Banach J. Math. Anal. 9 (2015), no. 3, 206–227
http://doi.org/10.15352/bjma/09-3-15
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

COMPACTNESS OF COMMUTATORS AND MAXIMAL COMMUTATORS OF MULTILINEAR SINGULAR INTEGRAL OPERATORS WITH NON-SMOOTH KERNELS ON MORREY SPACE

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Communicated by M. A. Ragusa

Abstract. In this paper, the behavior for commutators and maximal commutators of a class of bilinear singular integral operators associated with non-smooth kernels on the products of Morrey space is studied. By some maximal operators and commutators, we proved that the commutators and maximal commutators of singular integral operators and CMO functions are bounded and compact.

1. Introduction

In recent years, considerable attention has been paid to the study of multilinear singular integral. In this article, we will address the m-linear operators T which defined on the m-fold product of Schwartz spaces and taking values into the space of tempered distributions,

$$T : \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n) \mapsto \mathcal{S}'(\mathbb{R}^n),$$

and associated with kernel $K(x, y_1, \ldots, y_m)$ in the sense that

$$T(f_1, \ldots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \cdots f_m(y_m) \mathrm{d}y_1 \cdots \mathrm{d}y_m, \quad (1.1)$$

Date: Received: Jul, 24, 2014; Accepted: Nov. 5, 2014.
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2010 *Mathematics Subject Classification.* Primary 42B20; Secondary 42B25, 47B07.
Key words and phrases. Morrey space, singular integral operator, maximal operator, commutator, compactness.

where $K(x, y_1, \dots, y_m)$ is a locally integral function defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, $x \notin \cap_{j=1}^m \operatorname{supp} f_j$ and $f_1, \dots, f_m$ are bounded functions with compact supports. The corresponding maximal singular integral operator T^* is defined by

$$T^*(f_1, \cdots, f_m)(x) = \sup_{\delta>0} |T_\delta(f_1, \cdots, f_m)(x)|,$$

where

$$T_\delta(f_1, \cdots, f_m)(x) = \int_{|x-y_1|^2+\cdots+|x-y_m|^2>\delta} K(x, y_1, \dots, y_m) \prod_{i=1}^m f_i(y_i) \mathrm{d}y_1 \cdots \mathrm{d}y_m.$$

A kernel K is called of class $m - CZK(A, \gamma)$ if there exist positive constants A and $\gamma \in (0, 1]$ such that K satisfies the size condition

$$|K(x, y_1, \dots, y_m)| \le \frac{A}{(|x - y_1| + \cdots + |x - y_m|)^{mn}}, \tag{1.2}$$

for all $(x, y_1, \dots, y_m) \in (\mathbb{R}^n)^{m+1}$ with $x \ne y_j$ for some $j \in \{1, 2, \dots, m\}$; and the smoothness condition

$$|K(x, y_1, \dots, y_j, \dots, y_m) - K(x', y_1, \dots, y_j, \dots, y_m)| \le \frac{A|x - x'|^\gamma}{(\sum_{i=1}^m |x - y_i|)^{mn+\gamma}},$$

whenever $|x - x^{'}| \le \frac{1}{2} \max_{1\le j\le m} |x - y_j|$, and also for each j,

$$|K(x, y_1, \dots, y_j, \dots, y_m) - K(x, y_1, \dots, y_j^{'}, \dots, y_m)| \le \frac{A|y_j - y_j^{'}|^\gamma}{(\sum_{i=1}^m |x - y_i|)^{mn+\gamma}}, \tag{1.3}$$

whenever $|y_j - y_j^{'}| \le \frac{1}{2} \max_{1\le j\le m} |x - y_j|$. Then the operator T associated with this kernel K is called the multilinear Calderón–Zygmund operator. For the multilinear Calderón–Zygmund operator T, in [17], Grafakos and Torres obtained the multilinear $T1$ theorem. Thus they show the boundedness of T on the products of Lebesgue spaces. Moreover, in [16], the weighted estimates with A_p weights for the operators T and T^* was considered. For more works about multilinear Calderón–Zygmund operator, the reader can refer to [16], [17], [21], [23] and the references therein.

In this article, we are interested in the commutators and the maximal commutators of multilinear singular integral operators. For the sake of convenience, we will only consider the bilinear setting, since the other multilinear case are quite similar.

For $b \in \mathrm{BMO}(\mathbb{R}^n)$, we define the following commutators

$$T_{b,1}(f_1, f_2)(x) = [b, T]_1(f_1, f_2)(x) = T(bf_1, f_2)(x) - bT(f_1, f_2)(x),$$
$$T_{b,2}(f_1, f_2)(x) = [b, T]_2(f_1, f_2)(x) = T(f_1, bf_2)(x) - bT(f_1, f_2)(x),$$
$$T^*_{b,1}(f_1, f_2)(x) = \sup_{\delta>0} |[b, T_\delta]_1(f_1, f_2)(x)| = \sup_{\delta>0} |(T_\delta(bf_1, f_2) - bT_\delta(f_1, f_2))(x)|,$$
$$T^*_{b,2}(f_1, f_2)(x) = \sup_{\delta>0} |[b, T_\delta]_2(f_1, f_2)(x)| = \sup_{\delta>0} |(T_\delta(f_1, bf_2) - bT_\delta(f_1, f_2))(x)|.$$

Also, for a vector function

$$\vec{b} = (b_1, b_2) \in \mathrm{BMO}(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n),$$

we consider the following two iterated commutators introduced in [24],

$$T_{\vec{b}}(f_1,f_2)(x)=[b_2,[b_1,T]_1]_2(f_1,f_2)(x),$$
$$T^*_{\vec{b}}(f_1,f_2)(x)=\sup_{\delta>0}|[b_2,[b_1,T_\delta]_1]_2(f_1,f_2)(x)|.$$

In the sense of (1.1), it is easy to see that

$$T_{b,1}(f_1,f_2)(x)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}K(x,y_1,y_2)(b(y_1)-b(x))f_1(y_1)f_2(y_2)\mathrm{d}y_1\mathrm{d}y_2,$$
$$T_{b,2}(f_1,f_2)(x)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}K(x,y_1,y_2)(b(y_2)-b(x))f_1(y_1)f_2(y_2)\mathrm{d}y_1\mathrm{d}y_2,$$
$$T_{\vec{b}}(f_1,f_2)(x)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}K(x,y_1,y_2)(b_1(y_1)-b_1(x))(b_2(y_2)-b_2(x))\prod_{i=1}^{2}f_i(y_i)\mathrm{d}\vec{y},$$
$$T^*_{b,1}(f_1,f_2)(x)=\sup_{\delta>0}\left|\int\int_{|x-y_1|^2+|x-y_2|^2>\delta}K(x,y_1,y_2)(b(y_1)-b(x))\prod_{i=1}^{2}f_i(y_i)\mathrm{d}\vec{y}\right|,$$
$$T^*_{b,2}(f_1,f_2)(x)=\sup_{\delta>0}\left|\int\int_{|x-y_1|^2+|x-y_2|^2>\delta}K(x,y_1,y_2)(b(y_2)-b(x))\prod_{i=1}^{2}f_i(y_i)\mathrm{d}\vec{y}\right|,$$
$$T^*_{\vec{b}}(f_1,f_2)(x)=\sup_{\delta>0}\left|\int\int_{\sum_{i=1}^{2}|x-y_i|^2>\delta}K(x,y_1,y_2)\prod_{j=1}^{2}(b_j(y_j)-b_j(x))\prod_{i=1}^{2}f_i(y_i)\mathrm{d}\vec{y}\right|.$$

The commutator of multilinear Calderón–Zygmund operator has been extensively studied in last decades. To know the history of this topic, the reader can refer to [7], [23], [24], [25], [26], [28] and the references therein. Particularly, in [24], the boundedness of iterated commutators of multilinear Calderón–Zygmund operators on product of weighted Lebesgue spaces with multiple weights was studied, and in [28], the weighted strong and end-point estimates of maximal commutators of multilinear Calderón–Zygmund operators were considered.

However, for some multilinear singular integral operators including the Calderón commutator, people found that their kernels do not satisfy (1.3) (see [12]). Here, the Calderón commutator is defined by

$$\mathcal{C}_{m+1}(f,a_1,\dots,a_m)(x)=\int_{\mathbb{R}}\frac{\prod_{j=1}^{m}(A_j(x)-A_j(y))}{(x-y)^{m+1}}f(y)\mathrm{d}y,$$

where $A'_j=a_j$. For this reason, in [12], via the generalized approximation to the identity, Duong et al. introduced a class of multilinear singular integral operators whose kernels satisfy certain "smoothness conditions" that are weaker than $K\in m-CZK(A,\gamma)$.

For $p_1,\dots,p_{m+1}\in[1,\infty]$ and $p\in(0,\infty)$ with $\frac{1}{p}=\sum_{j=1}^{m+1}\frac{1}{p_j}$, the following weak type estimate was established in [12],

$$\|\mathcal{C}_{m+1}(f,a_1,\dots,a_m)\|_{L^{p,\infty}(\mathbb{R})}\le C\|f\|_{L^{p_{m+1}}(\mathbb{R})}\prod_{j=1}^{m}\|a_j\|_{L^{p_k}(\mathbb{R})}.$$

If $\min_{1\leq j\leq m+1} p_j > 1$, the strong type estimate was also established in [12]. Also, the weighted case, including the multiple weights, of the maximal Calderón commutator were considered in [11] and [15]. Moreover, there are a lot of works related to singular integral operators with non-smooth kernels. The reader may refer [11], [12], [13], [15], [18] and [19].

In this article, we are interested in the compactness of the commutator and maximal commutator of bilinear singular integral operators with non-smooth kernels and CMO functions on Morrey space, where CMO denotes the closure of C_c^∞ in the BMO topology and C_c^∞ is the set of C^∞ functions with compact support. In particular, we will study the compactness on the Morrey space for the bilinear singular integral operator T associated with a kernel K which satisfies (1.2) in the sense (1.1), and

(1) T is bounded from $L^1(\mathbb{R}^n)\times L^1(\mathbb{R}^n)$ to $L^{1/2,\infty}(\mathbb{R}^n)$;

(2) for $x, x', y_1, y_2 \in \mathbb{R}^n$ with $8|x-x'| < \min_{1\leq j\leq 2}|x-y_j|$,

$$|K(x,y_1,y_2) - K(x',y_1,y_2)| \leq \frac{C\tau^\gamma}{(|x-y_1|+|x-y_2|)^{2n+\gamma}}, \tag{1.4}$$

where C is a constant and τ is a number such that $2|x-x'| < \tau$ and $4\tau < \min_{1\leq j\leq 2}|x-y_j|$. It was pointed out in [19] that the kernel satisfying condition (1.2) and (1.4) includes the non-smooth kernel introduced by Doung et al. in [11], [12]. Hence, the study of the commutator and maximal commutator of bilinear singular integral operators with above non-smooth kernels is significative.

