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TWO METHODS FOR THE CHARACTERIZATION OF COMPACT OPERATORS BETWEEN BK SPACES

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Abstract. We establish some identities or inequalities for the Hausdorff measure of noncompactness for operators $L \in \mathcal{B}(X, Y)$ when $X = \ell_p$ $(1 \leq p < \infty)$ and $Y = c$; $X = \ell_p$ $(1 < p < \infty)$ and $Y = \ell_\infty$; $X = bv_0$ and $Y = c$; $X = c_0(\Delta), c(\Delta), \ell_\infty(\Delta)$ and $Y = \ell_\infty$. These identities and estimates are used to establish necessary and sufficient conditions for the operators to be compact. Furthermore, we apply a result by Sargent to establish necessary and sufficient conditions for operators in $\mathcal{B}(bv_0, \ell_\infty)$ and $\mathcal{B}(\ell_1, Y)$ to be compact, where $Y = w_\infty, v_\infty, [c]_\infty$.

1. Introduction

There are two important and useful methods for the characterizations of compact linear operators between BK spaces, namely the application of the Hausdorff measure of noncompactness and a result by Sargent [16].

The first method is based on the fundamental result by Goldenštein, Gohberg and Markus, for instance in [18, Theorem 4.2], and has been used in several recent papers, for instance in [3, 4, 8, 9, 12, 13, 14, 15]. It can, however, only be applied when the final space has a Schauder basis. In such cases, many authors modified the proof of [10, Theorem 2.15] to obtain sufficient conditions for the compactness of operators which are not necessary, in general.

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One prominent case where the first method fails is the characterization of compact linear operators from ℓ_1, the space of absolutely convergent series, into ℓ_∞, the space of bounded sequences. The characterization of those compact operators was given in [16, Theorem 5]. To the best of the authors' knowledge this result or a modified version have only been used in a few papers, for instance in [11, Theorem 4.8 (vi), (b)], [9, Proposition 2.5] and [3, Lemma 3.13, Corollary 3.14 **7.**].

We are going to both use the first method and apply [16, Theorem 5] or some modification to establish new characterizations of compact linear operators. To be able to give a more detailed survey, we need a few useful and customary notations.

A recent characterization of compact operators on ℓ_∞ can be found in [1, Lemma 4.1(a)].

Let ω be the set of all complex sequences $x = (x_k)_{k=1}^\infty$. By ℓ_∞, c, c_0 and ϕ, we denote the sets of all bounded, convergent, null and finite sequences, respectively; we also write $\ell_p = \{x \in \omega : \sum_{k=1}^\infty |x_k|^p < \infty\}$ for $1 \le p < \infty$, and bs and cs for the sets of all bounded and convergent series.

We write $\mathcal{B}(X, Y)$ for the set of all bounded linear operators between the normed spaces X and Y; the operator norm of $L \in \mathcal{B}(X, Y)$ is

$$\|L\| = \sup_{\|x\|=1} \|L(x)\|.$$

Applying the first method, we establish identities or estimates for operators $L \in \mathcal{B}(X, Y)$ when $X = \ell_p$ $(1 \le p < \infty)$ and $Y = c$; $X = \ell_p$ $(1 < p < \infty)$ and $Y = \ell_\infty$; $X = bv_0$, the intersection of c_0 and the space of all difference sequences in ℓ_1 and $Y = c$; and $X = c_0(\Delta), c(\Delta), \ell_\infty(\Delta)$, the spaces of all difference sequences in c_0, c and ℓ_∞ and $Y = \ell_\infty$. These identities and estimates are used to establish necessary and sufficient conditions for the operators to be compact.

The second method is applied to establish necessary and sufficient conditions for operators in $\mathcal{B}(bv_0, \ell_\infty)$ and $\mathcal{B}(\ell_1, Y)$ where $Y = w_\infty, v_\infty, [c]_\infty$ to be compact, where w_∞, v_∞ and $[c]_\infty$ are the sets of all strongly C_1 bounded sequences, of all difference sequences in w_∞, and of all strongly bounded sequences, respectively.

We also give simple proofs for the characterizations of the classes of matrix transformations from X into Y, when $X = c_0(\Delta), c(\Delta), \ell_\infty(\Delta)$ and $Y = \ell_\infty$; and $X = \ell_1$ and $Y = w_\infty, v_\infty, [c]_\infty$.

2. Notations and Known Results

We list the most important notations and known results which will be used throughout the paper.

Let e and $e^{(n)}$ $(n = 1, 2, \dots)$ be the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ $(k \ne n)$.

Let X be any subset of ω and $z \in \omega$ be given. Then we write $z^{-1} * X = \{a \in \omega : a \cdot z = (a_k z_k)_{k=1}^\infty \in X\}$ and $X^\beta = \bigcap_{x \in X} x^{-1} * cs$ for the β–dual of X, that is, $a \in X^\beta$ if and only if $\sum_{k=1}^\infty a_k x_k$ converges for all $x \in X$.

A BK space is a Banach sequence space X with continuous coordinates P_n $(n = 1, 2, \dots)$ where $P_n(x) = x_n$ for each sequence $x = (x_k)_{k=1}^\infty \in X$; a BK space

$X \supset \phi$ is said to have AK if $x^{[m]} = \sum_{k=1}^{m} x_k e^{(k)} \to x$ $(m \to \infty)$ for every sequence $x = (x_k)_{k=1}^{\infty} \in X$; $x^{[m]}$ is called the m–section of the sequence x.

Let $(X, \|\cdot\|)$ be a normed space, and $S_X = \{x \in X : \|x\| = 1\}$ and $\overline{B_X} = \{x \in X : \|x\| \leq 1\}$ denote the unit sphere and closed unit ball in X. If $(X, \|\cdot\|)$ is a normed sequence space, then we write $\|a\|_X^* = \sup_{x \in \overline{B_X}} |\sum_{k=1}^{\infty} a_k x_k|$ for $a \in \omega$ provided the expression on the right–hand side exists and is finite which is the case whenever X is a BK space and $a \in X^\beta$ [19, Theorem 7.2.9].

