Bernoulli **10**(3), 2004, 503–548

Rate of convergence in probability to the Marchenko–Pastur law

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It is shown that the Kolmogorov distance between the spectral distribution function of a random covariance $(1/p)\mathbf{X}\mathbf{X}^{\mathrm{T}}$, where $\mathbf{X}$ is an $n \times p$ matrix with independent entries and the distribution function of the Marchenko–Pastur law is of order $O(n^{-1/2})$ in probability. The bound is explicit and requires that the twelfth moment of the entries of the matrix is uniformly bounded and that p/n is separated from 1.

Keywords: independent random variables; random matrix; spectral distributions

1. Introduction and results

Let X_{ij}, $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant n$, be independent random variables with $\mathrm{E}X_{ij} = 0$ and $\mathrm{E}X_{ij}^2 = 1$, and $\mathbf{X}_p = (X_{ij})_{\{1 \leqslant i \leqslant p, 1 \leqslant j \leqslant n\}}$. Denote by $\lambda_1 \leqslant \ldots \leqslant \lambda_p$ the eigenvalues of the symmetric matrix

$$\mathbf{W} := \mathbf{W}_p := \frac{1}{n}\mathbf{X}_p\mathbf{X}_p^{\mathrm{T}}$$

and defined its empirical distribution by

$$F_p(x) = \frac{1}{p}\sum_{k=1}^{p} I_{[\lambda_k \leqslant x\}},$$

where $I_{\{B\}}$ denotes the indicator of an event B. We shall investigate the rate of convergence of the expected spectral distribution $\mathrm{E}F_p(x)$ as well as $F_p(x)$ to the Marchenko–Pastur distribution function $F_y(x)$ with density

$$f_y(x) = \frac{1}{2xy\pi}\sqrt{(b-x)(x-a)}\,I_{\{[a,b]\}}(x) + I_{\{[1,\infty)\}}(y)(1-y^{-1})\delta(x),$$

where $y \in (0, \infty)$ and $a = (1-\sqrt{y})^2$, $b = (1+\sqrt{y})^2$. Here we denote by $\delta(x)$ the Dirac delta function and by $I_{\{[a,b]\}}(x)$ the indicator function of the interval $[a, b]$. As in Marchenko and Pastur (1967) and Pastur (1973), assume that X_{ij}, $i, j \geqslant 1$, are independent and identically distributed (i.i.d.) random variables such that

$$\mathrm{E}X_{ij} = 0, \quad \mathrm{E}X_{ij}^2 = 1, \quad \mathrm{E}|X_{ij}|^4 \leqslant \infty, \qquad \text{for all } i, j.$$

Then $\mathrm{E}F_p \to F_y$ and $F_p \to F_y$ in probability, where $y = \lim_{n\to\infty} y_p := \lim_{n\to\infty}(p/n) \in (0, \infty)$. Yin (1986) has shown that the result holds in the i.i.d. case assuming $\mathrm{E}X_{ij}^2 = \sigma^2$ only. Wachter (1978) proved the result for independent X_{ij} with $\mathrm{E}X_{ij} = 0$, $\mathrm{E}X_{ij}^2 = 1$ and $\mathrm{E}|X_{ij}|^{2+\varepsilon} \leqslant C < \infty$, for any $\varepsilon > 0$.

Let $y := y_p := p/n$. We introduce the following distance between the distributions $\mathrm{E}F_p(x)$ and $F_y(x)$,

$$\Delta_p := \sup_x |\mathrm{E}F_p(x) - F_y(x)|,$$

as well as another distance between the distributions $F_p(x)$ and $F_y(x)$,

$$\Delta_p^* := \sup_x |F_p(x) - F_y(x)|.$$

We shall use the notation $\xi_n = O_P(a_n)$ if, for any $\varepsilon > 0$, there exists an $L > 0$ such that $P\{|\xi_n| \geqslant La_n\} \leqslant \varepsilon$. Note that, for any $L > 0$,

$$P\left\{\sup_x |F_p(x) - F_y(x)| \geqslant L\right\} \leqslant \frac{\Delta_p^*}{L}.$$

Hence bounds for Δ_p^* provide bounds for the rate of convergence in probability of the quantity $\sup_x |F_p(x) - F_y(x)|$ to zero. Using our techniques it is straightforward, though technical, to prove that the rate of almost sure convergence is at least $O(n^{-1/2+\epsilon})$, for any $\epsilon > 0$. In view of the length of the proofs for the results stated above we refrain from including the details in this paper as well.

Bai (1993b) proved that $\Delta_p = O(n^{-1/4})$, assuming $\mathrm{E}X_{ij} = 0$, $\mathrm{E}X_{ij}^2 = 1$, $\sup_n \sup_{i,j} \mathrm{E}X_{ij}^4 I_{\{|X_{ij}|>M\}} \to 0$ as $M \to \infty$, and

$$y \in (\theta, \Theta) \text{ such that } 0 < \theta < \Theta < 1 \text{ or } 1 < \theta < \Delta < \infty. \tag{1.1}$$

If y is close to 1 the limit density and the Stieltjes transform of the limit density have a singularity. In this case the investigation of the rate of convergence is more difficult. Bai (1993b) showed that, if $0 < \theta \leqslant y_p \leqslant \Theta < \infty$, $\Delta_p = O(n^{-5/48})$. Recently Bai *et al.* (2003) have shown, for y_p equal to 1 or asymptotically near 1, that $\Delta_p = O(n^{-1/8})$. It is clear that the case $y_p \approx 1$ requires different techniques. Recent results of the authors show that for Gaussian random variables X_{ij} the rate $\Delta_p = O(n^{-1})$ is the correct rate of approximation.

In the present paper we shall consider bounds for Δ_p in the case (1.1) only. By C (with or without an index) we shall denote generic absolute constants, whereas $C(\cdot, \cdot)$ will denote positive constants depending on arguments. For $k \geqslant 1$, we introduce the notation

$$M_k := M_k^{(n)} := \sup_{1\leqslant j,k\leqslant n} \mathrm{E}|X_{jk}|^k.$$

Our main results are the following:

Theorem 1.1. *Let* $0 < \Theta_1 \leqslant y \leqslant \Theta_2 < \infty$ *and* $|y-1| \geqslant \theta > 0$. *Assume that* X_{ij} *satisfies the conditions above and that*

$$M_8 := \sup_{1 \leqslant j,k \leqslant n} \mathrm{E}|X_{jk}|^8 \leqslant \infty. \tag{1.2}$$

Then there exists an absolute constant $C(\theta, \Theta_1, \Theta_2) > 0$ *such that*

$$\Delta_p \leqslant C(\theta, \Theta_1, \Theta_2) M_8^{1/4} n^{-1/2}.$$

Theorem 1.2. *Let* $0 < \Theta_1 \leqslant y \leqslant \Theta_2 < \infty$ *and* $|y-1| \geqslant \theta > 0$. *Assume that* X_{ij} *satisfies the conditions above and condition* (1.2)*, and that*

$$M_{12} := \sup_{1 \leqslant j,k \leqslant n} \mathrm{E}|X_{jk}|^{12} < \infty. \tag{1.3}$$

Then there exists an absolute constant $C(\theta, \Theta_1, \Theta_2) > 0$ *such that*

$$\mathrm{E}\Delta_p^* = \mathrm{E}\sup_x |F_p(x) - G(x)| \leqslant C(\theta, \Theta_1, \Theta_2) M_{12}^{1/6} n^{-1/2}.$$

2. Inequalities for the distance between distributions via Stieltjes transforms

We define the Stieltjes transform $s(z)$ of a random variable ξ with distribution function $F(x)$ (the Stieltjes transform $s(z)$ of distribution function $F(x)$) by

$$s(z) := \mathrm{E}\frac{1}{\xi - z} = \int_{-\infty}^{\infty} \frac{1}{x-z}\,\mathrm{d}F(x), \qquad z = u + \mathrm{i}v,\ v > 0.$$

Given $\varepsilon > 0$, we introduce the intervals $I_\varepsilon = [a+\varepsilon, b-\varepsilon]$ and $I'_\varepsilon = [a + \frac{1}{2}\varepsilon, b - \frac{1}{2}\varepsilon]$. Recall that $a = a(y) = (1-\sqrt{y})^2$ and $b = b(y) = (1+\sqrt{y})^2$.

Lemma 2.1. *Let* F *be a distribution function and let* F_y *denote the Marchenko–Pastur distribution function. Denote their Stieltjes transforms by* $s(z)$ *and* $s_y(z)$ *respectively. Assume that* $\int_{-\infty}^{\infty} |F(x) - F_y(x)|\mathrm{d}x < \infty$. *Let* $v > 0$, *and* d *and* ε *be positive numbers such that*

$$\gamma = \frac{1}{\pi}\int_{|u| \leqslant d} \frac{1}{u^2+1}\,\mathrm{d}u = \frac{3}{4}, \tag{2.1}$$

and

$$\varepsilon > 2vd. \tag{2.2}$$

Assume that $|y-1| \geqslant \theta > 0$. *Then there exist some constants* $C_1(\theta), C_2(\theta), C_3(\theta)$, *depending only on* θ, *such that*

$$\Delta(F, F_y) := \sup_x |F(x) - F_y(x)|$$
$$\leqslant C_1 \sup_{x \in I'_\varepsilon} \left| \mathrm{Im}\left(\int_{-\infty}^{x} (s(z) - s_y(z))\mathrm{d}u \right) \right| + C_2 v + C_3 \varepsilon^{3/2},$$

where $z = u + \mathrm{i}v$.

A proof of Lemma 2.1 is given in Götze and Tikhomirov (2000; 2003).

Corollary 2.2. *The following inequality holds:*

$$\Delta(F,\, F_y) \leqslant C_1 \int_{-\infty}^{\infty} |(s(u + \mathrm{i}V) - s_y(u + \mathrm{i}V))| \mathrm{d}u + C_2 v + C_3 \varepsilon^{3/2}$$
$$+ C_1 \sup_{x \in I'_\varepsilon} \left| \mathrm{Im} \left\{ \int_v^V (s(x + \mathrm{i}u) - s_y(x + \mathrm{i}u)) \mathrm{d}u \right\} \right|. \tag{2.3}$$

3. The main lemma

We shall follow the notation of Bai (1993b). Let

$$s_y(z) = -\frac{y + z - 1 - \sqrt{(y + z - 1)^2 - 4yz}}{2yz}, \qquad s_p(z) = \int_{-\infty}^{\infty} \frac{1}{x - z}\, \mathrm{d}\mathrm{E}F_p(x). \tag{3.1}$$

Note that, for $z = u + \mathrm{i}v$ such that $a \leqslant u \leqslant b$,

$$|s_y(z)| \leqslant \frac{C|y - 1|}{|z|} + \frac{C}{\sqrt{|z|}} \leqslant \frac{C}{\sqrt{a}}. \tag{3.2}$$

By definition of $F_p(x)$, we can write

$$s_p(z) = \mathrm{E}\left(\frac{1}{p} \sum_{j=1}^{p} \frac{1}{\lambda_j - z} \right) = \frac{1}{p} \mathrm{E}\,\mathrm{tr}\,\mathbf{R} = \frac{1}{p} \sum_{j=1}^{p} \mathrm{E} R(j,\, j), \tag{3.3}$$

where $\mathbf{R} := \mathbf{R}(z) := (\mathbf{W} - z\,\mathbf{I}_p)^{-1} = (R(j,\, k))_{j,k=1}^{p}$. Here $\mathbf{I}_p$ denotes the $p \times p$ identity matrix.

Set $\mathbf{W}(k) = (1/n)\mathbf{X}(k)\mathbf{X}(k)^{\mathrm{T}}$, where $\mathbf{X}(k)$ denotes the matrix obtained from $\mathbf{X}$ by deleting the kth row, and let $\mathbf{x}_k^{\mathrm{T}} = (X_{k1}, \ldots, X_{kn})$. Set $\mathbf{a}_k = (1/n)\mathbf{X}(k)\mathbf{x}_k$. Write

$$\varepsilon_k = \frac{1}{n} \sum_{j=1}^{n} (X_{kj}^2 - 1) + y + yzs_p(z) - \mathbf{a}_k^{\mathrm{T}} (\mathbf{W}(k) - z\,\mathbf{I}_{p-1})^{-1} \mathbf{a}_k. \tag{3.4}$$

We introduce the scalar

$$\delta_p(z) = -\frac{1}{n} \sum_{k=1}^{p} \mathrm{E} \frac{\varepsilon_k}{(y + z - 1 + yzs_p(z))(y + z - 1 + yzs_p(z) - \varepsilon_k)} \tag{3.5}$$

and the matrix

$$\mathbf{R}_k = (\mathbf{W}(k) - z\,\mathbf{I}_{p-1})^{-1}.$$

For readers' convenience we state here two algebraic lemmas which are proved in Bai (1993a) and in Götze and Tikhomirov (2003). Let $\mathbf{A} = (a_{kj})$ denote a matrix of order n and

$\mathbf{A}_k$ denote the principal submatrix of order $n-1$, that is, $\mathbf{A}_k$ is obtained from $\mathbf{A}$ by deleting the kth row and the kth column. Let $\mathbf{A}^{-1} = (a^{jk})$. Let $\mathbf{a}_k^{\mathrm{T}}$ denote the vector obtained from the kth row of $\mathbf{A}$ by deleting the kth entry and $\mathbf{b}_k$ the vector from the kth column by deleting the kth entry. Let $\mathbf{I}$, with or without a subscript, denote the identity matrix of corresponding order.

Lemma 3.1. *Assume that* $\mathbf{A}$ *and* $\mathbf{A}_k$ *are non-singular. Then*

$$a^{kk} = \frac{1}{a_{kk} - \mathbf{a}_k^{\mathrm{T}} \mathbf{A}_k^{-1} \mathbf{b}_k}.$$

Lemma 3.2. *Let* $z = u + \mathrm{i}v$, *and* $\mathbf{A}$ *be an* $n \times n$ *symmetric matrix. Then*

$$\begin{aligned} \operatorname{tr}(\mathbf{A} - z\mathbf{I}_n)^{-1} - \operatorname{tr}(\mathbf{A}_k - z\mathbf{I}_{n-1})^{-1} &= \frac{1 + \mathbf{a}_k^{\mathrm{T}}(\mathbf{A}_k - z\mathbf{I}_{n-1})^{-2}\mathbf{a}_k}{a_{kk} - z - \mathbf{a}_k^{\mathrm{T}}(\mathbf{A}_k - z\mathbf{I}_{n-1})^{-1}\mathbf{a}_k} \\ &= (1 + \mathbf{a}_k^{\mathrm{T}}(\mathbf{A}_k - z\mathbf{I}_{n-1})^{-2}\mathbf{a}_k)a^{kk} \end{aligned} \tag{3.6}$$

and

$$|\operatorname{tr}(\mathbf{A} - z\mathbf{I}_n)^{-1} - \operatorname{tr}(\mathbf{A}_k - z\mathbf{I}_{n-1})^{-1}| \leqslant v^{-1}. \tag{3.7}$$

Applying Lemma 3.1 with $\mathbf{A} = \mathbf{W}$ and relation (3.3), we may write

$$\begin{aligned} R(j, j) &= -\frac{1}{y + z - 1 + yzs_p(z) - \varepsilon_j} \\ &= -\frac{1}{y + z - 1 + yzs_p(z)} - \frac{\varepsilon_j}{(y + z - 1 + yzs_p(z))(y + z - 1 + yzs_p(z) - \varepsilon_j)}. \end{aligned} \tag{3.8}$$

This implies that

$$s_p(z) = -\frac{1}{y + z - 1 + yzs_p(z)} + \delta_p(z). \tag{3.9}$$

To prove Theorem 1.1 we shall use the result of Corollary 2.2.

The following inequality (3.10) was proved in Bai (1993b), but for readers' convenience we repeat its proof here. Throughout this paper we shall consider $z = u + \mathrm{i}v$ with $a \leqslant u \leqslant b$ and $0 < v < C$.

Lemma 3.3. *Under the conditions of Theorem* 1.1, *for any* $v > 0$ *and for any* $k = 1, \ldots, n$, *we have*

$$|\mathrm{E}\varepsilon_k| \leqslant \frac{C}{nv}. \tag{3.10}$$

Proof. Let $\mathrm{E}^{(k)}$ denote the conditional expectation given X_{ij}, $i \neq k$. Note that the random vector $\mathbf{x}_k$ and the random matrices $\mathbf{W}(k)$, $\mathbf{X}_p(k)$ are independent. Using the definition of a vector $\mathbf{a}_k$ and taking into account the above-mentioned independence, we obtain

$$\begin{aligned}\mathrm{E}^{(k)}\mathbf{a}_k^{\mathrm{T}}(\mathbf{W}(k)-z\,\mathbf{I}_{p-1})^{-1}\mathbf{a}_k &= n^{-2}\mathrm{E}^{(k)}\mathbf{x}^{\mathrm{T}}(k)\mathbf{X}_p^{\mathrm{T}}(k)(\mathbf{W}(k)-z\,\mathbf{I}_{p-1})^{-1}\mathbf{X}_p(k)\mathbf{x}(k)\\ &= n^{-2}\,\mathrm{tr}\,\mathbf{X}_p^{\mathrm{T}}(k)(\mathbf{W}(k)-z\,\mathbf{I}_{p-1})^{-1}\mathbf{x}_p(k)\\ &= \frac{1}{n}\,\mathrm{tr}(\mathbf{W}(k)-z\,\mathbf{I}_{p-1})^{-1}\mathbf{W}(k).\end{aligned}$$

Furthermore, since

$$(\mathbf{W}(k)-z\,\mathbf{I}_{p-1})^{-1}\mathbf{W}(k) = z(\mathbf{W}(k)-z\,\mathbf{I}_{p-1})^{-1}+\mathbf{I}_{p-1},$$

we obtain

$$\begin{aligned}\mathrm{E}^{(k)}\mathbf{a}_k^{\mathrm{T}}(\mathbf{W}(k)-z\,\mathbf{I}_{p-1})^{-1}\mathbf{a}_k &= \frac{p-1}{n}+zyp^{-1}\,\mathrm{tr}(\mathbf{W}(k)-z\,\mathbf{I}_{p-1})^{-1}\\ &= y-\frac{1}{n}+zyp^{-1}\,\mathrm{tr}(\mathbf{W}(k)-z\,\mathbf{I}_{p-1})^{-1}.\end{aligned} \tag{3.11}$$

Equation (3.11) implies that

$$|\mathrm{E}\varepsilon_k| = \frac{|zy|}{p}|\mathrm{E}[\mathrm{tr}(\mathbf{W}(k)-z\,\mathbf{I}_{p-1})^{-1}-\mathrm{tr}(\mathbf{W}-z\,\mathbf{I}_p)^{-1}]|+\frac{1}{n}.$$

Using Lemma 3.2 with $\mathbf{A}=\mathbf{W}$ and $\mathbf{A}_k=\mathbf{W}(k)$, we obtain inequality (3.10). □

Without loss of generality, we may assume that $v \geqslant \beta\Delta_p$ with some constant $0<\beta<1$ depending on θ only. Thus, using inequality (3.2), we immediately obtain, for $z=u+\mathrm{i}v$ such that $a \leqslant u \leqslant b$,

$$|s_p(z)| \leqslant |s_y(z)|+|s_p(z)-s_p(z)| \leqslant C_1(\theta)(1+\beta^{-1}) \leqslant C(\theta)\beta^{-1}.$$

The main result of this section is the following:

Lemma 3.4. *Let*

$$\mathrm{Im}\{yz\delta_p(z)+z\} \geqslant 0.$$

Then there exists a positive constant a_1 depending on θ, Θ_1, Θ_2 and β such that

$$|z+y-1+yzs_p(z)| \geqslant a_1.$$

Proof. Assume that $|z| \leqslant |y-1|/2(1+yC(\theta)\beta^{-1})$. This immediately implies that

$$|z+y-1+yzs_p(z)| \geqslant |y-1|-|z|(1+y|s_p(z)|) \geqslant \frac{|y-1|}{2} \geqslant a_1 > 0.$$