Before stating our results, we briefly describe the background and our motivation. The Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ (see Definition 1.1) which was introduced by Morrey (see [22]) in 1938, is connected to certain problems in elliptic PDE. We note that in the linear setting, the compactness of commutator on Morrey space has been studied in [6]. Recently, Bényi and Torres posed a concept of compactness (see Definition 1.2) in the bilinear setting in [2] which was equivalent to the concept proposed by Calderón in [4]. Bényi and Torres then extended the result of compactness for linear singular integrals by Uchiyama [27] to the bilinear setting and obtained that $[b,T]_1$, $[b,T]_2$, $[b_2,[b_1,T]_1]_2$ are compact bilinear operators from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $b, b_1, b_2 \in \mathrm{CMO}(\mathbb{R}^n)$. Later, Ding et al. considered the compactness of the commutators of bilinear Calderón–Zygmund operators and CMO functions on the products of $L^{p,\lambda}(\mathbb{R}^n)$ space (see [10]), as well as the compactness of the maximal commutators of bilinear Calderón–Zygmund operators and CMO functions on the products of $L^p(\mathbb{R}^n)$ space (see [9]). We note that in [9] and [10], T is a Calderón–Zygmund operator. In the article, we will consider the compactness of commutators and maximal commutators by assuming that T is an operator associated with non-smooth kernel. This assumption raises essential difficulties if we following the proofs in [9] and [10].

To formulate the main hypotheses on the commutators under consideration, we need to define the John-Nirenberg space ([20]) of functions with Bounded Mean Oscillation (BMO).

Let f be a locally integrable function in $\mathbb{R}^n$. We say that f belongs to BMO if

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} = \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B |f(x) - f_B|dx < \infty$$

where B ranges in the class of balls of $\mathbb{R}^n$. Hereafter, f_B stands for the integral average

$$\frac{1}{|B|}\int_B |f(x)|dx$$

of the function f over the set B.

Definition 1.1. For $1 \le p < \infty$, $n > 1$ and $0 < \lambda < n$, the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ is defined by

$$L^{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L^p_{\mathrm{loc}} : \|f\|_{p,\lambda} = \sup_{y\in\mathbb{R}^n, r>0} \left(\frac{1}{r^\lambda}\int_{B(y,r)} |f(x)|^p \mathrm{d}x \right)^{1/p} < \infty \right\},$$

where $B(y, r)$ denotes the ball centered at y with radius $r > 0$. The spaces $L^{p,\lambda}(\mathbb{R}^n)$ becomes a Banach space with norm $\|\cdot\|_{p,\lambda}$. If $\lambda = 0$ and $\lambda = n$, then $L^{p,0}(\mathbb{R}^n)$ and $L^{p,n}(\mathbb{R}^n)$ coincide with the space $L^p(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$.

We denote the closed ball of radius r centered at the origin in the normed space X as $B_{r,X} = \{x \in X : \|x\| \le r\}$.

Definition 1.2. A bilinear operator $T : X\times Y \mapsto Z$ is called compact if $T(B_{1,X} \times B_{1,Y})$ is precompact in Z.

Definition 1.3. A weight w belongs to the class A_p, $1 < p < \infty$, if

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w(y)\mathrm{d}y \right) \left(\frac{1}{|Q|}\int_Q w(y)^{1-p'}\mathrm{d}y \right)^{p-1} < \infty.$$

A weight w belongs to the class A_1 if there is a constant C such that

$$\frac{1}{|Q|}\int_Q w(y)\mathrm{d}y \le C \inf_{x\in Q} w(x).$$

Definition 1.4. Let $\vec{p} = (p_1, p_2)$ and $1/p = 1/p_1 + 1/p_2$ with $1 \le p_1, p_2 < \infty$. Given $\vec{w} = (w_1, w_2)$, set $\nu_{\vec{w}} = \prod_{j=1}^2 w_j^{p/p_j}$. We say that $\vec{w}$ satisfies the $A_{\vec{p}}$ condition if

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \nu_{\vec{w}} \right)^{1/p} \prod_{j=1}^{2} \left(\frac{1}{|Q|}\int_Q w_j^{1-p_j'} \right)^{1/p_j'} < \infty.$$

Here, $\left(\frac{1}{|Q|}\int_Q w_j^{1-p_j'} \right)^{1/p_j'}$ is understood as $(\inf_Q w_j)^{-1}$,when $p_j = 1$.

The following theorems are our main results:

Theorem 1.5. *Let T be a bilinear operator that bounded from $L^1(\mathbb{R}^n) \times L^1(\mathbb{R}^n)$ to $L^{1/2,\infty}(\mathbb{R}^n)$ and its kernel K satisfies (1.2), (1.4). Assume $b \in \mathrm{CMO}(\mathbb{R}^n)$, $0 < \lambda, \lambda_1, \lambda_2 < n$, $p_1, p_2 \in (1,\infty)$, $p \in (1,\infty)$ such that $1/p = 1/p_1 + 1/p_2$ and $\lambda/p = \lambda_1/p_1 + \lambda_2/p_2$. Then $T^*_{b,1}$, $T^*_{b,2}$ are compact from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to $L^{p,\lambda}(\mathbb{R}^n)$.*

Corollary 1.6. *Let T be a bilinear operator that bounded from $L^1(\mathbb{R}^n)\times L^1(\mathbb{R}^n)$ to $L^{1/2,\infty}(\mathbb{R}^n)$ and its kernel K satisfies (1.2), (1.4). Assume $b\in \mathrm{CMO}(\mathbb{R}^n)$, $0<\lambda,\lambda_1,\lambda_2<n$, $p_1,p_2\in(1,\infty)$, $p\in(1,\infty)$ such that $1/p=1/p_1+1/p_2$ and $\lambda/p=\lambda_1/p_1+\lambda_2/p_2$. Then $T_{b,1}$, $T_{b,2}$ are compact from $L^{p_1,\lambda_1}(\mathbb{R}^n)\times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to $L^{p,\lambda}(\mathbb{R}^n)$.*

Theorem 1.7. *Let T be a bilinear operator that bounded from $L^1(\mathbb{R}^n)\times L^1(\mathbb{R}^n)$ to $L^{1/2,\infty}(\mathbb{R}^n)$ and its kernel K satisfies (1.2), (1.4). Assume $b\in \mathrm{BMO}(\mathbb{R}^n)$, $0<\lambda,\lambda_1,\lambda_2<n$, $p_1,p_2\in(1,\infty)$, $p\in(1,\infty)$ such that $1/p=1/p_1+1/p_2$ and $\lambda/p=\lambda_1/p_1+\lambda_2/p_2$. Then*

$$\|T^*_{b,1}(f_1,f_2)\|_{p,\lambda},\|T^*_{b,2}(f_1,f_2)\|_{p,\lambda}\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{p_1,\lambda_1}\|f_2\|_{p_2,\lambda_2}.$$

Remark 1.8. Theorem 1.5 and Corollary 1.6 are also true for the iterated commutators $T^*_{\vec{b}}$ and $T_{\vec{b}}$, and their proofs are similar to those of Theorem 1.5 and Corollary 1.6. We leave the detail to the interested reader. We will only write out the proof of Theorem 1.5, because the proof of Corollary 1.6 is quite similar. From [3] and Lemma 2.3, we can see that Theorem 1.7 is also true for the operators $T_{b,1}(f_1,f_2)$ and $T_{b,2}(f_1,f_2)$.

We make some conventions. In this paper, we always denote a positive constant by C which is independent of the main parameters and its value may differ from line to line. We use the symbol $A\lesssim B$ to denote that there exists a constant C such that $A\le CB$. For a measurable set E, χ_E denotes its characteristic function. For a fixed p with $p\in[1,\infty)$, p' denotes the dual index of p. We also denote $\vec{f}=(f_1,\cdots,f_m)$ with scalar functions f_j, $j=1,2,...,m$. Let $B(s,t)$ denote the ball centered at s with radius $t>0$. Given $\alpha>0$ and a ball $B(s,t)$, $\alpha B(s,t)$ denotes the ball which is centered at s with radius αt. Let us define f_Q as the average of f over Q and Q is a cube in $\mathbb{R}^n$. Let M be the standard Hardy-Littlewood maximal operator.

2. Proof of Theorem 1.7

The proof of Theorem 1.7 needs some maximal functions in the following. For $0<\eta<\infty$, M_η is the maximal operator

$$M_\eta f(x)=M(|f|^\eta)^{1/\eta}(x)=\left(\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)|^\eta\mathrm{d}y\right)^{1/\eta},$$

and $M^\#$ is the sharp maximal operator defined by Fefferman and Stein [14],

$$M^\# f(x)=\sup_{Q\ni x}\inf_c\frac{1}{|Q|}\int_Q|f(y)-c|\mathrm{d}y\approx\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)-f_Q|\mathrm{d}y,$$

and

$$M^\#_\eta f(x)=M^\#(|f|^\eta)^{1/\eta}(x).$$

Moreover, when $0<p,\eta<\infty$, $w\in A_\infty(\mathbb{R}^n)$, then there exists $C>0$ such that

$$\int_{\mathbb{R}^n}(M_\eta f(x))^p w(x)\mathrm{d}x\le C\int_{\mathbb{R}^n}(M^\#_\eta f(x))^p w(x)\mathrm{d}x,$$

for every function f for which the left-hand side is finite.

