(t, \tilde{b}_0), \qquad (4.4)$$

$$\frac{\partial \tilde{\Lambda}(\tilde{x}_0, \tilde{y}_0, t)}{\partial x} = 0,$$

$$\frac{\partial \tilde{\Lambda}(\tilde{x}_0, \tilde{y}_0, t)}{\partial y} = 0.$$

We now present the following proposition and theorem whose proofs are given in the Appendix.

Proposition 4.1. *Let t belong to the support of X. Suppose $t \neq 0$, and let conditions* (C1)–(C2) *be satisfied. Then, for any n satisfying $[n\alpha] \geq 1$ and $n - 2[n\alpha] \geq 3$, $\Lambda_s(t, b)$ attains its minimum at some interior point $b_0(t)$ and $\tilde{\Lambda}(x, y, t)$ attains its global minimum at the finite point $(\tilde{x}_0, \tilde{y}_0)$ which satisfies* (4.4).

Remark 4.1. Proposition 4.1 guarantees that the saddlepoint equation (4.4) always has a solution under conditions (C1)–(C2). The conditions $[n\alpha] \geq 1$, $n - 2[n\alpha] \geq 3$ guarantee that $q_{r,s:n}(x, y)$ is meaningful.

Theorem 4.1. *Let t belong to the support of X. Suppose $t \neq 0$. In addition to conditions* (C1)–(C2), *we make the following assumptions:*

  (i) $f(x) = F'(x)$ *and $f''(x)$ exists.*

 (ii) *For any n satisfying $[n\alpha] \geq 1$ and $n - 2[n\alpha] \geq 3$, $(\tilde{x}_0, \tilde{y}_0)$ is unique, that is, $\tilde{\Lambda}(x, y, t) > \tilde{\Lambda}(\tilde{x}_0, \tilde{y}_0, t)$ if $(x, y) \neq (\tilde{x}_0, \tilde{y}_0)$, and $\tilde{\Delta}(\tilde{x}_0, \tilde{y}_0, t)$ is positive definite. In addition, the minimum point $b_0(t)$ is unique as (x, y) varies in A_{B,B_0} (cf. also* (4.4)).

 (iii) $|\mathrm{E}e^{i\eta_1 X + i\eta_2 X^2}|^{v_1} \in L(\mathbb{R}^2)$ *for some $v_1 > 0$.*

 (iv) *There exist $w_1 > 0$, $w_2 > 0$ such that both $|x|(F(x))^{w_1}$ and $y(1 - F(y))^{w_2}$ are bounded when $x < 0$ and $y > 0$.*

Then we have

$$\tilde{g}(t) = \tilde{g}_{\mathrm{sp}}(t)\{1 + m^{-1}\tilde{R}_n(t)\},$$

where

$$\tilde{g}_{\mathrm{sp}}(t) = \sqrt{\frac{2\pi}{m - 2}} \frac{J(t, \tilde{b}_0)}{G^{1/2}(t, \tilde{b}_0)|\tilde{\Delta}(\tilde{x}_0, \tilde{y}_0, t)|^{1/2}} \exp(-(m - 2)\tilde{\Lambda}(\tilde{x}_0, \tilde{y}_0, t)),$$

where $\tilde{x}_0(t)$, $\tilde{y}_0(t)$, $\tilde{d}_0(t)$, $\tilde{u}_0(t)$ and $\tilde{b}_0(t)$ are the solutions to equations (4.4).

Note that conditions (i)–(iii) in Theorem 4.1 are similar to those in Theorem 3.1. The first three conditions guarantee that $f_{(\overline{Y}, \overline{Z})}(y, z)$ has a uniform saddlepoint approximation as x and y vary in some compact set A_{B, B_0}. Since $(\overline{Y}, \overline{Z}) \rightarrow (a(\overline{Y}, \overline{Z}), b(\overline{Y}, \overline{Z}))$ is a one-to-one and differentiable transformation, $f_{a(\overline{Y}, \overline{Z})}(t)$ has a uniform saddlepoint approximation as x and y vary in A_{B, B_0}, that is, $\tilde{r}_m(x, y, t)$ is bounded as x and y vary. Condition (iv) implies that the random variable X will have finite moments of arbitrarily small order. It is used in the proof of Lemma A.13. We conjecture that it can be removed.

4.3. Saddlepoint approximation to the tail probability of the studentized trimmed mean

To conclude this section, we shall derive a saddlepoint approximation to the tail probability of the studentized trimmed mean by integrating the density approximation obtained in Theorem 4.1. To simplify our notation, let

$$\tilde{a}(t) = \sqrt{\frac{2\pi}{m-2}} J(t, \tilde{b}_0) G^{-1/2}(t, \tilde{b}_0) |\tilde{\Delta}(\tilde{x}_0, \tilde{y}_0, t)|^{-1/2},$$

$$\tilde{h}(t) = \tilde{\Lambda}(\tilde{x}_0, \tilde{y}_0, t).$$

Then, we can rewrite $\tilde{g}_{\mathrm{sp}}(t)$ from Theorem 4.1 as

$$\tilde{g}_{\mathrm{sp}}(t) = \tilde{a}(t) \exp\{-(m-2)\tilde{h}(t)\}. \tag{4.5}$$

From the proof of Theorem 4.1, we see that $\tilde{h}(t) = \tilde{\Lambda}(\tilde{x}_0, \tilde{y}_0, t)$ achieves its minimum at $t = t_0$. Let

$$v = \sqrt{2(\tilde{h}(t) - \tilde{h}(t_0))} \, \mathrm{sgn}(t - t_0), \tag{4.6}$$

$$\tilde{\psi}(v) = \sqrt{\frac{2\pi}{m-2}} \tilde{a}(t(v)) \exp\{-(m-2)\tilde{h}(t_0)\} \left| \frac{\mathrm{d}t}{\mathrm{d}v} \right|. \tag{4.7}$$

Then we have the following theorem whose proof is provided in the Appendix.

Theorem 4.2. *Under the conditions of Theorem 4.1, we have*

$$P(T \geqslant t) = 1 - \Phi(v\sqrt{m-2}) - \frac{\phi(v\sqrt{m-2})}{\sqrt{m-2}} \left(\frac{\tilde{\psi}(0) - \tilde{\psi}(v)}{v\tilde{\psi}(0)} + O(m^{-1}) \right), \tag{4.8}$$

where v and $\tilde{\psi}(v)$ are given in (4.6) *and* (4.7).

5. Numerical results

In this section, we present some numerical evidence on the quality of our saddlepoint approximations. For simplicity, we shall do so only for ordinary trimmed means (cf.

Theorems 3.1 and 3.2). Different distributions F and varying trimming proportions α and β are chosen in the simulations. The results are presented in Figures 1–4. In these figures, the left-hand panels give the right-tail probabilities $1 - G(x)$, $1 - G_{\mathrm{LR}}(x)$ and $1 - G_{\mathrm{ss}}(x)$. The right-hand panels display absolute relative errors; that is, we plot $|G_{\mathrm{ss}}(x) - G(x)|/(1 - G(x))$

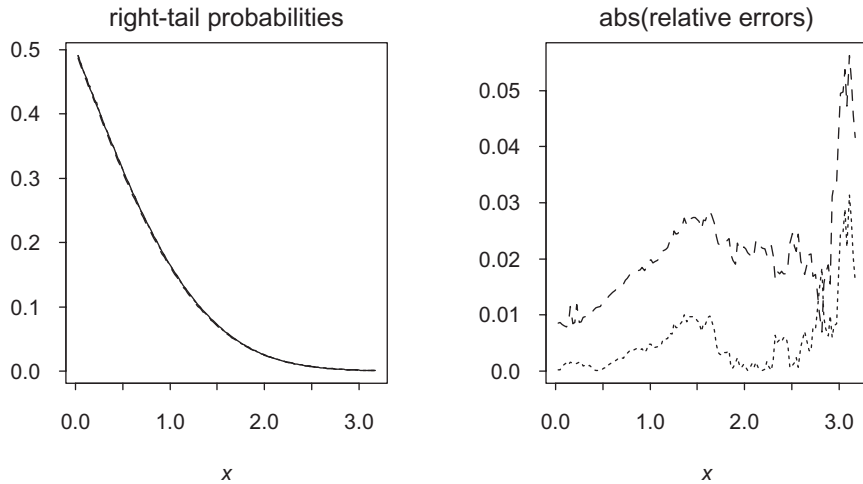**Figure 1.** Simulation for standard normal distribution, $\alpha = \beta = 0.1$, $n = 20$: (left) $1 - G(x)$ (solid, $N = 10^6$), $1 - G_{\mathrm{LR}}(x)$ (dotted), $1 - G_{\mathrm{ss}}(x)$ (dashed); (right) relative errors with respect to exact $1 - G(x)$

Figure 2. Simulation for a mixture of normal distributions $0.9\Phi(x) + 0.1\Phi(x/5))$, $\alpha = \beta = 0.25$, $n = 20$: (left) $1 - G(x)$ (solid, $N = 10^6$), $1 - G_{\mathrm{LR}}(x)$ (dotted), $1 - G_{\mathrm{ss}}(x)$ (dashed); (right) relative errors with respect to exact $1 - G(x)$

right-tail probabilities   abs(relative errors)