Now let

$$|z| \geqslant \frac{|y-1|}{2(1+yC(\theta)\beta^{-1})}.$$

Equation (3.9) and the assumption of Lemma 3.4 together imply that

$$\mathrm{Im}\{z + yzs_p(z)\} \geqslant -\mathrm{Im}\left\{\frac{yz}{z+y-1+yzs_p(z)}\right\}.$$

Note that

$$\mathrm{Im}\{z + yzs_p(z)\} = v + y(v\,\mathrm{Re}\{s_p(z)\} + u\,\mathrm{Im}\{s_p(z)\}) = v + yv\mathrm{E}\,\mathrm{tr}\,\mathbf{W}|\mathbf{R}|^2 > 0.$$

Hence,

$$|z+y-1+yzs_p(z)|^2 \geqslant \frac{-\mathrm{Im}\{yz(\bar{z}+y-1+y\bar{z}\overline{s_p(z)})\}}{\mathrm{Im}\{z+yzs_p(z)\}}, \tag{3.12}$$

where $\overline{w}$ denotes the complex conjugate of w. Furthermore,

$$\begin{aligned} -\mathrm{Im}\{yz(\bar{z}+y-1+y\bar{z}\overline{s_p(z)})\} &= -y\,\mathrm{Im}\{|z|^2 + z(y-1) + y|z|^2\overline{s}_p(z)\} \\ &= y^2|z|^2\,\mathrm{Im}\{s_p(z)\} + y(1-y)v. \end{aligned} \tag{3.13}$$

If $y \leqslant 1$, we have

$$\frac{-\mathrm{Im}\{yz(\bar{z}+y-1+y\bar{z}\overline{s_p(z)})\}}{\mathrm{Im}\{z+yzs_p(z)\}} = \frac{y(1-y)v + y^2|z|^2\,\mathrm{Im}\{s_p(z)\}}{v + yv\mathrm{Re}\{s_p(z)\} + yu\,\mathrm{Im}\{s_p(z)\}}. \tag{3.14}$$

Assuming $\mathrm{Im}\{s_p(z)\} \leqslant v$, we obtain

$$\begin{aligned} \frac{-\mathrm{Im}\{yz(\bar{z}+y-1+y\bar{z}\overline{s_p(z)})\}}{\mathrm{Im}\{z+yzs_p(z)\}} &\geqslant \frac{y(1-y)v}{v(1+yC(\theta)\beta^{-1}+yb)} \\ &= y(y-1)(1+yC(\theta)\beta^{-1}+yb)^{-1} \geqslant a_1 > 0. \end{aligned} \tag{3.15}$$

If $\mathrm{Im}\{s_p(z)\} \geqslant v$, then

$$\begin{aligned} \frac{-\mathrm{Im}\{yz(\bar{z}+y-1+y\bar{z}\overline{s_p(z)})\}}{\mathrm{Im}\{z+yzs_p(z)\}} &\geqslant \frac{y^2|z|^2\,\mathrm{Im}\{s_p(z)\}}{(1+yC(\theta)\beta^{-1}+yb)\mathrm{Im}\{s_p(z)\}} \\ &\geqslant y^2a^2(1+yC(\theta)\beta^{-1}+yb)^{-1} \geqslant a_1 > 0. \end{aligned} \tag{3.16}$$

Inequalities (3.12), (3.15) and (3.16) together complete the proof for $y \leqslant 1$.

Consider the case $y \geqslant 1$. Assuming $\mathrm{Im}\{s_p(z)\} \geqslant 2v(y-1)/ya^2 \geqslant 2v(y-1)/y|z|^2$, we obtain

$$\frac{-\mathrm{Im}\{yz(\bar{z}+y-1+y\bar{z}\overline{s_p(z)})\}}{\mathrm{Im}\{z+yzs_p(z)\}} \geqslant \frac{\frac{1}{2}y^2|z|^2\,\mathrm{Im}\{s_p(z)\}}{v+yv\,\mathrm{Re}\{s_p(z)\}+yu\,\mathrm{Im}\{s_p(z)\}}$$
$$\geqslant \frac{\frac{1}{2}y^2|z|^2\,\mathrm{Im}\{s_p(z)\}}{((1+yC(\theta)\beta^{-1})(ya^2/2(y-1))+yb)\mathrm{Im}\{s_p(z)\}}$$
$$\geqslant \frac{\frac{1}{2}y^2|a|^2}{(1+yC(\theta)\beta^{-1})(ya^2/2(y-1))+yb}.$$

Write $B=2(y-1)/ya^2$ and assume that

$$\mathrm{Im}\,s_p(z)\leqslant Bv. \tag{3.17}$$

If $\mathrm{Im}\{\delta_p(z)\}\geqslant 0$, then (3.9) implies

$$\mathrm{Im}\{s_p(z)\}\geqslant \frac{v+\mathrm{Im}\{yzs_p(z)\}}{|z+y-1+yzs_p(z)|^2}. \tag{3.18}$$

Since $\mathrm{Im}(yzs_p(z))\geqslant 0$, inequalities (3.17) and (3.18) together imply

$$|z+y-1+yzs_p(z)|\geqslant B^{-1/2}\geqslant a_1>0. \tag{3.19}$$

If $\mathrm{Im}\,\delta_p(z)\leqslant 0$ the condition $\mathrm{Im}(z+yz\delta_p(z))>0$ implies

$$|\mathrm{Im}\,\delta_p(z)|\leqslant v\frac{1+|\delta_p(z)|}{yu}. \tag{3.20}$$

From (3.9) it follows that

$$|\delta_p(z)|\leqslant |z+y-1+yzs_p(z)|^{-1}+\beta^{-1}. \tag{3.21}$$

Without loss of generality, we may assume that

$$|z+y-1+yzs_p(z)|\leqslant \frac{ya}{2}.$$

Thus inequalities (3.9), (3.20) and (3.21) together imply that

$$\mathrm{Im}\{s_p(z)\}\geqslant \frac{v}{|z+y-1+yzs_p(z)|^2}-\frac{v}{ya|z+y-1+yzs_p(z)|}-\frac{(1+\beta^{-1})v}{ya}.$$

From this inequality and assumption (3.17) it follows that

$$|z+y-1+yzs_p(z)|\geqslant y^{-1}a^{-1}\left(B+\frac{1+\beta^{-1}}{ya}\right)^{-1}.$$

This completes the proof of Lemma 3.4. □

4. Bounds for the function $\delta_p(z)$

In this section we shall assume that there exist some positive constants a_1 and a_2 depending only on θ, Θ_1, Θ_2 and β such that

$$a_1 \leqslant |z + y - 1 + yzs_p(z)| \leqslant a_2. \tag{4.1}$$

We introduce in addition the following notation:

$$\varepsilon_j^{(1)} = \frac{1}{n}\sum_{l=1}^{n}(X_{jl}^2 - 1), \qquad \varepsilon_j^{(2)} = -\frac{1}{n}(\mathbf{x}_j^{\mathrm{T}}\mathbf{X}(j)^{\mathrm{T}}\mathbf{R}_j\mathbf{X}(j)\mathbf{x}_j - \operatorname{tr}\mathbf{R}_j\mathbf{W}(j)),$$

$$\varepsilon_j^{(3)} = -\frac{z}{n}(\operatorname{tr}\mathbf{R} - \operatorname{tr}\mathbf{R}_j), \qquad \varepsilon_j^{(4)} = \frac{1}{n}, \qquad \varepsilon_j^{(5)} = \frac{z}{n}(\operatorname{tr}\mathbf{R} - \mathrm{E}\operatorname{tr}\mathbf{R}).$$

Note that

$$\operatorname{tr}\mathbf{R}_j\mathbf{W}(j) = \operatorname{tr}\mathbf{I}_{p-1} + z\operatorname{tr}\mathbf{R}_j. \tag{4.2}$$

Using (4.2), we obtain the representation

$$\varepsilon_j = \sum_{\nu=1}^{5}\varepsilon_j^{(\nu)}. \tag{4.3}$$

This representation implies, for $j = 1, \ldots, p$,

$$\mathrm{E}|\varepsilon_j|^2 \leqslant 5\sum_{\nu=1}^{5}\mathrm{E}|\varepsilon_j^{(\nu)}|^2. \tag{4.4}$$

Lemma 4.1. *Under condition* (4.1) *there exist some constants* $C_1(a_1, a_2)$ *and* $C_2(a_1, a_2)$ *depending on* a_1 *and* a_2 *only, such that, for* $u \in [a, b]$ *and* $1 \geqslant v \geqslant C_1(a_1, a_2)\sqrt{M_4}n^{-1/2}$,

$$\mathrm{E}|\varepsilon_j^{(1)}|^2 \leqslant \frac{CM_4}{n}.$$

Proof. The proof of this bound is trivial. □

Lemma 4.2. *Under condition* (4.1) *there exist some constants* $C_1(a_1, a_2)$ *and* $C_2(a_1, a_2)$ *depending on* a_1 *and* a_2 *only, such that, for* $u \in [a, b]$ *and* $1 \geqslant v \geqslant C_1(a_1, a_2)\sqrt{M_4}n^{-1/2}$,

$$\mathrm{E}|\varepsilon_j^{(2)}|^2 \leqslant \frac{C_2(a_1, a_2)M_4}{nv}. \tag{4.5}$$

Proof. Since $\mathbf{x}_j$ and $\mathbf{X}(j)$ are independent, we have

$$\mathrm{E}|\varepsilon_j^{(2)}|^2 \leqslant \frac{CM_4}{n^2}\mathrm{E}\operatorname{tr}|\mathbf{R}_j\mathbf{W}(j)|^2. \tag{4.6}$$

Here and in what follows we use the notation $|\mathbf{A}|^2 := \mathbf{A}\overline{\mathbf{A}}^{\mathrm{T}}$, for any complex matrix $\mathbf{A}$. It is easy to check that

$$\mathrm{tr}|\mathbf{R}_j\mathbf{W}(j)|^2 = v^{-1}\,\mathrm{Im}\{\mathrm{tr}\,\mathbf{R}_j\mathbf{W}(j)^2\}. \tag{4.7}$$

Using (4.7), we obtain

$$\mathrm{E}\,\mathrm{tr}|\mathbf{R}_j\mathbf{W}(j)|^2 \leqslant (p-1) + v^{-1}|z^2||\mathrm{E}\,\mathrm{tr}\,\mathbf{R}_j|. \tag{4.8}$$

Since, by (3.6),

$$|\mathrm{E}\,\mathrm{tr}\,\mathbf{R}_j| \leqslant |\mathrm{E}\,\mathrm{tr}\,\mathbf{R}| + v^{-1}, \tag{4.9}$$

from (4.4) and (4.6) we obtain

$$\mathrm{E}|\varepsilon_j^{(2)}|^2 \leqslant \frac{CM_4}{n^2 v}|\mathrm{E}\,\mathrm{tr}\,\mathbf{R}| + \frac{CM_4}{n^2v^2} + \frac{CM_4}{nv}. \tag{4.10}$$

Inequality (4.10) and assumption (4.1) together imply

$$\mathrm{E}|\varepsilon_j^{(2)}|^2 \leqslant \frac{C(a_1, a_2)M_4}{nv} + \frac{C(a_1, a_2)M_4}{n^2v^2}. \tag{4.11}$$

This concludes the proof. □

Lemma 4.3. *Under condition* (4.1) *there exist some constants C such that, for $u \in [a, b]$ and $1 \geqslant v > 0$,*

$$\mathrm{E}|\varepsilon_j^{(3)}|^2 \leqslant \frac{CM_4}{nv}.$$

Proof. The proof follows from inequality (3.7) with $\mathbf{A} = \mathbf{W}$ and $\mathbf{A}_k = \mathbf{W}(k)$. □

Lemma 4.4. *Under condition* (4.1) *there exist some constants $C_1(a_1, a_2)$ and $C_2(a_1, a_2)$ depending on a_1 and a_2 only, such that, for $u \in [a, b]$ and $1 \geqslant v \geqslant C_1(a_1, a_2)M_8^{1/4}n^{-1/2}$,*

$$\mathrm{E}|\varepsilon_j^{(5)}|^2 \leqslant \frac{C_2(a_1, a_2)M_4}{n^2v^3}.$$

Proof. Note that $\varepsilon_j^{(5)}$ does not depend on $j = 1, \ldots, n$. To obtain a bound for $\mathrm{E}|\varepsilon_j^{(5)}|^2$ we shall use the method of martingale differences which was first used for random matrices in Girko (1989); see also Girko (1990). Let

$$\sigma_k = \mathrm{tr}(\mathbf{W} - z\,\mathbf{I}_p)^{-1} - \mathrm{tr}(\mathbf{W}(k) - z\,\mathbf{I}_{p-1})^{-1} \tag{4.12}$$

and

$$\gamma_k = \mathrm{E}_{k-1}\sigma_k - \mathrm{E}_k\sigma_k. \tag{4.13}$$

Here and in what follows let E_k denote the conditional expectation given X_{jl} with $k \leqslant j \leqslant p$, $1 \leqslant l \leqslant n$. It is easy to check that

$$\mathrm{E}|\mathrm{tr}(\mathbf{W} - z\,\mathbf{I}_p)^{-1} - \mathrm{E}\,\mathrm{tr}(\mathbf{W} - z\,\mathbf{I}_p)|^2 = \sum_{k=1}^{p}\mathrm{E}|\gamma_k|^2. \tag{4.14}$$

By (3.6), we have

$$\sigma_k = \{(1 + \mathbf{a}_k^{\mathrm{T}}(\mathbf{W}(k) - z\,\mathbf{I}_{p-1})^{-2}\mathbf{a}_k)R(k,\, k). \tag{4.15}$$

Write

$$\sigma_k = \sigma_k^{(1)} + \sigma_k^{(2)} + \sigma_k^{(3)}, \tag{4.16}$$

where

$$\sigma_k^{(1)} = \frac{1 + (1/n)\mathrm{tr}(\mathbf{W}(k) - z\,\mathbf{I}_{p-1})^{-2}\mathbf{W}(k)}{z + y + yzs_p(z)}, \qquad \sigma_k^{(2)} = \frac{\varepsilon_k \sigma_k}{z + y + yzs_p(z)},$$

$$\sigma_k^{(3)} = \frac{\mathbf{a}^{\mathrm{T}}(k)(\mathbf{W}(k) - z\,\mathbf{I}_{p-1})^{-2}\mathbf{a}_k - (1/n)\mathrm{tr}(\mathbf{W}(k) - z\,\mathbf{I}_{p-1})^{-2}\mathbf{W}(k)}{z + y + yzs_p(z)}.$$

Since $\mathrm{E}_{k-1}\sigma_k^{(1)} - \mathrm{E}_k\sigma_d^{(1)} = 0$, we obtain

$$\begin{aligned}\mathrm{E}|\gamma_k|^2 &\leqslant \mathrm{E}|\sigma_k^{(2)}|^2 + \mathrm{E}|\sigma_k^{(3)}|^2 \\ &\leqslant C(a_1,\, a_2)\left(v^{-2}\mathrm{E}|\varepsilon_k|^2 + \mathrm{E}\left|\mathbf{a}^{\mathrm{T}}(k)(\mathbf{W}(k) - z\,\mathbf{I}_{p-1})^{-2}\mathbf{a}_k - \frac{1}{n}\mathrm{tr}(\mathbf{W}(k) - z\,\mathbf{I}_{p-1})^{-2}\mathbf{W}(k)\right|^2\right).\end{aligned} \tag{4.17}$$

By the representation (4.3) of ε_k and by Lemmas 4.1–4.3, we have

$$\mathrm{E}|\varepsilon_k|^2 \leqslant 5\sum_{\nu=1}^{4}|\mathrm{E}\varepsilon_k^{(\nu)}|^2 + 5\,\mathrm{E}|\varepsilon_k^{(5)}|^2 \leqslant \frac{C(a_1,\, a_2)M_4}{nv} + 5\,\mathrm{E}|\varepsilon_k^{(5)}|^2. \tag{4.18}$$

Similarly to the proof of Lemma 4.3, we obtain

$$\mathrm{E}\left|\mathbf{a}^{\mathrm{T}}(k)(\mathbf{W}(k) - z\,\mathbf{I}_{p-1})^{-2}\mathbf{a}(k) - \frac{1}{n}\,\mathrm{tr}(\mathbf{W}(k) - z\,\mathbf{I}_{p-1})^{-2}\mathbf{W}(k)\right|^2 \leqslant \frac{CM_4}{n^2}\mathrm{E}\,\mathrm{tr}\,\mathbf{G}^{(1)}(k)\overline{\mathbf{G}^{(1)}}(k), \tag{4.19}$$

where $\mathbf{G}^{(1)}(k) = (\mathbf{W}(k) - z\,\mathbf{I}_{p-1})^{-2}\mathbf{W}(k)$. It is easy to check that

$$\mathrm{E}\,\mathrm{tr}\,\mathbf{G}^{(1)}(k)\overline{\mathbf{G}^{(1)}(k)} \leqslant v^{-3}\,\mathrm{Im}\{\mathrm{E}\,\mathrm{tr}((\mathbf{W}(k) - z\,\mathbf{I}_{p-1})^{-1}\mathbf{W}^2(k))\}.$$

Using (4.7), we obtain

$$\mathrm{E}\,\mathrm{tr}\,\mathbf{G}^{(1)}(k)\overline{\mathbf{G}^{(1)}(k)} \leqslant v^{-3}(v(p-1) + |z|^2|\mathrm{E}\,\mathrm{tr}\,\mathbf{R}_k|). \tag{4.20}$$

Inequalities (4.1) and (4.20) together imply

$$\mathrm{E}\,\mathrm{tr}\,\mathbf{G}^{(1)}(k)\overline{\mathbf{G}^{(1)}(k)} \leqslant v^{-3}(v(p-1) + nC(a_1,\, a_2))). \tag{4.21}$$

From (4.19) and (4.21), it follows that

$$\mathrm{E}\left|\mathbf{a}^{\mathrm{T}}(k)(\mathbf{W}(k) - z\,\mathbf{I}_{p-1})^{-2}\mathbf{a}(k)\text{-}\frac{1}{n}\,\mathrm{tr}(\mathbf{W}(k) - z\,\mathbf{I}_{p-1})^{-2}\mathbf{W}(k)\right|^2 \leqslant \frac{C(a_1,\, a_2)M_4}{nv^2}\left(1 + \frac{1}{v}\right). \tag{4.22}$$

The relations (4.14), (4.17), (4.18) and (4.22) together imply

$$\mathrm{E}|\varepsilon_k^{(5)}|^2 \leqslant \frac{C(a_1, a_2)M_4}{n^2 v^3} + \frac{C(a_1, a_2)M_4}{n v^2}\mathrm{E}|\varepsilon_k^{(5)}|^2. \tag{4.23}$$

Inequality (4.23) implies that, for some positive constant $C_1(a_1, a_2)$ and for $v \geqslant C_1(a_1, a_2)\sqrt{M_4}n^{-1/2}$,

$$\frac{1}{n^2}\mathrm{E}|\mathrm{tr}(\mathbf{W} - z\mathbf{I}_p)^{-1} - \mathrm{E}\,\mathrm{tr}(\mathbf{W} - z\mathbf{I}_p)^{-1}|^2 \leqslant C(a_1, a_2)M_4 n^{-2} v^{-3}, \tag{4.24}$$

which proves the lemma. □

Let us introduce the matrices

$$\begin{aligned}
\mathbf{G} &= (G(j, k))_{j,k=1}^{p} = \frac{1}{\mathrm{n}}\mathbf{X}_p^{\mathrm{T}}(W_p - z\mathbf{I}_p)^{-1}\mathbf{X}_p,\\
\mathbf{G}(k) &= (\mathrm{G}_k(j, l)) = \frac{1}{n}\mathbf{X}_p^{\mathrm{T}}(k)(\mathbf{W}(k) - z\mathbf{I}_{p-1})^{-1}\mathbf{X}_p(k),\\
\mathbf{W}(k, d) &= \frac{1}{n}\mathbf{X}_p(k, d)\mathbf{X}_p^{\mathrm{T}}(k, d),\\
\mathbf{R}_{kd} &= (\mathbf{W}(k, d) - z\mathbf{I}_{p-2})^{-1},\\
\mathbf{G}(k, d) &= (G_{kd}(j, l)) = \frac{1}{n}\mathbf{X}_p^{\mathrm{T}}(k, d)\mathbf{R}_{kd}\mathbf{X}_p(k, d),
\end{aligned}$$

where $\mathbf{X}_p(k, d)$ is obtained from $\mathbf{X}_p$ by deleting the kth and dth rows. Note that

$$\mathrm{tr}\,\mathbf{G} = \mathrm{tr}(\mathbf{W} - z\mathbf{I}_p)^{-1}\mathbf{W}, \qquad \mathrm{tr}\,\mathbf{G}(k) = \mathrm{tr}(\mathbf{W}(k) - z\mathbf{I}_{p-1})^{-1}\mathbf{W}(k).$$

The next lemma is similar to Lemma 5.4 in Götze and Tikhomirov (2003).