Now let $b \in \mathrm{BMO}(\mathbb{R}^n)$, define the commutator M_b of the Hardy-Littlewood maximal operator with b by

$$M_b f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |b(x) - b(y)||f(y)| \mathrm{d}y$$

for all $x \in \mathbb{R}^n$. For $\Phi(t) = t(1 + \log^+ t)$ and a ball B in $\mathbb{R}^n$,

$$\|f\|_{L(\log L),B} = \inf \left\{ \lambda > 0 : \frac{1}{|B|} \int_B \Phi\left(\frac{|f(x)|}{\lambda} \right) \mathrm{d}x \leq 1 \right\}.$$

Define the maximal operator $M_{L(\log L)}$ by

$$M_{L(\log L)} f(x) = \sup_{B \ni x} \|f\|_{L(\log L),B},$$

where the supremum is taken over all the balls containing x. By the generalized Jensen's inequlity, we have

$$M_{L(\log L)} f(x) \lesssim M_\eta f(x) \tag{2.1}$$

for any $\eta > 1$.

Lemma 2.1. *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$, then for any $0 < \eta < \frac{1}{2}$*

$$M_\eta^{\#}(M_b f)(x) \lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} M_{L(\log L)} f(x),$$

for any bounded functions f with compact support.

This Lemma was proved in [1].

Furthermore, Grafakos, Liu, and Yang [15] introduced a kind of new bilinear maximal operator which was defined as:

$$\mathcal{M}_{2,1}(\vec{f})(x) = \sup_{B \ni x} \sum_{k=0}^{\infty} 2^{-kn} \left(\frac{1}{|B|} \int_B |f_1(y_1)| \mathrm{d}y_1 \right) \left(\frac{1}{|2^k B|} \int_{2^k B} |f_2(y_2)| \mathrm{d}y_2 \right),$$

$$\mathcal{M}_{2,2}(\vec{f})(x) = \sup_{B \ni x} \sum_{k=0}^{\infty} 2^{-kn} \left(\frac{1}{|B|} \int_B |f_2(y_2)| \mathrm{d}y_2 \right) \left(\frac{1}{|2^k B|} \int_{2^k B} |f_1(y_1)| \mathrm{d}y_1 \right),$$

where $\vec{f} = (f_1, f_2)$ are locally integrable functions. And the following boundedness for $\mathcal{M}_{2,1}$ and $\mathcal{M}_{2,2}$ were proved in [15].

Lemma 2.2. *Let $1 < p_1, p_2 < \infty$, $\frac{1}{p} = \sum_{j=1}^{2} \frac{1}{p_j}$, and $\vec{w} = (w_1, w_2) \in A_{\vec{p}}(\mathbb{R}^{2n})$. Then $\mathcal{M}_{2,1}$ and $\mathcal{M}_{2,2}$ are bounded from $L^{p_1}(w_1) \times L^{p_2}(w_2)$ to $L^p(\nu_{\vec{w}})$.*

Lemma 2.3. *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. Suppose that T is as in Theorem 1.5, then for any $x \in \mathbb{R}^n$, $\delta > 0$ and $0 < \eta < 1/2$,*

$$\begin{aligned} |[b, T_\delta]_1(f_1, f_2)(x)| \lesssim\ & M_b(f_1)(x) M(f_2)(x) + M_\eta(T_{b,1}(f_1, f_2)(x)) \\ & + [M_b(T(f_1, f_2)^\eta(x))][M_\eta(T(f_1, f_2)(x))]^{1-\eta} \\ & + M_\eta(T_{b,1}(f_1 \chi_{B(x,\delta^{1/2})}, f_2 \chi_{B(x,\delta^{1/2})})(x)) \\ & + \Big([M_b(T(f_1 \chi_{B(x,\delta^{1/2})}, f_2 \chi_{B(x,\delta^{1/2})})^\eta(x))] \\ & \quad \times [M_\eta(T(f_1 \chi_{B(x,\delta^{1/2})}, f_2 \chi_{B(x,\delta^{1/2})})(x))]^{1-\eta} \Big), \end{aligned}$$

where f_1, f_2 are any bounded functions with compact support.

Proof. We adopt some ideas of [5] and [16]. Let $\delta > 0$ and function $\vec{g} = (g_1, g_2)$ which is chosen later. From the size condition (1.2), we deduce that

$$
\begin{aligned}
&|T_\delta(g_1,g_2)(x)| = \Big|\int\int_{|x-y_1|^2+|x-y_2|^2>\delta} K(x,y_1,y_2)g_1(y_1)g_2(y_2)\mathrm{d}y_1\mathrm{d}y_2\Big| \\
&\le \Big|\int\int_{\substack{|x-y_1|^2+|x-y_2|^2>\delta \\ \max(|x-y_1|,|x-y_2|)<\delta^{1/2}}} K(x,y_1,y_2)g_1(y_1)g_2(y_2)\mathrm{d}y_1\mathrm{d}y_2\Big| \\
&\quad + \Big|\int\int_{\max(|x-y_1|,|x-y_2|)\ge\delta^{1/2}} K(x,y_1,y_2)g_1(y_1)g_2(y_2)\mathrm{d}y_1\mathrm{d}y_2\Big| \\
&\lesssim M(g_1)(x)M(g_2)(x) + \Big|\int\int_{\max(|x-y_1|,|x-y_2|)\ge\delta^{1/2}} K(x,y_1,y_2)g_1(y_1)g_2(y_2)\mathrm{d}\vec{y}\Big| \\
&\lesssim M(g_1)(x)M(g_2)(x) + \Big|\int\int_{\min(|x-y_1|,|x-y_2|)\ge\delta^{1/2}} K(x,y_1,y_2)g_1(y_1)g_2(y_2)\mathrm{d}\vec{y}\Big| \\
&\quad + \Big|\int_{|x-y_1|>\delta^{1/2}}\int_{|x-y_2|<\delta^{1/2}} K(x,y_1,y_2)g_1(y_1)g_2(y_2)\mathrm{d}y_1\mathrm{d}y_2\Big| \\
&\quad + \Big|\int_{|x-y_2|>\delta^{1/2}}\int_{|x-y_1|<\delta^{1/2}} K(x,y_1,y_2)g_1(y_1)g_2(y_2)\mathrm{d}y_1\mathrm{d}y_2\Big| \\
&\lesssim M(g_1)(x)M(g_2)(x) + \Big|\int\int_{\min(|x-y_1|,|x-y_2|)\ge\delta^{1/2}} K(x,y_1,y_2)g_1(y_1)g_2(y_2)\mathrm{d}\vec{y}\Big|,
\end{aligned}
$$

where the last inequality holds because

$$
\begin{aligned}
&\Big|\int_{|x-y_1|>\delta^{1/2}}\int_{|x-y_2|<\delta^{1/2}} K(x,y_1,y_2)g_1(y_1)g_2(y_2)\mathrm{d}y_1\mathrm{d}y_2\Big| \\
&\lesssim \int_{|x-y_1|>\delta^{1/2}}\int_{|x-y_2|<\delta^{1/2}} \frac{|g_1(y_1)||g_2(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}}\mathrm{d}y_1\mathrm{d}y_2 \\
&\lesssim \int_{|x-y_1|>\delta^{1/2}} \frac{|g_1(y_1)|}{|x-y_1|^{n+1}}\mathrm{d}y_1 \int_{|x-y_2|<\delta^{1/2}} \frac{|g_2(y_2)|}{|x-y_2|^{n-1}}\mathrm{d}y_2 \\
&\lesssim M(g_1)(x)M(g_2)(x).
\end{aligned}
$$

Denote

$$\widetilde{T}_\delta(g_1,g_2)(x) = \int\int_{\min(|x-y_1|,|x-y_2|)\ge\delta^{1/2}} K(x,y_1,y_2)g_1(y_1)g_2(y_2)\mathrm{d}y_1\mathrm{d}y_2.$$

Fix $\delta > 0$ and let $z \in B(x, \frac{\delta^{1/2}}{8})$, we obtain

$$
\begin{aligned}
&\int\int_{\min(|x-y_1|,|x-y_2|)\ge\delta^{1/2}} K(z,y_1,y_2)g_1(y_1)g_2(y_2)\mathrm{d}y_1\mathrm{d}y_2 \\
&= T(\vec{g})(z) - T\big(g_1\chi_{B(x,\delta^{1/2})}, g_2\chi_{B(x,\delta^{1/2})}\big)(z) - T\big(g_1\chi_{B(x,\delta^{1/2})}, g_2\chi_{B^c(x,\delta^{1/2})}\big)(z) \\
&\quad - T\big(g_1\chi_{B^c(x,\delta^{1/2})}, g_2\chi_{B(x,\delta^{1/2})}\big)(z).
\end{aligned}
$$

Because $|z-y_2| = |z-x+x-y_2| \geq |x-y_2| - |z-x|$, we know that

$$\begin{aligned}
&|T\big(g_1\chi_{B(x,\delta^{1/2})}, g_2\chi_{B^c(x,\delta^{1/2})}\big)(z)| \\
&\leq \int_{|x-y_1|<\delta^{1/2}} \int_{|x-y_2|>\delta^{1/2}} \frac{|g_1(y_1)||g_2(y_2)|}{(|z-y_1|+|z-y_2|)^{2n}} \mathrm{d}y_1 \mathrm{d}y_2 \\
&\lesssim \int_{|x-y_1|<\delta^{1/2}} |g_1(y_1)|\mathrm{d}y_1 \sum_{k=1}^{\infty} \int_{2^{k-1}\delta^{1/2}<|x-y_2|<2^k\delta^{1/2}} \frac{|g_2(y_2)|}{|x-y_2|^{2n}} \mathrm{d}y_2 \\
&\lesssim \sum_{k=1}^{\infty} 2^{-kn} \frac{1}{|B(x,\delta^{1/2})|} \int_{B(x,\delta^{1/2})} |g_1(y_1)|\mathrm{d}y_1 \frac{1}{|B(x,2^k\delta^{1/2})|} \int_{B(x,2^k\delta^{1/2})} |g_2(y_2)|\mathrm{d}y_2 \\
&\lesssim \mathcal{M}_{2,1}(g_1,g_2)(x),
\end{aligned}$$

similarly, we get

$$|T\big(g_1\chi_{B^c(x,\delta^{1/2})}, g_2\chi_{B(x,\delta^{1/2})}\big)(z)| \lesssim \mathcal{M}_{2,2}(g_1,g_2)(x).$$