Let $A = (a_{nk})_{n,k=1}^{\infty}$ be an infinite matrix of complex numbers, X and Y be subsets of ω and $x \in \omega$. We write $A_n = (a_{nk})_{k=1}^{\infty}$ and $A^k = (a_{nk})_{n=1}^{\infty}$ for the sequences in the nth row and kth column of A, $A_n x = \sum_{k=1}^{\infty} a_{nk} x_k$, $Ax = (A_n x)_{n=1}^{\infty}$ (provided all the series $A_n x$ converge), and $X_A = \{x \in \omega : Ax \in X\}$ for the matrix domain of A in X. Also (X, Y) is the class of all matrices A such that $X \subset Y_A$; so $A \in (X, Y)$ if and only if $A_n \in X^\beta$ for all n and $Ax \in Y$ for all $x \in X$.

Let $T = (t_{nk})_{n,k=1}^{\infty}$ be a triangle, that is, $t_{nn} \neq 0$ for all n and $t_{nk} = 0$ for $k > n$, and X be a normed sequence space with $\|\cdot\|$. Then $\|\cdot\|_{X_T}$ is defined by $\|x\|_{X_T} = \|Tx\|$ for all $x \in X_T$; it is clear that $\|\cdot\|_{X_T}$ is a norm on X_T. Moreover, if X is a BK space with $\|\cdot\|$ then so is X_T with $\|\cdot\|_{X_T}$ by [18, Theorem 4.3.12]. If z is a sequence with $z_k \neq 0$ for all k, and T is the diagonal matrix with z on the diagonal, then $z^{-1} * X = X_T$ becomes a BK space with $\|x\|_{z^{-1} * X} = \|z \cdot x\|$ for all $x \in X$, whenever X is a BK space.

The following well–known result gives the relation between $\mathcal{B}(X, Y)$ and (X, Y) for BK spaces X and Y, and will frequently be used throughout.

Proposition 2.1. ([19, Theorem 4.2.8], [5, Theorem 1.9]) *Let X and Y be BK spaces.*
(a) Then we have $(X, Y) \subset \mathcal{B}(X, Y)$, that is, every matrix $A \in (X, Y)$ defines an operator $L_A \in \mathcal{B}(X, Y)$, where $L_A(x) = Ax$ for all $x \in X$.
(b) If X has AK then we have $\mathcal{B}(X, Y) \subset (X, Y)$, that is, every operator $L \in \mathcal{B}(X, Y)$ is given by a matrix $A \in (X, Y)$, where $Ax = L(x)$ for all $x \in X$.

We recall that a linear operator L between infinite dimensional Banach spaces X and Y is said to be compact if its domain is all of X and, for every bounded sequence (x_n) is X, the sequence $(L(x_n))$ has a convergent subsequence in Y.

Let (X, d) be a complete metric space and $\mathcal{M}_X$ denote the class of all bounded subsets of X. We write $B_r(x_0) = \{x \in X : d(x, x_0) < r\}$ for the open ball of radius $r > 0$ and center in $x_0 \in X$. The function $\chi : \mathcal{M}_X \to [0, \infty)$ with

$$\chi(Q) = \inf \left\{ \varepsilon > 0 : Q \subset \bigcup_{k=1}^{n} B_{r_k}(x_k),\ x_k \in X,\ r_k < \varepsilon\ (k = 1, 2, \dots, n; n \in \mathbb{N}) \right\}$$

is called the Hausdorff measure or ball measure of noncompactness ([18, Definition 2.1]). It has the following known properties for all $Q, Q_1, Q_2 \in \mathcal{M}_X$ ([18, Proposition 2.3 (d), (f), (i)])

$$Q_1 \subset Q_2 \text{ implies } \chi(Q_1) \leq \chi(Q_2) \text{ (monotonicity)}; \tag{2.1}$$

$$\chi(Q) = 0 \text{ for every compact set } Q\ ; \tag{2.2}$$

if X is a Banach space, then χ also satisfies

$$\chi(Q_1 + Q_2) \leq \chi(Q_1) + \chi(Q_2) \text{ (algebraic semi–additivity).} \tag{2.3}$$

Let X and Y be Banach spaces and χ_1 and χ_2 be Hausdorff measures of noncompactness on X and Y. Then an operator $L : X \to Y$ is called (χ_1, χ_2)–bounded if $L(Q) \in \mathcal{M}_Y$ for all $Q \in \mathcal{M}_X$, and if there exists a constant $C > 0$ such that

$$\chi_2(L(Q)) \leq C \cdot \chi_1(Q) \text{ for all } Q \in \mathcal{M}_X. \tag{2.4}$$

If an operator L is (χ_1, χ_2)–bounded then the number

$$\|L\|_{(\chi_1,\chi_2)} = \inf\{C > 0 : \text{(2.4) holds}\} \tag{2.5}$$

is called the (χ_1, χ_2)–measure of noncompactness of L.
We write $\|L\|_\chi = \|L\|_{(\chi_1,\chi_2)}$, for short, and call $\|L\|_\chi$ the Hausdorff measure of noncompactness of the operator L ([10, Definition 2.24]).

It is known that if X and Y are infinite dimensional Banach spaces and $L \in \mathcal{B}(X, Y)$, then

$$\|L\|_\chi = \chi(L(\overline{B_X})) = \chi(L(S_X)) \quad \text{([10, Theorem 2.25])} \tag{2.6}$$

and

$$L \text{ is compact if and only if } \|L\|_\chi = 0 \quad \text{([10, Corollary 2.26 (2.58)]).} \tag{2.7}$$

3. Main Results

Throughout, let q denote the conjugate number of p for $1 \leq p \leq \infty$, that is, $q = \infty$ for $p = 1$, $q = p/(p-1)$ for $1 < p < \infty$ and $q = 1$ for $p = \infty$. We write $\|\cdot\|_p$ for the BK norms on ℓ_p $(1 \leq p \leq \infty)$, that is,

$$\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} \quad \text{for } 1 < p < \infty \text{ and } \|x\|_\infty = \sup_k |x_k| \text{ for } p = \infty.$$

If $A = (a_{nk})_{n,k=1}^\infty$ is any infinite matrix, we write $A^{<r>}$ and $A^{>r<}$ for the matrices with the first r rows and the first r columns replaced by zero sequences, respectively.