Lemma 4.5. *There exist positive constants $C_1(a_1, a_2)$ and $C_2(a_1, a_2)$ such that, for any $1 \geqslant v \geqslant C_1(a_1, a_2)\sqrt{M}n^{-1/2}$,*

$$\frac{1}{p}\sum_{k=1}^{p}\mathrm{E}|R(k, k)|^2 \leqslant C_2(a_1, a_2).$$

Proof. Equation (3.5), condition (4.1) and the representation (4.3) together imply

$$\mathrm{E}|R(k, k)|^2 \leqslant C(a_1, a_2)\left(1 + \sum_{\nu=1}^{5}\mathrm{E}|\varepsilon_k^{(\nu)}|^2|R(k, k)|^2\right). \tag{4.25}$$

It is obvious that

$$\mathrm{E}|\varepsilon_k^{(1)}|^2|R(k, k)|^2 \leqslant \frac{CM_4}{nv^2}, \qquad \mathrm{E}|\varepsilon_k^{(4)}|^2|R(k, k)|^2 \leqslant \frac{1}{n^2 v^2}. \tag{4.26}$$

By Lemma 3.2, we have

$$\mathrm{E}|\varepsilon_k^{(3)}|^2|R(k,\,k)|^2 \leqslant \frac{1}{n^2v^4}. \tag{4.27}$$

Using Rosenthal's inequality for quadratic forms or direct calculation, we obtain

$$\mathrm{E}|\varepsilon_k^{(2)}|^4 \leqslant \frac{CM_8}{n^4}\mathrm{E}\left(\sum_{\substack{l,m=1\\ l\neq k, m\neq k}}^{n} |G_k(l,\,m)|^2\right)^2.$$

We may write

$$\mathrm{E}\left(\sum_{\substack{l,m=1\\ l\neq k, m\neq k}}^{n} |G_k(l,\,m)|^2\right)^2 \leqslant |\mathrm{E}\,\mathrm{tr}|\mathbf{G}(k)|^2|^2 + \mathrm{E}|\mathrm{tr}|\mathbf{G}(k)|^2 - \mathrm{E}\,\mathrm{tr}|\mathbf{G}(k)|^2|^2.$$

By relations (4.2), (4.7) and condition (4.1), we have

$$\mathrm{E}\,\mathrm{tr}|\mathbf{G}(k)|^2 \leqslant \frac{C(a_1,\,a_2)n}{v}. \tag{4.28}$$

Similarly as in the bounds for $\mathrm{E}|\mathrm{tr}\,\mathbf{R} - \mathrm{E}\,\mathrm{tr}\,\mathbf{R}|^2$, we introduce the random variables

$$\tilde{\gamma}_d(k) = \mathrm{E}_{d-1}\,\mathrm{tr}|\mathbf{G}(k)|^2 - \mathrm{E}_d\,\mathrm{tr}|\mathbf{G}(k)|^2 = \mathrm{E}_{d-1}\tilde{\sigma}_k(\mathrm{d}) - \mathrm{E}_d\tilde{\sigma}_k(d),$$

with $\tilde{\sigma}_d(k) = \mathrm{tr}|\mathbf{G}(k)|^2 - \mathrm{tr}|\mathbf{G}(k,\,d)|^2$. Since the $\tilde{\gamma}_d(k)$ are orthogonal, for $d = 1,\,\ldots,\,p$, we obtain

$$\frac{1}{n^4}\mathrm{E}|\mathrm{tr}|\mathbf{G}(k)|^2 - \mathrm{E}\,\mathrm{tr}|\mathbf{G}(k)|^2|^2 \leqslant \frac{1}{n^4}\sum_{d=1}^{p}\mathrm{E}|\tilde{\gamma}_d(k)|^2.$$

Note that, according to (4.7),

$$|\mathrm{tr}|\mathbf{G}(k)|^2 - \mathrm{tr}|\mathbf{G}(k,\,d)|^2| = \frac{1}{v}|\mathrm{Im}\{z^2(\mathrm{tr}\,\mathbf{R}_k - \mathrm{tr}\,\mathbf{R}_{k,d}) + z\}| \leqslant \frac{C}{v^2}.$$

This implies that $|\tilde{\gamma}_k(\mathrm{d})| \leqslant Cv^{-2}$ and

$$\frac{1}{n^4}\mathrm{E}|\mathrm{tr}|\mathbf{G}(k)|^2 - \mathrm{E}\,\mathrm{tr}|\mathbf{G}(k)|^2|^2 \leqslant \frac{C}{n^3v^4}. \tag{4.29}$$

Inequalities (4.27)–(4.29) together imply that, for $v \geqslant C(a_1,\,a_2)M_8^{1/4}n^{-1/2}$,

$$\mathrm{E}|\varepsilon_k^{(2)}|^4 \leqslant \frac{C\sqrt{M_8}}{n^2v^2}. \tag{4.30}$$

Using Cauchy's inequality, we obtain

$$\frac{1}{p}\sum_{k=1}^{p}\mathrm{E}|\varepsilon_k^{(2)}|^2|R(k,\,k)|^2 \leqslant v^{-1}\left(\frac{1}{p}\sum_{k=1}^{p}\mathrm{E}|\varepsilon_k^{(2)}|^4\right)^{1/2}\left(\frac{1}{p}\sum_{k=1}^{p}\mathrm{E}|R(k,\,k)|^2\right)^{1/2}$$

$$\leqslant \frac{CM_8^{1/4}}{nv^2}\left(\frac{1}{p}\sum_{k=1}^{p}\mathrm{E}|R(k,\,k)|^2\right)^{1/2}. \qquad (4.31)$$

Notice that

$$\frac{1}{p}\sum_{k=1}^{p}\mathrm{E}|\varepsilon_k^{(5)}|^2|R(k,\,k)|^2 = \mathrm{E}|\varepsilon_1^{(5)}|^2\left(\frac{1}{p}\sum_{k=1}^{p}|R(k,\,k)|^2\right)$$

$$\leqslant \mathrm{E}|\varepsilon_1^{(5)}|^2\left(\frac{1}{p}\sum_{k,j=1}^{p}|\mathrm{R}(k,\,j)|^2\right) = v^{-1}\mathrm{E}|\varepsilon_1^{(5)}|^2\,\mathrm{Im}\left\{\frac{1}{p}\,\mathrm{tr}\,\mathbf{R}\right\}$$

$$\leqslant \frac{1}{vp}\mathrm{E}|\varepsilon_1^{(5)}|^2|\mathrm{tr}\,\mathbf{R}-\mathrm{E}\,\mathrm{tr}\,\mathbf{R}| + \frac{|\mathrm{E}\,\mathrm{tr}\,\mathbf{R}|}{pv}\mathrm{E}|\varepsilon_1^{(5)}|^2. \qquad (4.32)$$

Furthermore,

$$\frac{1}{vp}\mathrm{E}|\varepsilon_1^{(5)}|^2|\mathrm{tr}\,\mathbf{R}-\mathrm{E}\,\mathrm{tr}\,\mathbf{R}| \leqslant \frac{C}{vn^3}\mathrm{E}|\mathrm{tr}\,\mathbf{R}-\mathrm{E}\,\mathrm{tr}\,\mathbf{R}|^3.$$

By Burkholder's inequality for martingales (see Hall and Heyde 1980, p. 24), we obtain

$$\mathrm{E}|\mathrm{tr}\,\mathbf{R}-\mathrm{E}\,\mathrm{tr}\,\mathbf{R}|^3 \leqslant C\sqrt{p}\sum_{k=1}^{p}\mathrm{E}|\gamma_k|^3, \qquad (4.33)$$

with γ_k defined in (4.13). Inequalities (4.17), (4.18) and (4.22) together imply that, for $1 \geqslant v \geqslant C_1(a_1,\,a_2)M_8^{1/4}n^{-1/2}$,

$$\mathrm{E}|\gamma_k|^2 \leqslant \frac{C(a_1,\,a_2)M_8^{1/2}}{nv^3}.$$

Since $|\gamma_k| \leqslant 2v^{-1}$, we obtain

$$\mathrm{E}|\gamma_k|^3 \leqslant \frac{C(a_1,\,a_2)}{nv^4}. \qquad (4.34)$$

From (4.33) and (4.34) we obtain

$$\frac{1}{vn^3}\mathrm{E}|\mathrm{tr}\,\mathbf{R}-\mathrm{E}\,\mathrm{tr}\,\mathbf{R}|^3 \leqslant \frac{C(a_1,\,a_2)M_8^{1/2}}{\sqrt{n^5v^{10}}}. \qquad (4.36)$$

Inequalities (4.32), (4.36), (4.1) and Lemma 4.4 together imply

$$\frac{1}{n}\sum_{k=1}^{n}\mathrm{E}|\varepsilon_k^{(5)}|^2|G_{kk}|^2 \leqslant \frac{C(a_1,\,a_2)M_8^{1/2}}{\sqrt{n^5v^{10}}} + \frac{C(a_1,\,a_2)M_8^{1/2}}{nv^2}. \qquad (4.37)$$

Inequalities (4.25)–(4.27), (4.31) and (4.37) finally yield, for $v \geqslant C(a_1,\,a_2)M_8^{1/4}n^{-1/2}$,

$$\frac{1}{p}\sum_{j=1}^{p}\mathrm{E}|R(j,j)|^2 \leqslant C_1(a_1,a_2)+C_2(a_1,a_2)\left(\frac{1}{p}\sum_{j=1}^{p}\mathrm{E}|R(j,j)|^2\right)^{1/2}.$$

From the last inequality it follows that, for $v \geqslant C(a_1,a_2)M_8^{1/4}n^{-1/2}$,

$$\frac{1}{p}\sum_{j=1}^{p}\mathrm{E}|R(j,j)|^2 \leqslant C(a_1,a_2),$$

which completes the proof. □

Lemma 4.6. *There exist positive constants $C_1(a_1,a_2)$ and $C_2(a_1,a_2)$ such that, for any $v \geqslant C_1(a_1,a_2)M_8^{1/4}n^{-1/2}$,*

$$|\delta_p(z)| \leqslant \frac{C_2(a_1,a_2)\sqrt{M_8}}{nv}.$$

Proof. By the definition of $\delta_p(z)$, (3.4) implies that

$$|\delta_p(z)| \leqslant |y+z-1+yzs_p(z)|^{-1}\left|\frac{1}{p}\sum_{j=1}^{p}\mathrm{E}\varepsilon_j R(j,j)\right|. \tag{4.38}$$

Taking into account that $\mathrm{E}\varepsilon_j^{(\nu)}=0$, for $\nu=1,2$, $|\mathrm{E}\varepsilon_j^{(\nu)}|\leqslant 1/nv$, for $\nu=3$, and $|\mathrm{E}\varepsilon^{\nu}|\leqslant 1/n$, for $\nu=4$, and expanding $R(j,j)$ into the parts ε_j defined in (4.3), we obtain

$$\begin{aligned}|\delta_p(z)| \leqslant\ & \sum_{\nu=1}^{3}\frac{C}{p}\sum_{j=1}^{p}\mathrm{E}|\varepsilon_j^{\nu}|^2|R(j,j)| \\ &+\sum_{\nu=1}^{3}\frac{C}{p}\left|\sum_{j=1}^{p}\mathrm{E}\varepsilon_j^{(\nu)}\varepsilon_j^{(5)}R(j,j)\right|+\frac{C}{nv}+\frac{C}{p}\left|\sum_{j=1}^{p}\mathrm{E}\varepsilon_j^{(5)}R(j,j)\right|.\end{aligned} \tag{4.39}$$

Since $|R(j,j)|\leqslant v^{-1}$ and $\mathrm{E}|\varepsilon_j^{(1)}|^2\leqslant C\sqrt{M_8}n^{-1}$, we obtain

$$\frac{1}{p}\sum_{j=1}^{p}\mathrm{E}|\varepsilon_j^{(1)}|^2|R(j,j)| \leqslant \frac{C\sqrt{M_8}}{nv}. \tag{4.40}$$

Lemma 3.2 and the definition of both $\varepsilon_j^{(3)}$ and $\varepsilon_j^{(4)}$ together imply that

$$\frac{1}{p}\sum_{j=1}^{p}\mathrm{E}|\varepsilon_j^{(3)}|^2|R(j,j)| \leqslant \frac{C}{n^2v^3}, \qquad \frac{1}{p}\sum_{j=1}^{p}\mathrm{E}|\varepsilon_j^{(4)}|^2|R(j,j)| \leqslant \frac{C}{n^2v}. \tag{4.41}$$

Applying Hölder's inequality, inequality (4.30) and Lemma 4.5, we obtain that

$$\frac{C}{p}\sum_{j=1}^{p}\mathrm{E}|\varepsilon_j^{(2)}|^2|R(j,j)| \leqslant \frac{C}{p}\left(\sum_{j=1}^{p}\mathrm{E}|\varepsilon_j^{(2)}|^4\right)^{1/2}\left(\sum_{j=1}^{p}\mathrm{E}|R(j,j)|^2\right)^{1/2} \leqslant \frac{C\sqrt{M_8}}{nv}. \tag{4.42}$$

Consider now the summand with $\varepsilon_j^{(5)}$. Write

$$\varepsilon_j^{(5)} = \varepsilon_j' + \varepsilon_j'',$$

where

$$\varepsilon_j' = \frac{1}{n}(\operatorname{tr}\mathbf{R}_j - \operatorname{E}\operatorname{tr}\mathbf{R}_j), \qquad \varepsilon_j'' = \frac{1}{n}(\operatorname{tr}\mathbf{R} - \operatorname{tr}\mathbf{R}_j) - \frac{1}{n}\operatorname{E}(\operatorname{tr}\mathbf{R} - \operatorname{tr}\mathbf{R}_j).$$

By (3.7), $|\varepsilon_j''| \leqslant 2(nv)^{-1}$. This inequality and Lemma 4.3 together imply that, for $\nu = 1, 2, 3, 4$,

$$\frac{1}{n}\left|\sum_{j=1}^{p}\operatorname{E}\varepsilon_j^{(\nu)}\varepsilon_j''R(j, j)\right| \leqslant \frac{C}{nv}\left(\frac{1}{n}\sum_{j=1}^{p}\operatorname{E}|\varepsilon_j^{(\nu)}|^2\right)^{1/2}\left(\frac{1}{n}\sum_{j=1}^{p}\operatorname{E}|R(j, j)|^2\right)^{1/2}$$

$$\leqslant \frac{C(a_1, a_2)}{nv}\left(\frac{1}{n}\sum_{j=1}^{p}\operatorname{E}|\varepsilon_j^{(\nu)}|^2\right)^{1/2} \leqslant \frac{C(a_1, a_2)\sqrt{M_8}}{nv}. \tag{4.43}$$

Since the random variables X_{jl}, $l = 1, \ldots, n$, and the random matrix $\mathbf{R}_j$ are independent, we obtain, for $\nu = 1, 2, 3$,

$$\operatorname{E}|\varepsilon_j^{(1)}\varepsilon_j'|^2 = \operatorname{E}|\varepsilon_j^{(1)}|^2\operatorname{E}|\varepsilon_j'|^2 \leqslant \frac{C\sqrt{M_8}}{n}\operatorname{E}|\varepsilon_j'|^2 \leqslant \frac{CM_8}{n^3v^3}, \tag{4.44}$$

$$\operatorname{E}|\varepsilon_j^{(2)}\varepsilon_j'|^2 \leqslant \frac{M_8C}{n^2}\operatorname{E}\operatorname{tr}|\mathbf{G}(j)|^2|\varepsilon_j'|^2, \tag{4.45}$$

$$\operatorname{E}|\varepsilon_j^{(3)}\varepsilon_j'|^2 \leqslant \frac{C}{n^2v^2}\operatorname{E}|\varepsilon_j'|^2. \tag{4.46}$$

By definition of the matrix $\mathbf{G}(j)$, we have

$$\operatorname{tr}|\mathbf{G}(j)|^2 = \frac{1}{v}\operatorname{Im}\{(\operatorname{tr}(\mathbf{W}(j) - z\,\mathbf{I}_{p-1})^{-1}\mathbf{W}(j)^2\}$$

$$= \frac{1}{v}\operatorname{Im}\{z^2\operatorname{tr}\mathbf{R}_j + z\operatorname{tr}\mathbf{I}_{p-1}\}. \tag{4.47}$$

The relations (4.45) and (4.47) together imply that

$$\operatorname{E}|\varepsilon_j^{(2)}\varepsilon_j'|^2 \leqslant \frac{C\sqrt{M_8}}{nv}\operatorname{E}|\varepsilon_j'|^2 + \frac{C\sqrt{M_8}}{n^2v}|\operatorname{E}\operatorname{tr}\mathbf{R}_j|\varepsilon_j'|^2|. \tag{4.48}$$

Using the definition of ε_j', we obtain

$$\operatorname{E}|\varepsilon_j^{(2)}\varepsilon_j'|^2 \leqslant \frac{C\sqrt{M_8}}{nv}\operatorname{E}|\varepsilon_j'|^2 + \frac{C\sqrt{M_8}}{n^2v}|\operatorname{E}\operatorname{tr}\mathbf{R}_j|\operatorname{E}|\varepsilon_j'|^2 + \frac{C\sqrt{M_8}}{nv}\operatorname{E}|\varepsilon_j'|^3. \tag{4.49}$$

Using (4.36) and (4.24), simple calculations yield

$$\operatorname{E}|\varepsilon_j^{(2)}\varepsilon_j'|^2 \leqslant \frac{CM_8}{n^3v^4} + \frac{CM_8}{\sqrt{n^7v^{10}}}. \tag{4.50}$$

Furthermore, by (4.46) and (4.24), we have, for $v \geqslant CM_8^{1/4}n^{-1/2}$,

$$\mathrm{E}|\varepsilon_j^{(3)}\varepsilon_j'|^2 \leqslant \frac{CM_8}{n^4 v^5}. \tag{4.51}$$

Applying Hölder's inequality and (4.44), (4.50) and (4.51), we obtain, for $\nu = 1, 2, 3$,

$$\begin{aligned}\frac{1}{n}\left|\sum_{k=1}^{p} \mathrm{E}\varepsilon_k^{(\nu)}\varepsilon_k' R(k, k)\right| &\leqslant \left(\frac{1}{n}\sum_{k=1}^{p} \mathrm{E}|\varepsilon_k^{(\nu)}\varepsilon_k'|^2\right)^{1/2} \left(\frac{1}{n}\sum_{k=1}^{p} \mathrm{E}|R_{kk}|^2\right)^{1/2} \\ &\leqslant \frac{C(a_1, a_2)\sqrt{M_8}}{nv}. \end{aligned}\tag{4.52}$$