Figure 3. Simulation for a mixture of normal distributions $(0.7\Phi(x) + 0.3\Phi(x/5))$, $\alpha = \beta = 0.1$, $n = 20$: (left) $1 - G(x)$ (solid, $N = 10^6$), $1 - G_{LR}(x)$ (dotted), $1 - G_{ss}(x)$ (dashed); (right) relative errors with respect to exact $1 - G(x)$
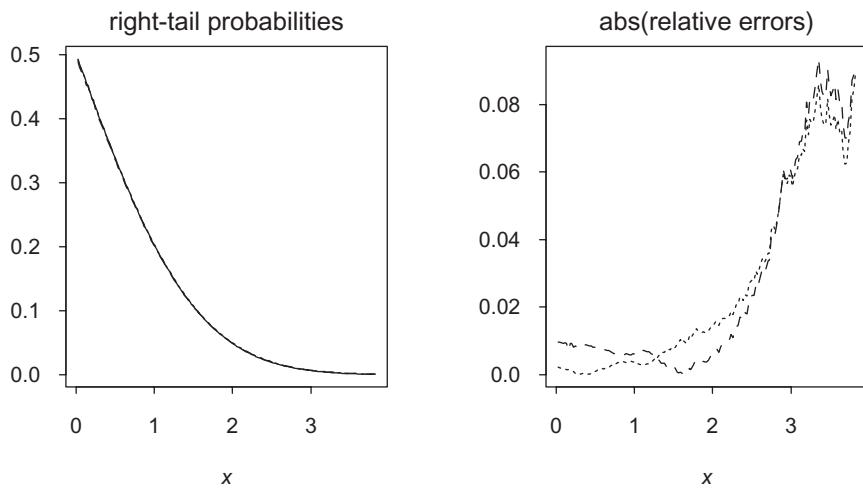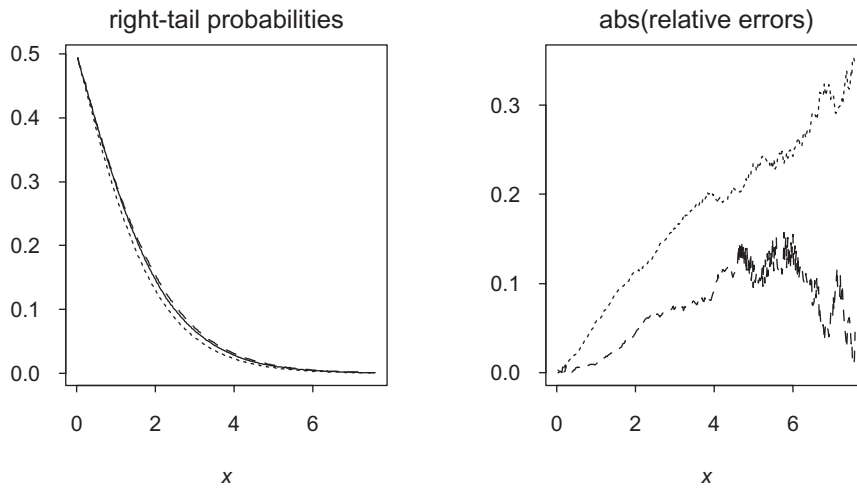
right-tail probabilities   abs(relative errors)

Figure 4. Simulation for Cauchy distribution, $\alpha = \beta = 0.25$, $n = 80$: (left) $1 - G(x)$ (solid, $N = 10^6$), $1 - G_{LR}(x)$ (dotted), $1 - G_{ss}(x)$ (dashed); (right) relative errors with respect to exact $1 - G(x)$
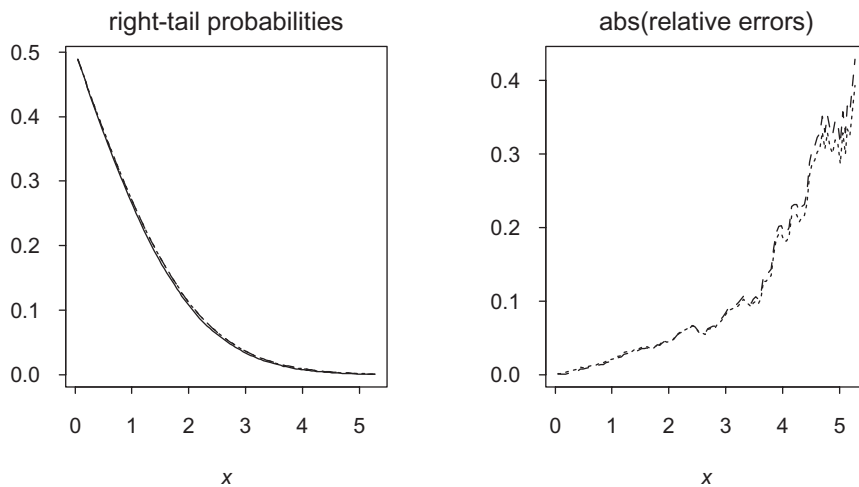
(dashed) and $|G_{LR}(x) - G(x)|/(1 - G(x))$ (dotted), where G denotes the exact distribution function computed by Monte Carlo using $N = 10^6$ samples from F, while G_{ss} is the integrated saddlepoint density (3.8), renormalized by dividing through its integral, which is computed by numerical integration; G_{LR} denotes the Lugannani–Rice type approximation

given in Theorem 3.2. We note that in our simulations, $\psi(0)$ in Theorem 3.2 is calculated approximately by $\psi(a)$ for some small value very close to zero.

Figure 1 deals with the case where F is standard normal, the trimming proportions α and β are both equal to 0.10, and the sample size $n = 20$. The results are very satisfactory. In Figure 2, we take F to be a normal mixture, namely $F(x) = 0.9\Phi(x) + 0.1\Phi(x/5)$, the trimming proportions α and β are both equal to 0.25, and the sample size is again $n = 20$. The results are again very satisfactory, though not as good as in the first example. In this example, we find that, for $|x| > 5.1206$, the determinant appearing in (3.8) becomes negative; the probability that the trimmed mean takes values outside the interval $(-5.1206, 5.1206)$ is 10^{-4} (estimated by Monte Carlo), so that the renormalization factor is in fact a little too small. In the first example (F normal) these difficulties do not arise, as the determinant in (3.8) is positive for all values of x.

Figures 3 and 4 depict two cases of interest for which we find that the resulting saddlepoint approximations behave much less well than those described in Figures 1 and 2. Figure 3 deals with the normal mixture $F(x) = 0.7\Phi(x) + 0.3\Phi(x/5)$, trimming proportions $\alpha = \beta = 0.1$ and sample size $n = 20$, while in Figure 4 we present results for the case where F is Cauchy, $\alpha = \beta = 0.25$, and $n = 80$. The reason for taking a sample size as large as 80 in the Cauchy example is that, for smaller sample sizes, the determinant appearing in (3.8) is positive only for a rather small interval of x-values; the probability that the trimmed mean takes values outside this interval is less than 10^{-6} (estimated by Monte–Carlo). One way to improve upon this would be the use of higher-order saddlepoint approximations to the trimmed mean.

Appendix

Throughout this appendix, we suppose that t is in the support of X, $x < t < y$. We will use (x, y) to denote a point or an open interval. They can be distinguished from the context.

We begin with six lemmas which will be used to prove Theorem 3.1.

Lemma A.1. *Under condition* (iii) *of Theorem 3.1, there exist some constant M and some even integer u such that*

$$\int_{-\infty}^{\infty} |F(y) - F(x)|^u |\mathrm{E}\mathrm{e}^{\mathrm{i}\eta Y_1}|^u \, \mathrm{d}\eta \leq 2\pi M.$$

Proof. Let u be the smallest even integer which is greater than or equal to v. Since $\mathrm{E}\mathrm{e}^{\mathrm{i}\eta X} \in L^v(\mathbb{R})$ and $|\mathrm{E}\mathrm{e}^{\mathrm{i}\eta X}| \leq 1$, we have

$$\mathrm{E}\mathrm{e}^{\mathrm{i}\eta X} \in L^u(\mathbb{R}). \tag{A.1}$$

Suppose $X_1, X_2, \ldots, X_u$ are i.i.d. with the same distribution as X. So $|\mathrm{E}\mathrm{e}^{\mathrm{i}\eta X}|^u$ is the characteristic function of $(X_1 + \ldots + X_{u/2}) - (X_{u/2+1} + \ldots + X_u)$. Expression (A.1) implies that $|\mathrm{E}\mathrm{e}^{\mathrm{i}\eta X}|^u \in L^1(\mathbb{R})$. Thus the density function $f_u(z)$ of $(X_1 + \ldots + X_{u/2})$

$-(X_{u/2+1} + \ldots + X_u)$ is bounded by some constant M (see Feller 1971, Chapter XV, Section 3). Now Parseval's inequality (see Feller 1971, Section XV.3) gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathrm{E}\exp \mathrm{i}\eta[(Y_1 + \ldots + Y_{u/2}) - (Y_{u/2+1} + \ldots + Y_u)]|\mathrm{e}^{-a^2\eta^2/2}\,\mathrm{d}\eta \qquad (A.2)$$

$$= \frac{1}{\sqrt{2\pi}a}\int_{-\infty}^{\infty}\mathrm{e}^{-z^2/2a^2}f_{u(x,y)}(z)\mathrm{d}z,$$

where $Y_1, \ldots, Y_u$ are i.i.d. with the same distribution as Y_1, and $f_{u(x,y)}(z)$ is the density function of $(Y_1 + \ldots + Y_{u/2}) - (Y_{u/2+1} + \ldots + Y_u)$ and a is some positive constant. Noting that

$$f_{u(x,y)}(z) \leq \frac{1}{[F(y) - F(x)]^u}f_u(z),$$

we have, from (A.2),

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(y) - F(x)|^u|\mathrm{E}\mathrm{e}^{\mathrm{i}\eta Y_1}|^u\mathrm{e}^{-a^2\eta^2/2}\,\mathrm{d}\eta \qquad (A.3)$$

$$\leq \frac{1}{\sqrt{2\pi}a}\int_{-\infty}^{\infty}\mathrm{e}^{-z^2/2a^2}f_u(z)\mathrm{d}z$$

$$\leq M.$$

Letting $a \to 0$ in (A.3) completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Denote the root of $K'_{Y_1}(\lambda) = t$ by $\tilde{\lambda}$. Let $Y(\tilde{\lambda})$ be the random variable with density function $f_{Y(\tilde{\lambda})}(z) = \mathrm{e}^{\tilde{\lambda}z}f(z)I(x \leq z \leq y)/\int_x^y \mathrm{e}^{\tilde{\lambda}z}f(z)\mathrm{d}z$. For each pair of positive numbers B and B_0 such that $B > B_0$, $B - B_0 \geq |t|$, define

$$A_{B,B_0} := \{(x, y) : -B \leq x \leq t - B_0, \ t + B_0 \leq y \leq B\}. \qquad (A.4)$$

Lemma A.2. *Under condition* (iii) *of Theorem 3.1, we have*

$$\sup_{(x,y)\in A_{B,B_0}}\int_{-\infty}^{\infty}|\mathrm{E}\mathrm{e}^{\mathrm{i}\eta Y(\tilde{\lambda})}|^u\,\mathrm{d}\eta < \infty, \qquad (A.5)$$

where u is the smallest even integer greater then or equal to v.