On the other hand, by taking infimum over τ in condition (1.4), we have

$$|K(x,y_1,y_2) - K(z,y_1,y_2)| \leq \frac{D|x-z|^{\gamma}}{(|x-y_1|+|x-y_2|)^{2n+\gamma}},$$

when $8|x-z| < \min_{1\leq j\leq 2} |x-y_j|$. Thus,

$$\begin{aligned}
&\left|\widetilde{T}_\delta(g_1,g_2)(x) - \int\int_{\min(|x-y_1|,|x-y_2|)\geq\delta^{1/2}} K(z,y_1,y_2)g_1(y_1)g_2(y_2)\mathrm{d}y_1\mathrm{d}y_2\right| \\
&\lesssim \int\int_{\min(|x-y_1|,|x-y_2|)\geq\delta^{1/2}} \frac{|x-z|^{\gamma}}{(|x-y_1|+|x-y_2|)^{2n+\gamma}} |g_1(y_1)||g_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2 \\
&\lesssim \delta^{\frac{\gamma}{2}} \int\int_{\min(|x-y_1|,|x-y_2|)\geq\delta^{1/2}} \frac{|g_1(y_1)||g_2(y_2)|}{(|x-y_1|+|x-y_2|)^{2n+\gamma}} \mathrm{d}y_1\mathrm{d}y_2 \\
&\lesssim M(g_1)(x)M(g_2)(x).
\end{aligned}$$

In summary, for $z \in B(x, \frac{\delta^{1/2}}{8})$, we obtain

$$\begin{aligned}
|T_\delta(g_1,g_2)(x)| \;\lesssim\; & M(g_1)(x)M(g_2)(x) + \sum_{i=1}^{2} \mathcal{M}_{2,i}(g_1,g_2)(x) \\
& + |T(g_1,g_2)(z) - T\big(g_1\chi_{B(x,\delta^{1/2})}, g_2\chi_{B(x,\delta^{1/2})}\big)(z)|.
\end{aligned}$$

For $0 < \eta < \frac{1}{2}$. Raising the above inequality to the power η, integrating over $z \in B(x, \frac{\delta^{1/2}}{8})$, and dividing by $|B(x, \frac{\delta^{1/2}}{8})|$, we get

$$\begin{aligned}
|T_\delta(g_1,g_2)(x)|^{\eta} \;\lesssim\; & \Big[\prod_{i=1}^{2} M(g_i)(x)\Big]^{\eta} + \sum_{i=1}^{2} [\mathcal{M}_{2,i}(g_1,g_2)(x)]^{\eta} \\
& + \frac{1}{|B(x,\frac{\delta^{1/2}}{8})|} \int_{B(x,\frac{\delta^{1/2}}{8})} |T(g_1,g_2)(z)|^{\eta}\mathrm{d}z \\
& + \frac{1}{|B(x,\frac{\delta^{1/2}}{8})|} \int_{B(x,\frac{\delta^{1/2}}{8})} |T\big(g_1\chi_{B(x,\delta^{1/2})}, g_2\chi_{B(x,\delta^{1/2})}\big)(z)|^{\eta}\mathrm{d}z.
\end{aligned}$$

Let $g_1(\cdot) = (b(x) - b(\cdot))f_1(\cdot)$ and $g_2(\cdot) = f_2(\cdot)$, we have

$$\begin{aligned}
&|T_\delta((b(x)-b)f_1, f_2)(x)| \\
&\lesssim M((b(x)-b)f_1)(x)M(f_2)(x) + \sum_{i=1}^{2} \mathcal{M}_{2,i}((b(x)-b)f_1, f_2)(x) \\
&\quad + \left[\frac{1}{|B(x,\frac{\delta^{1/2}}{8})|}\int_{B(x,\frac{\delta^{1/2}}{8})} |T((b(x)-b)f_1, f_2)(z)|^\eta \mathrm{d}z\right]^{\frac{1}{\eta}} \\
&\quad + \left[\frac{1}{|B(x,\frac{\delta^{1/2}}{8})|}\int_{B(x,\frac{\delta^{1/2}}{8})} |T\big((b(x)-b)f_1\chi_{B(x,\delta^{1/2})}, f_2\chi_{B(x,\delta^{1/2})}\big)(z)|^\eta \mathrm{d}z\right]^{\frac{1}{\eta}} \\
&\lesssim M_b(f_1)(x)M(f_2)(x) + \sum_{i=1}^{2} \mathcal{M}_{2,i}((b(x)-b)f_1, f_2)(x) \\
&\quad + \left[\frac{1}{|B(x,\frac{\delta^{1/2}}{8})|}\int_{B(x,\frac{\delta^{1/2}}{8})} |T((b(x)-b(z)+b(z)-b)f_1, f_2)(z)|^\eta \mathrm{d}z\right]^{\frac{1}{\eta}} \\
&\quad + \left[\frac{1}{|B(x,\frac{\delta^{1/2}}{8})|}\int_{B(x,\frac{\delta^{1/2}}{8})} |T\big((b(x)-b(z)+b(z)-b) \right. \\
&\qquad \left. \times f_1\chi_{B(x,\delta^{1/2})}, f_2\chi_{B(x,\delta^{1/2})}\big)(z)|^\eta \mathrm{d}z\right]^{\frac{1}{\eta}}.
\end{aligned}$$

Hence, by Hölder's inequality and the fact that

$$\sum_{i=1}^{2} \mathcal{M}_{2,i}((b(x)-b)f_1, f_2)(x) \lesssim M_b(f_1)(x)M(f_2)(x),$$

we know that

$$\begin{aligned}
|[b,T_\delta]_1(f_1,f_2)(x)| \lesssim\ & M_b(f_1)(x)M(f_2)(x) + M_\eta(T_{b,1}(f_1,f_2)(x)) \\
&+[M_b(T(f_1,f_2)^\eta(x))][M_\eta(T(f_1,f_2)(x))]^{1-\eta} \\
&+M_\eta(T_{b,1}(f_1\chi_{B(x,\delta^{1/2})}, f_2\chi_{B(x,\delta^{1/2})})(x)) \\
&+\Big([M_b(T(f_1\chi_{B(x,\delta^{1/2})}, f_2\chi_{B(x,\delta^{1/2})})^\eta(x))] \\
&\quad \times[M_\eta(T(f_1\chi_{B(x,\delta^{1/2})}, f_2\chi_{B(x,\delta^{1/2})})(x))]^{1-\eta}\Big).
\end{aligned}$$

□

Remark 2.4. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$, T is as in Theorem 1.5. Then for any $x \in \mathbb{R}^n$, $\delta > 0$ and $0 < \eta < 1/2$, $[b,T_\delta]_2(f_1,f_2)(x)$ has the similar estimate as $[b,T_\delta]_1(f_1,f_2)(x)$ in Lemma 2.3.

Lemma 2.5. *Let $0 < \lambda, \lambda_1, \lambda_2 < n$, $p_1, p_2 \in (1,\infty)$, $p \in (1,\infty)$ such that $1/p = 1/p_1 + 1/p_2$ and $\lambda/p = \lambda_1/p_1 + \lambda_2/p_2$. Suppose that T is bounded from $L^{p_1}(w_1) \times L^{p_2}(w_2)$ to $L^p(\nu_{\vec{w}})$ for $\vec{w} = (w_1, w_2) \in A_1(\mathbb{R}^n) \times A_1(\mathbb{R}^n)$ and $\nu_{\vec{w}} \in A_1(\mathbb{R}^n)$. Then T is bounded from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to $L^{p,\lambda}(\mathbb{R}^n)$.*

Proof. For any fixed ball $B=B(s,r)$ we decompose f_1, f_2 as $f_1=f_1^0+\sum_{k=1}^{\infty} f_1^k = f_1\chi_B+\sum_{k=1}^{\infty} f_1^k\chi_{2^kB\backslash 2^{k-1}B}$, $f_2=f_2^0+\sum_{j=1}^{\infty} f_2^j = f_2\chi_B+\sum_{j=1}^{\infty} f_2^j\chi_{2^jB\backslash 2^{j-1}B}$. Hence,

$$\begin{aligned}
&\left(\frac{1}{r^\lambda}\int_B |T(f_1^0,f_2^0)(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}\\
&\lesssim \frac{1}{r^{\lambda/p}}\left(\int_B |T(f_1^0,f_2^0)(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}\\
&\lesssim \|T\|_{L^{p_1}(w_1)\times L^{p_2}(w_2)\to L^p(\nu_{\vec{w}})} \frac{1}{r^{\lambda/p}}\left(\int_B |f_1^0(x)|^{p_1}\mathrm{d}x\right)^{\frac{1}{p_1}}\left(\int_B |f_2^0(x)|^{p_2}\mathrm{d}x\right)^{\frac{1}{p_2}}\\
&\lesssim \|T\|_{L^{p_1}(w_1)\times L^{p_2}(w_2)\to L^p(\nu_{\vec{w}})}\|f_1\|_{p_1,\lambda_1}\|f_2\|_{p_2,\lambda_2}.
\end{aligned}$$

Let $\max(\frac{\lambda_1}{n},\frac{\lambda_2}{n})<\theta<1$, then $(M\chi_B(x))^\theta\in A_1(\mathbb{R}^n)$ (see [8]). By the fact that $M\chi_B(x)\lesssim 2^{-jn}$ when $x\in 2^{j+1}B\setminus 2^jB$, we have

$$\begin{aligned}
&\sum_{j=1}^{\infty}\left(\frac{1}{r^\lambda}\int_B |T(f_1^0,f_2^j)(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} \lesssim \sum_{j=1}^{\infty}\frac{1}{r^{\lambda/p}}\left(\int_{\mathbb{R}^n} |T(f_1^0,f_2^j)(x)|^p (\chi_B(x))^\theta \mathrm{d}x\right)^{\frac{1}{p}}\\
&\lesssim \sum_{j=1}^{\infty}\frac{1}{r^{\lambda/p}}\left(\int_{\mathbb{R}^n} |T(f_1^0,f_2^j)(x)|^p (M\chi_B(x))^\theta \mathrm{d}x\right)^{\frac{1}{p}}\\
&\lesssim \|T\|_{L^{p_1}(w_1)\times L^{p_2}(w_2)\to L^p(\nu_{\vec{w}})}\sum_{j=1}^{\infty}\frac{1}{r^{\lambda/p}}\left(\int_B |f_1(x)|^{p_1}(M\chi_B(x))^\theta\mathrm{d}x\right)^{\frac{1}{p_1}}\\
&\times\left(\int_{2^jB\backslash 2^{j-1}B} |f_2(x)|^{p_2}(M\chi_B(x))^\theta\mathrm{d}x\right)^{\frac{1}{p_2}}\\
&\lesssim \|T\|_{L^{p_1}(w_1)\times L^{p_2}(w_2)\to L^p(\nu_{\vec{w}})}\sum_{j=1}^{\infty}\frac{1}{r^{\lambda/p}}\left(\int_B |f_1(x)|^{p_1}\mathrm{d}x\right)^{\frac{1}{p_1}}\\
&\times 2^{\frac{-jn\theta}{p_2}}\left(\int_{2^jB} |f_2(x)|^{p_2}\mathrm{d}x\right)^{\frac{1}{p_2}}\\
&\lesssim \|T\|_{L^{p_1}(w_1)\times L^{p_2}(w_2)\to L^p(\nu_{\vec{w}})}\sum_{j=1}^{\infty}2^{-\frac{j}{p_2}(n\theta-\lambda_2)}\|f_1\|_{p_1,\lambda_1}\|f_2\|_{p_2,\lambda_2}\\
&\lesssim \|T\|_{L^{p_1}(w_1)\times L^{p_2}(w_2)\to L^p(\nu_{\vec{w}})}\|f_1\|_{p_1,\lambda_1}\|f_2\|_{p_2,\lambda_2},
\end{aligned}$$