Inequalities (4.52) and (4.39)–(4.43) conclude the proof. □

5. Proof of Theorem 1.1

The rest of the proof of Theorem 1.1 is similar to the proof of the results for a Wigner matrix in Götze and Tikhomirov (2003). First we prove that there exists some constant C such that, for any

$$v \geqslant v_0 := \max\{\beta\Delta_n, CM_8^{1/4} n^{-1/2}\}, \tag{5.1}$$

the inequality $\mathrm{Im}\{z + yz\delta_p(z)\} > 0$ holds. Assume that

$$\mathrm{Im}\{z + yz\delta_p(z)\} = 0. \tag{5.2}$$

Then according to Lemma 3.2, there exists some constant $a_1 > 0$ such that

$$|y + 1 - z + yzs_p(z)| \geqslant a_1. \tag{5.3}$$

In addition, we have

$$\begin{aligned}|s_p(z) - s_y(z)| &= \left|\int_{-\infty}^{\infty} \frac{1}{x - z}\, \mathrm{d}(\mathrm{E}F_p(x) - F_y(x))\right| \\ &= \left|\int_{-\infty}^{\infty} \frac{\mathrm{E}F_p(x) - F_y(x)}{(x - z)^2}\, \mathrm{d}x\right| \leqslant \frac{\Delta_n}{v} \leqslant \frac{1}{\beta}.\end{aligned}$$

Since $|s_y(z)| \leqslant C(\theta, \Theta, a, b) = C$, we obtain

$$|y + z - 1 + yzs_p(z)| \leqslant C(\theta, \Theta, a, b, \beta) = a_2. \tag{5.4}$$

By Lemma 4.6, there exist constants $C_1(a_1, a_2)$ and $C_2(a_1, a_2)$ such that, for $v \geqslant C_1(a_1, a_2)M_8^{1/4} n^{-1/2}$,

$$|\delta_p(z)| \leqslant \frac{C_2(a_1, a_2)\sqrt{M_8}}{nv}. \tag{5.5}$$

Recall that $v_0 = \max\{\beta\Delta_n, n^{-1/2}C_1\sqrt{M_8}\}$, with $1 > \beta > 0$ to be chosen later. The constant C_1 is chosen such that, for any $1 \geqslant v \geqslant v_0$, we have

$$|\delta_p(z)| \leqslant \frac{v}{2\Theta(b+1)}. \tag{5.6}$$

This inequality contradicts condition (5.2), since it implies that $|\delta_p(z)| \geqslant v/\Theta(b+1)$. Now choose $v = 1$. It is easy to see that

$$|y + z - 1 + yzs_p(z)| \geqslant \operatorname{Im}\{y + z - 1 + yzs_p(z)\} \geqslant 1, \tag{5.7}$$

and since $|s_p(z)| \leqslant v^{-1} \leqslant 1$,

$$|z + y - 1 + yzs_p(z)| \leqslant (b+1)(\Theta + 1). \tag{5.8}$$

By Lemma 4.6,

$$|\delta_p(z)| \leqslant \frac{C}{n}. \tag{5.9}$$

This inequality implies, for $v = 1$,

$$\operatorname{Im}(z + yz\delta_p(z)) > 0, \tag{5.10}$$

and since $\operatorname{Im}(z + yz\delta_p(z)) \neq 0$, we obtain that (5.10) holds, for $v \geqslant v_0$.

By Lemma 3.2, for $z = u + \mathrm{i}v$ with $u \in [a, b]$ and $1 \geqslant v \geqslant v_0$, we have

$$|z + y - 1 + yzs_p(z)| \geqslant C_1(\beta, \theta, \Theta_1, \Theta_2). \tag{5.11}$$

Using inequality (5.11) and (5.4), by Lemma 4.6, we obtain that

$$|\delta_p(z)| \leqslant \frac{C\sqrt{M_8}}{nv}.$$

It is straightforward to check that

$$|s_p(z) - s_y(z)| \leqslant |\delta_p(z)||y + z - 1 + yzs_p(z) + yzs_y(z)|^{-1}. \tag{5.12}$$

Since $\operatorname{Im}(yzs_y(z)) > 0$ and $\operatorname{Im}(yzs_p(z)) > 0$, we obtain

$$|y + z - 1 + yzs_p(z) + yzs_y(z)| \geqslant \operatorname{Im}\{y + z - 1 + yzs_p(z) + yzs_y(z)\} \geqslant v.$$

These inequalities together imply that

$$|s_p(z) - s_y(z)| \leqslant \frac{C\sqrt{M_8}}{nv^2}.$$

Integrating this inequality yields

$$\int_{v_0}^{V} |s_p(u + \mathrm{i}v) - s_y(u + \mathrm{i}v)|\mathrm{d}v \leqslant \frac{C}{nv_0}. \tag{5.13}$$

Choose $V = 1$ and consider the first integral in (2.3). By (5.12), we obtain

$$\int_{-\infty}^{\infty} |s_p(z) - s_y(z)|\mathrm{d}u \leqslant \int_{-\infty}^{\infty} |\delta_p(z)|\mathrm{d}u. \tag{5.14}$$

By definition of $\delta_p(z)$, we have

$$|\delta_p(z)| \leqslant C|z+y-1+yzs_p(z)|^{-2}\left(\frac{1}{p}\sum_{k=1}^{p}|\mathrm{E}\varepsilon_k| + \frac{1}{p}\sum_{k=1}^{p}\mathrm{E}|\varepsilon_k|^2|R(k,\,k)|\right). \tag{5.15}$$

Using Lemma 3.1 and inequality (3.31) in Bai (1993b), we obtain

$$|\delta_p(z)| \leqslant \frac{C}{n}|z+y-1+yzs_p(z)|^{-2}.$$

Finally, applying (3.6) gives

$$|\delta_p(z)| \leqslant \frac{C}{n}(|s_p(z)|^2 + |\delta_p(z)|^2). \tag{5.16}$$

Without loss of generality we may assume that $|\delta_p(z)| \leqslant 1/4$. Inequalities (5.14)–(5.16) together imply that

$$\int_{-\infty}^{\infty}|\delta_p(z)|\mathrm{d}u \leqslant \frac{C}{n}\int_{-\infty}^{\infty}|s_p(z)|^2\mathrm{d}u. \tag{5.17}$$

It is easy to check that

$$\int_{-\infty}^{\infty}|s_p(z)|^2\mathrm{d}u \leqslant \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{(u-x)^2+v^2}\,\mathrm{d}u\,d\mathrm{E}\,F_p(x) \leqslant \frac{1}{v}.$$

Applying this inequality, for $v=1$, we obtain

$$\int_{-\infty}^{\infty}|s_p(z)-s_y(z)|\mathrm{d}u \leqslant \frac{C}{n}. \tag{5.18}$$

Now choose $\varepsilon = v_0^{2/3}$ and apply Lemma 2.1. We obtain that

$$\Delta_n \leqslant \frac{C_1\sqrt{M_8}}{nv_0} + \frac{C_2}{n} + C_3 v_0.$$

Note that the constant C_3 does not depend on β. We choose $\beta < 1(2C_3)^{-1}$ and $v_0 = CM_8^{1/4}n^{-1/2}$. We finally conclude

$$\Delta_n \leqslant \frac{CM_8^{1/4}}{\sqrt{n}},$$

which proves Theorem 1.1. □

6. An improved bound for $\mathrm{E}|\mathrm{tr}\,\mathbf{R} - \mathrm{E}\,\mathrm{tr}\,\mathbf{R}|^2$

Recall that

$$\mathbf{W} = \frac{1}{p}\mathbf{X}\mathbf{X}^{\mathrm{T}}, \qquad \mathbf{R} = (\mathbf{W} - z\,\mathbf{I}_p)^{-1}, \qquad \mathbf{W}(k) = \frac{1}{p}\mathbf{X}(k)\mathbf{X}(k)^{\mathrm{T}},$$

$$\mathbf{R}_k = (\mathbf{W}(k) - z\,\mathbf{I}_{p-1})^{-1}, \qquad s_p(z) = \frac{1}{p}\mathrm{E}\,\mathrm{tr}\,\mathbf{R}, \qquad s_{pk}(z) = \frac{1}{p}\mathrm{E}\,\mathrm{tr}\,\mathbf{R}_k$$

and

$$R(j,j) = -\frac{1}{y+z-1+yzs_p(z)} + \frac{\varepsilon_j}{y+z-1+yzs_p(z)} R(j,j), \tag{6.1}$$

where

$$\varepsilon_j = \frac{1}{n}\sum_{j=1}^{n}(X_{kj}^2 - 1) + y + yzs_p(z) - \mathbf{a}^{\mathrm{T}}(k)(\mathbf{W}_p(k) - z\,\mathbf{I}_{p-1})^{-1}\mathbf{a}_k.$$

From this representation it follows that

$$\frac{1}{p}\operatorname{tr}\mathbf{R} = -\frac{1}{y+z-1+yzs_p(z)} + \delta_p(z), \tag{6.2}$$

where

$$\delta_p(z) = \frac{1}{p(y+z-1+yzs_p(z))}\sum_{j=1}^{p}\varepsilon_j R(j,j).$$

Write

$$\varepsilon_j = \varepsilon_j^{(1)} + \varepsilon_j^{(2)} + \varepsilon_j^{(3)} + \varepsilon_j^{(4)}, \tag{6.3}$$

with

$$\varepsilon_j^{(1)} = \frac{1}{n}\sum_{j=1}^{n}(X_{kj}^2 - 1), \qquad \varepsilon_j^{(2)} = -\left(\mathbf{a}_j^{\mathrm{T}}\mathbf{R}_j\mathbf{a}_j - \frac{1}{n}\operatorname{tr}\mathbf{R}_j\mathbf{W}(j)\right),$$

$$\varepsilon_j^{(3)} = \frac{1}{n}(\operatorname{tr}\mathbf{RW} - \operatorname{tr}\mathbf{R}_j\mathbf{W}(j)), \qquad \varepsilon_j^{(4)} = \frac{1}{n}\operatorname{tr}\mathbf{RW} - y - yzs_p(z).$$

Recall that $y = p/n$. Note that $\operatorname{tr}\mathbf{RW} = \operatorname{tr}\mathbf{I}_p + z\operatorname{tr}\mathbf{R}$. These relations and the definition of $s_p(z)$ imply that

$$\frac{1}{n}\mathrm{E}\operatorname{tr}\mathbf{RW} = y + yzs_p(z).$$

We can now write

$$\varepsilon_j^{(4)} = \frac{z}{n}(\operatorname{tr}\mathbf{R} - \mathrm{E}\operatorname{tr}\mathbf{R}) = \frac{yz}{p}(\operatorname{tr}\mathbf{R} - \mathrm{E}\operatorname{tr}\mathbf{R}). \tag{6.4}$$

Furthermore,

$$\operatorname{tr}\mathbf{RW} - \operatorname{tr}\mathbf{R}_j\mathbf{W}(j) = 1 + z(\operatorname{tr}\mathbf{R} - \operatorname{tr}\mathbf{R}_j). \tag{6.5}$$

Hence, it follows that

$$\varepsilon_j^{(3)} = \varepsilon_j^{(5)} + \varepsilon_j^{(6)}, \tag{6.6}$$

where

$$\varepsilon_j^{(5)} = \frac{1}{n}, \qquad \varepsilon_j^{(6)} = \frac{z}{n}(\operatorname{tr}\mathbf{R} - \operatorname{tr}\mathbf{R}_j).$$

We introduce the notation

$$M_q = \sup_{1\leqslant j,k\leqslant n} \mathrm{E}|X_{jk}|^q.$$

Proposition 6.1. *Assuming the conditions of Theorem* 1.1, *we have, for any* $v > v_0 := \gamma M_{12}^{1/6} n^{-1/2}$ *with sufficiently large* $\gamma > 0$,

$$\frac{1}{n^2}\mathrm{E}|\operatorname{tr}\mathbf{R} - \mathrm{E}\operatorname{tr}\mathbf{R}|^2 \leqslant \frac{CM_8}{n^2 v^2 |y + z - 1 + 2yzs(z)|^2}. \tag{6.7}$$

Proof. To prove Proposition 6.1 we shall use the following facts: for $v \geqslant v_0$, we have

$$\frac{1}{n}\sum_{j=1}^{p} \mathrm{E}|R(j,j)|^2 \leqslant C \tag{6.8}$$

and

$$\mathrm{E}|\operatorname{tr}\mathbf{R} - \mathrm{E}\operatorname{tr}\mathbf{R}|^2 \leqslant CM_4 v^{-3}. \tag{6.9}$$

These inequalities were proved in Lemmas 4.4 and 4.5. We shall use also the following lemma:

Lemma 6.2. *Under the conditions of Theorem* 1.2 *we have, for any* $q \geqslant 4$ *and for* $v \geqslant v_0$,

$$\mathrm{E}\left|\frac{1}{p}(\operatorname{tr}\mathbf{R} - \mathrm{E}\operatorname{tr}\mathbf{R})\right|^q \leqslant \frac{C(q)M_{2q}}{(n^2 v^3)^{q/2}}. \tag{6.10}$$

Proof. By Burkholder's inequality for martingales, we have

$$\mathrm{E}|\operatorname{tr}\mathbf{R} - \mathrm{E}\operatorname{tr}\mathbf{R}|^q \leqslant C(q) n^{q/2-1} \sum_{j=1}^{n} \mathrm{E}|\gamma_j|^q, \tag{6.11}$$

where the martingale difference γ_j is defined in (4.12) and (4.13). Furthermore, using the representation (4.16) which is similar to inequality (4.17), we obtain

$$\begin{aligned} \mathrm{E}|\gamma_j|^q &\leqslant C(q)\mathrm{E}|\sigma_j^{(2)}|^q + C(q)\mathrm{E}|\sigma_j^{(3)}|^q \\ &\leqslant C(q)\frac{1}{n^q}\mathrm{E}|\mathbf{a}(j)^{\mathrm{T}}\mathbf{R}_j^2\mathbf{a}(j) - \operatorname{tr}\mathbf{R}_j^2\mathbf{W}(j)|^q + C(q)\mathrm{E}|\varepsilon_j(\operatorname{tr}\mathbf{R} - \operatorname{tr}\mathbf{R}_j)|^q. \end{aligned} \tag{6.12}$$

By the definition (6.3) of ε_j, using Rosenthal's inequality for quadratic forms (see, for example, Götze and Tikhomirov 2003), we obtain

$$\begin{aligned}\mathrm{E}|\gamma_j|^q \leqslant \frac{C(q)M_{2q}}{n^q}\mathrm{E}(\mathrm{tr}(|\mathbf{R}_j^2|\mathbf{W}(j))^2)^{q/2} \\ + \mathrm{E}\left|\frac{1}{n}(\mathbf{X}_j^{\mathrm{T}}\mathbf{G}_j\mathbf{X}_j - \mathrm{tr}\,\mathbf{G}_j)\right|^q |\mathrm{tr}\,\mathbf{R} - \mathrm{tr}\,\mathbf{R}_j|^q \\ + \frac{C(q)}{n^q}\mathrm{E}\left|\sum_{k=1}^n (X_{kj}^2 - 1)\right|^q |\mathrm{tr}\,\mathbf{R} - \mathrm{tr}\,\mathbf{R}_j|^q \\ + \frac{C(q)}{n^q}\mathrm{E}|\mathrm{tr}\,\mathbf{R} - \mathrm{E}\,\mathrm{tr}\,\mathbf{R}|^q|\mathrm{tr}\,\mathbf{R} - \mathrm{tr}\,\mathbf{R}_j|^q \\ + \frac{1}{n^q}\mathrm{E}|\mathrm{tr}\,\mathbf{R} - \mathrm{tr}\,\mathbf{R}_j|^{2q}. \end{aligned} \tag{6.13}$$

Inequalities (6.11), (6.13) and the inequality $|\mathrm{tr}\,\mathbf{R} - \mathrm{tr}\,\mathbf{R}_j| \leqslant v^{-1}$ together imply

$$\begin{aligned}\mathrm{E}|\mathrm{tr}\,\mathbf{R} - \mathrm{E}\,\mathrm{tr}\,\mathbf{R}|^q \leqslant \frac{C(q)M_{2q}}{n^{q/2}} \max_{1\leqslant j\leqslant n} \mathrm{E}(\mathrm{tr}(|\mathbf{R}_j^2|\mathbf{W}(j))^2)^{q/2} + \frac{C(q)M_{2q}}{v^q} \\ + \frac{C(q)}{n^{q/2}v^{2q}} + \frac{C(q)M_{2q}}{n^{q/2}vq} \max_{1\leqslant j\leqslant n} \mathrm{E}(\mathrm{tr}(|\mathbf{R}_j|^2\mathbf{W}(j)^2)^{q/2}) \\ + \frac{C(q)}{n^{q/2}v^q}\mathrm{E}|\mathrm{tr}\,\mathbf{R} - \mathrm{E}\,\mathrm{tr}\,\mathbf{R}|^q. \end{aligned} \tag{6.14}$$

From (6.14) we obtain for $v \geqslant v_0$,

$$\begin{aligned}\mathrm{E}|\mathrm{tr}\,\mathbf{R} - \mathrm{E}\,\mathrm{tr}\,\mathbf{R}|^q \leqslant \frac{C(q)M_{2q}}{n^{q/2}} \max_{1\leqslant j\leqslant n} \mathrm{E}(\mathrm{tr}(|\mathbf{R}_j^2|\mathbf{W}(j))^2)^{q/2} \\ + \frac{C(q)M_{2q}}{v^q} + \frac{C(q)M_{2q}}{n^{q/2}v^q} \max_{1\leqslant j\leqslant n} \mathrm{E}(\mathrm{tr}|\mathbf{R}_j|^2\mathbf{W}(j)^2)^{q/2}. \end{aligned} \tag{6.15}$$

Note that

$$\mathrm{tr}|\mathbf{R}_j|^4\mathbf{W}(j)^2 \leqslant v^{-2}\mathrm{tr}|\mathbf{R}_j|^2\mathbf{W}(j)^2 = v^{-3}\,\mathrm{Im}\{\mathrm{tr}\,\mathbf{R}_j\mathbf{W}(j)^2\}. \tag{6.16}$$

Furthermore, it is easy to check that

$$\mathrm{tr}\,\mathbf{R}_j\mathbf{W}(j)^2 = z^2\mathrm{tr}\,\mathbf{R}_j + z\,\mathrm{tr}\,\mathbf{I}_{p-1} + \mathrm{tr}\,\mathbf{W}(j). \tag{6.17}$$

Relations (6.16) and (6.17) imply that

$$\mathrm{E}(\mathrm{tr}|\mathbf{R}_j|^4\mathbf{W}^2)^{q/2} \leqslant C(q)v^{-3q/2}(\mathrm{E}|\mathrm{tr}\,\mathbf{R}_j|^{q/2} + (\mathrm{tr}\,\mathbf{I}_{p-1})^{q/2}). \tag{6.18}$$

Analogously,

$$\begin{aligned}\mathrm{E}(\mathrm{tr}|\mathbf{R}_j|^2\mathbf{W}(j)^2)^{q/2} \leqslant C(q)v^{-q/2}\mathrm{E}|\mathrm{tr}\,\mathbf{R}_j\mathbf{W}(j)^2|^{q/2} \\ \leqslant C(q)v^{-q/2}(\mathrm{E}|\mathrm{tr}\,\mathbf{R}_j|^{q/2} + |\mathrm{tr}\,\mathbf{I}_{p-1}|^{q/2}). \end{aligned} \tag{6.19}$$

By Burkholder's inequality for the martingales, we have, for $v \geqslant v_0$,

$$\mathrm{E}|\operatorname{tr}\mathbf{R}_j - \mathrm{E}\operatorname{tr}\mathbf{R}_j|^{q/2} \leqslant C(q)n^{q/4-1}\sum_{\substack{l=1\\ l\neq j}}^{n}\mathrm{E}|\gamma_{jl}|^{q/2} \leqslant \frac{C(q)n^{q/2}}{n^{q/4}v^{q/2}} \leqslant C(q)n^{q/2}. \tag{6.20}$$

Inequality (6.20) implies that for, $v \geqslant v_0$,

$$\mathrm{E}|\operatorname{tr}\mathbf{R}_j|^{q/2} \leqslant C(q)|\mathrm{E}\operatorname{tr}\mathbf{R}_j|^{q/2} + C(q)\mathrm{E}|\operatorname{tr}\mathbf{R}_j - \mathrm{E}\operatorname{tr}\mathbf{R}_j|^{q/2} \leqslant C(q)n^{q/2}. \tag{6.21}$$

From inequalities (6.17)–(6.21) it follows that

$$\mathrm{E}\Big(\operatorname{tr}|\mathbf{R}_j|^4\mathbf{W}(j)^2)^{q/2} \leqslant C(q)v^{-3q/2}n^{q/2}, \qquad \mathrm{E}(\operatorname{tr}|\mathbf{R}_j|^2\mathbf{W}(j)^2)^{q/2} \leqslant C(q)v^{-q/2}n^{q/2}. \tag{6.22}$$

Combining the inequalities (6.15) and (6.22), we obtain, for $v \geqslant v_0$,

$$\mathrm{E}|\operatorname{tr}\mathbf{R} - \mathrm{E}\operatorname{tr}\mathbf{R}|^q \leqslant \frac{C(q)M_{2q}}{v^{3q/2}} + \frac{C(q)M_{2q}}{v^q} \leqslant \frac{C(q)M_{2q}}{v^{3q/2}}, \tag{6.23}$$

which concludes the proof of the lemma. □

We may prove a rougher bound than (6.10), assuming $M_8 < \infty$ only.