Proof. Since $K'_{Y_1}(\tilde{\lambda}) = t$, we have

$$\int_x^y(z - t)\mathrm{e}^{\tilde{\lambda}z}f(z)\mathrm{d}z = 0.$$

Let $p(x, y, \lambda) = \int_x^y(z - t)\mathrm{e}^{\lambda z}f(z)\mathrm{d}z$. Since $p(x, y, \tilde{\lambda}) = 0$ and $\partial p(x, y, \tilde{\lambda})/\partial\lambda = \int_x^y(z - t)^2\mathrm{e}^{\tilde{\lambda}z}f(z)\mathrm{d}z > 0$, it follows from the implicit function theorem that there exists some $\epsilon > 0$ such that $\tilde{\lambda} = \tilde{\lambda}(x, y)$ is a continuous function on $A_\epsilon(x_1, y_1) = \{(x, y) : |x - x_1| \leq \epsilon, |y - y_1| \leq \epsilon\}$ for each point $(x_1, y_1) \in A_{B,B_0}$. Hence $\tilde{\lambda}$ is bounded on $A_\epsilon(x_1, y_1)$.

Define

$$\varphi(i\eta; x, y) := \frac{1}{F(y) - F(x)} \int_x^y e^{i\eta z} f(z) dz.$$

Lemma A.1 shows that $\varphi(i\eta; x, y) \in L^u(\mathbb{R})$, where u is the smallest even integer greater than or equal to v. By changing the path of integration, we have

$$\int_{-\infty}^{\infty} \varphi^u(i\eta; x, y) d\eta = \frac{1}{i} \int_{-i\infty}^{i\infty} \varphi^u(\eta; x, y) d\eta$$

$$= \frac{1}{i} \int_{\tilde{\lambda}(x,y)-i\infty}^{\tilde{\lambda}(x,y)+i\infty} \varphi^u(\eta; x, y) d\eta$$

$$= \int_{-\infty}^{\infty} \varphi^u(i\eta + \tilde{\lambda}(x, y); x, y) d\eta.$$

By the definition of Lebesgue integrability, $\varphi^u(i\eta + \tilde{\lambda}(x, y)) \in L^1(\mathbb{R})$. Hence

$$E e^{i\eta Y(\tilde{\lambda})} \in L^u(\mathbb{R}). \tag{A.6}$$

Suppose $\tilde{Y}_1, \tilde{Y}_2, \ldots, \tilde{Y}_u$ are i.i.d. with the same distribution as $Y(\tilde{\lambda})$. Thus $|E e^{i\eta Y(\tilde{\lambda})}|^u$ is the characteristic function of $(\tilde{Y}_1 + \ldots + \tilde{Y}_{u/2}) - (\tilde{Y}_{u/2+1} + \ldots + \tilde{Y}_u)$. Expression (A.6) implies that $|E e^{i\eta Y(\tilde{\lambda})}|^u \in L^1(\mathbb{R})$. Now Parseval's inequality gives

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |E \exp i\eta[(\tilde{Y}_1 + \ldots + \tilde{Y}_{u/2}) - (\tilde{Y}_{u/2+1} + \ldots + \tilde{Y}_u)]| e^{-a^2\eta^2/2} d\eta \tag{A.7}$$

$$= \frac{1}{\sqrt{2\pi}a} \int_{-\infty}^{\infty} e^{-z^2/2a^2} f_{u(\tilde{\lambda};x,y)}(z) dz,$$

where $f_{u(\tilde{\lambda};x,y)}(z)$ is the density function of $(\tilde{Y}_1 + \ldots + \tilde{Y}_{u/2}) - (\tilde{Y}_{u/2+1} + \ldots + \tilde{Y}_u)$. Note that $f_{u(\tilde{\lambda};x,y)}(z)$ is the convolution of $f_{\tilde{Y}_1}(z), \ldots, f_{\tilde{Y}_{u/2}}(z), f_{-\tilde{Y}_{u/2+1}}(z), \ldots, f_{-\tilde{Y}_u}(z)$, where $f_{\tilde{Y}_1}(z) = \ldots = f_{\tilde{Y}_{u/2}}(z) = f_{Y(\tilde{\lambda})}(z)$ and $f_{-\tilde{Y}_{u/2+1}}(z) = \ldots = f_{-\tilde{Y}_u}(z) = f_{Y(\tilde{\lambda})}(-z)$; and that $f_u(z)$ is the convolution of $f_{X_1}(z), \ldots, f_{X_{u/2}}(z), f_{-X_{u/2+1}}(z), \ldots, f_{-X_u}(z)$, where $f_{X_1}(z) = \ldots = f_{X_{u/2}}(z) = f(z)$ and $f_{-X_{u/2+1}}(z) = \ldots = f_{-X_u}(z) = f(-z)$.

Since $f_{Y(\tilde{\lambda})}(z) = e^{\tilde{\lambda}z} f(z) I(x \leqslant z \leqslant y) / \int_x^y e^{\tilde{\lambda}z} f(z) dz \leqslant C f(z)$ for some absolute constant C as (x, y) varies in A_{B,B_0} by the boundedness of $\tilde{\lambda}$, we have $f_{u(\tilde{\lambda};x,y)}(z) \leqslant C^u f_u(z)$. Hence, by (A.7),

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |E e^{i\eta Y(\tilde{\lambda})}|^u e^{-a^2\eta^2/2} d\eta \tag{A.8}$$

$$\leqslant \frac{1}{\sqrt{2\pi}a} \int_{-\infty}^{\infty} e^{-z^2/2a^2} C^u f_u(z) dz$$

$$\leqslant C^u M,$$

where M is the same as in Lemma A.1. Letting $a \to 0$ in (A.8) gives

$\int_{-\infty}^{\infty} |Ee^{i\eta Y(\tilde{\lambda})}|^u \, d\eta \leqslant 2\pi C^u M$. Thus $\sup_{(x,y)\in A_\epsilon(x_1, y_1)} \int_{-\infty}^{\infty} |Ee^{i\eta Y(\tilde{\lambda})}|^u \, d\eta < \infty$. Since A_{B,B_0} can be covered by a finite number of $A_\epsilon(x_1, y_1)$, the proof is complete. $\qquad\square$

Lemma A.3. *Let* $f(x) = F'(x)$. *For arbitrary* $\epsilon_1 > 0$, *there exists* $\delta > 0$ *such that, if* $|\eta| \geqslant \delta$,

$$\sup_{(x,y)\in A_{B,B_0}} |Ee^{i\eta Y_1 + \tilde{\lambda} Y_1} / Ee^{\tilde{\lambda} Y_1}| \leqslant \epsilon_1.$$

Proof. Define $\hat{f}(\eta) = \int_x^y e^{i\eta z} e^{\tilde{\lambda} z} f(z) dz$. Then

$$\hat{f}(\eta) = -\int_x^y e^{i\eta(z+\pi/\eta)} e^{\tilde{\lambda} z} f(z) dz$$

$$= -\int_{x+\pi/\eta}^{y+\pi/\eta} e^{i\eta z} e^{\tilde{\lambda}(z-\pi/\eta)} f\left(z - \frac{\pi}{\eta}\right) dz.$$

So

$$2\hat{f}(\eta) = \int_{-\infty}^{\infty} \left[f(z)e^{\tilde{\lambda} z} I(x \leqslant z \leqslant y) - f\left(z - \frac{\pi}{\eta}\right) e^{\tilde{\lambda}(z-\pi/\eta)} I\left(x \leqslant z - \frac{\pi}{\eta} \leqslant y\right) \right] e^{i\eta z} dz$$

$$= \int_{-\infty}^{\infty} [f(z)e^{\tilde{\lambda} z} - f(z)e^{\tilde{\lambda}(z-\pi/\eta)} I(x \leqslant z \leqslant y) dz$$

$$+ \int_{-\infty}^{\infty} f(z)e^{\tilde{\lambda}(z-\pi/\eta)} I(x \leqslant z \leqslant y) - f\left(z - \frac{\pi}{\eta}\right) e^{\tilde{\lambda}(z-\pi/\eta)} I\left(x \leqslant z - \frac{\pi}{\eta} \leqslant y\right) e^{i\eta z} dz$$

$$= \int_x^y f(z)e^{\tilde{\lambda} z}(1 - e^{-\tilde{\lambda}\pi/\eta}) e^{i\eta z} dz \tag{A.9}$$

$$+ \int_x^{x+\pi/\eta} f(z)e^{\tilde{\lambda}(z-\pi/\eta)} e^{i\eta z} dz - \int_y^{y+\pi/\eta} f\left(z - \frac{\pi}{\eta}\right) e^{\tilde{\lambda}(z-\pi/\eta)} e^{i\eta z} dz$$

$$+ \int_{x+\pi/\eta}^y \left[f(z) - f\left(z - \frac{\pi}{\eta}\right) e^{\tilde{\lambda}(z-\pi/\eta)} e^{i\eta z} dz.$$

From the proof of Lemma A.2, we know that $\tilde{\lambda}$ is a continuous function on $A_\epsilon(x_1, y_1)$. Hence $\tilde{\lambda}$ is bounded on each $A_\epsilon(x_1, y_1)$. The compactness of A_{B,B_0} shows that $\tilde{\lambda}$ is also bounded on A_{B,B_0}. Thus $1 - e^{-\tilde{\lambda}\pi/\eta} \to 0$ uniformly on A_{B,B_0} as $|\eta| \to \infty$. This implies that

$$\int_x^y f(z)e^{\tilde{\lambda} z}(1 - e^{-\tilde{\lambda}\pi/\eta}) e^{i\eta z} dz \to 0 \tag{A.10}$$

uniformly on A_{B,B_0} as $|\eta| \to \infty$. Since $F(x)$ is absolutely continuous with respect to the Lebesgue measure, it follows from Theorem 6.11 of Rudin (1987) that

$$\int_x^{x+\pi/\eta} f(z) dz \to 0$$

uniformly in x as $|\eta| \to \infty$. Hence

$$\left| \int_x^{x+\pi/\eta} f(z) e^{\tilde{\lambda}(z-\pi/\eta)} e^{i\eta z} \, dz \right| \leq \int_x^{x+\eta} f(z) dz \sup_{(x,y) \in A_{B,B_0}, x \leq z \leq y} e^{\tilde{\lambda}(z-\pi/\eta)} \tag{A.11}$$

$$\to 0$$

uniformly on A_{B,B_0} as $|\eta| \to \infty$. Similarly,

$$\int_y^{y+\pi/\eta} f\left(z - \frac{\pi}{\eta}\right) e^{\tilde{\lambda}(z-\pi/\eta)} e^{i\eta z} \, dz \to 0 \tag{A.12}$$

uniformly on A_{B,B_0} as $|\eta| \to \infty$. Since

$$\int_{x+\pi/\eta}^y \left[f(z) - f\left(z - \frac{\pi}{\eta}\right) \right] e^{\tilde{\lambda}(z-\pi/\eta)} e^{i\eta z} \, dz \leq \sup_{(x,y) \in A_{B,B_0}, x \leq z \leq y} e^{\tilde{\lambda}(z-\pi/\eta)} \int_{-\infty}^\infty \left| f(z) - f\left(z - \frac{\pi}{\eta}\right) \right| dz$$

and $\int_{-\infty}^\infty |f(z) - f(z - \pi/\eta)| dz \to 0$ as $|\eta| \to \infty$ (Rudin 1987, Theorem 9.5), we have

$$\int_{x+\pi/\eta}^y \left[f(z) - f\left(z - \frac{\pi}{\eta}\right) \right] e^{\tilde{\lambda}(z-\pi/\eta)} e^{i\eta z} \, dz \to 0 \tag{A.13}$$

uniformly on A_{B,B_0} as $|\eta| \to \infty$.