similarly, we get

$$\sum_{k=1}^{\infty}\left(\frac{1}{r^\lambda}\int_B |T(f_1^k,f_2^0)(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} \lesssim \|T\|_{L^{p_1}(w_1)\times L^{p_2}(w_2)\to L^p(\nu_{\vec{w}})}\|f_1\|_{p_1,\lambda_1}\|f_2\|_{p_2,\lambda_2},$$

and

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{r^\lambda}\int_B |T(f_1^k,f_2^j)(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} \lesssim \|T\|_{L^{p_1}(w_1)\times L^{p_2}(w_2)\to L^p(\nu_{\vec{w}})}\|f_1\|_{p_1,\lambda_1}\|f_2\|_{p_2,\lambda_2}.$$

In summary, we obtain

$$
\begin{aligned}
\|T(f_1,f_2)\|_{p,\lambda} &\lesssim \|T(f_1^0,f_2^0)\|_{p,\lambda}+\sum_{j=1}^{\infty}\|T(f_1^0,f_2^j)\|_{p,\lambda}\\
&\quad+\sum_{k=1}^{\infty}\|T(f_1^k,f_2^0)\|_{p,\lambda}+\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\|T(f_1^k,f_2^j)\|_{p,\lambda}\\
&\lesssim \|T\|_{L^{p_1}(w_1)\times L^{p_2}(w_2)\to L^p(\nu_{\vec{w}})}\|f_1\|_{p_1,\lambda_1}\|f_2\|_{p_2,\lambda_2}.
\end{aligned}
$$

□

Remark 2.6. Lemma 2.5 is also true for the linear case.

Now, we are ready to prove Theorem 1.7.

Proof. We only write out the proof of $T^*_{b,1}(\vec{f})(x)$, the other can be obtained by symmetry. From Lemma 2.3, we deduce that

$$
\begin{aligned}
\|T^*_{b,1}(\vec{f})\|_{p,\lambda} &\lesssim \|M_b(f_1)(x)M(f_2)(x)\|_{p,\lambda}+\|M_\eta(T_{b,1}(f_1,f_2)(x))\|_{p,\lambda}\\
&\quad+\|[M_b(T(f_1,f_2)^\eta(x))][M_\eta(T(f_1,f_2)(x))]^{1-\eta}\|_{p,\lambda}\\
&\quad+\sup_{\delta>0}\|M_\eta(T_{b,1}(f_1\chi_{B(x,\delta^{1/2})},f_2\chi_{B(x,\delta^{1/2})})(x))\|_{p,\lambda}\\
&\quad+\sup_{\delta>0}\big\|\big([M_b(T(f_1\chi_{B(x,\delta^{1/2})},f_2\chi_{B(x,\delta^{1/2})})^\eta(x))]\\
&\qquad\qquad\times[M_\eta(T(f_1\chi_{B(x,\delta^{1/2})},f_2\chi_{B(x,\delta^{1/2})})(x))]^{1-\eta}\big)\big\|_{p,\lambda}.
\end{aligned}
$$

By Hölder's inequality, Lemma 2.1, Lemma 2.5 and (2.1), we know that, there exists a $\eta_0>1$ such that

$$
\begin{aligned}
&\|M_b(f_1)(x)M(f_2)(x)\|_{p,\lambda}\\
&\lesssim \|M_b(f_1)(x)\|_{p_1,\lambda_1}\|M(f_2)(x)\|_{p_2,\lambda_2}\\
&\lesssim \|M_b\|_{L^{p_1}(w)\to L^{p_1}(w)}\|f_1\|_{p_1,\lambda_1}\|f_2\|_{p_2,\lambda_2}\\
&\lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|M_{\eta_0}\|_{L^{p_1}(w)\to L^{p_1}(w)}\|f_1\|_{p_1,\lambda_1}\|f_2\|_{p_2,\lambda_2}\\
&\lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{p_1,\lambda_1}\|f_2\|_{p_2,\lambda_2}.
\end{aligned}
$$

where $w\in A_1(\mathbb{R}^n)$. From [3] and Lemma 2.5, we get

$$
\begin{aligned}
\|M_\eta(T_{b,1}(f_1,f_2)(x))\|_{p,\lambda} &\lesssim \|T_{b,1}(f_1,f_2)(x)\|_{p,\lambda}\\
&\lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{p_1,\lambda_1}\|f_2\|_{p_2,\lambda_2}.
\end{aligned}
$$

By Hölder's inequality, Lemma 2.3 and Lemma 2.5, we have

$$
\begin{aligned}
&\|[M_b(T(f_1,f_2)^\eta(x))][M_\eta(T(f_1,f_2)(x))]^{1-\eta}\|_{p,\lambda}\\
&\lesssim \|[M_b(T(f_1,f_2)^\eta(x))]\|_{p/\eta,\lambda}\|[M_\eta(T(f_1,f_2)(x))]^{1-\eta}\|_{p/(1-\eta),\lambda}\\
&\lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|T(f_1,f_2)^\eta(x)\|_{p/\eta,\lambda}\|M_\eta(T(f_1,f_2)(x))\|_{p,\lambda}^{1-\eta}\\
&\lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{p_1,\lambda_1}^{\eta}\|f_2\|_{p_2,\lambda_2}^{\eta}\|f_1\|_{p_1,\lambda_1}^{1-\eta}\|f_2\|_{p_2,\lambda_2}^{1-\eta}\\
&\lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{p_1,\lambda_1}\|f_2\|_{p_2,\lambda_2}.
\end{aligned}
$$

Similarly, we obtain

$$\begin{aligned}
&\sup_{\delta>0}\|M_\eta(T_{b,1}(f_1\chi_{B(x,\delta^{1/2})},f_2\chi_{B(x,\delta^{1/2})})(x))\|_{p,\lambda}\lesssim\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{p_1,\lambda_1}\|f_2\|_{p_2,\lambda_2},\\
&\sup_{\delta>0}\|\big([M_b(T(f_1\chi_{B(x,\delta^{1/2})},f_2\chi_{B(x,\delta^{1/2})})^\eta(x))]\\
&\qquad\times[M_\eta(T(f_1\chi_{B(x,\delta^{1/2})},f_2\chi_{B(x,\delta^{1/2})})(x))]^{1-\eta}\big)\|_{p,\lambda}\\
&\quad\lesssim\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{p_1,\lambda_1}\|f_2\|_{p_2,\lambda_2}.
\end{aligned}$$

□

3. Proof of Theorem 1.5

Lemma 3.1. *Let $0<\lambda,\lambda_1,\lambda_2<n$, $p_1,p_2\in(1,\infty)$, $p\in(1,\infty)$ such that $1/p=1/p_1+1/p_2$ and $\lambda/p=\lambda_1/p_1+\lambda_2/p_2$. Suppose that T is as in Theorem 1.5, then for all $0<\eta<1/2$, $x\in\mathbb{R}^n$ and all $\vec{f}$ in the product of $L^{p_j,\lambda_j}(\mathbb{R}^n)$,*

$$T^*(\vec{f})(x)\lesssim\big(M_\eta(T(\vec{f}))(x)\big)+\sum_{i=1}^{2}\mathcal{M}_{2,i}(\vec{f})(x)+M(f_1)(x)M(f_2)(x).$$

The proof of the Lemma 3.1 is similar to that of [16, Theorem 1] and Lemma 2.3, so we leave it to the interested reader. From Lemma 2.2, Lemma 2.5, Lemma 3.1 and [3], it is easy to find that

$$\|T^*(\vec{f})(x)\|_{p,\lambda}\lesssim\|f_1\|_{p_1,\lambda_1}\|f_2\|_{p_2,\lambda_2}\tag{3.1}$$

for $\lambda,\lambda_1,\lambda_2,p,p_1,p_2$ in Lemma 3.1.

Lemma 3.2. *Let $1\le p<\infty$, $0<\lambda<n$ and $\mathcal{H}\subset L^{p,\lambda}(\mathbb{R}^n)$, if*

(1) $\sup_{f\in\mathcal{H}}\|f\|_{p,\lambda}<\infty$;
(2) $\lim_{A\to\infty}\|f\chi_{\{|x|>A\}}\|_{p,\lambda}=0$ *uniformly for* $f\in\mathcal{H}$;
(3) $\lim_{t\to0}\|f(\cdot+t)-f(\cdot)\|_{p,\lambda}=0$ *uniformly for* $f\in\mathcal{H}$.

Then $\mathcal{H}$ is strongly precompact set in $L^{p,\lambda}(\mathbb{R}^n)$.

This Lemma was given in [6].

Now, we are ready to prove Theorem 1.5.