Remark 6.1. We have, for $v \geqslant v_0$ and for $q \geqslant 4$,

$$\mathrm{E}\left|\frac{1}{n}(\operatorname{tr}\mathbf{R} - \mathrm{E}\operatorname{tr}\mathbf{R})\right|^q \leqslant \frac{CM_8}{(\sqrt{nv})^{q-4}}\frac{1}{n^4v^6}. \tag{6.24}$$

Proof. To prove (6.24) we use Burkholder's inequality for martingales. We obtain

$$\mathrm{E}\left|\frac{1}{n}(\operatorname{tr}\mathbf{R} - \mathrm{E}\operatorname{tr}\mathbf{R})\right|^q \leqslant \frac{C}{n^{q/2+1}}\sum_{j=1}^{n}\mathrm{E}|\gamma_j|^q \leqslant \frac{C}{n^{q/2-2}v^{q-4}}\left(\frac{1}{n^3}\sum_{j=1}^{n}\mathrm{E}|\gamma_j|^4\right).$$

Applying now arguments similar to the relations (6.13)–(6.18) for $(1/n^3)\sum_{j=1}^{n}\mathrm{E}|\gamma_j|^4$, we obtain inequality (6.24). □

We now continue with our proof of Proposition 6.1. In order to simplify the exposition we introduce the following notation:

$$\Delta_{\mathrm{E}}(\mathbf{R}) := \overline{\operatorname{tr}\mathbf{R} - \mathrm{E}\operatorname{tr}\mathbf{R}}, \qquad \Delta_E^{(j)}(\mathbf{R}) := \operatorname{tr}\mathbf{R}_j - \mathrm{E}\operatorname{tr}\mathbf{R}_j,$$

$$\mathcal{X}_j = \sum_{l=1}^{n}(X_{jl}^2 - 1), \qquad \mathcal{D}_j(\mathbf{R}) := \operatorname{tr}\mathbf{R} - \operatorname{tr}\mathbf{R}_j, \qquad \mathcal{C}(\mathbf{R}) := \left(\frac{1}{n}\sum_{l=1}^{n}\mathrm{E}|R(j,j)|^2\right)^{1/2},$$

$$\mathcal{Q}_j(\mathbf{R}) := \mathbf{a}_j^{\mathrm{T}}\mathbf{R}_j\mathbf{a}_j - \operatorname{tr}\mathbf{R}_j\mathbf{W}(j), \qquad \mathcal{Q}_j(\mathbf{R}^2) := \mathbf{a}_j^{\mathrm{T}}\mathbf{R}_j^2\mathbf{a}_j - \operatorname{tr}\mathbf{R}_j^2\mathbf{W}(j),$$

$$a_n(z) := (yzs_p(z) + \mathrm{z} + y - 1)^{-1}, \qquad b_n(z) := (2yzs_p(z) + y + z - 1)^{-1}.$$

Using this notation, we have

$$\varepsilon_j = \frac{1}{n}\mathcal{X}_j - \frac{1}{n}\mathcal{Q}_j(\mathbf{R}) + \frac{1}{n} + \frac{z}{n}\mathcal{D}_j(\mathbf{R}) - \frac{z}{n}\Delta_E(\mathbf{R}). \tag{6.25}$$

Consider the representation

$$\begin{aligned}\mathrm{E}|\mathrm{tr}\,\mathbf{R} - \mathrm{E}\,\mathrm{tr}\,\mathbf{R}|^2 &= \mathrm{E}(\overline{\mathrm{tr}\,\mathbf{R} - \mathrm{E}\,\mathrm{tr}\,\mathbf{R}})(\mathrm{tr}\,\mathbf{R} - \mathrm{E}\,\mathrm{tr}\,\mathbf{R}) \\ &= \mathrm{E}(\mathrm{tr}\,\mathbf{R} - \mathrm{E}\,\mathrm{tr}\,\mathbf{R})\mathrm{tr}\,\mathbf{R} = a_n(z)\sum_{j=1}^{p}\mathrm{E}\Delta_E(\mathbf{R})\varepsilon_j R(j,j).\end{aligned}$$

Using (6.25), we may rewrite this equality as follows

$$\frac{1}{n^2}\mathrm{E}|\mathrm{tr}\,\mathbf{R} - \mathrm{E}\,\mathrm{tr}\,\mathbf{R}|^2 = a_n(z)(A_1 + A_2 + zA_3 + zA_4 + zA_5), \tag{6.26}$$

where

$$A_1 = \frac{1}{n^3}\sum_{j=1}^{p}\mathrm{E}\Delta_E(\mathbf{R})\mathcal{X}_j R(j,j), \qquad A_2 = -\frac{1}{n^3}\sum_{j=1}^{p}\mathrm{E}\Delta_E(\mathbf{R})\mathcal{Q}_j(\mathbf{R})R(j,j),$$

$$A_3 = \frac{1}{n^3}\sum_{j=1}^{p}\mathrm{E}\Delta_E(\mathbf{R})\mathcal{D}_j(\mathbf{R})R(j,j), \qquad A_4 = -\frac{1}{n^3}\sum_{j=1}^{p}\mathrm{E}|\Delta_E(\mathbf{R})|^2 R(j,j),$$

$$A_5 = \frac{1}{n^3}\sum_{j=1}^{p}\Delta_E(\mathbf{R})R(j,j) = \frac{1}{n^3}\mathrm{E}|\Delta_E(\mathbf{R})|^2.$$

We first consider A_4.

Lemma 6.3. *Assuming the conditions of Theorem* 1.2*, the following representation holds:*

$$A_4 = y(a_n(z) - \delta_n(z)yzs_p(z)b_n(z))\mathrm{E}\left|\frac{1}{n}(\Delta_E(\mathbf{R}))\right|^2 + b_n(z)(A_6 + A_7 + zA_8) + \Gamma_1, \tag{6.27}$$

where

$$A_6 = -\frac{1}{n^4}\sum_{j=1}^{p}\mathrm{E}|\Delta_E(\mathbf{R})|^2\mathcal{X}_j R(j,j),$$

$$A_7 = \frac{1}{n^4}\sum_{j=1}^{p}\mathrm{E}|\Delta_E(\mathbf{R})|^2\mathcal{Q}_j(\mathbf{R})R(j,j),$$

$$A_8 = -\frac{1}{n^4}\sum_{j=1}^{p}\mathrm{E}|\Delta_E(\mathbf{R})|^2\mathcal{D}_j(\mathbf{R})R(j,j),$$

and Γ_1 *satisfies the inequality*

$$|\Gamma_1| \leqslant \frac{CM_8|b_n(z)|}{n^4 v^6}. \tag{6.28}$$

Proof. We have

$$A_4 = -\mathrm{E}\left|\frac{1}{n}\Delta_E(\mathbf{R})\right|^2\left(\frac{1}{n}\operatorname{tr}\mathbf{R}\right).$$

Adding and subtracting $ys_p(z) = \frac{1}{n}\mathrm{E}\operatorname{tr}\mathbf{R}$, we rewrite this equality as

$$A_4 = A_9 + A_{10}, \tag{6.29}$$

where

$$A_9 = -ys_p(z)\mathrm{E}\left|\frac{1}{n}\Delta_E(\mathbf{R})\right|^2, \qquad A_{10} = -\mathrm{E}\left|\frac{1}{n}\Delta_E(\mathbf{R})\right|^2\left(\frac{1}{n}\Delta_E(\mathbf{R})\right).$$

To investigate the asymptotics of A_{10} we derive some recursion relations. Using the representation (6.2), we obtain

$$A_{10} = yA_{11} + a_n(z)(A_6 + A_7 + zA_8 + zA_{12}), \tag{6.30}$$

where

$$A_{11} = \frac{a_n(z) + s_p(z)}{n^2}\mathrm{E}|\Delta_E(\mathbf{R})|^2,$$

$$A_{12} = \frac{1}{n^4}\sum_{j=1}^{p}\mathrm{E}|\Delta_E(\mathbf{R})|^2\overline{\Delta}_E(\mathbf{R})R(j,j) = \frac{1}{n^4}\mathrm{E}|\Delta_E(\mathbf{R})|^2\overline{\Delta}_E(\mathbf{R})\operatorname{tr}\mathbf{R}.$$

Adding and subtracting $ys_p(z)$ again, we write the term A_{12} in the form

$$\begin{aligned} A_{12} &= \frac{1}{n^3}\left(ys_p(z)\mathrm{E}|\Delta_E(\mathbf{R})|^2\overline{\Delta}_E(\mathbf{R}) + \frac{1}{n}\mathrm{E}|\Delta_E(\mathbf{R})|^2(\overline{\Delta}_E(\mathbf{R}))^2\right) \\ &= -ys_p(z)A_{10} + \frac{1}{n}\mathrm{E}|\Delta_E(\mathbf{R})|^2(\overline{\Delta}_E(\mathbf{R}))^2. \end{aligned} \tag{6.31}$$

Comparing (6.30) and (6.31), we obtain

$$A_{10} = yzs_p(z)a_n(z)A_{10} + a_n(z)\left(\frac{z}{n^4}\mathrm{E}|\Delta_E(\mathbf{R})|^2(\Delta_E(\mathbf{R}))^2 + A_6 + A_7 + zA_8\right) + yA_{11}.$$

Combining the left-hand side with the first term on the right-hand side, we obtain, after multiplication with $(a_n(z))^{-1}b_n(z) = (1 - yzs_p(z)a_n(z))^{-1}$,

$$A_{10} = b_n(z)\left(y(a_n(z))^{-1}A_{11} + \mathrm{E}\left|\frac{1}{n}\Delta_E(\mathbf{R})\right|^2\left(\frac{1}{n}\overline{\Delta}_E(\mathbf{R})\right)^2 + A_6 + A_7 + zA_8\right). \tag{6.32}$$

From the definition of A_{11} and (6.2), it follows that

$$A_{11} = \delta_p(z)\mathrm{E}\left|\frac{1}{n}(\Delta_E(\mathbf{R}))\right|^2.$$

This relation and (6.2), (6.31), (6.32) together imply that

$$
\begin{aligned}
A_4 &= -ys_p(z)\mathrm{E}\left|\frac{1}{n}(\Delta_E(\mathbf{R}))\right|^2 + A_{10} \\
&= ya_n(z)\mathrm{E}\left|\frac{1}{n}(\Delta_E(\mathbf{R}))\right|^2 - y\delta_p(z)\mathrm{E}\left|\frac{1}{n}(\Delta_E(\mathbf{R}))\right|^2 + A_{10} \\
&= y(a_n(z) - \delta_p(z)yzs_p(z)b_n(z))\mathrm{E}\left|\frac{1}{n}(\Delta_E(\mathbf{R}))\right|^2 \\
&\quad + b_n(z)(A_6 + A_7 + zA_8) + \Gamma_1,
\end{aligned} \tag{6.33}
$$

where

$$\Gamma_1 = \frac{a_n(z)b_n(z)}{n^4}\mathrm{E}|\Delta_E(\mathbf{R})|^2(\overline{\Delta}_E(\mathbf{R}))^2.$$

According to Remark 6.1 and inequality (5.11), we obtain

$$|\Gamma_1| \leqslant \frac{CM_8|b_n(z)|}{n^4 v^6}. \tag{6.34}$$

The relations (6.33) and (6.34) conclude the proof of the lemma. □

We now turn to A_1.

Lemma 6.4. *Under the conditions of Theorem* 1.2 *the following inequality holds, for* $v \geqslant v_0$*:*

$$|A_1| \leqslant \frac{\sqrt{M_8}}{n^2 v^2}. \tag{6.35}$$

Proof. Using (6.1), we obtain

$$A_1 = \frac{1}{n^3}\sum_{j=1}^{p} \mathrm{E}\Delta_E(\mathbf{R})\mathcal{X}_j R(j,j) = A_{13} + A_{14}, \tag{6.36}$$

where

$$A_{13} = -\frac{a_n(z)}{n^3}\sum_{j=1}^{p} \mathrm{E}\Delta_E(\mathbf{R})\mathcal{X}_j, \qquad A_{14} = \frac{a_n(z)}{n^3}\sum_{j=1}^{p} \mathrm{E}\Delta_E(\mathbf{R})\mathcal{X}_j\varepsilon_j R(j,j).$$

Using the equality $\mathcal{D}_j(\mathbf{R}) = (1 + \mathbf{a}_j^{\mathrm{T}} R_j^2 \mathbf{a}_j)R(j,j)$, which follows from (3.6), we obtain

$$A_{13} = -a_n(z)\frac{1}{n^3}\sum_{j=1}^{p} \mathrm{E}\mathcal{D}_j(\mathbf{R})\mathcal{X}_j = -a_n(z)\frac{1}{n^3}\sum_{j=1}^{p} \mathrm{E}\left(1 + \frac{1}{n}\mathbf{a}_j^{\mathrm{T}}\mathbf{R}_j^2\mathbf{a}_j\right)\mathcal{X}_j R(j,j). \tag{6.37}$$

Applying (6.1) and (6.25), we can write

$$A_{13} = A_{15} + A_{16} + A_{17} + A_{18} + A_{19} + A_{20}, \tag{6.38}$$

where

$$A_{15} = a_n^2(z)\frac{1}{n^3}\sum_{j=1}^{p} \mathrm{E}\left(1 + \frac{1}{n}\mathbf{a}_j^{\mathrm{T}}\mathbf{R}_j^2\mathbf{a}_j\right)\mathcal{X}_j,$$

$$A_{16} = a_n^2(z)\frac{1}{n^4}\sum_{j=1}^{p} \mathrm{E}\left(1 + \frac{1}{n}\mathbf{a}_j^{\mathrm{T}}\mathbf{R}_j^2\mathbf{a}_j\right)\mathcal{X}_j^2 R(j, j),$$

$$A_{17} = a_n^2(z)\frac{1}{n^4}\sum_{j=1}^{p} \mathrm{E}\left(1 + \frac{1}{n}\mathbf{a}_j^{\mathrm{T}}\mathbf{R}_j^2\mathbf{a}_j\right)\mathcal{X}_j \mathrm{D}_j(\mathbf{R})R(j, j),$$

$$A_{18} = a_n^2(z)\frac{1}{n^4}\sum_{j=1}^{p} \mathrm{E}\left(1 + \frac{1}{n}\mathbf{a}_j^{\mathrm{T}}\mathbf{R}_j^2\mathbf{a}_j\right)\mathcal{X}_j \mathcal{Q}_j(\mathbf{R})R(j, j),$$

$$A_{19} = a_n^2(z)\frac{1}{n^4}\sum_{j=1}^{p} \mathrm{E}\left(1 + \frac{1}{n}\mathbf{a}_j^{\mathrm{T}}\mathbf{R}_j^2\mathbf{a}_j\right)\mathcal{X}_j \overline{\Delta}_{\mathrm{E}}(\mathbf{R})R(j, j),$$

$$A_{20} = a_n^2(z)\frac{1}{n^4}\sum_{j=1}^{p} \mathrm{E}\left(1 + \frac{1}{n}\mathbf{a}_j^{\mathrm{T}}\mathbf{R}_j^2\mathbf{a}_j\right)\mathcal{X}_j R(j, j).$$

Note that

$$A_{15} = a_n(z)^2\frac{1}{n^4}\sum_{j=1}^{p} \mathrm{E}\mathbf{a}_j^{\mathrm{T}}\mathbf{R}_j^2\mathbf{a}_j \mathrm{X}_j = a_n(z)^2\frac{1}{n^4}\sum_{j=1}^{p}\sum_{j=1}^{n} \mathrm{E}(X_{jl}^2 - 1)X_{jl}^2 \mathrm{E}\,\mathbf{G}_j(l, l), \tag{6.39}$$

where $\mathbf{G}_j = \mathbf{X}(j)^{\mathrm{T}}\mathbf{R}_j^2\mathbf{X}(j)$. Since $\mathrm{tr}|\mathbf{G}_j| = \mathrm{tr}|\mathbf{R}_j^2\mathbf{W}(j)| \leqslant Cnv^{-1}$, we obtain, from (6.39),

$$|A_{15}| \leqslant \frac{CM_4}{n^2 v}. \tag{6.40}$$

To bound A_{16}–A_{20} we use (3.6) again. It is easy to check that

$$|A_{16}| \leqslant \frac{CM_4}{n^2 v} \tag{6.41}$$

Furthermore, the inequality $|\mathcal{D}_j(\mathbf{R})| \leqslant v^{-1}$, which follows from (3.7), implies

$$|A_{17}| \leqslant \frac{C}{n^{5/2}v^2} \leqslant \frac{C}{n^2 v^2}. \tag{6.42}$$

Applying Cauchy's inequality gives

$$|A_{18}| \leqslant \frac{C}{n^4 v}\sum_{j=1}^{p} \mathrm{E}^{1/2}|\mathcal{Q}_j(\mathbf{R})|^2 \mathrm{E}^{1/2}|\mathcal{X}_j|^2 \leqslant \frac{C}{n^2 v^2}. \tag{6.43}$$

Using Cauchy's inequality and (6.9), we obtain

$$|A_{19}| \leqslant \frac{C}{n^4 v}\sum_{j=1}^{p} \mathrm{E}^{1/2}|\Delta_E(\mathbf{R})|^2 \mathrm{E}^{1/2}|\mathcal{X}_j|^2 \leqslant \frac{C}{n^{5/2}v^{5/2}} \leqslant \frac{C}{n^2 v^2}. \tag{6.44}$$

For the term A_{20} we have a similar bound:

$$|A_{20}| \leqslant \frac{C}{n^4 v}\sum_{j=1}^{p} \mathrm{E}^{1/2}|\mathcal{X}_j|^2 \leqslant \frac{C}{n^2 v^2}. \tag{6.45}$$

Inequalities (6.39)–(6.45) together imply that

$$|A_{13}| \leqslant \frac{C}{n^2 v^2}. \tag{6.46}$$

We write the term A_{14} as

$$A_{14} = \frac{a_n(z)}{n^3}\sum_{j=1}^{p} \mathrm{E}\Delta_E(\mathbf{R})\mathcal{X}_j\varepsilon_j R(j,j) = A_{21} + A_{22} + A_{23} + A_{24}, \tag{6.47}$$

where

$$A_{21} = \frac{1}{n^4}\sum_{j=1}^{p} \mathrm{E}\Delta_E(\mathbf{R})\mathcal{X}_j^2 R(j,j), \qquad A_{22} = -\frac{1}{n^4}\sum_{j=1}^{p} \mathrm{E}\Delta_E(\mathbf{R})\mathcal{X}_j\mathcal{Q}_j(\mathbf{R})R(j,j),$$

$$A_{23} = \frac{1}{n^4}\sum_{j=1}^{p} \mathrm{E}\Delta_E(\mathbf{R})\mathcal{X}_j\mathcal{D}_j(\mathbf{R})R(j,j), \qquad A_{24} = -\frac{1}{n^4}\sum_{j=1}^{p} \mathrm{E}|\Delta_E(\mathbf{R})|^2\mathcal{X}_j R(j,j).$$