Combining (A.9)–(A.13), we see that $\hat{f}(\eta) \to 0$ uniformly on A_{B,B_0} as $|\eta| \to \infty$. Since $E\, e^{\tilde{\lambda} Y_1}$ is bounded away from 0 as $(x, y) \in A_{B,B_0}$, the proof is complete. $\qquad\square$

Lemma A.4. *Suppose conditions* (i) *and* (iii) *of Theorem 3.1 hold. Then* $f_{\overline{Y}}(t)$ *has a uniform saddlepoint approximation as* (x, y) *varies in* A_{B,B_0}, *that is,*

$$f_{\overline{Y}}(t) = \sqrt{\frac{m}{2\pi K_{Y_1}''(\tilde{\lambda})}} \exp\{-m[\tilde{\lambda} t - K_{Y_1}(\tilde{\lambda})]\}(1 + m^{-1} r_m(x, y, t)), \tag{A.14}$$

where $|r_m(x, y, t)|$ *is bounded by some absolute constant* C_0.

Proof. Denote the mean and variance of $Y(\tilde{\lambda})$ by $\tilde{\mu}$ and $\tilde{\sigma}^2$, respectively. Define

$$T(\tilde{\lambda}) = \frac{1}{\sqrt{m}\tilde{\sigma}} \sum_{j=1}^m (Y_j(\tilde{\lambda}) - \tilde{\mu}),$$

where $Y_1(\tilde{\lambda}), \ldots, Y_m(\tilde{\lambda})$ are i.i.d. with the same distribution as $Y(\tilde{\lambda})$. In order to prove (A.14), it suffices to prove that the Edgeworth expansion of the density $f_{T(\tilde{\lambda})}(t)$ of $T(\tilde{\lambda})$ has a uniform error as (x, y) varies in A_{B,B_0}, that is,

$$f_{T(\tilde{\lambda})}(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \left[1 + \frac{\tilde{\mu}_3}{6\tilde{\sigma}^3 \sqrt{m}} (t^3 - 3t) \right] + m^{-1} r_m(t), \tag{A.15}$$

where $\tilde{\mu}_3 = E(Y(\lambda) - \tilde{\mu})^3$ and $|r_m(t)|$ is bounded by some finite constant C_1 as (x, y) varies in A_{B,B_0}.

Lemma A.1 guarantees that

$$|m^{-1} r_m(t)| \leqslant N_m := \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \varphi_1^m \left(\frac{i\eta}{\tilde{\sigma}\sqrt{m}} \right) - e^{-\eta^2/2} - \frac{\tilde{\mu}_3}{6\tilde{\sigma}^3 \sqrt{m}} (i\eta)^3 e^{-\eta^2/2} \right| d\eta, \qquad \text{(A.16)}$$

where $\varphi_1(i\eta) = E e^{i\eta(Y(\tilde{\lambda}) - \bar{\mu})}$.

By Lemma A.3, for all $\epsilon_1 < 1$, there exists $\delta > 0$ such that if $|\eta| \geqslant \delta$, $\sup_{(x,y) \in A_{B,B_0}} |\varphi_1(i\eta)| \leqslant \epsilon_1$. Hence the contribution of the interval $(-\infty, -\delta\tilde{\sigma}\sqrt{m} \cup (\delta\tilde{\sigma}\sqrt{m}, +\infty)$ to the integral in (A.16) is at most

$$\delta^{m-u} \int_{-\infty}^{\infty} \left| \varphi_1 \left(\frac{i\eta}{\tilde{\sigma}\sqrt{m}} \right) \right|^u d\eta + \int_{|\eta| > \delta\tilde{\sigma}\sqrt{m}} e^{-\eta^2/2} \left(1 + \left| \frac{\tilde{\mu}_3 \eta^3}{\tilde{\sigma}^3} \right| \right) d\eta,$$

which decreases to 0 faster than any power of $1/m$ if we note Lemma A.2 and the fact that $\tilde{\mu}_3$ is uniformly bounded and $\tilde{\sigma}^2$ is uniformly bounded away from 0 and ∞ as (x, y) varies in A_{B,B_0}.

Define $\psi(\eta) = \log \varphi_1(i\eta) + \frac{1}{2}\tilde{\sigma}^2 \eta^2$. Thus we have

$$N_m = \frac{1}{2\pi} \int_{|\eta| \leqslant \delta\tilde{\sigma}\sqrt{m}} e^{-\eta^2/2} \left| \exp \left(m\psi \left(\frac{\eta}{\tilde{\sigma}\sqrt{m}} \right) \right) - 1 - \frac{\tilde{\mu}_3}{6\tilde{\sigma}^3 \sqrt{m}} (i\eta)^3 \right| d\eta + o \left(\frac{1}{m} \right) \qquad \text{(A.17)}$$

uniformly on A_{B,B_0} as $m \to \infty$.

The integrand can be estimated by the inequality (see Feller 1971)

$$|e^{\alpha} - 1 - \beta| \leqslant |e^{\alpha} - e^{\beta} + e^{\beta} - 1 - \beta| \qquad \text{(A.18)}$$

$$\leqslant (|\alpha - \beta| + \tfrac{1}{2}\beta^2) e^{\gamma},$$

where $\gamma \geqslant \max(|\alpha|, |\beta|)$.

The function $\psi(\eta)$ is four times continuously differentiable, and $\psi(0) = \psi'(0) = \psi''(0) = 0$, $\psi'''(0) = i^3 \tilde{\mu}_3$. Since $\psi^{(4)}(\eta)$ is continuous, we can choose δ such that if $|\eta| < \delta$, $|\psi^{(4)}(\eta)|$ is uniformly bounded by some finite constant as $(x, y) \in A_{B,B_0}$. By the four-term Taylor expansion we have

$$|\psi(\eta) - \tfrac{1}{6}\tilde{\mu}_3(i\eta)^3| \leqslant C_2 \tilde{\sigma}^4 |\eta|^4, \qquad |\eta| \leqslant \delta, \qquad \text{(A.19)}$$

for some finite constant C_2 as $(x, y) \in A_{B,B_0}$.

Next we shall choose sufficiently small δ so that

$$|\psi(\eta)| \leqslant \tfrac{1}{4}\tilde{\sigma}^2 \eta, \qquad |\tfrac{1}{6}\tilde{\mu}_3(i\eta)^3| \leqslant \tfrac{1}{4}\tilde{\sigma}^2 \eta^2, \qquad |\eta| \leqslant \delta, \qquad \text{(A.20)}$$

uniformly as $(x, y) \in A_{B,B_0}$.

Thus if δ is so small that (A.19) and (A.20) hold, the integrand is at most

$$e^{-\eta^2/4} \left(\frac{C_2}{m} \eta^4 + \frac{\tilde{\mu}_3^2}{72m} \eta^6 \right).$$

This shows that (A.15) holds. $\square$

Lemma A.5. *If $F(x)$ is continuous at $x = t$, where t is in the support of X, then for any n satisfying $[n\alpha] \geqslant 2$, $[n\beta] \geqslant 2$ and $n - [n\alpha] - [n\beta] \geqslant 1$, $\Lambda(x, y, t)$ attains its minimum at*

some finite point (x_0, y_0) *such that not only both* x_0 *and* y_0 *satisfy* (3.5)*, but also both* $F(x_0)$ *and* $F(y_0)$ *are unique.*

Proof. Suppose (x_n, y_n) is an arbitrary sequence in $\Omega(t)$. We will prove the following five assertions:

(i) If $x_n \to -\infty$, $y_n \to y_0$, where $t < y_0 \leqslant \infty$, then $\Lambda(x_n, y_n, t) \to \infty$.
(ii) If $x_n \to x_0$, $y_n \to +\infty$, where $-\infty \leqslant x_0 < t$, then $\Lambda(x_n, y_n, t) \to \infty$.
(iii) If $x_n \to t$, $y_n \to t$, then $\Lambda(x_n, y_n, t) \to \infty$.
(iv) If $x_n \to t$, $y_n \to y_0$, where $t < y_0 \leqslant \infty$, then $\Lambda(x_n, y_n, t) \to \infty$.
(v) If $x_n \to x_0$, $y_n \to t$, where $-\infty \leqslant x_0 < t$, then $\Lambda(x_n, y_n, t) \to \infty$.

Since

$$\Lambda(x, y, t) = \tilde{\lambda}t - \log \int_x^y e^{\tilde{\lambda}z}\, \mathrm{d}F(z) - m^{-1} \log(C_{n\alpha\beta}[F(x)]^{r-2}[1 - F(y)]^{n-s-1}),$$

noting that

$$x_n \to -\infty \quad \text{implies} \quad F(x_n) \to 0,$$
$$y_n \to +\infty \quad \text{implies} \quad F(y_n) \to 1,$$
$$x_n \to t \quad \text{and} \quad y_n \to t \quad \text{imply} \quad \tilde{\lambda} \to t \quad \text{and} \quad \int_x^y e^{\tilde{\lambda}z}\, \mathrm{d}F(z) \to 0,$$

we have assertions (i)–(iii).