Proposition 3.1. *Let $1 \leq p < \infty$ and $Y = c$ or $Y = \ell_\infty$.*
If $L \in \mathcal{B}(\ell_p, Y)$, then L is given by an infinite matrix $A \in (\ell_p, Y)$ as in Proposition 2.1 (b).
(a) If $Y = c$, then the Hausdorff measure of noncompactness of L satisfies the inequalities.

$$\frac{1}{2} \cdot \lim_{r\to\infty} \left(\sup_n \|B_n^{<r>}\|_q\right) \leq \|L\|_\chi \leq \lim_{r\to\infty} \left(\sup_n \|B_n^{<r>}\|_q\right), \tag{3.1}$$

where B is the matrix with the rows $B_n = A_n - (\alpha_k)_{k=1}^\infty$ for $n = 1, 2, \ldots$ and $\alpha_k = \lim_{n\to\infty} a_{nk}$ for $k = 1, 2, \ldots$.
(b) If $Y = \ell_\infty$, then the Hausdorff measure of noncompactness of L satisfies the inequality

$$0 \leq \|L\|_\chi \leq \lim_{r\to\infty} \left(\sup_n \|A_n^{<r>}\|_q\right). \tag{3.2}$$

Proof. Since ℓ_p is a BK space with AK for each p with $1 \le p < \infty$, the first part is clear.

(a) If follows by [2, Theorem 3.4] and the the fact that ℓ_p^* and ℓ_q are norm isomorphic that

$$\frac{1}{2} \cdot \limsup_{r\to\infty} \left(\sup_n \|B_n^{<r>}\|_q \right) \le \|L\|_\chi \le \limsup_{r\to\infty} \left(\sup_n \|B_n^{<r>}\|_q \right).$$

Since obviously $\|B_n^{<r>}\|_q \ge \|B_n^{<r+1>}\|_q \ge 0$ ($r = 1, 2, \dots$) for each $n \in \mathbb{N}$, the limits exist in the last two inequalities. Hence we have established the inequalities in (3.1).

(b) Let $\mathcal{P}_r : \ell_\infty \to \ell_\infty$ for $r \in \mathbb{N}$ be defined by $P_r(x) = x^{(r)}$ for all $x \in \ell_\infty$ and $\mathcal{R}_r = I - \mathcal{P}_r$, where I is the identity on ℓ_∞. We also write $\overline{B} = \overline{B_{\ell_p}}$, for short. Then it follows by (2.6), the monotonicity (2.1), algebraic semi–additivity (2.3) of χ and (2.2), and by [10, Theorem 1.23 (b)] and the fact that ℓ_p^* and ℓ_q are norm isomorphic

$$\begin{aligned} 0 &\le \|L\|_\chi = \chi(L(\overline{B})) \le \chi(\mathcal{P}_r(L(\overline{B}))) + \chi(\mathcal{R}_r(L(\overline{B}))) = \chi(\mathcal{R}_r(L(\overline{B})) \\ &\le \sup_{x\in\overline{B}} \|\mathcal{R}_r(L(x))\| = \|\mathcal{R}_r \circ L\| = \|L_{A^{<r+1>}}\| = \sup_n \|A_n^{<r+1>}\|_{\ell_p^*} \\ &= \sup_n \|A_n^{<r+1>}\|_q \text{ for all } r \in \mathbb{N}. \end{aligned}$$

This yields the inequalities in (3.2). $\square$

Applying (2.7) and Proposition 3.1, we obtain

Corollary 3.2. *Let $1 \le p < \infty$. We use the notations of Proposition 3.1.*
(a) If $L \in \mathcal{B}(\ell_p, c)$, then L is compact if and only if

$$\lim_{r\to\infty} \left(\sup_n \|B_n^{<r>}\|_q \right) = 0.$$

(b) If $L \in \mathcal{B}(\ell_p, \ell_\infty)$ and

$$\lim_{r\to\infty} \left(\sup_n \|A_n^{<r>}\|_q \right) = 0, \tag{3.3}$$

then L is compact.

We note that the condition in (3.3) is only sufficient, but not necessary, in general, for an operator $L \in \mathcal{B}(\ell_p, \ell_\infty)$ to be compact. To see this, let $L : \ell_p \to \ell_\infty$ be defined by $L(x) = x_1 \cdot e$ for all $x \in \ell_p$, hence L is compact. Also L is given by the matrix A with the rows $A_n = e^{(1)}$ for $n = 1, 2, \dots$, but the limit in (3.3) is equal to 1, since $\sup_n \|A_n^{<r>}\|_q = 1$ for all $r \in \mathbb{N}$.

If $1 < p < \infty$ the result of Corollary (3.2) (b) can be improved by applying [19, Theorem 8.3.9] and using the transpose A^T of the matrix A that represents the operator $L \in \mathcal{B}(\ell_p, \ell_\infty)$.

Proposition 3.3. *Let $1 < p < \infty$. If $L \in \mathcal{B}(\ell_p, \ell_\infty)$, then L is compact if and only if*

$$\lim_{r\to\infty} \left(\sup_n \|A_n^{>r<}\|_q \right) = 0. \tag{3.4}$$

Proof. Let $L \in \mathcal{B}(\ell_p, \ell_\infty)$ and L be given by the matrix $A \in (\ell_p, \ell_\infty)$. We apply [19, Theorem 8.3.9] with $X = \ell_p$ and $Z = \ell_1$, BK spaces with AK, $Y = Z^\beta = \ell_\infty$ and $X^\beta = \ell_q$ to obtain $A \in (\ell_p, \ell_\infty)$ if and only if $C = A^T \in (\ell_1, \ell_q)$; also $\|L\| = \|L_C\|$ by [16, Lemma 2]. Now $A \in (\ell_p, \ell_\infty)$ implies $\|L\| = \sup_n \|A_n\|_{\ell_p^*} = \sup_n \|A_n\|_q$ as before in the proof of Proposition 3.1 (b). It follows by (2.6), [10, Theorem 2.15] and since obviously $C^{<r>} = (A^{>r<})^T$ that