Note that

$$\begin{aligned}|A_{21}| &\leqslant \frac{C\mathcal{C}(\mathbf{R})}{n^3}\left(\frac{1}{n}\sum_{j=1}^{p} \mathrm{E}|\Delta_E(\mathbf{R})|^2|\mathcal{X}_j|^4\right)^{1/2} \\ &\leqslant \frac{CM_8^{1/2}}{n}\left(\mathrm{E}^{1/2}\left|\frac{1}{n}\Delta_E(\mathbf{R})\right|^2 + \frac{1}{nv}\right) \leqslant \frac{CM_8^{1/2}}{n^2 v^2}.\end{aligned} \tag{6.48}$$

Using the inequality $|\Delta_{\mathrm{E}}(\mathbf{R})| \leqslant |\Delta_{\mathrm{E}}(\mathbf{R}_j)| + v^{-1}$, we obtain

$$\begin{aligned}|A_{22}| &\leqslant \frac{C\mathcal{C}(\mathbf{R})}{n^3}\left(\frac{1}{n}\sum_{j=1}^{p} \mathrm{E}|\Delta_E(\mathbf{R})|^2|\mathcal{Q}_j(\mathbf{R})|^2\mathcal{X}_j^2\right)^{1/2} \\ &\leqslant \frac{C}{n^3}\left\{\left(\frac{1}{n}\sum_{j=1}^{p} \mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^2|\mathcal{Q}_j(\mathbf{R})|^2\mathcal{X}_j^2\right)^{1/2}\right. \\ &\qquad \left. + v^{-1}\left(\frac{1}{n}\sum_{j=1}^{p} \mathrm{E}|\mathcal{Q}_j(\mathbf{R})|^2\mathcal{X}_j^2\right)^{1/2}\right\}.\end{aligned} \tag{6.49}$$

Applying Cauchy's inequality, we obtain

$$\mathrm{E}\{|\mathcal{Q}_j(\mathbf{R})|^2\mathcal{X}_j^2|\mathbf{R}_j\} \leqslant \mathrm{E}^{1/2}\{|\mathcal{Q}_j(\mathbf{R})|^4|\mathbf{R}_j\}\mathrm{E}^{1/2}\mathcal{X}_j^4$$
$$\leqslant CM_8\, n\,\mathrm{tr}|\mathbf{R}_j|^2 \leqslant CM_8 \mathrm{n}v^{-1}|\mathrm{tr}\,\mathbf{R}_j|. \tag{6.50}$$

Furthermore,

$$\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^2|\mathrm{tr}\,\mathbf{R}_j| \leqslant Cn|s_{pj}(z)\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^2 + C\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^3$$
$$\leqslant Cn\mathrm{E}|\Delta_E(\mathbf{R})|^2 + C\mathrm{E}|\Delta_E(\mathbf{R})|^3 + Cv^{-3} + Cnv^{-2}. \tag{6.51}$$

By Burkholder's inequality for martingales, we have, for $v \geqslant v_0$,

$$\mathrm{E}\left|\frac{1}{n}\Delta_E(\mathbf{R})\right|^3 \leqslant \frac{C}{\sqrt{nv}}\,\mathrm{E}\left|\frac{1}{n}\Delta_E(\mathbf{R})\right|^2 \leqslant C\mathrm{E}\left|\frac{1}{n}\Delta_E(\mathbf{R})\right|^2. \tag{6.52}$$

Inequalities (6.50)–(6.52) together imply that, for $v \geqslant v_0$,

$$|A_{22}| \leqslant \frac{CM_8^{1/2}}{n^2v^2}. \tag{6.53}$$

Using the inequality $|\mathcal{D}_j(\mathbf{R})| \leqslant v^{-1}$ and Cauchy's inequality, we obtain that, for $v \geqslant v_0$,

$$|A_{23}| \leqslant \frac{C\mathcal{C}(\mathbf{R})}{n^3v}\left(\frac{1}{n}\sum_{j=1}^{p}\mathrm{E}|\Delta_E(\mathbf{R})|^2\mathcal{X}_j^2\right)^{1/2}. \tag{6.54}$$

Since $\Delta_E^{(j)}$ and $\mathcal{X}_j$ are independent,

$$\mathrm{E}|\Delta_E(\mathbf{R})|^2\mathcal{X}_j^2 \leqslant C\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^2\mathrm{E}\mathcal{X}_j^2 + \frac{C}{v^2}\mathrm{E}\mathcal{X}_j^2 \leqslant CnM_8(\mathrm{E}|\Delta_E(\mathbf{R})|^2 + v^{-2}). \tag{6.55}$$

Inequalities (6.8), (6.9), (6.54) and (6.55) together imply

$$|A_{23}| \leqslant \frac{CM_8^{1/2}M_4^{1/2}}{n^{5/2}v^{5/2}} \leqslant \frac{CM_8^{1/2}}{n^2v^2}. \tag{6.56}$$

Using Cauchy's inequality,

$$|A_{24}| \leqslant \frac{C\mathcal{C}(\mathbf{R})}{n^3}\left(\frac{1}{n}\sum_{j=1}^{p}\mathrm{E}|\Delta_E(\mathbf{R})|^4\mathcal{X}_j^2\right)^{1/2} \leqslant \frac{C}{n^3}\left(\frac{1}{n}\sum_{j=1}^{p}\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^4\mathrm{E}\mathcal{X}_j^2 + v^{-4}\mathrm{E}\mathcal{X}_j^2\right)^{1/2}$$
$$\leqslant \frac{CM_4^{1/2}}{n^{5/2}}(\mathrm{E}|\Delta_E(\mathbf{R})|^4 + v^{-4})^{1/2} \leqslant \frac{CM_8^{1/2}M_4^{1/2}}{n^{5/2}v^3} \leqslant \frac{CM_8^{1/2}}{n^2v^2}. \tag{6.57}$$

The relations (6.36), (6.46), (6.47), (6.48), (6.53), (6.56) and (6.57) together imply that, for $v \geqslant v_0$,

$$|\mathrm{A}_1| \leqslant \frac{\sqrt{M_8}}{n^2v^2}. \tag{6.58}$$

This concludes the proof of the lemma. □

Furthermore, using Cauchy's inequality and Lemma 4.5, we obtain

$$|\mathrm{A}_3| \leqslant \frac{\mathcal{C}(\mathbf{R})}{nv}\mathrm{E}^{1/2}\left|\frac{1}{n}\Delta_E(\mathbf{R})\right|^2 \leqslant \frac{C}{nv}\mathrm{E}^{1/2}\left|\frac{1}{n}\Delta_E(\mathbf{R})\right|^2. \tag{6.59}$$

Using Cauchy's inequality again,

$$|A_6| \leqslant \frac{C\mathcal{C}(\mathbf{R})}{\sqrt{n}}\left(\frac{1}{n}\sum_{j=1}^{p}\mathrm{E}\left|\frac{1}{n}(\Delta_E(\mathbf{R}))\right|^4\mathcal{X}_j^2\right)^{1/2}. \tag{6.60}$$

Applying the inequality $|\Delta(\mathbf{R})| \leqslant |\Delta^{(j)}(\mathbf{R})| + 2v^{-1}$,

$$\begin{aligned}\mathrm{E}\left|\frac{1}{n}(\Delta_E(\mathbf{R}))\right|^4\mathcal{X}_j^2 &\leqslant C\mathrm{E}\left|\frac{1}{n}(\Delta_E^{(j)}(\mathbf{R}))\right|^4\mathrm{E}\mathcal{X}_j^2 + \frac{C}{n^4v^4}\mathrm{E}\mathcal{X}_j^2\\ &\leqslant C\mathrm{E}\left|\frac{1}{n}(\Delta_E(\mathbf{R}))\right|^4\mathrm{E}\mathcal{X}_j^2 + \frac{CM_4}{n^3v^4}.\end{aligned} \tag{6.61}$$

Hence inequalities (6.8), (6.60), (6.61) and Remark 6.1 imply that, for $v \geqslant v_0$,

$$|\mathrm{A}_6| \leqslant \frac{CM_8^{1/2}}{n^2v^2}. \tag{6.62}$$

We have the bound

$$\begin{aligned}|\mathrm{A}_8| &= \frac{1}{n^4}\sum_{j=1}^{p}\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^2\mathcal{D}_jR(j,j)|\\ &\leqslant \frac{C\mathcal{C}(\mathbf{R})}{n^3v}\left(\frac{1}{n}\sum_{j=1}^{p}\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^4\right)^{1/2} \leqslant \frac{C\sqrt{M_8}}{n^3v^4} \leqslant \frac{C}{n^2v^2}.\end{aligned} \tag{6.63}$$

Now consider A_7. We may write

$$A_7 = -\frac{1}{n^4}\sum_{j=1}^{p}\mathrm{E}|\Delta_E(\mathbf{R})|^2\mathcal{Q}_j(\mathbf{R})R(j,j) = A_{25} + A_{26}, \tag{6.64}$$

where

$$A_{25} = \frac{1}{n^4}\sum_{j=1}^{p}\mathrm{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^2\mathcal{Q}_j(\mathbf{R})R(j,j),$$

$$A_{26} = \frac{1}{n^4}\sum_{j=1}^{p}\mathrm{E}(|\Delta_E(\mathbf{R})|^2 - |\Delta_E^{(j)}(\mathbf{R})|^2)\mathcal{Q}_j(\mathbf{R})R(j,j).$$

Lemma 6.5. *Under the conditions of Theorem* 1.2*, there exists some absolute constant* C *such that, for* $v \geqslant v_0$*,*

$$|A_{26}| \leqslant \frac{CM_8^{1/2}}{n^2 v^2}.$$

Proof. We decompose A_{26} into

$$A_{26} = A_{27} + A_{28} + A_{29} + A_{30} + A_{31}, \tag{6.65}$$

where

$$A_{27} = \frac{1}{n^4} \sum_{j=1}^{p} \mathrm{E}|\mathcal{D}_j(\mathbf{R})|^2 \mathcal{Q}_j(\mathbf{R}) R(j, j),$$

$$A_{28} = \frac{1}{n^4} |(\mathrm{E}\mathcal{D}_j(\mathbf{R}))|^2 \mathrm{E}\mathcal{Q}_j(\mathbf{R}) R(j, j),$$

$$A_{29} = -\frac{2}{n^4} \sum_{j=1}^{p} \mathrm{E}[\mathrm{Re}\{\overline{\mathcal{D}}_j(\mathbf{R})\}\mathrm{E}\mathcal{D}_j(\mathbf{R})\}]\mathcal{Q}_j(\mathbf{R})\{R(j, j)\},$$

$$A_{30} = \frac{2}{n^4} \sum_{j=1}^{p} \mathrm{E}[\mathrm{Re}\{\Delta_E^{(j)}(\mathbf{R})\overline{\mathcal{D}}_j(\mathbf{R})\}]\mathcal{Q}_j(\mathbf{R}) R(j, j),$$

$$A_{31} = -\frac{2}{n^4} \sum_{j=1}^{p} \mathrm{E}[\mathrm{Re}\{\Delta_E^{(j)}(\mathbf{R})\mathrm{E}\mathcal{D}_j(\mathbf{R})\}]\mathcal{Q}_j(\mathbf{R}) R(j, j).$$

Using Cauchy's inequality and Lemma 4.5, we obtain that, for $v \geqslant v_0$,

$$\max\{|A_{27}|, |A_{28}|, |A_{29}|\} \leqslant \frac{C\mathcal{C}(\mathbf{R})}{n^3 v^2} \left(\frac{1}{n} \sum_{j=1}^{p} \mathrm{E}|\mathcal{Q}_j(\mathbf{R})|^2 \right)^{1/2} \leqslant \frac{C\sqrt{M_4}}{(nv)^2}. \tag{6.66}$$

Furthermore, using Cauchy's inequality again,

$$\max\{|\mathrm{A}_{30}|, |\mathrm{A}_{31}|\} \leqslant \frac{C\mathcal{C}(\mathbf{R})}{n^3 v} \left(\frac{1}{n} \sum_{j=1}^{p} \mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^2 |\mathcal{Q}_j(\mathbf{R})|^2 \right)^{1/2}. \tag{6.67}$$

Since $\mathbf{x}_j$ and $\mathbf{R}_j \mathbf{W}(j)$ are independent, we have

$$\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^2 |\mathcal{Q}_j(\mathbf{R})|^2 \leqslant CM_4 \mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^2 (\mathrm{tr}|\mathbf{R}_j|^2 \mathbf{W}(j)^2). \tag{6.68}$$

Note that

$$\mathrm{tr}|\mathbf{R}_j|^2 \mathbf{W}(j)^2 = \frac{1}{v} \mathrm{Im}\{\mathrm{tr}\,\mathbf{R}_j \mathbf{W}(j)^2\} = \frac{1}{v} \mathbf{I}\{\mathrm{tr}\,\mathbf{W}(j) + z\,\mathrm{tr}\,\mathrm{Im}_{p-1} + z\,\mathrm{tr}\,\mathbf{R}_j\}. \tag{6.69}$$

With a similar argument to (6.50), we obtain, for $v \geqslant v_0$,

$$\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^2|\mathcal{Q}_j(\mathbf{R})|^2 \leqslant \frac{CM_4}{v}\{(n(|s_{pj}(z)|+1)\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^2+\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^3\}$$
$$\leqslant \frac{CM_4}{v}\left\{n\mathrm{E}|\Delta_E(\mathbf{R})|^2+\mathrm{E}|\Delta_E(\mathbf{R})|^3+\frac{n}{v^2}\right\} \leqslant \frac{CM_4^2 n}{v^4}. \qquad (6.70)$$

Inequalities (6.8), (6.67) and (6.70) together imply that, for $v \geqslant v_0$ ($\sqrt{n}v \geqslant C > 0$),

$$\max\{|\mathrm{A}_{30}|, |A_{31}|\} \leqslant \frac{CM_4}{n^{5/2}v^3} \leqslant \frac{C\sqrt{M_8}}{n^2v^2} \qquad (6.71)$$

From inequalities (6.66) and (6.71) it follows that, for $v \geqslant v_0$,

$$|A_{26}| \leqslant \frac{C\sqrt{M_8}}{n^2v^2}. \qquad (6.72)$$

The last inequality concludes the proof. □

We continue with A_{25}.

Lemma 6.6. *Under conditions of Theorem* 1.2*, there exists some constant* C_1 *such that, for* $v \geqslant v_0$*,*

$$|A_{25}| \leqslant \frac{C_1 M_8^{1/2}}{n^2v^2}.$$

Proof. We express A_{25} in the form

$$A_{25} = \frac{1}{n^4}\sum_{j=1}^{p}\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^2\mathcal{Q}_j(\mathbf{R})R(j,j) = A_{32}+A_{33}+A_{34}+A_{35}, \qquad (6.73)$$

where

$$A_{32} = -\frac{1}{n^5}\sum_{j=1}^{p}\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^2\mathcal{Q}_j(\mathbf{R})\mathcal{X}_j(R(j,j),$$

$$A_{33} = \frac{1}{n^5}\sum_{j=1}^{p}\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^2(\mathcal{Q}_j(\mathbf{R}))^2R(j,j),$$

$$A_{34} = -\frac{1}{n^5}\sum_{j=1}^{p}\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^2\mathcal{Q}_j(\mathbf{R})\mathcal{D}_j(\mathbf{R})R(j,j)\},$$

$$A_{35} = \frac{1}{n^5}\sum_{j=1}^{p}\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^2\Delta_E(\mathbf{R})\mathcal{Q}_j(\mathbf{R})R(j,j).$$

Using Cauchy's inequality,

$$|A_{32}| \leqslant \frac{C\mathcal{C}(\mathbf{R})}{n^4}\left(\frac{1}{n}\sum_{j=1}^{p}\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^4|\mathcal{Q}_j(\mathbf{R})|^2\mathcal{X}_j^2\right)^{1/2}. \tag{6.74}$$

Applying inequality (6.50), we obtain

$$\frac{1}{n^8}\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^4|\mathcal{Q}_j(\mathbf{R})|^2\mathcal{X}_j^2 \leqslant \frac{CM_8}{n^2 v}\left(\mathrm{E}\left|\frac{1}{n}\Delta_E^{(j)}(\mathbf{R})\right|^4 + \mathrm{E}\left|\frac{1}{n}\Delta_E^{(j)}(\mathbf{R})\right|^5\right).$$

This inequality and (6.74), (6.24) and together imply

$$|A_{32}| \leqslant \frac{C\sqrt{M_8}}{n^2 v^2}. \tag{6.75}$$

Similarly,

$$|A_{33}| \leqslant \frac{C\mathcal{C}(\mathbf{R})}{n^4}\left(\frac{1}{n}\sum_{j=1}^{p}\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^4|\mathcal{Q}_j(\mathbf{R})|^4\right)^{1/2}. \tag{6.76}$$

According to Rosenthal's inequality for quadratic forms, we have

$$\mathrm{E}\{|\mathcal{Q}_j(\mathbf{R})|^4|X(j)\} \leqslant CM_8(\mathrm{tr}|\mathbf{R}_j|^2\mathbf{W}(j)^2)^2.$$

Similar to inequality (6.70), we obtain that, for $v \geqslant v_0$,

$$\begin{aligned}\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^4|\mathcal{Q}_j(\mathbf{R})|^4 &\leqslant \frac{CM_8}{v^2}\left(|ns_{nj}(z)|^2\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^4 + \mathrm{E}\left|\Delta_E^{(j)}(\mathbf{R})\right|^6\right)\\ &\leqslant \frac{CM_8 n^2}{v^2}\left(\mathrm{E}|\Delta_E(\mathbf{R}))|^4 + \frac{1}{n^2}\mathrm{E}|\Delta_E(\mathbf{R})|^6 + \frac{1}{v^4}\right).\end{aligned} \tag{6.77}$$

Using the last inequality, the relations (6.69), (6.24) and the inequalities (6.74)–(6.77), we obtain, for $v \geqslant v_0$,

$$|A_{33}| \leqslant \frac{CM_8}{n^3 v^4} \leqslant \frac{C\sqrt{M_8}}{n^2 v^2}. \tag{6.78}$$

For A_{34} the following bound holds:

$$|A_{34}| \leqslant \frac{C\mathcal{C}(\mathbf{R})}{n^4 v}\left(\frac{1}{n}\sum_{j=1}^{p}\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^4|\mathcal{Q}_j(\mathbf{R})|^2\right)^{1/2}.$$

Analogously to (6.77), (6.78), we obtain that, for $v \geqslant v_0$,

$$|A_{34}| \leqslant \frac{CM_4\sqrt{M_8}}{n^{7/2}v^{9/2}} \leqslant \frac{C\sqrt{M_8}}{n^2 v^2}. \tag{6.79}$$