We now turn to the proof of (iv). Since $K'_{Y_1}(\tilde{\lambda}) = t$, we have

$$\int_x^y (t - z)e^{\tilde{\lambda}z}\, \mathrm{d}F(z) = 0. \tag{A.21}$$

For each (x_n, y_n), we have a solution $\tilde{\lambda}_n$ to (A.21). Hence, we have a sequence $\{\tilde{\lambda}_n, n \geqslant 1\}$. Now consider a convergent subsequence $\{\tilde{\lambda}_{n_k}, k = 1, 2, \ldots\}$ of $\{\tilde{\lambda}_n, n \geqslant 1\}$. Hence, we suppose $\tilde{\lambda}_{n_k} \to \lambda_0$. From (A.21), we have

$$\int_{x_{n_k}}^t (t - z)e^{\tilde{\lambda}_{n_k}z}\, \mathrm{d}F(z) = \int_t^{y_{n_k}} (z - t)e^{\tilde{\lambda}_{n_k}z}\, \mathrm{d}F(z). \tag{A.22}$$

If λ_0 is finite, the left-hand side of (A.22) goes to 0 but the right-hand side of (A.22) goes to some positive number as $x_{n_k} \to t$. If λ_0 is $+\infty$, we can consider the formula

$$\int_{x_{n_k}}^t (t - z)e^{\tilde{\lambda}_{n_k}(z-t)}\, \mathrm{d}F(z) = \int_t^{y_{n_k}} (z - t)e^{\tilde{\lambda}_{n_k}(z-t)}\, \mathrm{d}F(z),$$

which is obtained from (A.22). The left-hand side of the above formula goes to 0 but the right-hand side goes to ∞. Therefore $\lambda_0 = -\infty$. And we can conclude that $\tilde{\lambda}_n \to -\infty$ as $n \to \infty$. Noting that $e^a \leqslant 1 + a\, e^a$ if $a \geqslant 0$, we have, for x_n sufficiently close to t,

$$\int_{x_n}^{t} e^{\tilde{\lambda}(z-t)} \, dF(z) \leqslant \int_{x_n}^{t} dF(z) + \int_{x_n}^{t} \tilde{\lambda}(z-t) e^{\tilde{\lambda}(z-t)} \, dF(z) \qquad (A.23)$$

$$= \int_{x_n}^{t} dF(z) - \int_{t}^{y_n} \tilde{\lambda}(z-t) e^{\tilde{\lambda}(z-t)} \, dF(z),$$

where in the last equality we have used (A.22). Since $\tilde{\lambda}(z-t)e^{\tilde{\lambda}(z-t)}$ is bounded and goes to 0 for each $z > t$, we have $\int_{t}^{y_n} \tilde{\lambda}(z-t)e^{\tilde{\lambda}(z-t)} \, dF(z) \to 0$ by the dominated convergence theorem. Hence, $\int_{x_n}^{t} e^{\tilde{\lambda}(z-t)} \, dF(z) \to 0$ as $x \to t$. Therefore,

$$\int_{x_n}^{y_n} e^{\tilde{\lambda}(z-t)} \, dF(z) \to 0 \qquad \text{as } x \to t. \qquad (A.24)$$

Observe that

$$\tilde{\lambda}t - \log \frac{\int_{x_n}^{y_n} e^{\tilde{\lambda}z} \, dF(z)}{F(y_n) - F(x_n)} = -\log \frac{\int_{x_n}^{y_n} e^{\tilde{\lambda}(z-t)} \, dF(z)}{F(y_n) - F(x_n)}.$$

Therefore we have proved (iv). The proof of (v) is the same as that of (iv).

Now (i)–(v) imply that $\Lambda(x, y, t)$ attains its minimum at some finite point (x_0, y_0) in $\Omega(t)$.

Finally, we will prove the uniqueness of $F(x_0)$ and $F(y_0)$. The above proof shows that

$$\Lambda(x_0, y_0, t) = \inf_{(x,y) \in \Omega(t)} \Lambda(x, y, t). \qquad (A.25)$$

We also know that

$$\Lambda_1(x, y, t) = \sup_{\lambda} \Lambda_1(x, y, \lambda, t) = \tilde{\lambda}t - K_{Y_1}(\tilde{\lambda}),$$

where $\Lambda_1(x, y, \lambda, t) = \lambda t - K_{Y_1}(\lambda)$ and $\tilde{\lambda}$ is uniquely determined by the equation $K'_{Y_1}(\tilde{\lambda}) = t$. Hence (A.25) can be re-expressed as

$$\Lambda(x_0, y_0, t) = \inf_{(x,y) \in \Omega(t)} \sup_{\lambda} \Lambda_1(x, y, \lambda, t) + \Lambda_2(x, y).$$

Assume that (x'_0, y'_0) is another point such that

$$\Lambda(x'_0, y'_0, t) = \inf_{(x,y) \in \Omega(t)} \sup_{\lambda} \Lambda_1(x, y, \lambda, t) + \Lambda_2(x, y).$$

Denote the solution of $K'_{Y_1}(\tilde{\lambda}) = t$ by $\tilde{\lambda}'_0$ when $x = x'_0$ and $y = y'_0$. Therefore,

$$\Lambda_1(x_0, y_0, \tilde{\lambda}_0, t) + \Lambda_2(x_0, y_0) = \inf_{(x,y) \in \Omega(t)} \Lambda_1(x, y, \tilde{\lambda}_0, t) + \Lambda_2(x, y)$$

$$\leqslant \Lambda_1(x'_0, y'_0, \tilde{\lambda}_0, t) + \Lambda_2(x'_0, y'_0)$$

$$\leqslant \sup_{\lambda} \Lambda_1(x'_0, y'_0, \lambda, t) + \Lambda_2(x'_0, y'_0)$$

$$= \Lambda_1(x'_0, y'_0, \tilde{\lambda}'_0, t) + \Lambda_2(x'_0, y'_0). \qquad (A.26)$$

If $\tilde{\lambda}_0 \neq \tilde{\lambda}_0'$, then

$$\Lambda_1(x_0', y_0', \tilde{\lambda}_0, t) + \Lambda_2(x_0', y_0') < \sup_{\lambda} \Lambda_1(x_0', y_0', \lambda, t) + \Lambda_2(x_0', y_0') \qquad \text{(A.27)}$$

by the fact that $\Lambda_1(x, y, \lambda, t)$ is a strictly concave function of λ. Hence,

$$\tilde{\lambda}_0 = \tilde{\lambda}_0'. \qquad \text{(A.28)}$$

Define

$$\Lambda_3(x, y, \xi, t) = \xi \frac{t}{y} - \log \frac{\int_x^y e^{\xi z/y} \, dF(z)}{F(y) - F(x)},$$

$$\xi_0 = \tilde{\lambda}_0 y_0, \qquad \xi_0' = \tilde{\lambda}_0 y_0'.$$

We thus have

$$\Lambda_3(x_0, y_0, \xi_0, t) + \Lambda_2(x_0, y_0) = \Lambda_3(x_0', y_0', \xi_0', t) + \Lambda_2(x_0', y_0')$$

$$= \inf_{(x,y) \in \Omega(t)} \sup_{\xi} \Lambda_3(x, y, \xi, t) + \Lambda_2(x, y).$$

Similarly to the proof of (A.25), we can obtain

$$\xi_0 = \xi_0', \qquad \text{(A.29)}$$

noting that $\Lambda_3(x, y, \xi, t)$ is a strictly concave function of ξ for fixed x, y, t.

If $\tilde{\lambda}_0 \neq 0$, then $y_0 = y_0'$, which implies the uniqueness of y_0. Similarly for x_0.

If $\tilde{\lambda}_0 = 0$, then $\Lambda(x_0, y_0, t) = \Lambda_2(x_0, y_0)$. The strict convexity of $\Lambda_2(x, y)$ as a function of $F(x)$ and $F(y)$ shows the uniqueness of $F(x_0)$ and $F(y_0)$. $\qquad \square$

Remark A.1. Since $\Lambda(x, y, t)$ is differentiable in both x and y, Lemma A.5 implies that (x_0, y_0) satisfies (3.5). So Lemma A.5 is just our Proposition 3.1. Also from the proof, we can see the uniqueness of (x_0, y_0) except in one particular case where $\tilde{\lambda}_0 = 0$.

Lemma A.6. *Under the conditions of Theorem 3.1, for suitably chosen B and B_0 which are independent of n,*

$$\iint_{\Omega(t) \setminus A_{B, B_0}} f_{\overline{Y}}(t) q_{r-1, s+1:n}(\mathrm{x}, \mathrm{y}) \mathrm{d}x \, \mathrm{d}y / \exp(-m\Lambda(x_0, y_0, t))$$

goes to 0 *faster than any power of* $1/m$.

Proof. From Lemma A.1, $|\mathrm{E} e^{i\eta Y_1}|^u$ is integrable. So we can apply the Fourier inversion theorem to obtain

$$f_{\overline{Y}}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\eta t} E e^{i\eta \overline{Y}} \, d\eta$$

$$= \frac{m}{2\pi} \int_{-\infty}^{\infty} e^{-i\eta m t} (E e^{i\eta Y_1})^m \, d\eta.$$

Again, Lemma A.1 shows that

$$|f_{\overline{Y}}(t)| \leq \frac{m}{2\pi} \int_{-\infty}^{\infty} |E e^{i\eta Y_1}|^m \, d\eta \leq \frac{Mm}{|F(y) - F(x)|^u}.$$

Hence

$$\iint_{\Omega(t) \setminus A_{B,B_0}} f_{\overline{Y}}(t) q_{r-1,s+1:n}(x, y) dx \, dy$$

$$\leq \iint_{\Omega(t) \setminus A_{B,B_0}} \frac{Mm}{[F(y) - F(x)]^u} q_{r-1,s+1:n}(x, y) dx \, dy$$

$$= \iint_{\Omega(t) \setminus A_{B,B_0}} Mm \exp\{-m\Lambda_3(x, y)\} f(x)f(y) dx \, dy, \tag{A.30}$$

where

$$\Lambda_3(x, y) = -m^{-1} \log(C_{n\alpha\beta}[F(x)]^{r-2}[F(y) - F(x)]^{m-u}[1 - F(y)]^{n-s-1}).$$

Since

$$\Lambda(x, y, t) = \tilde{\lambda} t - \log \int_x^y e^{\tilde{\lambda} z} \, dF(z) - m^{-1} \log(C_{n\alpha\beta}[F(x)]^{r-2}[1 - F(y)]^{n-s-1})$$

$$\leq \Lambda'(x, y, t)$$

$$:= \tilde{\lambda} t - \log \int_x^y e^{\tilde{\lambda} z} \, dF(z) - m^{-1} \log C_{n\alpha\beta} - \frac{2\alpha}{1 - \alpha - \beta} \log F(x) - \frac{2\beta}{1 - \alpha - \beta} \log(1 - F(y))$$

for sufficiently large m, we have

$$\Lambda(x_0, y_0, t) \leq \Lambda'(x_0', y_0', t) := \inf_{x < t < y} \Lambda'(x, y, t).$$

The existence and finiteness of (x_0', y_0') can be proved in the same way as that of (x_0, y_0). From the expression for $\Lambda'(x, y, t)$, we see that x_0', y_0' are independent of n. Noting that $\lim_{x \to -\infty} F(x) = 0$, $\lim_{y \to \infty} F(y) = 1$, $\lim_{x \to t, y \to t}(F(y) - F(x)) = 0$. $\forall \epsilon' > 0$, we can choose positive numbers B and B_0 independent of n such that

$$\inf_{(x,y) \in \Omega(t) \setminus A_{B,B_0}} \Lambda_3(x, y) > \Lambda'(x_0', y_0', t) + \epsilon' \geq \Lambda(x_0, y_0, t) + \epsilon'. \tag{A.31}$$

It follows from (A.26) that

$$\frac{1}{\exp\{-m\Lambda(x_0,\ y_0,\ t)\}}\iint_{\Omega(t)\backslash A_{B,B_0}} Mm\exp\{-m\Lambda_3(x,\ y)\}f(x)f(y)\mathrm{d}x\,\mathrm{d}y$$

goes to 0 faster than any power of m^{-1}. By (A.25), the proof is complete.