$$\begin{aligned}\|L_C\|_\chi = \chi(L_C(\overline{B_{\ell_1}})) &= \lim_{r\to\infty}\left(\sup_{x\in\overline{B_{\ell_1}}} \|\mathcal{R}_r(L_C(x))\|_q\right) = \lim_{r\to\infty}\|\mathcal{R}_r \circ L_C\| \\ &= \lim_{r\to\infty}\|L_{C^{<r>}}\| = \lim_{r\to\infty}\left\|L_{(C^{<r>})^T}\right\| = \lim_{r\to\infty}\|L_{A^{>r<}}\| \\ &= \lim_{r\to\infty}\left(\sup_n \|A_n^{>r<}\|_q\right).\end{aligned}$$

Since L_A is compact if and only if L_C is compact by [16, Theorem 3], (3.4) now follows from (2.7). □

We note that the statement of Proposition 3.3 is known and can be found, for instance, in [16, (b), p. 85].

The use of the transpose of the matrix A fails in the case of $p = 1$, since $\ell_1^\beta = \ell_\infty$ and $\ell_\infty^\beta = \ell_1$. In this case, the characterization of compact operators is given in [16, Theorem 5]. Since we are going to use this result, we state it here for the reader's convenience in a slightly modified version.

Lemma 3.4. ([16, Theorem 5]) *If $L \in \mathcal{B}(\ell_1, \ell_\infty)$, then L is compact if and only if*

$$\lim_{m\to\infty}\sup_{1\le n\le m}|a_{n,k_1} - a_{n,k_2}| = \sup_n |a_{n,k_1} - a_{n,k_2}| \quad \text{uniformly in } k_1 \text{ and } k_2\ (1 \le k_1, k_2 < \infty). \tag{3.5}$$

Now we consider the space $bv_0 = \{x \in \omega : \sum_{k=1}^\infty |x_k - x_{k-1}| < \infty\} \cap c_0$, where $x_0 = 0$; we use the convention that any term with an index less than or equal to 0 is equal to zero.

Theorem 3.5. *(a) If $L \in \mathcal{B}(bv_0, c)$, then L is given by an infinite matrix A as in Proposition 2.1 (b), and the Hausdorff measure of noncompactness of L satisfies the inequality*

$$\frac{1}{2}\cdot\lim_{r\to\infty}\left(\sup_n \|C_n^{<r>}\|_\infty\right) \le \|L\|_\chi \le \lim_{r\to\infty}\left(\sup_n \|C_n^{<r>}\|_\infty\right), \tag{3.6}$$

where $C = (c_{nk})_{n,k=1}^\infty$ is the matrix with $c_{nk} = \sum_{j=1}^k a_{nj} - \gamma_k$ for $n, k = 1, 2, \ldots$ and $\gamma_k = \lim_{n\to\infty}\sum_{j=1}^k a_{nj}$ for $k = 1, 2, \ldots$.
(b) If $L \in \mathcal{B}(bv_0, c)$, then L is compact if and only if

$$\lim_{r\to\infty}\left(\sup_n \|C_n^{<r>}\|_\infty\right) = 0. \tag{3.7}$$

(c) If $L \in \mathcal{B}(bv_0, \ell_\infty)$, then L is compact if and only if

$$\lim_{m\to\infty}\sup_{1\le n\le m}\left|\sum_{j=1}^{k_1} a_{nj}-\sum_{j=1}^{k_2} a_{nj}\right| = \sup_n\left|\sum_{j=1}^{k_1} a_{nj}-\sum_{j=1}^{k_2} a_{nj}\right|$$
$$\text{uniformly in } k_1 \text{ and } k_2 \ (1\le k_1,k_2<\infty). \quad (3.8)$$

Proof. (a) Since bv_0 is a BK space with AK with respect to the norm defined by $\|x\| = \sum_{k=1}^{\infty}|x_k - x_{k+1}|$ $(x\in bv_0)$ by [19, Theorem 7.3.5 (i)], it follows from Proposition 2.1 (b) that $L\in\mathcal{B}(bv_0,c)$ is given by a matrix A. By [19, Example 8.4.2A], we have $A\in(bv_0,c)$ if and only if

$$\sup_{n,k}\left|\sum_{j=1}^{k} a_{nj}\right| < \infty \text{ and } \alpha_k = \lim_{n\to\infty} a_{nk} \text{ exists for each } k.$$

We define the matrix $\tilde{C} = (\tilde{c}_{nk})_{n,k=1}^{\infty}$ by $\tilde{c}_{nk} = \sum_{j=1}^{k} a_{nj}$ for $n,k=1,2,\dots$. Obviously the limits α_k exist if and only if the limits $\gamma_k = \lim_{n\to\infty}\tilde{c}_{nk}$ exist for each k. Thus we obtain by [19, Example 8.4.1A] that $A\in(bv_0,c)$ if and only if $\tilde{C}\in(\ell_1,c)$. Furthermore, Abel's summation by parts yields for each $m\in\mathbb{N}$

$$\sum_{k=1}^{m} a_{nk}x_k = \sum_{k=1}^{m-1}\tilde{c}_{nk}(x_k - x_{k+1}) + \tilde{c}_{nm}x_m \text{ for each fixed } n\in\mathbb{N} \text{ and each } x\in bv_0.$$

Since $\tilde{C}_n\in\ell_\infty$ for each $n\in\mathbb{N}$ and $x\in bv_0\subset c_0$, we obtain $A_n x = \tilde{C}_n x$ for all $n\in\mathbb{N}$ and all $x\in bv_0$, where $y=(x_k-x_{k+1})_{k=1}^{\infty}$, hence

$$L_A(x) = L_{\tilde{C}}(y) \text{ for all } x\in bv_0, \text{ where } y=(x_k-x_{k+1})_{k=1}^{\infty}. \quad (3.9)$$

Also by [18, 7.3.4], bv_0 and ℓ_1 are equivalent, and also $x\in S_{bv_0}$ if and only if $y\in S_{\ell_1}$, since $\|x\| = \|y\|_1$. Thus we obtain from (2.6) and (3.9)

$$\|L\|_\chi = \|L_A\|_\chi = \chi\left(L_A(S_{bv_0})\right) = \chi\left(L_{\tilde{C}}(S_{\ell_1})\right) = \|L_{\tilde{C}}\|_\chi.$$

Finally, (3.1) with C in place of B and $p=1$, that is, $q=\infty$, yields (3.6).