Applying Cauchy's inequality, Rosenthal's inequality for quadratic forms and inequalities (6.24) and (6.68), we obtain that, for $v \geqslant v_0$

$$
\begin{aligned}
|A_{35}| &= \frac{1}{n^5}\sum_{j=1}^{p} \mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^2\Delta_E(\mathbf{R})\mathcal{Q}_j(\mathbf{R})R(j,\,j)\\
&\leqslant \frac{1}{n^5}\left(\sum_{j=1}^{p} \mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^3|\mathcal{Q}_j(\mathbf{R})\|R(j,\,j)| + v^{-1}\sum_{j=1}^{p} \mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^2|\mathcal{Q}_j(\mathbf{R})\|R(j,\,j)|\right)\\
&\leqslant \frac{\mathcal{C}(\mathbf{R})}{n^4}\left(\left(\frac{1}{n}\sum_{j=1}^{p} \mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^6|\mathcal{Q}_j(\mathbf{R})|^2\right)^{1/2} + v^{-1}\left(\frac{1}{n}\sum_{j=1}^{p} \mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^4|\mathcal{Q}_j(\mathbf{R})|^2\right)^{1/2}\right)\\
&\leqslant \frac{\mathcal{C}(\mathbf{R})}{n^4}\left(\left(\frac{1}{n}\sum_{j=1}^{p} \mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^6\mathrm{tr}|\mathbf{R}_j|^2\mathbf{W}(j)^2\right)^{1/2} + v^{-1}\left(\frac{1}{n}\sum_{j=1}^{p} \mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^4\mathrm{tr}|\mathbf{R}_j|^2\mathbf{W}(j)^2\right)^{1/2}\right)\\
&\leqslant \frac{C\sqrt{M_8}}{n^4 v^{1/2}}\left(\left(n\mathrm{E}|\Delta_E(\mathbf{R})|^6 + \mathrm{E}|\Delta_E(\mathbf{R})|^7 + \frac{n}{v^6} + \frac{1}{v^7}\right)^{1/2}\right.\\
&\qquad \left. + v^{-1}\left(n\mathrm{E}|\Delta_E(\mathbf{R})|^4 + \mathrm{E}|\Delta_E(\mathbf{R})|^5 + \frac{n}{v^4} + \frac{1}{v^5}\right)^{1/2}\right)\\
&\leqslant \frac{C\sqrt{M_8 M_{12}}}{n^{7/2}v^5} = \frac{C\sqrt{M_8}}{n^2v^2}\frac{\sqrt{M_{12}}}{n^{3/2}v^3} \leqslant \frac{C\sqrt{M_8}}{n^2v^2}.
\end{aligned}
\tag{6.80}
$$

The relations (6.75), (6.78), (6.79), and (6.80) together imply

$$
|A_{25}| \leqslant \frac{C\sqrt{M_8}}{n^2v^2}. \tag{6.81}
$$

This concludes the proof. □

Lemmas 6.5 and 6.6 imply that, for $v \geqslant v_0$,

$$
|A_7| \leqslant \frac{CM_8^{1/2}}{n^2v^2} \tag{6.82}
$$

We now continue with A_2 as follows:

$$
A_2 = -\frac{1}{n^3}\sum_{j=1}^{p} \mathrm{E}\Delta_{\mathrm{E}}(\mathbf{R})\mathcal{Q}_j(\mathbf{R})R(j,\,j) = A_{36} + a_n(z)(A_{37} + A_{38} + zA_{39} + zA_{40}), \tag{6.83}
$$

where

$$A_{36} = -\frac{1}{n^3}\sum_{j=1}^{p} \mathrm{E}(\overline{\mathcal{D}_j}(\mathbf{R}) - \mathrm{E}\overline{\mathcal{D}_j}(\mathbf{R}))\mathcal{Q}_j(\mathbf{R})R(j,j),$$

$$A_{37} = -\frac{1}{n^4}\sum_{j=1}^{p} \mathrm{E}\Delta_E^{(j)}(\mathbf{R})\mathcal{Q}_j(\mathbf{R})\mathcal{X}_j R(j,j),$$

$$A_{38} = \frac{1}{n^4}\sum_{j=1}^{p} \mathrm{E}\Delta_E^{(j)}(\mathbf{R})\mathcal{Q}_j(\mathbf{R})^2 R(j,j),$$

$$A_{39} = -\frac{1}{n^4}\sum_{j=1}^{p} \mathrm{E}\Delta_E^{(j)}(\mathbf{R})\mathcal{Q}_j(\mathbf{R})\mathcal{D}_j(\mathbf{R})R(j,j),$$

$$A_{40} = \frac{1}{n^4}\sum_{j=1}^{p} \mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^2\mathcal{Q}_j(\mathbf{R})R(j,j).$$

Lemma 6.7. *Under the conditions of Theorem* 1.2*, for* $v \geqslant v_0$*,*

$$|A_{36}| \leqslant \frac{CM_8^{1/2}}{n^2 v^2}.$$

Proof. Write

$$A_{36} = A_{41} + A_{42}, \tag{6.84}$$

where

$$A_{41} = \frac{1}{n^3}\sum_{j=1}^{p} \mathrm{E}\overline{\mathcal{D}_j}(\mathbf{R})\mathcal{Q}_j(\mathbf{R})R(j,j), \qquad A_{42} = -\frac{1}{n^3}\sum_{j=1}^{p} \mathrm{E}\overline{\mathcal{D}_j}(\mathbf{R})\mathrm{E}\,\mathcal{Q}_j(\mathbf{R})R(j,j).$$

Using (6.1) and (6.25), we obtain

$$A_{42} = a_n(z)(A_{43} + A_{44} + zA_{45} + zA_{46}), \tag{6.85}$$

where

$$A_{43} = \frac{1}{n^4}\sum_{j=1}^{p} \mathrm{E}\overline{\mathcal{D}_j}(\mathbf{R})\mathrm{E}\,\mathcal{Q}_j(\mathbf{R})\mathcal{X}_j R(j,j),$$

$$A_{44} = -\frac{1}{n^4}\sum_{j=1}^{p} \mathrm{E}\overline{\mathcal{D}_j}(\mathbf{R})\mathrm{E}(\mathcal{Q}_j(\mathbf{R}))^2 R(j,j),$$

$$A_{45} = -\frac{1}{n^4}\sum_{j=1}^{p} \mathrm{E}\overline{\mathcal{D}_j}(\mathbf{R})\mathrm{E}\,\mathcal{Q}_j(\mathbf{R})\mathcal{D}_j(\mathbf{R})R(j,j),$$

$$A_{46} = -\frac{1}{n^4}\sum_{j=1}^{p} \mathrm{E}\overline{\mathcal{D}_j}(\mathbf{R})\mathrm{E}\,\mathcal{Q}_j(\mathbf{R})\Delta_E(\mathbf{R})R(j,j).$$

Applying Cauchy's inequality and (6.49),

$$|A_{43}| \leqslant \frac{C\mathcal{C}(\mathbf{R})}{n^3 v}\left(\frac{1}{n}\sum_{j=1}^{p}\mathrm{E}|\mathcal{Q}_j(\mathbf{R})|^2\mathcal{X}_j^2\right)^{1/2} \leqslant \frac{C\sqrt{M_8}}{n^2 v^{3/2}} \leqslant \frac{C\sqrt{M_8}}{n^2 v^2}. \tag{6.86}$$

Furthermore,

$$|A_{44}| \leqslant \frac{C\mathcal{C}(\mathbf{R})}{n^3 v}\left(\frac{1}{n}\sum_{j=1}^{p}\mathrm{E}|\mathcal{Q}_j(\mathbf{R})|^4\right)^{1/2} \leqslant \frac{C\sqrt{M_8}}{n^2 v^2}. \tag{6.87}$$

Analogously we obtain, for $v \geqslant v_0$,

$$|A_{45}| \leqslant \frac{C\mathcal{C}(\mathbf{R})}{n^3 v^2}\left(\frac{1}{n}\sum_{j=1}^{p}\mathrm{E}|\mathcal{Q}_j(\mathbf{R})|^2\right)^{1/2} \leqslant \frac{C\sqrt{M_4}}{n^2 v^2}. \tag{6.88}$$

Finally, for A_{46} we have the following bounds:

$$|A_{46}| \leqslant \frac{C\mathcal{C}(\mathbf{R})}{n^3 v}\left(\frac{1}{n}\sum_{j=1}^{p}\mathrm{E}|\mathcal{Q}_j(\mathbf{R})|^2|\Delta_E(\mathbf{R})|^2\right)^{1/2}$$

$$\leqslant \frac{C}{n^3 v}\left(\frac{1}{n}\sum_{j=1}^{p}\mathrm{E}|\mathcal{Q}_j(\mathbf{R})|^2|\Delta_E^{(j)}(\mathbf{R})|^2\right)^{1/2}+\frac{C\sqrt{M_4}}{n^2v^2}$$

$$\leqslant \frac{C\sqrt{M_4}}{n^3 v^{3/2}}\left(\frac{1}{n}\sum_{j=1}^{p}\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^2\,\mathrm{Im}\{\mathrm{tr}\,\mathbf{R}_j\mathbf{W}(j)^2\}\right)^{1/2}+\frac{C\sqrt{M_4}}{n^2v^2}$$

$$\leqslant \frac{C\sqrt{M_4}}{n^3 v^{3/2}}\left(\frac{1}{n}\sum_{j=1}^{p}\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^2(v(p-1)+|z^2||\mathrm{tr}\,\mathrm{R}_j|)\right)^{1/2}+\frac{C\sqrt{M_4}}{n^2v^2}$$

$$\leqslant \frac{C\sqrt{M_4}}{(nv)^{3/2}}\mathrm{E}^{1/2}\left|\frac{1}{n}\Delta_E(\mathbf{R})\right|^2+\frac{C\sqrt{M_8}}{n^2v^2}\leqslant\frac{C\sqrt{M_8}}{n^2v^2}. \tag{6.89}$$

We now turn to the estimation of A_{41}. Using Lemma 3.3, we may write

$$A_{41}=\frac{1}{n^3}\sum_{j=1}^{p}\mathrm{E}\left(1+\frac{1}{n}\mathbf{a}_j^{\mathrm{T}}\overline{\mathbf{R}}_j^2\mathbf{a}_j\right)\mathcal{Q}_j(\mathbf{R})|R(j,j)|^2. \tag{6.90}$$

By (4.6), we obtain

$$A_{41}=A_{47}+A_{48}+A_{49}+A_{50}, \tag{6.91}$$

where

$$A_{47}=\frac{|a_n(z)|^2}{n^3}\sum_{j=1}^{p}\mathrm{E}\left(1+\frac{1}{n}\mathbf{a}_j^{\mathrm{T}}j^{\mathrm{T}}\overline{\mathbf{R}}_j^2\mathbf{a}_j\right)\mathcal{Q}_j(\mathbf{R}),$$

$$A_{48}=\frac{\overline{a_n(z)}}{n^3}\sum_{j=1}^{p}\mathrm{E}\left(1+\frac{1}{n}\mathbf{a}_j^{\mathrm{T}}j^{\mathrm{T}}\overline{\mathbf{R}}_j^2\mathbf{a}_j\right)\mathrm{Q}_j(\mathbf{R})\varepsilon_jR(j,j),$$

$$A_{49}=\frac{a_n(z)}{n^3}\sum_{j=1}^{p}\mathrm{E}\left(1+\frac{1}{n}\mathbf{a}_j^{\mathrm{T}}j^{\mathrm{T}}\overline{\mathbf{R}}_j^2\mathbf{a}_j\right)\mathcal{Q}_j(\mathbf{R})\overline{\varepsilon_j}\overline{R(j,j)},$$

$$A_{50}=\frac{1}{n^3}\sum_{j=1}^{p}\mathrm{E}\left(1+\frac{1}{n}\mathbf{a}_j^{\mathrm{T}}j^{\mathrm{T}}\overline{\mathbf{R}}_j^2\mathbf{a}_j\right)\mathcal{Q}_j(\mathbf{R})|\varepsilon_j|^2|R(j,j)|^2.$$

Using Cauchy's inequality,

$$|A_{47}|\leqslant\frac{C}{n^4}\sum_{j=1}^{p}\mathrm{E}^{1/2}|\mathcal{Q}_j(\mathbf{R}^2)|^2\mathrm{E}^{1/2}|\mathcal{Q}_j(\mathbf{R})|^2\leqslant\frac{CM_4}{n^2v^2}. \tag{6.92}$$

Since $|\operatorname{tr}\mathbf{R}-\operatorname{tr}\mathbf{R}_j|=|1+(1/n)\mathbf{a}_j^{\mathrm{T}}\mathbf{R}_j^2\mathbf{a}_j||R(j,j)|\leqslant v^{-1}$, we have

$$\max\{|A_{48}|,|A_{49}|\}\leqslant\frac{C}{n^2v}\max_{1\leqslant j\leqslant n}\{\mathrm{E}^{1/2}|\mathcal{Q}_j(\mathbf{R})|^2\mathrm{E}^{1/2}|\varepsilon_j|^2\}\leqslant\frac{CM_4}{n^2v^2}. \tag{6.93}$$

Furthermore, using (6.25),

$$|A_{50}|\leqslant A_{51}+A_{52}+A_{53}+A_{54}, \tag{6.94}$$

where

$$A_{51}=\frac{C}{n^5v}\sum_{j=1}^{p}\mathrm{E}|\mathcal{Q}_j(\mathbf{R})|\mathcal{X}_j^2|R(j,j)|,$$

$$A_{52}=\frac{C}{n^5v}\sum_{j=1}^{p}\mathrm{E}|\mathcal{Q}_j(\mathbf{R})|^3|R(j,j)|,$$

$$A_{53}=\frac{C}{n^5v}\sum_{j=1}^{p}\mathrm{E}|\mathcal{Q}_j(\mathbf{R})||\mathcal{D}_j(\mathbf{R})|^2|R(j,j)|,$$

$$A_{54}=\frac{C}{n^5v}\sum_{j=1}^{p}\mathrm{E}|\mathcal{Q}_j(\mathbf{R})||\Delta_{\mathrm{E}}(\mathbf{R})|^2|R(j,j)|.$$

For A_{51} we have the obvious bound, for $v\geqslant v_0$,

$$\begin{aligned}A_{51}&\leqslant\frac{C}{n^4v^2}\max_{1\leqslant j\leqslant n}\mathrm{E}|\mathcal{Q}_j(\mathbf{R})|\mathcal{X}_j^2\leqslant\frac{C}{n^4v^2}\max_{1\leqslant j\leqslant n}\mathrm{E}^{1/2}|\mathcal{Q}_j(\mathbf{R})|^2\mathrm{E}^{1/2}\mathcal{X}_j^4\\&\leqslant\frac{C\sqrt{M_8}}{n^3v^2}\mathrm{E}^{1/2}\operatorname{tr}|\mathbf{R}_j|^2\mathbf{W}(j)^2\leqslant\frac{C\sqrt{M_8}}{n^3v^{5/2}}|\mathrm{E}\operatorname{tr}\mathbf{R}_j\mathbf{W}(j)^2|^{1/2}\\&\leqslant\frac{C\sqrt{M_8}}{n^{5/2}v^{5/2}}\leqslant\frac{C\sqrt{M_8}}{n^2v^2}.\end{aligned} \tag{6.95}$$

Applying Cauchy's inequality, we obtain, for $v\geqslant v_0$,

$$A_{52}\leqslant\frac{C\mathcal{C}(\mathbf{R})^{1/2}}{n^4v\sqrt{v}}\left(\frac{1}{n}\sum_{j=1}^{p}\mathrm{E}|\mathcal{Q}_j(\mathbf{R})|^4\right)^{3/4}\leqslant\frac{CM_8^{3/4}}{(nv)^{5/2}\sqrt{v}}\leqslant\frac{C\sqrt{M_8}}{n^2v^2}. \tag{6.96}$$

Analogously we obtain, for $v\geqslant v_0$,

$$A_{53}\leqslant\frac{C\mathcal{C}(\mathbf{R})}{n^4v^3}\left(\frac{1}{n}\sum_{j=1}^{p}\mathrm{E}|\mathcal{Q}_j(\mathbf{R})|^2\right)^{1/2}\leqslant\frac{C\sqrt{M_4}}{n^2v^2}, \tag{6.97}$$

and

$$A_{54} \leqslant \frac{C\mathcal{C}(\mathbf{R})}{n^4 v}\left(\frac{1}{n}\sum_{j=1}^{p} \mathrm{E}|\mathcal{Q}_j(\mathbf{R})|^2|\Delta_E(\mathbf{R})|^4\right)^{1/2} \leqslant \frac{C\sqrt{M_8}}{n^2 v^2}. \tag{6.98}$$

Inequalities (6.71)–(6.85) together imply that, for $v \geqslant v_0$,

$$|A_{36}| \leqslant \frac{C\sqrt{M_8}}{n^2 v^2}. \tag{6.99}$$

This completes the proof. □

Lemma 6.8. *Under the conditions of Theorem* 1.2*, there exists constants C such that*

$$\max\{|A_{37}|, |A_{40}|\} \leqslant \frac{C\sqrt{M_8}}{n^2 v^2},$$

$$\max\{|A_{38}|, |A_{39}|\} \leqslant \frac{C\sqrt{M_8}}{n^2 v^2} + \frac{C\sqrt{M_8}}{nv}\mathrm{E}^{1/2}\left|\frac{1}{n}\Delta_E(\mathbf{R})\right|^2.$$

Proof. Using Cauchy's inequality, we obtain

$$|A_{37}| = \frac{1}{n^4}\left|\sum_{j=1}^{p} \mathrm{E}\Delta_E^{(j)}(\mathbf{R})\mathcal{Q}_j(\mathbf{R})\mathcal{X}_j R(j,j)\right|$$

$$\leqslant \frac{C\mathcal{C}(\mathbf{R})}{n^3}\left(\frac{1}{n}\sum_{j=1}^{p} \mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^2|\mathcal{Q}_j(\mathbf{R})|^2\mathcal{X}_j^2\right)^{1/2}. \tag{6.100}$$

Inequalities (6.47), (6.8), and (6.85) together imply that, for $v \geqslant v_0$,

$$|A_{37}| \leqslant \frac{C\sqrt{M_8}}{n^{3/2}\sqrt{v}}\left(n\mathrm{E}\left|\frac{1}{n}\Delta_E(\mathbf{R})\right|^2 + \frac{M_4}{n^2 v^3}\right)^{1/2} \leqslant \frac{C\sqrt{M_8}}{n^2 v^2}. \tag{6.101}$$

Analogously to this inequality, we obtain

$$|A_{38}| = \frac{1}{n^4}\sum_{j=1}^{p} \mathrm{E}\Delta_E^{(j)}(\mathbf{R})\mathcal{Q}_j(\mathbf{R})^2 R(j,j)$$

$$\leqslant \frac{\mathcal{C}(\mathbf{R})}{n^3}\left(\frac{1}{n}\sum_{j=1}^{p} \mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^2|\mathcal{Q}_j(\mathbf{R})|^4\right)^{1/2}.$$

Since $\mathbf{R}_j$ and $\mathbf{x}_j$ are independent, we have

$$\frac{1}{n^6}\mathrm{E}|\Delta_{\mathrm{E}}^{(j)}(\mathbf{R})|^2|\mathcal{Q}_j(\mathbf{R})|^4 \leqslant \frac{CM_8}{n^6}\mathrm{E}|\Delta_{\mathrm{E}}^{(j)}(\mathbf{R})|^2(\mathrm{tr}|\mathbf{R}_j|^2\mathbf{W}(j)^2)^2.$$

Using (6.68),

$$\frac{1}{n^6}\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^2|\mathcal{Q}_j(\mathbf{R})|^4 \leqslant \frac{CM_8}{n^6}(\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^2(v^{-2}|\mathrm{E}\,\mathrm{tr}\,R_j|^2+n^2)+v^{-2}\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^4).$$

The last three inequalities and (6.19) together imply that, for $v \geqslant v_0$,

$$|A_{38}| \leqslant \frac{C\sqrt{M_8}}{n^2v^2}+\frac{C\sqrt{M_8}}{nv}\mathrm{E}^{1/2}\left|\frac{1}{n}\Delta_E(\mathbf{R})\right|^2+\frac{CM_8}{n^3v^4}. \tag{6.102}$$

For A_{39} the following inequality holds:

$$\begin{aligned}|A_{39}| &= -\frac{1}{n^4}\sum_{j=1}^{p}\mathrm{E}\Delta_E^{(j)}(\mathbf{R})\mathcal{Q}_j(\mathbf{R})\mathcal{D}_j(\mathbf{R})R(j,j)\\ &\leqslant \frac{\mathcal{C}(\mathbf{R})}{n^3v}\left(\frac{1}{n}\sum_{j=1}^{p}\mathrm{E}|\Delta_E^{(j)}(\mathbf{R})|^2|\mathcal{Q}_j(\mathbf{R})|^2\right)^{1/2}\\ &\leqslant \frac{C\sqrt{M_4}}{n^2v^2}+\frac{C\sqrt{M_4}}{nv}\mathrm{E}^{1/2}\left|\frac{1}{n}\Delta_E(\mathbf{R})\right|^2.\end{aligned} \tag{6.103}$$

For A_{40} the same bound holds as for A_{25} (see (6.61) and (6.70)). □

Equation (6.82) and Lemmas 6.7 and 6.8 together imply that, for $v \geqslant v_0$,

$$|A_2| \leqslant \frac{C\sqrt{M_8}}{n^2v^2}+\frac{C\sqrt{M_8}}{nv}\mathrm{E}^{1/2}\left|\frac{1}{n}(\Delta_E(\mathbf{R}))\right|^2. \tag{6.104}$$

From Lemma 6.3 and from relations (6.62), (6.63), (6.82) we conclude that

$$A_4 = y(a_n(z)-yz\delta_n(z)b_n(z))\mathrm{E}\left|\frac{1}{n}(\Delta_E(\mathbf{R}))\right|^2+\theta\frac{C|b_n(z)|\sqrt{M_8}}{n^2v^2}, \tag{6.105}$$

with some θ such that $|\theta| \leqslant 1$. Lemma 6.4 and the relations (6.26), (6.59), (6.104), (6.105) together imply that, for $v \geqslant v_0$,

$$\begin{aligned}\mathrm{E}\left|\frac{1}{n}\Delta_E(\mathbf{R})\right|^2 &= yza_n(z)(a_n(z)-yz\delta_n(z)b_n(z))\mathrm{E}\left|\frac{1}{n}(\Delta_E(\mathbf{R}))\right|^2\\ &\quad + C\theta\left(\frac{|b_n(z)|\sqrt{M_8}}{n^2v^2}+\frac{1}{nv}\mathrm{E}^{1/2}\left|\frac{1}{n}\Delta_E(\mathbf{R})\right|^2\right),\end{aligned} \tag{6.106}$$

with some θ such that $|\theta| \leqslant 1$. Finally, we investigate the quantity $\varkappa_n(z) = 1-yza_n^2(z)$.