Proof. We will work with the commutator $T^*_{b,1}$ first, and the proof of commutator $T^*_{b,2}$ can be got by symmetry. By Theorem 1.7, it suffices to show the result for $b\in C_c^\infty(\mathbb{R}^n)$. Suppose f_1,f_2 belong to

$$B_1(L^{p_1,\lambda_1})\times B_1(L^{p_2,\lambda_2})=\{(f_1,f_2):\|f_1\|_{p_1,\lambda_1},\|f_2\|_{p_2,\lambda_2}\le1\}.$$

We need to prove the following three conditions hold:

(a) $T^*_{b,1}(B_1(L^{p_1,\lambda_1})\times B_1(L^{p_2,\lambda_2}))$ is bounded in $L^{p,\lambda}(\mathbb{R}^n)$;
(b) For every $s\in\mathbb{R}^n$ and $r>0$, $\lim_{A\to\infty}\frac{1}{r^\lambda}\int_{B(s,r)}|T^*_{b,1}(f_1,f_2)(x)\chi_{\{|x|>A\}}(x)|^p\mathrm{d}x=0$;

(c) Given $0 < \xi < 1/8$, there exists a sufficiently small $t_0 (t_0 < \xi^2)$ such that for all $0 < |t| < t_0$, we have

$$\|T^*_{b,1}(f_1, f_2)(\cdot) - T^*_{b,1}(f_1, f_2)(\cdot + t)\|_{p,\lambda} \leq C\xi. \tag{3.2}$$

It is easy to find that the condition (a) holds because of the boundedness of $T^*_{b,1}$ in Theorem 1.7. Now, we prove the condition (b) first. We pick $A > 1$ sufficiently large so that $|x| > A$ implies $x \notin \operatorname{supp} b$. In particular, let $R > 0$ be large enough such that $\operatorname{supp} b \subset B(0, R)$, $A \gg \max(2R, 1)$ and $1 < p_0 < \min\{p_1, p_2\}$,

$$\begin{aligned}
&|T^*_{b,1}(f_1, f_2)(x)| \\
&= \sup_{\delta>0} \left| \int\int_{|x-y_1|^2+|x-y_2|^2>\delta} K(x, y_1, y_2)(b(y_1) - b(x))f_1(y_1)f_2(y_2)\mathrm{d}\vec{y} \right| \\
&\lesssim \|b\|_{L^\infty} \int_{|y_1|\leq R} \int_{\mathbb{R}^n} \frac{|f_1(y_1)||f_2(y_2)|}{(|x-y_1| + |x-y_2|)^{2n}}\mathrm{d}y_1\mathrm{d}y_2 \\
&\lesssim \int_{|x-y_1|\leq R} \int_{\mathbb{R}^n} \frac{|f_1(x-y_1)||f_2(x-y_2)|}{(|y_1| + |y_2|)^{2n}}\mathrm{d}y_1\mathrm{d}y_2 \\
&\lesssim \int_{|x-y_1|\leq R} \frac{|f_1(x-y_1)|}{|y_1|^{n/p_0}}\mathrm{d}y_1 \int_{\mathbb{R}^n} \frac{|f_2(x-y_2)|}{(1+|y_2|)^{n+n/p_0'}}\mathrm{d}y_2 \\
&\lesssim R^{n/p_1'} \left(\int_{|x-y_1|\leq R} \frac{|f_1(x-y_1)|^{p_1}}{|y_1|^{np_1/p_0}}\mathrm{d}y_1 \right)^{1/p_1} \left(\int_{\mathbb{R}^n} \frac{|f_2(x-y_2)|}{(1+|y_2|)^{n+n/p_0'}}\mathrm{d}y_2 \right).
\end{aligned}$$

Hence, for every $s \in \mathbb{R}^n$ and $r > 0$, by Hölder's inequality and Minkowski's inequality

$$\begin{aligned}
&\left(\frac{1}{r^\lambda} \int_{B(s,r)} |T^*_{b,1}(f_1, f_2)(x)|^p \chi_{\{|x|>A\}}(x)\mathrm{d}x \right)^{1/p} \\
&\lesssim R^{n/p_1'} \left(\frac{1}{r^\lambda} \int_{B(s,r)} \left(\int_{|x-y_1|\leq R} \frac{|f_1(x-y_1)|^{p_1}\chi_{\{|x|>A\}}(x)}{|y_1|^{np_1/p_0}}\mathrm{d}y_1 \right)^{p/p_1} \right. \\
&\quad \times \left. \left(\int_{\mathbb{R}^n} \frac{|f_2(x-y_2)|}{(1+|y_2|)^{n+n/p_0'}}\mathrm{d}y_2 \right)^p \mathrm{d}x \right)^{1/p} \\
&\lesssim R^{n/p_1'} \left(\frac{1}{r^{\lambda_1}} \int_{B(s,r)} \int_{|x-y_1|\leq R} \frac{|f_1(x-y_1)|^{p_1}\chi_{\{|x|>A\}}(x)}{|y_1|^{np_1/p_0}}\mathrm{d}y_1\mathrm{d}x \right)^{1/p_1} \\
&\quad \times \left(\frac{1}{r^{\lambda_2}} \int_{B(s,r)} \left(\int_{\mathbb{R}^n} \frac{|f_2(x-y_2)|}{(1+|y_2|)^{n+n/p_0'}}\mathrm{d}y_2 \right)^{p_2} \mathrm{d}x \right)^{1/p_2} \\
&\lesssim R^{n/p_1'} \left(\int_{|x-y_1|\leq R} \frac{1}{r^{\lambda_1}} \int_{B(s,r)} |f_1(x-y_1)|^{p_1}\chi_{\{|x|>A\}}(x)\mathrm{d}x \frac{1}{|y_1|^{np_1/p_0}}\mathrm{d}y_1 \right)^{1/p_1} \\
&\quad \times \left(\int_{\mathbb{R}^n} \left(\frac{1}{r^{\lambda_2}} \int_{B(s,r)} |f_2(x-y_2)|^{p_2}\mathrm{d}x \right)^{1/p_2} \frac{1}{(1+|y_2|)^{n+n/p_0'}}\mathrm{d}y_2 \right)
\end{aligned}$$

$$\lesssim R^{n/p_1'}\Big(\int_{|y_1|\geq A-R}\frac{1}{|y_1|^{np_1/p_0}}\mathrm{d}y_1\Big)^{1/p_1}\Big(\int_{\mathbb{R}^n}\frac{1}{(1+|y_2|)^{n+n/p_0'}}\mathrm{d}y_2\Big)$$
$$\times\|f_1\|_{p_1,\lambda_1}\|f_2\|_{p_2,\lambda_2}$$
$$\lesssim R^{n/p_1'}(A-R)^{n(\frac{1}{p_1}-\frac{1}{p_0})}\to 0,$$

when $A\to\infty$.

So, it suffices to verify condition (c). Denote

$$K^\delta(x,y_1,y_2)=K(x,y_1,y_2)\chi_{|x-y_1|^2+|x-y_2|^2>\delta}.$$

To prove (3.2), we need to decompose the expression inside the $L^{p,\lambda}(\mathbb{R}^n)$ norm as follows

$$\begin{aligned}
&T^*_{b,1}(f_1,f_2)(x)-T^*_{b,1}(f_1,f_2)(x+t)\\
&\lesssim \sup_{\delta>0}\Big|\int\int_{\min(|x-y_1|,|x-y_2|)>\frac{|t|}{\xi}}K^\delta(x,y_1,y_2)(b(x+t)-b(x))\prod_{j=1}^2 f_j(y_j)\mathrm{d}\vec{y}\Big|\\
&+\sup_{\delta>0}\Big|\int\int_{\min(|x-y_1|,|x-y_2|)>\frac{|t|}{\xi}}(K^\delta(x,y_1,y_2)-K^\delta(x+t,y_1,y_2))\\
&\quad\times(b(y_1)-b(x+t))\prod_{j=1}^2 f_j(y_j)\mathrm{d}\vec{y}\Big|\\
&+\sup_{\delta>0}\Big|\int\int_{\min(|x-y_1|,|x-y_2|)<\frac{|t|}{\xi}}K^\delta(x,y_1,y_2)(b(y_1)-b(x))\prod_{j=1}^2 f_j(y_j)\mathrm{d}\vec{y}\Big|\\
&+\sup_{\delta>0}\Big|\int\int_{\min(|x-y_1|,|x-y_2|)<\frac{|t|}{\xi}}K^\delta(x+t,y_1,y_2)(b(x+t)-b(y_1))\prod_{j=1}^2 f_j(y_j)\mathrm{d}\vec{y}\Big|\\
&=I_1+I_2+I_3+I_4.
\end{aligned}$$

It is easy to see that

$$I_1 \;\leq\; \|\nabla b\|_{L^\infty}|t|\sup_{\delta>0}\Big|\int\int_{\substack{\min(|x-y_1|,|x-y_2|)>\frac{|t|}{\xi}\\ |x-y_1|^2+|x-y_2|^2>\delta}}K(x,y_1,y_2)f_1(y_1)f_2(y_2)\mathrm{d}\vec{y}\Big|,$$

and

$$\begin{aligned}
&\Big|\int\int_{|x-y_1|^2+|x-y_2|^2>\delta}K(x,y_1,y_2)f_1(y_1)f_2(y_2)\mathrm{d}y_1\mathrm{d}y_2\\
&-\int\int_{\substack{\min(|x-y_1|,|x-y_2|)>\frac{|t|}{\xi}\\ |x-y_1|^2+|x-y_2|^2>\delta}}K(x,y_1,y_2)f_1(y_1)f_2(y_2)\mathrm{d}y_1\mathrm{d}y_2\Big|\\
&=\Big|\int\int_{\substack{\min(|x-y_1|,|x-y_2|)<\frac{|t|}{\xi}\\ |x-y_1|^2+|x-y_2|^2>\delta}}K(x,y_1,y_2)f_1(y_1)f_2(y_2)\mathrm{d}y_1\mathrm{d}y_2\Big|
\end{aligned}$$

$$\begin{aligned}
&\leq \int\int_{\substack{\min(|x-y_1|,|x-y_2|)<\frac{|t|}{\xi} \\ |x-y_1|^2+|x-y_2|^2>\delta}} \frac{|f_1(y_1)||f_2(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}}\mathrm{d}y_1\mathrm{d}y_2 \\
&\leq \int_{|x-y_1|<\frac{|t|}{\xi}}\int_{|x-y_2|>\frac{|t|}{\xi}} \frac{|f_1(y_1)||f_2(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}}\mathrm{d}y_1\mathrm{d}y_2 \\
&\quad+\int_{|x-y_1|>\frac{|t|}{\xi}}\int_{|x-y_2|<\frac{|t|}{\xi}} \frac{|f_1(y_1)||f_2(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}}\mathrm{d}y_1\mathrm{d}y_2 \\
&\lesssim \int_{|y_1|<\frac{|t|}{\xi}}\int_{|y_2|>\frac{|t|}{\xi}} \frac{|f_1(x-y_1)||f_2(x-y_2)|}{(|y_1|+|y_2|)^{2n}}\mathrm{d}y_1\mathrm{d}y_2,
\end{aligned}$$