(b) The statement is an immediate consequence of (3.6) and (2.7).

(c) As in the proof of Part (a), we obtain $A\in(bv_0,\ell_\infty)$ if and only if $\tilde{C}\in(\ell_1,\ell_\infty)$ and $L_{\tilde{C}}$ is compact by Lemma 3.4 if and only if (3.5) holds with $\tilde{c}_{n,k_1}$ and $\tilde{c}_{n,k_2}$ in place of a_{n,k_1} and a_{n,k_2}, which is (3.8). □

Remark 3.6. Similarly as in the proof of Proposition 3.3, we may apply [19, Theorem 8.3.9] with $X=bv_0$ and $Z=\ell_1$, BK spaces with AK, $Y=Z^\beta=\ell_\infty$ and $X^\beta = bs$ (by [19, Theorem 7.3.5 (ii)]) to obtain $A\in(bv_0,\ell_\infty)$ if and only if $C=A^T\in(\ell_1,bs)$. Since ℓ_1 and bs are BK spaces, we have by [10, Theorem 3.8 (a)] that $C\in(\ell_1,bs)$ if and only if $D\in(\ell_1,\ell_\infty)$, where D is the matrix with the rows $D_n=\sum_{j=1}^{n}C_j$ for $n=1,2,\dots$. So L_D is compact by Lemma 3.4 if and only if (3.5) holds with d_{n,k_1} and d_{n,k_2} in place of a_{n,k_1} and a_{n,k_2}, that is,

$$\lim_{m\to\infty}\sup_{1\le n\le m}\left|\sum_{j=1}^{n} a_{k_1 j}-\sum_{j=1}^{n} a_{k_2 j}\right| = \sup_n\left|\sum_{j=1}^{n} a_{k_1 j}-\sum_{j=1}^{n} a_{k_2 j}\right|$$
$$\text{uniformly in } k_1 \text{ and } k_2 \ (1\le k_1,k_2<\infty).$$

Next we consider some spaces of difference sequences. Let $\Delta = (\delta_{nk})_{n,k=1}^{\infty}$ be the matrix of the first order difference, that is, $\delta_{nn} = 1$, $\delta_{n,n-1} = -1$ and $\delta_{nk} = 0$ otherwise. Its inverse is the matrix $\Sigma = (\sigma_{nk})_{n,k=1}^{\infty}$ with $\sigma_{nk} = 1$ for $1 \le k \le n$ and $\sigma_{nk} = 0$ for $k > n$ $(n = 1, 2, \dots)$. If X is any of the spaces c_0, c or ℓ_∞, then we write $X(\Delta) = X_\Delta$. We need the following results.

Proposition 3.7. *Let R denote the transpose of the matrix Σ and $\mathbf{n} = (n)_{n=1}^{\infty}$. Then we have*
(a) $a \in (c_0(\Delta))^\beta$ if and only if

$$Ra \in \ell_1 \text{ and } Ra \in \mathbf{n}^{-1} * \ell_\infty; \tag{3.10}$$

(b) $a \in (\ell_\infty(\Delta))^\beta$ if and only if

$$Ra \in \ell_1 \text{ and } Ra \in \mathbf{n}^{-1} * c_0; \tag{3.11}$$

furthermore $(c(\Delta))^\beta = (\ell_\infty(\Delta))^\beta$.
(c) If $a \in (X(\Delta))^\beta$, then we have

$$\sum_{k=1}^{\infty} a_k x_k = \sum_{k=1}^{\infty} (R_k a)(\Delta_k x) \text{ for all } x \in X. \tag{3.12}$$

Proof. (a), (b) If $X = c_0$ or $X = \ell_\infty$, we apply [11, Theorem 3.2 and Remark 3.3 (a)] with $T = \Delta$ and $S = \Sigma$ to obtain $a \in (X(\Delta))^\beta$ if and only if $Ra \in X^\beta = \ell_1$, which is the first condition in (3.10) and (3.11), and $W \in (X, c_0)$, where W is the matrix with the rows $W_m = R_m a \cdot e^{[m]}$ for $m = 1, 2, \dots$. Now we have by [19, Example 8.4.5A] $W \in (c_0, c_0)$ if and only if

$$\sup_m \sum_{k=1}^{\infty} |w_{mk}| = \sup_m (m|R_m a|) < \infty,$$

which is the second condition in (3.10), and

$$\lim_{m\to\infty} w_{mk} = \lim_{m\to\infty} R_m a = 0,$$

which is redundant. Furthermore, we have by [17, (21.1)] $W \in (\ell_\infty, c_0)$ if and only if $\lim_{m\to\infty} \sum_{k=1}^{\infty} |w_{mk}| = \lim_{m\to\infty} m|R_m a| = 0$, which is the second condition in (3.11). Thus we have shown (a) and (b) for c_0 and ℓ_∞.
The sufficiency of the conditions in (3.11) for $a \in (c(\Delta))^\beta$ follows from the fact that $c(\Delta) \subset \ell_\infty(\Delta)$ implies $(\ell_\infty(\Delta))^\beta \subset (c(\Delta))^\beta$.
Conversely, we assume $a \in (c(\Delta))^\beta$. Since $(c(\Delta))^\beta \subset (c_0(\Delta))^\beta$, the first condition in (3.11) holds by (a). Also $\mathbf{n} \in c(\Delta)$ implies the convergence of the series $\sum_{n=1}^{\infty} k a_k$. This implies the second condition in (3.11) by [10, Corollary 3.16].