Proof of Theorem 3.1. Lemma A.6 assures us of the exponential smallness of

$$\iint_{\Omega(t)\backslash A_{B,B_0}} f_{\overline{Y}}(t)q_{r-1,s+1:n}(x,\ y)\mathrm{d}x\,\mathrm{d}y/\exp(-m\Lambda(x_0,\ y_0,\ t)).$$

To complete the proof, we need to consider the asymptotic expansion of

$$\iint_{A_{B,B_0}} f_{\overline{Y}}(t)q_{r-1,s+1:n}(x,\ y)\mathrm{d}x\,\mathrm{d}y.$$

Lemma A.4 gives

$$\iint_{A_{B,B_0}} f_{\overline{Y}}(t)q_{r-1,s+1:n}(x,\ y)\,\mathrm{d}x\,\mathrm{d}y$$

$$=\iint_{A_{B,B_0}}\sqrt{\frac{m}{2\pi K''_{Y_1}(\tilde{\lambda})}}f(x)f(y)\exp[-m\Lambda(x,\ y,\ t)]\mathrm{d}x\,\mathrm{d}y\{1+O(m^{-1})\}.$$

We thus obtain a double integral of Laplace type. Conditions (i) and (ii) of the theorem guarantee that we can use formula (8.2.55) of Bleistein and Handelsman (1986) to obtain

$$\iint_{A_{B,B_0}}\sqrt{\frac{m}{2\pi K''_{Y_1}(\tilde{\lambda})}}f(x)f(y)\exp\left[-m\Lambda(x,\ y,\ t)\right]\mathrm{d}x\,\mathrm{d}y\{1+O(m^{-1})\}$$

$$=\sqrt{\frac{2\pi}{m}}\frac{\exp\{-m\Lambda(x_0(t),\ y_0(t),\ t)\}}{\sqrt{K''_{Y_1}(\tilde{\lambda}_0(t))|_{x=x_0(t),y=y_0(t)}|\Delta_0(t)|}}\left(1+O\left(\frac{1}{m}\right)\right),$$

and we are done. $\square$

The next eight lemmas will be used to prove Theorem 4.1. Since the proofs of the first four of these are similar to the proofs of Lemmas A.1–A.4, we shall omit the details here.

Lemma A.7. *Under condition* (iii) *of Theorem 4.1, there exist some constant M_1 and some even integer u_1 such that*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F(y)-F(x)|^{u_1}|\mathrm{E}\mathrm{e}^{\mathrm{i}\eta_1 Y_1+\mathrm{i}\eta_2 Y_1^2}|^{u_1}\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\leqslant(2\pi)^2 M_1.$$

Let d_0, u_0 be the solutions to the equations

$$\frac{\partial K(d_0,\ u_0)}{\partial d}=\overline{Y}(t,\ b),\qquad\frac{\partial K(d_0,\ u_0)}{\partial u}=\overline{Z}(t,\ b).$$

Lemma A.8. *Under condition* (iii) *of Theorem 4.1, we have*

$$\sup_{(x,y)\in A_{B,B_0}} \iint_{\mathbb{R}^2} |\exp(K(d_0 + i\eta_1, u_0 + i\eta_2) - K(d_0, u_0))|^{u_1} \, d\eta_1 \, d\eta_2 < \infty, \qquad (A.32)$$

where u_1 is the smallest even integer greater than or equal to v_1.

Lemma A.9. *Let $f(x) = F'(x)$. For arbitrary $\epsilon_1' > 0$, there exists $\delta' > 0$ such that if $|\eta_1| + |\eta_2| \geqslant \delta'$, then*

$$\sup_{(x,y)\in A_{B,B_0}} |\exp(K(d_0 + i\eta_1, u_0 + i\eta_2) - K(d_0, u_0))| \leqslant \epsilon_1'. \qquad (A.33)$$

Lemma A.10. *Suppose conditions* (i) *and* (iii) *of Theorem 4.1 hold. Then the density function $f_{(\overline{Y}, \overline{Z})}(\overline{Y}(t, b), \overline{Z}(t, b))$ has a uniform saddlepoint approximation as (x, y) varies in A_{B,B_0}, that is,*

$$f_{(\overline{Y}, \overline{Z})}(\overline{Y}(t, b), \overline{Z}(t, b)) = \frac{m-2}{2\pi} \Delta_s^{-1/2}(t, b)\exp[-(m-2)\Lambda_s(t, b)](1 + m^{-1}\bar{r}_m(x, y, t)), \qquad (A.34)$$

where $|\bar{r}_m(x, y, t)|$ is bounded by some absolute constant C_0'.

Lemma A.11.

$$\sup_{d,u}[d\overline{Y} + u\overline{Z} - K(d, u)] \to \infty \qquad \text{as } \overline{Y} \to x \text{ from the right.} \qquad (A.35)$$

$$\sup_{d,u}[d\overline{Y} + u\overline{Z} - K(d, u)] \to \infty \qquad \text{as } \overline{Y} \to y \text{ from the left.} \qquad (A.36)$$

Proof. We only give a proof of (A.35); that of (A.36) is similar.

Since

$$\sup_{d,u}[d\overline{Y} + u\overline{Z} - K(d, u)] \geqslant \sup_d[d\overline{Y} - K(d, 0)] = \tilde{d}\overline{Y} - K(\tilde{d}, 0),$$

where $\tilde{d}$ satisfies the equation

$$\int_x^y \overline{Y} e^{\tilde{z}} \, dF(z) = \int_x^y z \, e^{\tilde{d}z} \, dF(z), \qquad (A.37)$$

it suffices to prove $\tilde{d}\overline{Y} - K(\tilde{d}, 0) \to \infty$ as $\overline{Y} \to x$ from the right.

Let $h(d, \overline{Y}) = \int_x^y (z - \overline{Y})e^{dz} \, dF(z)$. Since $h(\tilde{d}, \overline{Y}) = 0$, we have

$$\frac{\partial \tilde{d}}{\partial \overline{Y}} = \frac{\displaystyle\int_x^y e^{\tilde{d}z} dF(z)}{\displaystyle\int_x^y (z - \overline{Y})^2 \, e^{\tilde{d}z} \, dF(z)}.$$

Hence, $\tilde{d}$ is a increasing function of $\overline{Y}$. Then if $\overline{Y} \to x$ from the right, we can suppose $\tilde{d} \to \tilde{d}_0$. From (A.37), we have

$$\int_x^{\overline{Y}} (\overline{Y} - z)e^{\tilde{d}z}\, dF(z) = \int_{\overline{Y}}^y (z - \overline{Y})e^{\tilde{d}z}\, dF(z). \tag{A.38}$$

If $\tilde{d}_0$ is finite, then the left-hand side of (A.38) goes to 0 but the right goes to some positive number as $\overline{Y} \to x$. Therefore $\tilde{d}_0 = -\infty$. Noting that $e^a \leqslant 1 + a\,e^a$ if $a \geqslant 0$, we have, for $\overline{Y}$ sufficiently close to x,

$$\int_x^{\overline{Y}} e^{\tilde{d}(z-\overline{Y})}\, dF(z) \leqslant \int_x^{\overline{Y}} dF(z) + \int_x^{\overline{Y}} \tilde{d}(z - \overline{Y})e^{\tilde{d}(z-\overline{Y})}\, dF(z) \tag{A.39}$$

$$= \int_x^{\overline{Y}} dF(z) - \int_{\overline{Y}}^y \tilde{d}(z - \overline{Y})e^{\tilde{d}(z-\overline{Y})}\, dF(z),$$

where in the last equality we have used (A.38). Since $\tilde{d}(z - \overline{Y})e^{\tilde{d}(z-\overline{Y})}$ is bounded and goes to 0 for each $z > \overline{Y}$, we have $\int_{\overline{Y}}^y \tilde{d}(z - \overline{Y})e^{\tilde{d}(z-\overline{Y})}\, dF(z) \to 0$ by the dominated convergence theorem. Hence $\int_x^{\overline{Y}} e^{\tilde{d}(z-\overline{Y})}\, dF(z) \to 0$ as $\overline{Y} \to x$ from the right. Therefore,

$$\tilde{d}\overline{Y} - K(\tilde{d}, 0) = -\log \int_x^y \frac{e^{\tilde{d}(z-\overline{Y})}}{F(y) - F(x)}\, dF(z) \to \infty \qquad \text{as } \overline{Y} \to x \text{ from the right.}$$

Remark A.2. Since

$$b \equiv b(\overline{Y}, \overline{Z}) = \left(\frac{m-2}{m}\overline{Y} + \frac{x+y}{m} \right),$$

Lemma A.11 implies that the equation

$$\frac{\partial \Lambda_s(t, b)}{\partial b}\Big|_{d=d_0(t), u=u_0(t)} = 0$$

has a solution $b = b_0(t) \in (x, y)$.

Lemma A.12. *If $F(x)$ is continuous at $x = t$, where t is in the support of X, then for any n satisfying $[n\alpha] \geqslant 1$ and $n - 2[n\alpha] \geqslant 3$, $\tilde{\Lambda}(x, y, t)$ attains its minimum at some finite point $(\tilde{x}_0, \tilde{y}_0)$.*

Proof. Suppose (x_n, y_n) is an arbitrary sequence in $\Omega(t)$. We will prove the following five assertions.

   (i′) If $x_n \to -\infty$, $y_n \to y_0$, where $t < y_0 \leqslant \infty$, then $\tilde{\Lambda}(x_n, y_n, t) \to \infty$.
  (ii′) If $x_n \to x_0$, $y_n \to +\infty$, where $-\infty \leqslant x_0 < t$, then $\tilde{\Lambda}(x_n, y_n, t) \to \infty$.
 (iii′) If $x_n \to t$, $y_n \to t$, then $\tilde{\Lambda}(x_n, y_n, t) \to \infty$.
 (iv′) If $x_n \to t$, $y_n \to y_0$, where $t < y_0 \leqslant \infty$, then $\tilde{\Lambda}(x_n, y_n, t) \to \infty$.
  (v′) If $x_n \to x_0$, $y_n \to t$, where $-\infty \leqslant x_0 < t$, then $\tilde{\Lambda}(x_n, y_n, t) \to \infty$.