Lemma 6.9. *Under the conditions of Theorem 1.2, there exists a positive constant C such that, for $v \geqslant v_0$,*

$$|\varkappa_n(z)|^{-1} \leqslant C|a_n(z)||y+z-1+2yzs_y(z)|. \tag{6.107}$$

Proof. We may write

$$\varkappa_n(z) = 1 + \frac{yzs_p(z)}{z + y - 1 + yzs_p(z)} + \theta Cyz|a_n(z)\delta_p(z)|$$
$$= a_n(z)(b_n(z))^{-1} + \theta C|yza_n(z)\delta_p(z)|. \tag{6.108}$$

Note that, for $z = u + iv$ such that $\sqrt{(u-a)(b-u)} \geqslant C\sqrt{v_0}$ and $v \geqslant v_0$ according to (5.1)–(5.6) we have $\operatorname{Im}(z \pm \delta_n(z)) < 0$. We can write that

$$s_p(z) = \frac{y + z - 1 - yz\delta_p(z)}{2yz} + \frac{\sqrt{(y + z - 1 - yz\delta_n(z))^2 + 4yz\delta_n(z) - 4yz}}{2yz}$$
$$= s_y(z + yz\delta_n(z)) - \frac{\delta_n(z)}{2}.$$

This implies that

$$|s_p(z) - s(z)| \leqslant |\delta_n(z)| + \frac{|\sqrt{(y + z - 1 + yz\delta_n(z))^2 - 4yz} - \sqrt{(y + z - 1)^2 - 4yz}|}{2|yz|}$$
$$\leqslant c|\delta_n(z)|\left(1 + \frac{|y + z - 1| + |\delta_n(z)|}{|\sqrt{(y + z - 1 + yz\delta_n(z))^2 - 4yz} + \sqrt{(y + z - 1)^2 - 4yz}|}\right).$$

It is not difficult to check that for $z = u + iv$ such that $|y + u - 1| \geqslant 3v$ and $v \geqslant v_0$,

$$\operatorname{sgn}\{\operatorname{Re}\{\sqrt{(z + y - 1)^2 - 4yz}\}\} = \operatorname{sgn}\{\sqrt{(zy - 1 + yz\delta_n(z))^2 - 4yz}\}.$$

This implies that, for such z,

$$|\sqrt{(y + z - 1 + yz\delta_n(z))^2 - 4yz} + \sqrt{(y + z - 1)^2 - 4yz}|$$
$$\geqslant |\sqrt{(y + z - 1)^2 - 4yz}| \geqslant \sqrt{v}.$$

In this case we have

$$|s_p(z) - s_y(z)| \leqslant \frac{C|\delta_n(z)|}{\sqrt{v}} \leqslant \frac{C}{nv^{3/2}}.$$

On the other hand, if $|y + u - 1| \leqslant 3v$ then $|y + z - 1| \leqslant 4v$ and $v \geqslant v_0$, and we have

$$|s_p(z) - s_y(z)| \leqslant C|\delta_n(z)|\left(1 + \frac{|\delta_n(z)|}{v}\right) \leqslant C|\delta_n(z)|\left(1 + \frac{1}{nv^2}\right) \leqslant C|\delta_n(z)|.$$

The last two inequalities imply that, for $v \geqslant v_0$,

$$|s_p(z) - s_y(z)| \leqslant \frac{C}{nv^{3/2}} \leqslant \gamma\sqrt{v_0}, \tag{6.109}$$

with sufficiently small γ. Furthermore, note that

$$z + y - 1 + 2yzs_y(z) = \sqrt{(y + z - 1)^2 - 4yz} = \sqrt{(a - z)(b - z)},$$

where $a = (1 - \sqrt{y})^2$, $b = (1 + \sqrt{y})^2$. This implies that there exists some positive constant C_1 such that, for $v \geqslant v_0$,

$$|z+y-1+2yzs_y(z)| \geqslant C_1\sqrt{v_0}. \tag{6.110}$$

These relations imply that

$$|b_n(z)|^{-1} = |z+y-1+2yzs_p(z)| \geqslant |z+y-1+2yzs_y(z)| - 2|yz||s_p(z)-s_y(z)|$$

$$\geqslant \frac{1}{2}|z+y-1+2yzs_y(z)|. \tag{6.111}$$

According to Lemma 4.6 and inequality (6.110), we have, for $v \geqslant v_0$,

$$|\delta_p(z)| \leqslant C_2 v_0 \leqslant C_2|z+y-1+2yzs_y(z)|v_0^{1/2}. \tag{6.112}$$

We may choose the constant in the definition v_0 such that

$$C_1 - C_2 v_0^{1/2} \geqslant C_3 > 0. \tag{6.113}$$

The relations (6.107), (6.110)–(6.113) together imply that, for $v \geqslant v_0$,

$$|\varkappa_n(z)| \geqslant \gamma |a_n(z)||y+z-1+2yzs(z)|. \tag{6.114}$$

This concludes the proof. □

Put $b(z) = (z+y-1+2yzs_y(z))^{-1}$. Equation (6.106) and Lemma 6.9 together imply that

$$\mathrm{E}\left|\frac{1}{n}\Delta_E(\mathbf{R})\right|^2 = \theta_1(z)C|b(z)b_n(z)||\delta_p(z)|\mathrm{E}\left|\frac{1}{n}\Delta_E(\mathbf{R})\right|^2$$

$$+ C\theta_2(z)\left(\frac{|b(z)|^2\sqrt{M_8}}{|a_n(z)|n^2v^2} + \frac{|b(z)|}{|a_n(z)|nv}\,\mathrm{E}^{1/2}\left|\frac{1}{n}\Delta_E(\mathbf{R})\right|^2\right), \tag{6.115}$$

with some functions $\theta_1(z)$ and $\theta_2(z)$ such that $|\theta_i(z)| \leqslant 1$, for $i = 1, 2$. Inequalities (6.111) and (6.112) together imply that, for $v \geqslant v_0$,

$$|C\theta_1(z)Cb_n(z)b(z)||\delta_n(z)| \leqslant \frac{1}{2}. \tag{6.116}$$

From (6.115), (6.116) and (5.11) we obtain the recursive inequality

$$\mathrm{E}\left|\frac{1}{n}\Delta_E(\mathbf{R})\right|^2 \leqslant \frac{C\sqrt{M_8}\,\mathrm{E}^{1/2}\left|\frac{1}{n}(\Delta_E(\mathbf{R}))\right|^2}{nv|y+z-1+2yzs_y(z)|} + \frac{\sqrt{M_8}}{n^2v^2|y+z-1+2yzs_y(z)|^2}. \tag{6.117}$$

For n sufficiently large the recursion (6.93) implies that, for $v \geqslant v_0$,

$$\mathrm{E}\left|\frac{1}{n}(\Delta_E(\mathbf{R}))\right|^2 \leqslant \frac{CM_8}{n^2v^2|y+z-1+2yzs(z)|^2}.$$

The last bound concludes the proof of Proposition 6.1. □

7. Proof of Theorem 1.2

We now consider a modification of the smoothing inequality in Corollary 2.2.

Lemma 7.1. *Let $F_p(x)$ be the empirical spectral distribution function of the matrix* $\mathbf{W}$ *and let $F_y(x)$ denote the Marchenko–Pastur distribution function. Denote their Stieltjes transforms by $m_p(z)$ and $s_y(z)$, respectively. Let v_0, d and ε be positive numbers such that*

$$\frac{1}{\pi}\int_{|y|\leqslant d}\frac{1}{u^2+1}\,\mathrm{d}u=\frac{3}{4},$$

and

$$\varepsilon<2v_0 d.$$

Then there exist constants $C_1, \ldots, C_4$ such that

$$\begin{aligned}
&\mathrm{E}\sup_x|F_p(x)-\mathrm{E}F_p(x)|\\
&\leqslant C_1\int_{-\infty}^{\infty}|(\mathrm{E}m_n(u+\mathrm{i}V)-s_y(u+\mathrm{i}V)|\mathrm{d}u+C_2v_0+C_3\varepsilon^{3/2}\\
&\quad+C_4\sup_{x\in I'_\varepsilon}\left|\mathrm{Im}\left\{\int_{v_0}^{V}(\mathrm{E}\,m_n(x+\mathrm{i}u)-s_y(x+\mathrm{i}u)\mathrm{d}u\right\}\right|\\
&\quad+C_1\int_{-\infty}^{\infty}\mathrm{E}\left|\frac{1}{n}(\mathrm{tr}\,R(u+\mathrm{i}V)-\mathrm{E}\,\mathrm{tr}\,R(u+\mathrm{i}V))\right|\mathrm{d}u\\
&\quad+C_1\int_{v_0}^{V}\mathrm{E}\left|\frac{1}{n}\,\mathrm{tr}\,\mathbf{R}(x_0+\mathrm{i}v)-\frac{1}{n}\,\mathrm{E}\,\mathrm{tr}\,\mathbf{R}(x_0+\mathrm{i}v))\right|\mathrm{d}v\\
&\quad+C_2\int_{v_0}^{V}\int_{x\in I_\varepsilon}\mathrm{E}\left|\left(\frac{1}{n}\,\mathrm{tr}\,\mathbf{R}^2(x+\mathrm{i}u)-\mathrm{E}\frac{1}{n}\,\mathrm{tr}\,\mathbf{R}^2(x+\mathrm{i}u)\right)\right|\mathrm{d}x\,\mathrm{d}u. \qquad (7.1)
\end{aligned}$$

Proof. Note that the Stieltjes transform $m_n(z)$ of distribution function $F_p(x)$ is equal to $(1/n)\mathrm{tr}\,\mathbf{R}$, and

$$m_n'(z)=\frac{1}{n}\,\mathrm{tr}\,\mathbf{R}^2(z). \qquad (7.2)$$

Applying Corollary 2.2 to the distribution functions $F_p(x)$ and $F_y(x)$, we obtain

$$\Delta_p^* := \sup_x |F_p(x) - F_y(x)|$$
$$\leqslant C_1 \int_{-\infty}^{\infty} |(m_n(u+\mathrm{i}V) - S_y(u+\mathrm{i}V)|\mathrm{d}u + C_2 v_0 + C_3 \varepsilon^{3/2}$$
$$+ C_1 \sup_{x\in I'_\varepsilon} \left| \mathrm{Im}\left\{ \int_{v_0}^{V} (m_n(x+\mathrm{i}v) - S_y(x+\mathrm{i}v))\mathrm{d}v \right\} \right|. \tag{7.3}$$

Furthermore, using the obvious inequality

$$|m_n(z) - s_y(z)| \leqslant |m_n(z) - \mathrm{E}m_n(z)| + |\mathrm{E}m_n(z) - s_y(z)|,$$

we obtain

$$\sup_x |F_p(x) - F_y(x)| \leqslant C_1 \int_{-\infty}^{\infty} |(\mathrm{E}\, m_n(u+\mathrm{i}V) - S_y(u+\mathrm{i}V)|\mathrm{d}u + C_2 v + C_3 \varepsilon^{3/2}$$
$$+ C_4 \sup_{x\in I'_\varepsilon} \left| \mathrm{Im}\left\{ \int_{\Rightarrow_0}^{V} (\mathrm{E}\, m_n(x+\mathrm{i}v) - S_y(x+\mathrm{i}v)\mathrm{d}v \right\} \right|$$
$$+ C_1 \int_{-\infty}^{\infty} |(m_n(u+\mathrm{i}V) - \mathrm{E}m_n(u+\mathrm{i}V)|\mathrm{d}u$$
$$+ C_1 \sup_{x\in I'_\varepsilon} \left| \mathrm{Im}\left\{ \int_{v_0}^{V} (m_n(x+\mathrm{i}u) - \mathrm{E}m_n(x+\mathrm{i}u))\mathrm{d}u \right\} \right|. \tag{7.4}$$

By Taylor's formula,

$$\sup_{x\in I_\varepsilon} |m_n(x+\mathrm{i}v) - \mathrm{E}m_n(u+\mathrm{i}v)|$$
$$\leqslant |m_n(x_0+\mathrm{i}v) - \mathrm{E}m_n(x_0+\mathrm{i}v)| + \int_{x\in \mathrm{I}_\varepsilon} |m'_n(u+\mathrm{i}v) - \mathrm{E}\, m'_n(u+\mathrm{i}v)|\mathrm{d}u. \tag{7.5}$$

Inequalities (7.2)–(7.5) together imply (7.1), thus proving the lemma. □

Note that, for $v \geqslant v_0 = \gamma M_{12}^{1/6} n^{-1/2}$ and for $\varepsilon \geqslant C v_0$, we have

$$C_1 \int_{-\infty}^{\infty} |(\mathrm{E}m_n(u+\mathrm{i}V) - s_y(u+\mathrm{i}V))|\mathrm{d}u + C_2 v + C_3 \varepsilon^{3/2}$$
$$+ C_4 \sup_{x\in I'_\varepsilon} \left| \mathrm{Im}\left\{ \int_{v}^{V} (\mathrm{E}m_n(x+\mathrm{i}u) - s_y(x+\mathrm{i}u))\mathrm{d}u \right\} \right| \leqslant C M_{12}^{1/6} n^{-1/2}. \tag{7.6}$$

Analogously to Section 4, we obtain that

$$\int_{-\infty}^{\infty} \mathrm{E}\left| \frac{1}{n}(\mathrm{tr}\,\mathbf{R}(u+\mathrm{i}V) - \mathrm{E}\,\mathrm{tr}\,\mathbf{R}(u+\mathrm{i}V)) \right| \mathrm{d}u \leqslant Cn^{-1}. \tag{7.7}$$

From Lemma 6.1 it follows that

$$\int_{v_0}^{V} \mathrm{E}\left|\frac{1}{n}\operatorname{tr}\mathbf{R}(x_0+\mathrm{i}v) - \frac{1}{n}\mathrm{E}\operatorname{tr}\mathbf{R}(x_0+\mathrm{i}v))\right|\mathrm{d}v$$

$$\leqslant C\int_{v_0}^{V} \mathrm{E}^{1/2}\left|\frac{1}{n}\operatorname{tr}\mathbf{R}(x_0+\mathrm{i}v) - \frac{1}{n}\mathrm{E}\operatorname{tr}\mathbf{R}(x_0+\mathrm{i}v))\right|^2\mathrm{d}v$$

$$\leqslant \int_{v_0}^{V}\left[\frac{C\sqrt{M_8}}{nv|z_0+y-1+2yz_0s_y(z_0)|}\right]\mathrm{d}v, \tag{7.8}$$

where $z_0 = x_0 + \mathrm{i}v$. Using the fact that $|z_0 + y - 1 + 2yz_0s_y(z_0)| \geqslant v$, we obtain after integration in v,

$$\int_{v_0}^{V} \mathrm{E}\left|\frac{1}{n}\operatorname{tr}\mathbf{R}(x_0+\mathrm{i}v) - \frac{1}{n}\mathrm{E}\operatorname{tr}\mathbf{R}(x_0+\mathrm{i}v))\right|\mathrm{d}v \leqslant \frac{C\sqrt{M_8}}{nv_0} \leqslant \frac{CM_8^{1/4}}{n^{1/2}} \leqslant \frac{M_{12}^{1/6}}{n^{1/2}}.$$

Let $z = \mathrm{x} + \mathrm{i}v$. Note that, for $v \geqslant v_0$,

$$\int_{x\in I_\varepsilon} \frac{1}{|y+\mathrm{z}-1+2yzs_y(z)|}\mathrm{d}u \leqslant C.$$

By Cauchy's theorem, we have

$$\left|\frac{1}{n}(\operatorname{tr}\mathbf{R}^2 - \mathrm{E}\operatorname{tr}\mathbf{R}^2)\right| \leqslant Cv^{-1}\sup_{\zeta\in\Gamma_v}\left|\frac{1}{n}\Delta_{\mathrm{E}}(\mathbf{R})\right|,$$

where $\Gamma_v = \{z : |\zeta - z| = v_0/2\}$. Applying Cauchy's inequality and Proposition 6.1 gives

$$\mathrm{E}\left|\frac{1}{n}(\operatorname{tr}\mathbf{R}^2 - \mathrm{E}\operatorname{tr}\mathbf{R}^2)\right| \leqslant Cv^{-1}\sup_{\zeta\in\Gamma_v}\mathrm{E}^{1/2}\left|\frac{1}{n}\Delta_{\mathrm{E}}(\mathbf{R})\right|^2$$

$$\leqslant Cv^{-1}\left[\frac{C\sqrt{M_8}}{nv|z+y-1+2yzs_y(z)|}\right].$$

After integration, we obtain

$$\int_{v_0}^{V}\int_{x\in I_\varepsilon}\mathrm{E}\left|\left(\frac{1}{n}\operatorname{tr}\mathbf{R}^2(x+\mathrm{i}u) - \mathrm{E}\,\frac{1}{n}\operatorname{tr}\mathbf{R}^2(x+\mathrm{i}u)\right)\right|\mathrm{d}x\,\mathrm{d}u \leqslant \frac{CM_8^{1/4}}{n^{1/2}} \leqslant \frac{CM_{12}^{1/6}}{n^{1/2}}.$$

This proves Theorem 1.2. □

Acknowledgements

The research of the first author was supported by the Deutsche Forschungsgemeinschaft Forschergruppe FOR 399/1-1 'Spektrale Analysis, Asymptotische Verteilungen und Stochastische Dynamiken'. That of the second author was partially supported by the Russian Foundation for Fundamental Research (grants NN 02-01-00233 and DFG-RFBR N99-01-04027).

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Received February 2002 and revised May 2003