(c) The statement for $X = c_0$ or $X = \ell_\infty$ follows from [11, Theorem 3.2 (3.4) and Remark 3.3 (a)], and also for $X = c$, since $(c(\Delta))^\beta = (\ell_\infty(\Delta))^\beta$, as we have just shown. □

Proposition 3.8. *We have*
(a) $A \in (c_0(\Delta), \ell_\infty)$ if and only if

$$\sup_n \|\hat{A}_n\|_1 < \infty, \text{ where } \hat{A}_n = RA_n = \left(\sum_{j=k}^{\infty} a_{nj}\right)_{k=1}^{\infty} \tag{3.13}$$

and

$$\sup_m |mR_m A_n| < \infty \text{ for each } n; \tag{3.14}$$

(b) $A \in (\ell_\infty(\Delta), \ell_\infty)$ *if and only if* (3.13) *holds and*

$$\lim_{m\to\infty} mR_m A_n = 0 \text{ for each } n; \tag{3.15}$$

furthermore $(c(\Delta), \ell_\infty) = (\ell_\infty(\Delta), \ell_\infty)$.
(c) If $A \in (X(\Delta), \ell_\infty)$, *then we have*

$$Ax = \hat{A}(\Delta x) \text{ for all } x \in X(\Delta) \tag{3.16}$$

and

$$\|L_A\| = \|L_{\hat{A}}\| \tag{3.17}$$

where $\hat{A} \in (X, \ell_\infty)$.

Proof. All the statements for $X = c_0$ or $X = \ell_\infty$ are an immediate consequence of [11, Theorem 3.4, Remark 3.5 (a), Theorem 3.6] and the fact that $(\ell_\infty, \ell_\infty) = (c_0, \ell_\infty)$.
The inclusion $(\ell_\infty(\Delta), \ell_\infty) \subset (c(\Delta), \ell_\infty)$ is clear.
Conversely we assume $A \in (c(\Delta), \ell_\infty)$. This clearly implies $A \in (c_0(\Delta), \ell_\infty)$ and so the condition in (3.13) holds. Also $A \in (c(\Delta), \ell_\infty)$ implies $A_n \in (c(\Delta))^\beta$ for all n, and since $(c(\Delta))^\beta = (\ell_\infty(\Delta))^\beta$ by Proposition 3.7 (b), the condition in (3.15) follows from the second condition in (3.11). Now the conditions in (3.13) and (3.15) imply $A \in (\ell_\infty(\Delta), \ell_\infty)$, and so we have also shown $(c(\Delta), \ell_\infty) \subset (\ell_\infty(\Delta), \ell_\infty)$.
The identity in (3.17) is a consequence of that in (3.16) and the fact that the BK norms of the spaces $c_0(\Delta)$, $c(\Delta)$ are the same; (3.17) for $c_0(\Delta)$ follows from [10, Theorem 3.6]; for $\ell_\infty(\Delta)$ from [10, Remark 3.5] (which gives (3.16) for $\ell_\infty(\Delta)$) and the definition of the norms of the operators L_A and $L_{\hat{A}}$. Finally, clearly (3.12) implies (3.16) for $c(\Delta)$ and (3.17) follows from the definition of the norms of the operators L_A and $L_{\hat{A}}$. Indeed, we have the relations

$$\begin{aligned}\|L_A\| = \sup_{z\in B_{c(\Delta)}} \|L_A z\|_{\ell_\infty} = \sup_{z\in B_{c(\Delta)}} \|L_{\hat{A}}(\Delta z)\|_{\ell_\infty} = \\ = \sup_{\Delta z\in B_c} \|L_{\hat{A}}(\Delta z)\|_{\ell_\infty} = \sup_{y\in B_c} \|L_{\hat{A}} y\|_{\ell_\infty} = \|L_{\hat{A}}\|\end{aligned} \tag{3.18}$$

and the proof is completed. □

Theorem 3.9. *Let* X *denote any of the spaces* c_0, c *and* ℓ_∞. *If* $A \in (X(\Delta), \ell_\infty)$ *then* L_A *is compact if and only if*

$$\lim_{r\to\infty}\left(\sup_n \left\|\hat{A}_n^{>r<}\right\|_1\right) = \lim_{r\to\infty}\left(\sup_n \sum_{k=r}^\infty \left|\sum_{j=k}^\infty a_{nj}\right|\right) = 0. \tag{3.19}$$

Proof. Let $A \in (X(\Delta), \ell_\infty)$. Then we have $\hat{A} \in (X, \ell_\infty)$ and $Ax = \hat{A}(\Delta x)$ for all $x \in X(\Delta)$ by Proposition 3.8 (c) and (3.16). Since $(X, \ell_\infty) = (c_0, \ell_\infty)$ by [19, Example 8.4.5A], we obtain $\hat{A} \in (X, \ell_\infty)$ if and only if $\hat{B} = \hat{A}^T \in (\ell_1, \ell_1)$ by [19, Theorem 8.3.9], and by [10, Theorem 2.28], $L_{\hat{B}} \in \mathcal{B}(\ell_1, \ell_1)$ is compact if and only if

$$\begin{aligned}\lim_{r\to\infty}\left(\sup_k \sum_{n=r}^{\infty} |\hat{b}_{nk}|\right) &= \lim_{r\to\infty}\left(\sup_k \sum_{n=r}^{\infty} |\hat{a}_{kn}|\right) \\ &= \lim_{r\to\infty}\left(\sup_k \left\|\hat{A}_k^{>r<}\right\|_1\right) = \lim_{r\to\infty}\left(\sup_n \left\|\hat{A}_n^{>r<}\right\|_1\right) \\ &= \lim_{r\to\infty}\left(\sup_n \sum_{k=r}^{\infty}\left|\sum_{j=k}^{\infty} a_{nj}\right|\right) = 0.\end{aligned}$$

Finally, since $L_{\hat{A}}$ is compact if and only if $L_{\hat{B}}$ is compact by [16, Theorem 3] and L_A is compact if and only if $L_{\hat{A}}$ is compact by (3.16) , we have proved the statement of the theorem. □