The proof of (i′)–(iii′) is similar to Lemma A.5.

Turning to the proof of (iv′), since

$$\tilde{\Lambda}_1(x,\,y,\,t) = d_0(t)\overline{Y}(t,\,b_0(t)) + u_0(t)\overline{Z}(t,\,b_0(t)) - K(d_0(t),\,u_0(t))$$

$$= \sup_{d,u}[d\,\overline{Y}(t,\,b_0(t)) + u\overline{Z}(t,\,b_0(t)) - K(d,\,u)]$$

$$\geq \sup_d[d\,\overline{Y}(t,\,b_0(t)) - K(d,\,0)],$$

it suffices to prove that

$$\sup_d[d\,\overline{Y}(t,\,b_0(t)) - K(d,\,0)] \to \infty$$

as $x_n \to t$ and $y_n \to y_0$. This can be proved similarly to Lemma A.5.

It remains to prove (v′). Since $\tilde{\Lambda}_1(x,\,y,\,t) \geq \sup_u[u\overline{Z}(t,\,b_0(t)) - K(0,\,u)]$, this again follows similar lines to Lemma A.5. (i′)–(v′) give the assertions of Lemma A.12.   □

Remark A.3. Lemma A.12 implies that $(\tilde{x}_0,\,\tilde{y}_0)$ satisfies (4.4). So (4.4) has at least one solution $\tilde{x}_0,\,\tilde{y}_0,\,\tilde{d}_0,\,\tilde{u}_0,\,\tilde{b}_0$. Combining Lemmas A.11 and A.12 gives Proposition 4.2.

Lemma A.13. *Under the conditions of Theorem 4.1, for suitably chosen B and B_0 which are independent of n,*

$$\iint_{\Omega(t)\backslash A_{B,B_0}} f_{a(\overline{Y},\overline{Z})}(t)q_{r,s:n}(x,\,y)\mathrm{d}x\,\mathrm{d}y / \exp(-(m-2)\tilde{\Lambda}(\tilde{x}_0,\,\tilde{y}_0,\,t))$$

goes to 0 faster than any power of $1/m$.

Proof. Lemma A.7 shows that $|\mathrm{E}\mathrm{e}^{\mathrm{i}\eta_1 Y_1 + \mathrm{i}\eta_2 Z_1}|^{u_1}$ is integrable. So we can apply the Fourier inversion theorem to obtain

$$f_{(\overline{Y},\overline{Z})}(z_1,\,z_2) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\eta_1 z_1 - \mathrm{i}\eta_2 z_2}\mathrm{E}\mathrm{e}^{\mathrm{i}\eta_1 \overline{Y} + \mathrm{i}\eta_2 \overline{Z}}\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2$$

$$= \frac{(m-2)^2}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(m-2)\eta_1 z_1 - \mathrm{i}(m-2)\eta_2 z_2}(\mathrm{E}\mathrm{e}^{\mathrm{i}\eta_1 Y_1 + \mathrm{i}\eta_2 Z_1})^{m-2}\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2.$$

Hence, using Lemma A.7, we have

$$f_{(\overline{Y},\overline{Z})}(z_1,\,z_2) \leq \frac{(m-2)^2}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\mathrm{E}\mathrm{e}^{\mathrm{i}\eta_1 Y_1 + \mathrm{i}\eta_2 \overline{Z}}|^{m-2}\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2$$

$$\leq \frac{M_1(m-2)^2}{|F(y) - F(x)|^{u_1}}.$$

So

$$f_{a(\overline{Y},\overline{Z})}(t) = \int_x^y f_{(\overline{Y},\overline{Z})}(\overline{Y}(t,\,b),\,\overline{Z}(t,\,b))|J|\mathrm{d}b$$

$$\leqslant \frac{M_1(m-2)^2}{|F(y)-F(x)|^{u_1}}\frac{2n(1-2\alpha)^2 m}{(m-2)^2 t^3}\int_x^y b^2\,\mathrm{d}b$$

$$= \frac{M_1(m-2)^2}{|F(y)-F(x)|^{u_1}}\frac{2n(1-2\alpha)^2 m}{(m-2)^2 t^3}\frac{y^3-x^3}{3}.$$

Thus

$$\iint_{\Omega(t)\backslash A_{B,B_0}} f_{a(\overline{Y},\overline{Z})}(t)q_{r,s,:n}(x,\,y)\mathrm{d}x\,\mathrm{d}y$$

$$\leqslant \iint_{\Omega(t)\backslash A_{B,B_0}} \frac{M_1(m-2)^2}{|F(y)-F(x)|^{u_1}}\frac{2n(1-2\alpha)^2 m}{(m-2)^2 t^3}\frac{y^3-x^3}{3}q_{r,s:n}(x,\,y)\mathrm{d}x\,\mathrm{d}y$$

$$\leqslant \frac{2M_1(1-2\alpha)^2}{3t^3}\iint_{\Omega(t)\backslash A_{B,B_0}} nm\exp\{-(m-2)\Lambda_3'(x,\,y)\}$$

$$\times (y^3-x^3)(F(x))^{3w_1}(1-F(y))^{3w_2}f(x)f(y)\mathrm{d}x\,\mathrm{d}y, \tag{A.40}$$

where

$$\Lambda_3'(x,\,y) = -(m-2)^{-1}\log(D_{n\alpha\beta}[F(x)]^{r-1-3w_1}[F(y)-F(x)]^{m-2-u_1}[1-F(y)]^{n-s-3w_2}).$$

Condition (iv) of Theorem 4.1 implies that $(y^3-x^3)(F(x))^{3w_1}(1-F(y))^{3w_2}$ is bounded. Hence, from (A.40),

$$\iint_{\Omega(t)\backslash A_{B,B_0}} f_{a(\overline{Y},\overline{Z})}(t)q_{r,s,:n}(x,\,y)\mathrm{d}x\,\mathrm{d}y$$

$$\leqslant M_2\iint_{\Omega(t)\backslash A_{B,B_0}} nm\exp\{-(m-2)\Lambda_3'(x,\,y)\}f(x)f(y)\mathrm{d}x\,\mathrm{d}y, \tag{A.41}$$

where M_2 is some absolute constant. As in the proof of Lemma A.6, given $\epsilon_2 > 0$, we can select B and B_0 which are independent of n such that, for n sufficiently large,

$$\inf_{\Omega(t)\backslash A_{B,B_0}}\Lambda_3'(x,\,y) > \tilde{\Lambda}(\tilde{x}_0,\,\tilde{y}_0,\,t) + \epsilon_2. \tag{A.42}$$

Combining (A.40)–(A.42) completes the proof. $\qquad\square$

Lemma A.14. *Given $t \neq 0$, under conditions (C_1)–(C_2), (i) and (ii) of Theorem 4.1, $f_{a(\overline{Y},\,\overline{Z})^{(t)}}$ has a uniform saddlepoint approximation as $(x,\,y)$ varies in A_{B,B_0}, that is,*

$$f_{a(\overline{Y}, \overline{Z})}(t) = \sqrt{\frac{m-2}{2\pi}} J(t, b_0(t)) G^{-1/2}(t, b_0(t)) \exp[-(m-2)\Lambda_s(t, b_0(t))]$$

$$\times \{1 + m^{-1}\tilde{r}_m(x, y, t)\},$$

where $|\tilde{r}_m(x, y, t)|$ is bounded as (x, y) varies in A_{B, B_0}.

Proof. First, we will show that

$$\frac{\partial^2 \Lambda_s(t, b_0(t))}{\partial b^2} > 0 \tag{A.43}$$

as (x, y) varies in A_{B, B_0}. Lemma A.11 and Remark A.2 imply that $\partial \Lambda_s(t, b_0(t))/\partial b = 0$. Simple calculations show that

$$\frac{\partial \Lambda_s(t, b_0(t))}{\partial b}$$

$$= d_0 \frac{m}{m-2} + u_0(m-2)^{-1} \left(\frac{2n(1-2\alpha)^2}{t^2} b_0(t) + 2(m-2r+2)b_0(t) + 2(r+1)(x+y) \right),$$

$$\frac{\partial^2 \Lambda_s(t, b_0(t))}{\partial b^2}$$

$$= u_0(m-2)^{-1} \left(\frac{2n(1-2\alpha)^2}{t^2} + 2(m-2r+2) \right).$$

Since

$$(m-2)^{-1} \left(\frac{2n(1-2\alpha)^2}{t^2} + 2(m-2r+2) \right) \rightarrow \frac{2(1-2\alpha)^2}{t^2(1-\alpha-\beta)} + 2 - \frac{4\alpha}{(1-\alpha-\beta)} > 0,$$

we see that $\partial^2 \Lambda_s(t, b_0(t))/\partial b^2 > 0$ if and only if $u_0 > 0$ for sufficiently large n. Now we suppose $u_0 = 0$. Then $\partial \Lambda_s(t, b_0(t))/\partial b = 0$ gives $d_0 = 0$. Define x_0, y_0, b_0, t_0 by the following formulae:

$$F(x_0) = \frac{r-1}{n-2}, \qquad 1 - F(y_0) = \frac{n-s}{n-2}, \tag{A.44}$$

$$b_0 = \left(\frac{(m-2)\displaystyle\int_{x_0}^{y_0} z\,dF(z)}{m(F(y_0) - F(x_0))} + \frac{x_0 + y_0}{m} \right), \tag{A.45}$$

$$t_0 = \sqrt{n}(1 - 2\alpha)b_0 \left(\frac{(m-2)\displaystyle\int_{x_0}^{y_0} z^2\,dF(z)}{F(y_0) - F(x_0)} \right.$$

$$- (m - 2r + 2) \left(\frac{(m-2)\displaystyle\int_{x_0}^{y_0} z\,dF(z)}{m(F(y_0) - F(x_0))} + \frac{x_0 + y_0}{m} \right)^2 + r(x_0 + y_0)^2$$

$$\left. -2(r-1)(x_0 + y_0) \left[\frac{(m-2)\displaystyle\int_{x_0}^{y_0} z\,dF(z)}{m(F(y_0) - F(x_0))} + \frac{x_0 + y_0}{m} \right] \right)^{-1/2}. \tag{A.46}$$

The solutions to (4.4) can be shown by calculation to be $\tilde{x}_0(t_0) = x_0$, $\tilde{y}_0(t_0) = y_0$, $\tilde{d}_0(t_0) = 0$, $\tilde{u}_0(t_0) = 0$ and $\tilde{b}_0(t_0) = b_0$. Now from (A.44), we can easily see that $x_0 - \xi_\alpha = O(n^{-1})$ and $y_0 - \xi_{1-\alpha} = O(n^{-1})$. Furthermore, we have $\xi_\alpha + \xi_{1-\alpha} = 0$. Then, from these equations and the definition of t_0, we obtain $|t_0| = O(n^{-1})$. This is a contradiction because t is a fixed non-zero number.