provided $(\delta/2)^{1/2} > |t|/\xi$. Then for every $s \in \mathbb{R}^n$ and $r > 0$, by Hölder's inequality, Minkowski's inequality and (3.1), we have

$$\begin{aligned}
&\left(\frac{1}{r^\lambda}\int_{B(s,r)}|I_1|^p\mathrm{d}x\right)^{1/p} \\
&\lesssim |t|\left(\frac{1}{r^\lambda}\int_{B(s,r)}\left(\int_{|y_1|<\frac{|t|}{\xi}}\int_{|y_2|>\frac{|t|}{\xi}}\frac{|f_1(x-y_1)||f_2(x-y_2)|}{(|y_1|+|y_2|)^{2n}}\mathrm{d}y_1\mathrm{d}y_2\right)^p\mathrm{d}x\right)^{1/p} \\
&\quad+|t|\|T^*\|_{p,\lambda} \\
&\lesssim |t|\int_{|y_1|<\frac{|t|}{\xi}}\int_{|y_2|>\frac{|t|}{\xi}}\left(\frac{1}{r^\lambda}\int_{B(s,r)}|f_1(x-y_1)|^p|f_2(x-y_2)|^p\mathrm{d}x\right)^{1/p}\frac{\mathrm{d}y_1\mathrm{d}y_2}{(|y_1|+|y_2|)^{2n}} \\
&\quad+|t|\|f_1\|_{p_1,\lambda_1}\|f_2\|_{p_2,\lambda_2} \\
&\lesssim |t|\|f_1\|_{p_1,\lambda_1}\|f_2\|_{p_2,\lambda_2}\int_{|y_1|<\frac{|t|}{\xi}}\int_{|y_2|>\frac{|t|}{\xi}}\frac{1}{(|y_1|+|y_2|)^{2n}}\mathrm{d}y_1\mathrm{d}y_2+|t| \\
&\lesssim |t|\int_{|y_1|<\frac{|t|}{\xi}}\frac{1}{|y_1|^{n-1}}\mathrm{d}y_1\int_{|y_2|>\frac{|t|}{\xi}}\frac{1}{|y_2|^{n+1}}\mathrm{d}y_2+|t| \\
&\lesssim |t|.
\end{aligned}$$

We obtain

$$\|I_1\|_{p,\lambda} \leq C|t|. \tag{3.3}$$

Now, we estimate I_2 using some ideas of [9], notice that

$$\begin{aligned}
I_2 \quad\leq\quad \sup_{\delta>0}\Bigg|\int\int_{\substack{\min(|x-y_1|,|x-y_2|)>\frac{|t|}{\xi} \\ |x-y_1|^2+|x-y_2|^2>\delta \\ |x+t-y_1|^2+|x+t-y_2|^2>\delta}} &(K(x,y_1,y_2)-K(x+t,y_1,y_2)) \\
&\times(b(y_1)-b(x+t))f_1(y_1)f_2(y_2)\mathrm{d}y_1\mathrm{d}y_2\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&+\sup_{\delta>0}\left|\int\int_{\substack{\min(|x-y_1|,|x-y_2|)>\frac{|t|}{\xi}\\|x-y_1|^2+|x-y_2|^2>\delta\\|x+t-y_1|^2+|x+t-y_2|^2<\delta}} K(x,y_1,y_2)(b(y_1)-b(x+t))\prod_{i=1}^{2}f_i(y_i)\mathrm{d}\vec{y}\right|\\
&+\sup_{\delta>0}\left|\int\int_{\substack{\min(|x-y_1|,|x-y_2|)>\frac{|t|}{\xi}\\|x-y_1|^2+|x-y_2|^2<\delta\\|x+t-y_1|^2+|x+t-y_2|^2>\delta}} K(x+t,y_1,y_2)(b(y_1)-b(x+t))\prod_{i=1}^{2}f_i(y_i)\mathrm{d}\vec{y}\right|\\
=\;& I_2^1+I_2^2+I_2^3.
\end{aligned}
$$

In order to estimate I_2^1, by a consequence of condition (1.4), we get

$$
|K(x,y_1,y_2)-K(x^{'},y_1,y_2)|\le\frac{D|x-x^{'}|^{\gamma}}{(|x-y_1|+|x-y_2|)^{2n+\gamma}},
$$

when $|x-x^{'}|\le\frac{1}{8}\min\{|x-y_1|,|x-y_2|\}$. Then

$$
\begin{aligned}
I_2^1 &\lesssim \|b\|_{L^\infty}\int\int_{\min(|x-y_1|,|x-y_2|)>\frac{|t|}{\xi}}\frac{|t|^{\gamma}}{(|x-y_1|+|x-y_2|)^{2n+\gamma}}|f_1(y_1)||f_2(y_2)|\mathrm{d}\vec{y}\\
&\lesssim |t|^{\gamma}\int_{|y_1|>\frac{|t|}{\xi}}\int_{|y_2|>\frac{|t|}{\xi}}\frac{|f_1(x-y_1)||f_2(x-y_2)|}{(|y_1|+|y_2|)^{2n+\gamma}}\mathrm{d}y_1\mathrm{d}y_2.
\end{aligned}
$$

Hence, for every $s\in\mathbb{R}^n$ and $r>0$, Hölder's inequality and Minkowski's inequality give that

$$
\begin{aligned}
&\left(\frac{1}{r^{\lambda}}\int_{B(s,r)}|I_2^1|^p\mathrm{d}x\right)^{1/p}\\
&\lesssim |t|^{\gamma}\left(\frac{1}{r^{\lambda}}\int_{B(s,r)}\left(\int_{|y_1|>\frac{|t|}{\xi}}\int_{|y_2|>\frac{|t|}{\xi}}\frac{|f_1(x-y_1)||f_2(x-y_2)|}{(|y_1|+|y_2|)^{2n+\gamma}}\mathrm{d}y_1\mathrm{d}y_2\right)^{p}\mathrm{d}x\right)^{1/p}\\
&\lesssim |t|^{\gamma}\int_{|y_1|>\frac{|t|}{\xi}}\int_{|y_2|>\frac{|t|}{\xi}}\left(\frac{1}{r^{\lambda}}\int_{B(s,r)}|f_1(x-y_1)|^p|f_2(x-y_2)|^p\mathrm{d}x\right)^{1/p}\\
&\qquad\times\frac{1}{(|y_1|+|y_2|)^{2n+\gamma}}\mathrm{d}y_1\mathrm{d}y_2\\
&\lesssim |t|^{\gamma}\|f_1\|_{p_1,\lambda_1}\|f_2\|_{p_2,\lambda_2}\int_{|y_1|>\frac{|t|}{\xi}}\int_{|y_2|>\frac{|t|}{\xi}}\frac{1}{(|y_1|+|y_2|)^{2n+\gamma}}\mathrm{d}y_1\mathrm{d}y_2\\
&\lesssim |t|^{\gamma}\int_{|y_1|>\frac{|t|}{\xi}}\frac{1}{|y_1|^{n+\gamma/2}}\mathrm{d}y_1\int_{|y_2|>\frac{|t|}{\xi}}\frac{1}{|y_2|^{n+\gamma/2}}\mathrm{d}y_2\\
&\lesssim \xi^{\gamma}.
\end{aligned}
$$

Therefore,

$$
\|I_2^1\|_{p,\lambda}\le C\xi^{\gamma}.
$$

For I_2^2, as $\min(|x-y_1|,|x-y_2|) > \frac{|t|}{\xi}$, $|x+t-y_1|^2+|x+t-y_2|^2 < \delta$, we have $|x-y_1|^2+|x-y_2|^2 \le \frac{1}{(1-\xi)^2}(|x+t-y_1|^2+|x+t-y_2|^2) \le \frac{\delta}{(1-\xi)^2}$. It follows that

$$\begin{aligned} I_2^2 &\lesssim \|b\|_{L^\infty} \sup_{\delta>0} \int\int_{\delta<|x-y_1|^2+|x-y_2|^2<\frac{\delta}{(1-\xi)^2}} \frac{|f_1(y_1)||f_2(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}} \mathrm{d}y_1\mathrm{d}y_2 \\ &\lesssim \sup_{\delta>0} \int\int_{\delta<|y_1|^2+|y_2|^2<\frac{\delta}{(1-\xi)^2}} \frac{|f_1(x-y_1)||f_2(x-y_2)|}{(|y_1|+|y_2|)^{2n}} \mathrm{d}y_1\mathrm{d}y_2. \end{aligned}$$

Thus, for every $s \in \mathbb{R}^n$ and $r > 0$, by Hölder's inequality and Minkowski's inequality we obtain

$$\begin{aligned} &\left(\frac{1}{r^\lambda}\int_{B(s,r)} |I_2^2|^p \mathrm{d}x\right)^{1/p} \\ &\lesssim \sup_{\delta>0}\left(\frac{1}{r^\lambda}\int_{B(s,r)}\left(\int\int_{\delta<|y_1|^2+|y_2|^2<\frac{\delta}{(1-\xi)^2}} \frac{|f_1(x-y_1)||f_2(x-y_2)|}{(|y_1|+|y_2|)^{2n}}\mathrm{d}\vec{y}\right)^p \mathrm{d}x\right)^{1/p} \\ &\lesssim \sup_{\delta>0} \int\int_{\delta<|y_1|^2+|y_2|^2<\frac{\delta}{(1-\xi)^2}} \left(\frac{1}{r^\lambda}\int_{B(s,r)} |f_1(x-y_1)|^p|f_2(x-y_2)|^p \mathrm{d}x\right)^{1/p} \\ &\qquad \times \frac{1}{(|y_1|+|y_2|)^{2n}}\mathrm{d}\vec{y} \\ &\lesssim \|f_1\|_{p_1,\lambda_1}\|f_2\|_{p_2,\lambda_2} \sup_{\delta>0}\int\int_{\delta<|y_1|^2+|y_2|^2<\frac{\delta}{(1-\xi)^2}} \frac{\mathrm{d}y_1\mathrm{d}y_2}{(|y_1|+|y_2|)^{2n}} \\ &\lesssim \sup_{\delta>0}\int_{|y_2|^2\le\delta}\int_{\delta-|y_2|^2<|y_1|^2<\delta/(1-\xi)^2-|y_2|^2} \frac{1}{(|y_1|+|y_2|)^{2n}}\mathrm{d}y_1\mathrm{d}y_2 \\ &\quad + \sup_{\delta>0}\int_{\delta\le|y_2|^2\le\delta/(1-\xi)^2}\int_{|y_1|^2\le\delta/(1-\xi)^2-|y_2|^2} \frac{1}{(|y_1|+|y_2|)^{2n}}\mathrm{d}y_1\mathrm{d}y_2 \\ &\lesssim (1-\xi)^{-n} - 1 \lesssim \xi. \end{aligned}$$