Finally, we consider the spaces w_∞ and $[c]_\infty$ related to the concepts of strong C_1 boundedness and strong boundedness, studied by Maddox [7] and Kuttner and Thorpe [6], respectively, where

$$w_\infty = \left\{x \in \omega : \sup_n \left(\frac{1}{n}\sum_{k=1}^{n} |x_k|\right) < \infty\right\} \text{ and } [c]_\infty = \mathbf{n}^{-1} * (w_\infty)_\Delta.$$

It is well known that w_∞ is a BK space with the norm $\|\cdot\|_{w_\infty}$ defined by

$$\|x\|_{w_\infty} = \sup_\nu \frac{1}{2^\nu}\sum_{k=2^\nu}^{2^{\nu+1}-1} |x_k| \text{ for all } x \in w_\infty.$$

We also consider the space $v_\infty = (w_\infty)_\Delta$. For $\nu = 0, 1, \ldots,$ we write $\sum_\nu$ and $\max_\nu$ for the sum and maximum taken over all indices k with $2^\nu \le k \le 2^{\nu+1} - 1$, and

$$\mathcal{W} = \{a \in \omega : \|a\|_{\mathcal{W}} = \sum_{\nu=0}^{\infty} 2^\nu \max_\nu |a_k| < \infty\}.$$

Theorem 3.10. *Writing $\sum_\mu$ for the sum taken over all indices n with $2^\mu \le n \le 2^{\mu+1} - 1$, we have*
(a) $A \in (\ell_1, w_\infty)$ if and only if

$$\sup_{\mu,k}\left(\frac{1}{2^\mu}\sum\nolimits_\mu |a_{nk}|\right) < \infty; \tag{3.20}$$

(b) $A \in (\ell_1, v_\infty)$ if and only if

$$\sup_{\mu,k}\left(\frac{1}{2^\mu}\sum\nolimits_\mu |a_{nk} - a_{n-1,k}|\right) < \infty; \tag{3.21}$$

(c) $A \in (\ell_1, [c]_\infty)$ if and only if

$$\sup_{\mu,k}\left(\frac{1}{2^\mu}\sum\nolimits_\mu |n a_{nk} - (n-1) a_{n-1,k}|\right) < \infty. \tag{3.22}$$

Proof. (a) We apply [19, Theorem 8.3.9] with $X = \ell_1$ and $Z = \mathcal{W}$, a BK space with AK with respect to $\|\cdot\|_{\mathcal{W}}$ by [4, Proposition 2.4 (b)], and $Y = Z^\beta = w_\infty$ by [3, Proposition 1.2 (c)] to obtain $A \in (\ell_1, w_\infty)$ if and only if $B = A^T \in (\mathcal{W}, \ell_\infty)$. Since, by [4, Proposition 2.4 (c)],

$$\|a\|^*_{\mathcal{W}} = \|a\|_{w_\infty} = \sup_\nu \left(\frac{1}{2^\nu}\sum\nolimits_\nu |a_k|\right) \text{ on } \mathcal{W}^\beta$$

we obtain by [10, Theorem 1.23 (b)] $B \in (\mathcal{W}, \ell_\infty)$ if and only if

$$\sup_n \|B_n\|^*_{\mathcal{W}} = \sup_{n,\nu}\left(\frac{1}{2^\nu}\sum\nolimits_\nu |b_{nk}|\right) = \sup_{n,\nu}\left(\frac{1}{2^\nu}\sum\nolimits_\nu |a_{kn}|\right) = \sup_{\mu,k}\left(\frac{1}{2^\mu}\sum\nolimits_\mu |a_{nk}|\right) < \infty.$$

(b) We have by [10, Theorem 3.8 (a)] $A \in (\ell_1, v_\infty)$ if and only if $C = (c_{nk})_{n,k=1}^\infty = \Delta \cdot A \in (\ell_1, w_\infty)$. Since $c_{nk} = a_{nk} - a_{n-1,k}$ for all n and k, (3.21) is an immediate consequence of (3.20).

(c) We have by [10, Theorem 3.8 (a)] $A \in (\ell_1, [c]_\infty)$ if and only in $D(\mathbf{n}) \cdot A \in (\ell_1, v_\infty)$ where $D(\mathbf{n})$ is the diagonal matrix with the sequence $\mathbf{n}$ on its diagonal. Now (3.22) is an immediate consequence of (3.21). □

Finally we give the characterizations for compact operators $L \in B(\ell_1, Y)$, where Y is any of the spaces w_∞, v_∞ or $[c]_\infty$.

Theorem 3.11. *Writing $\sum_{\mu_1}$ and $\sum_{\mu_2}$ for the sums taken over all indices n with $2^{\mu_1} \le n \le 2^{\mu_1+1}-1$ and $2^{\mu_2} \le n \le 2^{\mu_2+1}-1$, we have:*
(a) If $L \in \mathcal{B}(\ell_1, w_\infty)$, then L is compact if and only if

$$\lim_{j\to\infty}\left(\sup_{1\le k\le j}\left|\frac{1}{2^{\mu_1}}\sum\nolimits_{\mu_1} a_{nk} - \frac{1}{2^{\mu_2}}\sum\nolimits_{\mu_2} a_{nk}\right|\right) = \sup_k \left|\frac{1}{2^{\mu_1}}\sum\nolimits_{\mu_1} a_{nk} - \frac{1}{2^{\mu_2}}\sum\nolimits_{\mu_2} a_{nk}\right|$$
$$\text{uniformly in } \mu_1 \text{ and } \mu_2\ (0 \le \mu_1, \mu_2 < \infty). \tag{3.23}$$