It is also impossible that $u_0 \to 0$ as $n \to \infty$. Otherwise $u_0 \to 0$ implies that $d_0 \to 0$. Equation (A.46) shows that $t_0 \to 0$.

Hence we can suppose $\partial^2 \Lambda_s(t, b_0(t))/\partial b^2$ is positive and bounded away from 0 as (x, y) varies in A_{B,B_0}.

Next we will show that there exists some fixed δ_f such that, for n sufficiently large,

$$\frac{\partial^2 \Lambda_s(t, b)}{\partial b^2} > 0$$

if $b \in (b_0(t) - \delta_f, b_0(t) + \delta_f)$ as (x, y) varies in A_{B,B_0}. Otherwise there exists a sequence $\{\delta_n\}$ such that $\delta_n \to 0$ as $n \to \infty$ and $\partial^2 \Lambda_s(t, b_0(t) + \delta_n)/\partial b^2 \leqslant 0$. Since A_{B,B_0} is compact, we can suppose $\lim_{n\to\infty}(b_0(t) + \delta_n) = b_0^*$. At the same time $\lim_{n\to\infty} b_0(t) = b_0^*$. Note the uniform convergence of $\partial^2 \Lambda_s(t, b)/\partial b^2$ in any compact set as $n \to \infty$ when $\partial^2 \Lambda_s(t, b)/\partial b^2$ is regarded as a function of x, y, b. We have $\partial^2 \Lambda_s(t, b_0^*)/\partial b^2 \leqslant 0$. But we have already shown that $\partial^2 \Lambda_s(t, b_0(t))/\partial b^2$ is positive and bounded away from 0 as (x, y) varies in A_{B,B_0} for sufficiently large n, thus we have a contradiction.

So

$$
f_{a(\overline{Y},\overline{Z})}(t)
$$

$$
= \int_x^y f_{(a(\overline{Y},\overline{Z}),b(\overline{Y},\overline{Z}))}(t,\,b)\mathrm{d}b
$$

$$
= \frac{m-2}{2\pi} \int_x^y \Delta_s^{-1/2}(t,\,b)J(t,\,b)\exp[-(m-2)\Lambda_s(t,\,b)](1+m^{-1}\tilde{r}_m(x,\,y,\,t))
$$

$$
= \frac{m-2}{2\pi} \left(\int_{|b-b_0(t)|\leqslant\delta_f} + \int_x^{b_0(t)-\delta_f} + \int_{b_0(t)+\delta_f}^y \right)
$$

$$
\Delta_s^{-1/2}(t,\,b)J(t,\,b)\exp\left[-(m-2)\Lambda_s(t,\,b)\right](1+m^{-1}\tilde{r}_m(x,\,y,\,t))\mathrm{d}b.
$$

Laplace approximation gives the result. The uniform error comes from the compactness of A_{B,B_0}.   □

Proof of Theorem 4.1. Lemma A.13 ensures the exponential smallness of

$$
\iint_{\Omega(t)\backslash A_{B,B_0}} f_{a(\overline{Y},\overline{Z})}(t)q_{r,s:n}(x,\,y)\mathrm{d}x\,\mathrm{d}y / \exp(-(m-2)\tilde{\Lambda}(\tilde{x}_0,\,\tilde{y}_0,\,t)).
$$

To complete the proof, we need to consider the asymptotic expansion of

$$
\iint_{A_{B,B_0}} f_{a(\overline{Y},\overline{Z})}(t)q_{r,s:n}(x,\,y)\mathrm{d}x\,\mathrm{d}y.
$$

Lemma A.14 implies that

$$
\iint_{A_{B,B_0}} f_{a(\overline{Y},\overline{Z})}(t)q_{r,s:n}(x,\,y)\mathrm{d}x\,\mathrm{d}y
$$

$$
= \iint_{A_{B,B_0}} \sqrt{\frac{m-2}{2\pi}}\, \frac{J(t,\,b_0(t))}{G^{1/2}(t,\,b_0(t))}\, \exp[-(m-2)\tilde{\Lambda}(x,\,y,\,t)]
$$

$$
\times f(x)f(y)\{1+m^{-1}\tilde{r}_m(x,\,y,\,t)\}\mathrm{d}x\,\mathrm{d}y.
$$

We thus obtain a double integral of Laplace type. Conditions (i) and (ii) of the theorem guarantee that we can use formula (8.2.55) of Bleistein and Handelsman (1986) to obtain

$$\iint_{A_{B,B_0}} \sqrt{\frac{m-2}{2\pi}} \frac{J(t, b_0(t))}{G^{1/2}(t, b_0(t))} \exp\left[-(m-2)\tilde{\Lambda}(x, y, t)\right]$$

$$\times f(x)f(y)\{1 + m^{-1}\tilde{r}_m(x, y, t)\}\mathrm{d}x\,\mathrm{d}y$$

$$= \sqrt{\frac{2\pi}{m-2}} \frac{J(t, \tilde{b}_0)}{G^{1/2}(t, \tilde{b}_0)|\tilde{\Delta}(\tilde{x}_0, \tilde{y}_0, t)|^{1/2}} \exp\left[-(m-2)\tilde{\Lambda}(\tilde{x}_0, \tilde{y}_0, t)\right]\{1 + m^{-1}\tilde{R}_n(t)\},$$

and we are done. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Proof of Theorem 4.2. Setting $v = 0$ in (4.7), we obtain

$$\tilde{\psi}(0) = \tilde{a}(t_0)\exp\{-(m-2)\tilde{h}(t_0)\}\left|\frac{\mathrm{d}t}{\mathrm{d}v}\right|_{v=0}.$$

Note that $\tilde{h}'(t) = \tilde{\Lambda}'_x\tilde{x}'_0(t) + \tilde{\Lambda}'_y\tilde{y}'_0(t) + \tilde{\Lambda}'_t = \tilde{\Lambda}'_t$. Using this and differentiating (4.6) results in

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\tilde{h}'(t)}{v} = \frac{\tilde{\Lambda}'_t}{v} = \frac{\tilde{\Lambda}'_{1t}}{v}$$

$$= \frac{1}{v} \cdot \frac{\mathrm{d}\Lambda_s(t, \tilde{b}_0(t))}{\mathrm{d}t} = v^{-1}(\Lambda'_{sa}(t, \tilde{b}_0(t)) + \Lambda'_{sb}(t, \tilde{b}_0(t))\tilde{b}'_0(t))$$

$$= v^{-1}\Lambda'_{sa}(t, \tilde{b}_0(t)) = \left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^{-1}\frac{\mathrm{d}\Lambda'_{sa}(t, \tilde{b}_0(t))}{\mathrm{d}t}$$

$$= \left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^{-1}(\Lambda''_{saa}(t, \tilde{b}_0(t)) + \Lambda''_{sab}(t, \tilde{b}_0(t))\tilde{b}'_0(t)).$$

Differentiating $\Lambda'_{sb}(t, \tilde{b}_0(t)) = 0$ with respect to t, we find

$$\frac{\mathrm{d}\tilde{b}_0(t)}{\mathrm{d}t} = -\frac{\Lambda''_{sab}(t, \tilde{b}_0(t))}{\Lambda''_{sbb}(t, \tilde{b}_0(t))}.$$

Therefore,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \left(\Lambda''_{saa}(t, \tilde{b}_0(t)) - \frac{(\Lambda'''_{sab}(t, \tilde{b}_0(t)))^2}{\Lambda''_{sbb}(t, \tilde{b}_0(t))}\right)^{1/2}$$

$$\tilde{\psi}(0) = \sqrt{\frac{2\pi}{m-2}} \frac{J(t_0, b_0) \cdot (\mathrm{d}t/\mathrm{d}v)|_{v=0} \cdot \exp\{-(m-2)\tilde{h}(t_0)\}}{|\Delta_s(t_0, b_0)|^{1/2} \cdot |\Lambda''_{sbb}(t_0, b_0)|^{1/2} \cdot |\tilde{\Delta}(t_0)|^{-1/2}}.$$

From (4.5)–(4.7) and using an integration by parts similarly to Theorem 3.2.1 of Jensen (1995), we obtain

$$\int_t^\infty \tilde{g}_{sp}(t)\mathrm{d}t = \int_v^\infty \tilde{\psi}(v)\exp\{-(m-2)v^2/2\}\mathrm{d}v,$$

$$= (1 - \Phi(v\sqrt{m-2}))\tilde{\psi}(0) + \int_v^\infty (\tilde{\psi}(v) - \tilde{\psi}(0))\exp\{-(m-2)v^2/2\}\mathrm{d}v.$$

$$= (1 - \Phi(v\sqrt{m-2}))\tilde{\psi}(0) - \frac{\phi(v\sqrt{m-2})}{\sqrt{m-2}}\left(\frac{\tilde{\psi}(0) - \tilde{\psi}(v)}{v} + O(m^{-1})\right).$$

From this we obtain $\int_{-\infty}^\infty \tilde{g}_{sp}(t)\mathrm{d}t = \tilde{\psi}(0)$. Finally, we have

$$P(T \geqslant t) = \int_t^\infty \tilde{g}_{sp}(t)\{1 + m^{-1}\tilde{R}_n(t)\}\mathrm{d}t$$

$$= \int_t^\infty \tilde{g}_{sp}(t)\mathrm{d}t \Big/ \int_{-\infty}^\infty \tilde{g}_{sp}(t)\mathrm{d}t.$$

$$= 1 - \Phi(v\sqrt{m-2}) - \frac{\phi(v\sqrt{m-2})}{\sqrt{m-2}}\left(\frac{\tilde{\psi}(0) - \tilde{\psi}(v)}{v\tilde{\psi}(0)} + O(m^{-1})\right),$$

where, in going from the first line to the second, we have used the relation between the integration of the saddlepoint density approximations and renormalization outlined in Jing and Robinson (1994). This completes our proof. □

Acknowledgement

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