We have

$$\|I_2^2\|_{p,\lambda} \le C\xi.$$

For I_2^3, we proceed in a similar way. It is easy to find that $\min(|x-y_1|,|x-y_2|) > \frac{|t|}{\xi}$, $|x+t-y_1|^2+|x+t-y_2|^2 > \delta$, then we have $\frac{\delta}{(1+\xi)^2} \le |x-y_1|^2+|x-y_2|^2 \le \delta$. Hence,

$$\begin{aligned} I_2^3 &\lesssim \|b\|_{L^\infty} \sup_{\delta>0} \int\int_{\delta/(1+\xi)^2\le|x-y_1|^2+|x-y_2|^2\le\delta} \frac{|f_1(y_1)||f_2(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}} \mathrm{d}y_1\mathrm{d}y_2 \\ &\lesssim \sup_{\delta>0} \int\int_{\delta/(1+\xi)^2\le|y_1|^2+|y_2|^2\le\delta} \frac{|f_1(x-y_1)||f_2(x-y_2)|}{(|y_1|+|y_2|)^{2n}} \mathrm{d}y_1\mathrm{d}y_2, \end{aligned}$$

consequently, for every $s \in \mathbb{R}^n$ and $r > 0$, by Hölder's inequality and Minkowski's inequality we get

$$\begin{aligned}
&\left(\frac{1}{r^\lambda}\int_{B(s,r)}|I_2^3|^p \mathrm{d}x\right)^{1/p} \\
&\lesssim \sup_{\delta>0}\left(\frac{1}{r^\lambda}\int_{B(s,r)}\left(\int\int_{\delta/(1+\xi)^2<|y_1|^2+|y_2|^2<\delta}\frac{|f_1(x-y_1)||f_2(x-y_2)|}{(|y_1|+|y_2|)^{2n}}\mathrm{d}\vec{y}\right)^p \mathrm{d}x\right)^{1/p} \\
&\lesssim \sup_{\delta>0}\int\int_{\delta/(1+\xi)^2<|y_1|^2+|y_2|^2<\delta}\left(\frac{1}{r^\lambda}\int_{B(s,r)}|f_1(x-y_1)|^p|f_2(x-y_2)|^p \mathrm{d}x\right)^{1/p} \\
&\qquad\times\frac{1}{(|y_1|+|y_2|)^{2n}}\mathrm{d}\vec{y} \\
&\lesssim \sup_{\delta>0}\int_{|y_2|^2\le\delta/(1+\xi)^2}\int_{\delta/(1+\xi)^2-|y_2|^2<|y_1|^2<\delta-|y_2|^2}\frac{1}{(|y_1|+|y_2|)^{2n}}\mathrm{d}y_1\mathrm{d}y_2 \\
&\quad+\sup_{\delta>0}\int_{\delta/(1+\xi)^2\le|y_2|^2\le\delta}\int_{|y_1|^2\le\delta-|y_2|^2}\frac{1}{(|y_1|+|y_2|)^{2n}}\mathrm{d}y_1\mathrm{d}y_2 \\
&\lesssim 1-(1+\xi)^{-n}\lesssim\xi.
\end{aligned}$$

Therefore,

$$\|I_2^3\|_{p,\lambda}\le C\xi.$$

In summary,

$$\|I_2\|_{p,\lambda}\le C\xi. \tag{3.4}$$

Next, we estimate I_3

$$\begin{aligned}
I_3 \quad&\lesssim\quad \|\nabla b\|_{L^\infty}\int_{|x-y_1|<\frac{|t|}{\xi}}\int_{|x-y_2|>\frac{|t|}{\xi}}\frac{|f_1(y_1)||f_2(y_2)|}{(|x-y_1|+|x-y_2|)^{2n-1}}\mathrm{d}y_1\mathrm{d}y_2 \\
&\qquad+\|\nabla b\|_{L^\infty}\int_{|x-y_1|>\frac{|t|}{\xi}}\int_{|x-y_2|<\frac{|t|}{\xi}}\frac{|f_1(y_1)||f_2(y_2)|}{(|x-y_1|+|x-y_2|)^{2n-1}}\mathrm{d}y_1\mathrm{d}y_2 \\
&\lesssim\quad \int_{|y_1|<\frac{|t|}{\xi}}\int_{|y_2|>\frac{|t|}{\xi}}\frac{|f_1(x-y_1)||f_2(x-y_2)|}{(|y_1|+|y_2|)^{2n-1}}\mathrm{d}y_1\mathrm{d}y_2,
\end{aligned}$$

provided $(\delta/2)^{1/2} > |t|/\xi$, then for every $s \in \mathbb{R}^n$ and $r > 0$, Hölder's inequality and Minkowski's inequality give that

$$\begin{aligned}
&\left(\frac{1}{r^\lambda}\int_{B(s,r)}|I_3|^p \mathrm{d}x\right)^{1/p} \\
&\lesssim\left(\frac{1}{r^\lambda}\int_{B(s,r)}\left(\int_{|y_1|<\frac{|t|}{\xi}}\int_{|y_2|>\frac{|t|}{\xi}}\frac{|f_1(x-y_1)||f_2(x-y_2)|}{(|y_1|+|y_2|)^{2n-1}}\mathrm{d}y_1\mathrm{d}y_2\right)^p \mathrm{d}x\right)^{1/p} \\
&\lesssim\int_{|y_1|<\frac{|t|}{\xi}}\int_{|y_2|>\frac{|t|}{\xi}}\left(\frac{1}{r^\lambda}\int_{B(s,r)}|f_1(x-y_1)|^p|f_2(x-y_2)|^p \mathrm{d}x\right)^{1/p}\frac{\mathrm{d}y_1\mathrm{d}y_2}{(|y_1|+|y_2|)^{2n-1}} \\
&\lesssim |t|/\xi.
\end{aligned}$$

We get

$$\|I_3\|_{p,\lambda} \leq C|t|/\xi. \tag{3.5}$$

Finally, for the last part I_4, we estimate it as I_3. By replacing the region of integration $\{(x, y_1, y_2) : \min(|x-y_1|, |x-y_2|) < |t|/\xi\}$ with a larger one $\{(x, y_1, y_2) : \min(|x+t-y_1|, |x+t-y_2|) < |t|/\xi + |t|\}$. We know that

$$\begin{aligned} I_4 \lesssim\ & \|\nabla b\|_{L^\infty} \int_{|x+t-y_1|<\frac{|t|}{\xi}+|t|} \int_{|x+t-y_2|>\frac{|t|}{\xi}} \frac{|f_1(y_1)||f_2(y_2)|}{(|x+t-y_1|+|x+t-y_2|)^{2n-1}} \mathrm{d}\vec{y} \\ & + \|\nabla b\|_{L^\infty} \int_{|x+t-y_1|>\frac{|t|}{\xi}} \int_{|x+t-y_2|<\frac{|t|}{\xi}+|t|} \frac{|f_1(y_1)||f_2(y_2)|}{(|x+t-y_1|+|x+t-y_2|)^{2n-1}} \mathrm{d}\vec{y} \\ \lesssim\ & \int_{|y_1|<\frac{|t|}{\xi}+|t|} \int_{|y_2|>\frac{|t|}{\xi}} \frac{|f_1(x+t-y_1)||f_2(x+t-y_2)|}{(|y_1|+|y_2|)^{2n-1}} \mathrm{d}\vec{y}, \end{aligned}$$

provided $(\delta/2)^{1/2} > |t|/\xi$, then for every $s \in \mathbb{R}^n$ and $r > 0$,

$$\begin{aligned} & \left(\frac{1}{r^\lambda} \int_{B(s,r)} |I_4|^p \mathrm{d}x\right)^{1/p} \\ & \lesssim \left(\frac{1}{r^\lambda} \int_{B(s,r)} \left(\int_{|y_1|<\frac{|t|}{\xi}+|t|} \int_{|y_2|>\frac{|t|}{\xi}} \frac{|f_1(x+t-y_1)||f_2(x+t-y_2)|}{(|y_1|+|y_2|)^{2n-1}} \mathrm{d}\vec{y}\right)^p \mathrm{d}x\right)^{1/p} \\ & \lesssim \int_{|y_1|<\frac{|t|}{\xi}+|t|} \int_{|y_2|>\frac{|t|}{\xi}} \left(\frac{1}{r^\lambda} \int_{B(s,r)} |f_1(x+t-y_1)|^p |f_2(x+t-y_2)|^p \mathrm{d}x\right)^{1/p} \\ & \qquad \times \frac{1}{(|y_1|+|y_2|)^{2n-1}} \mathrm{d}y_1 \mathrm{d}y_2 \\ & \lesssim \|f_1\|_{p_1,\lambda_1} \|f_2\|_{p_2,\lambda_2} \int_{|y_1|<\frac{|t|}{\xi}+|t|} \frac{1}{|y_1|^{n-2}} \mathrm{d}y_1 \int_{|y_2|>\frac{|t|}{\xi}} \frac{1}{|y_2|^{n+1}} \mathrm{d}y_2 \\ & \lesssim |t|/\xi + |t|. \end{aligned}$$

Consequently,

$$\|I_4\|_{p,\lambda} \leq C|t|/\xi + |t|. \tag{3.6}$$

Then inequalities (3.3), (3.4), (3.5) and (3.6) imply (3.2), and in this way, we can conclude that $T^*_{b,1}$ is compact. By symmetry, $T^*_{b,2}$ is also compact. □

Acknowledgement. This research was supported by the NNSF of China (Grant No. 11271330) and the NSF of Henan Province (Grant No. 122300410117).

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