(b) If $L \in \mathcal{B}(\ell_1, v_\infty)$, then L is compact if and only if

$$\lim_{j\to\infty}\left(\sup_{1\le k\le j}\left|\frac{1}{2^{\mu_1}}(a_{2^{\mu_1+1}-1,k} - a_{2^{\mu_1}-1,k}) - \frac{1}{2^{\mu_2}}(a_{2^{\mu_2+1}-1,k} - a_{2^{\mu_2}-1,k})\right|\right) =$$
$$\sup_k\left|\frac{1}{2^{\mu_1}}(a_{2^{\mu_1+1}-1,k} - a_{2^{\mu_1}-1,k}) - \frac{1}{2^{\mu_2}}(a_{2^{\mu_2+1}-1,k} - a_{2^{\mu_2}-1,k})\right|$$
$$\text{uniformly in } \mu_1 \text{ and } \mu_2\ (0 \le \mu_1, \mu_2 < \infty). \tag{3.24}$$

(c) If $L \in \mathcal{B}(\ell_1, [c]_\infty)$, then L is compact if and only if

$$\lim_{j\to\infty}\left(\sup_{1\le k\le j}\left|\frac{1}{2^{\mu_1}}\left((2^{\mu_1+1}-1)a_{2^{\mu_1+1}-1,k} - (2^{\mu_1}-1)a_{2^{\mu_1}-1,k}\right) -\right.\right.$$
$$\left.\left. -\frac{1}{2^{\mu_2}}\left((2^{\mu_2+1}-1)a_{2^{\mu_2+1}-1,k} - (2^{\mu_2}-1)a_{2^{\mu_2}-1,k}\right)\right|\right) =$$
$$\sup_k\left|\frac{1}{2^{\mu_1}}\left((2^{\mu_1+1}-1)a_{2^{\mu_1+1}-1,k} - (2^{\mu_1}-1)a_{2^{\mu_1}-1,k}\right) -\right.$$
$$\left. -\frac{1}{2^{\mu_2}}\left((2^{\mu_2+1}-1)a_{2^{\mu_2+1}-1,k} - (2^{\mu_2}-1)a_{2^{\mu_2}-1,k}\right)\right|$$

$$\textit{uniformly in } \mu_1 \textit{ and } \mu_2 \ (0 \le \mu_1, \mu_2 < \infty). \quad (3.25)$$

Proof. (a) This is the case $p = 1$ in [3, Corollary 3.14 7.(7.1)$^+$].

(b) By Theorem 3.10 (a) and (b), we have $A \in (\ell_1, v_\infty)$ if and only if $C = \Delta \cdot A \in (\ell_1, w_\infty)$ and so the condition in (3.24) is an immediate consequence of that in (3.23).

(c) By Theorem 3.10 (b) and (c), we have $A \in (\ell_1, [c]_\infty)$ if and only if $D(\mathbf{n}) \cdot A \in (\ell_1, v_\infty)$ and so the condition in (3.25) is an immediate consequence of that in (3.24). □

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References

1. E.A. Alekhno, *The lower Weyl spectrum of a positive operator*, Integral Equations Operator Theory **67** (2010), no. 3, 301–326.
2. I. Djolović and E. Malkowsky, *A note on compact operators on matrix domains*, J. Math. Anal. Appl. **340** (2008), 291–303.
3. I. Djolović and E. Malkowsky, *Compact operators into the spaces of strongly C_1 summable and bounded sequences*, Nonlinear Anal. **74** (2011), 3736–3750.
4. I. Djolović and E. Malkowsky, *Banach algebras of matrix transformations between some sequence spaces related to Λ–strong convergence and boundedness*, Appl. Math. Comp. **219** (2013), 8779–8789.
5. A.M. Jarrah and E. Malkowsky, *Ordinary, absolute and strong summability*, Filomat **17** (2003), 59–78.
6. B. Kuttner and B. Thorpe, *Strong convergence*, J. Reine Angew. Math. **311/312** (1979), 42–55.
7. I.J. Maddox, *On Kuttner's theorem*, J. London Math. Soc. **43** (1968) 285–290.
8. B. de Malafosse, E. Malkowsky and V. Rakočević, *Measure of noncompactness of operators and matrices on the spaces c and c_0*, Int. J. Math. Math. Sci. (2006), Issue 1, Article ID 46930, 5 pages.
9. B de Malafosse and V. Rakočević, *Application of measure of noncompactness in operators on the spaces s_α, s_α^0, $s_\alpha^{(c)}$, ℓ_α^p*, J. Math. Anal. Appl. **323** (2006), no. 1, 131–145.
10. E. Malkowsky and V. Rakočević, *A*n Introduction into the Theory of Sequence Spaces and Measures of Noncompactness, Zb. Rad. (Beogr.) **9**(17) (2000), 143–234.
11. E. Malkowsky and V. Rakočević, *On matrix domains of triangles*, Appl. Math. Comp. **189** (2007) 1146–1163.
12. M. Mursaleen and S.A. Mohiuddine, *Applications of measures of noncompactness to infinite systems of differential equations in ℓ_p spaces*, Nonlinear Anal. **75** (2012), 2111–2115.
13. M. Mursaleen and A.M. Noman, *Compactness by the Hausforff measure of noncompactness*, Nonlinear Anal. **73** (2010) 2541–2557.
14. M. Mursaleen and A.M. Noman, *Applications of the Hausdorff measure of noncompactness in some sequence spaces of weighted means*, Comput. Math. Appl. **60** (2010) 1245–1258.
15. M. Mursaleen and V. Karakaya, H. Polat and N. Simşek, *Measure of noncompactness of matrix operators in some difference sequences of weighted means*, Comput. Math. Appl. **62** (2011) 814–820.
16. W.L.C. Sargent, *On compact matrix transformations between sectionally bounded BK–spaces*, J. London Math. Soc. **41** (1966), 79–87.

17. M. Stieglitz and H. Tietz, *Matrixtransformationen in Folgenräumen*, Math. Z. **154** (1977), 1–16.
18. J.M. Ayerbee Toledano, T. Dominguez Benavides and G. Lopez Acedo, *Measures of Noncompactness in Metric Fixed Point Theory*, Operator Theory, Advances and Applications, Vol 99, Birkhäuser, Basel, Boston, Berlin, 1997.
19. A. Wilansky, *Summability Through Functional Analysis*, North-Holland Mathematics Studies 85, Amsterdam, 1984.

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