q-FREQUENTLY HYPERCYCLIC OPERATORS

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ABSTRACT. We introduce q-frequently hypercyclic operators and derive a sufficient criterion for a continuous operator to be q-frequently hypercyclic on a locally convex space. Applications are given to obtain q-frequently hypercyclic operators with respect to the norm-, F-norm- and weak*- topologies. Finally, the frequent hypercyclicity of the non-convolution operator T_μ defined by $T_\mu(f)(z) = f'(\mu z)$, $|\mu| \geq 1$ on the space $H(\mathbb{C})$ of entire functions equipped with the compact-open topology is shown.

1. INTRODUCTION

The main theme in the dynamics of linear operators is the notion of hypercyclicity which plays an important role in the study of the invariant subset problem in Banach spaces. This notion was initiated by S. Rolewicz [22] in the setting of infinite dimensional Banach spaces in 1969, though the examples of translation and differential operators on the space of entire functions equipped with the compact-open topology were known to be hypercyclic in an earlier work of G. D. Birkhoff [8] and G. R. MacLane [20]. Now a vast literature dealing with hypercyclicity of operators as well as other related notions in linear dynamics is available in [5], [14], and [15].

In 2006, F. Bayart and S. Grivaux [3] further strengthened this concept to frequent hypercyclicity, which quantifies the frequency with which the iterates of a given linear operator at a point visit each non-empty open set. After the

appearance of this work, several results on frequently hypercyclic operators have been established, for instance one may refer to [9], [10], [11], [13] and [24]. In this paper we introduce q-frequent hypercyclicity which lies between hypercyclicity and frequent hypercyclicity, where q is a fixed natural number. The case $q = 1$ coincides with frequent hypercyclicity. We prove a sufficient criterion for a continuous linear operator to be q-frequently hypercyclic on a locally convex space and give applications to obtain q-frequently hypercyclic operators with respect to the norm on Banach spaces, the F-norm on F-spaces and the weak*-topology on dual of Banach spaces. We also provide examples of hypercyclic operators that are not q-frequently hypercyclic for any $q \in \mathbb{N}$.

2. Preliminaries

Let X be a separable topological vector space, and $\mathcal{L}(X)$ denote the space of all continuous linear operators on X. An operator $T \in \mathcal{L}(X)$ is said to be *hypercyclic* if there exists a vector $x \in X$ such that the orbit $\{T^n(x) : n \geq 0\}$ is dense in X. Such a vector x is called a *hypercyclic vector* for T. As mentioned in the previous section, Birkhoff's translation operator $T_a(f)(z) = f(z+a)$, for nonzero $a \in \mathbb{C}$ and MacLane's differentiation operator $D(f) = f'$ on the space $H(\mathbb{C})$ are hypercyclic. Also, Rolewicz proved the hypercyclicity of the operator λB on ℓ^p or c_0 for $1 \leq p < \infty$ and $|\lambda| > 1$, where B is the unweighted backward shift defined by $B(e_n) = e_{n-1}, n \geq 1$, with $e_0 = 0$ and $e_n = \{0, 0, .., 1, 0..\}$, 1 being placed at the nth coordinate. Generalizing this result, H.N. Salas [23] proved that the weighted shift B_w associated to a weight sequence (w_n) of positive reals, given by $B_w(e_n) = w_n e_{n-1}, n \geq 1$ is hypercyclic on ℓ^p or c_0 if and only if $\limsup_{n \to \infty}(w_1 w_2 \cdots w_n) = \infty$.

For testing the hypercyclicity of a linear operator, a sufficient criterion known as the hypercyclicity criterion, initially obtained by Kitai [18], has appeared in different forms and the one which is given below is due to H. Petterson [21]. This is useful even for linear operators defined on non-metrizable topological vector spaces. For the definition of F-norm, we refer to [11], p. 385.

Theorem 2.1 (Hypercyclicity criterion)**.** *Let (X, τ) be a separable topological vector space. Suppose further that X carries an F-norm $||.||$ with respect to which it is complete and that $||.||$-topology is stronger than τ. If T is an operator continuous with respect to the F-norm, $D \subset X$ a countable τ-dense set and $S_n : D \to X$ maps such that, for all $x \in D$,*

1. *$||T^n(x)|| \to 0$ and $||S_n(x)|| \to 0$ as $n \to \infty$; and*
2. *$T^n S_n(x) = x$, for each $n \in \mathbb{N}$,*

then the operator T is τ-hypercyclic.

An operator $T \in \mathcal{L}(X)$ is called *frequently hypercyclic* if there exists an x, called a *frequently hypercyclic vector* for T such that for every nonempty open set U in X, the set $\mathbf{N}(x, U) = \{n \in \mathbb{N} : T^n(x) \in U\}$ has positive lower density; where the *lower density* of a subset A of $\mathbb{N}$, the set of natural numbers, is defined

as

$$\underline{\text{dens}}(A) = \liminf_{N \to \infty} \frac{\text{card}\{n \in A : n \leq N\}}{N}, \tag{2.1}$$

the symbol $\text{card}(B)$ being used to denote the cardinality of the set B.

Let us note that $\underline{\text{dens}}(B) \in [0,1]$ for any subset B of $\mathbb{N}$. Clearly, the lower density of any finite set is zero and that of $\mathbb{N}$ is 1. If A is a strictly increasing sequence (n_k), the lower density of A is characterized as [11]

$$\underline{\text{dens}}(n_k) = \liminf_{k \to \infty} \frac{k}{n_k}. \tag{2.2}$$

Alternatively, an operator $T \in L(X)$ is frequently hypercyclic if there is some $x \in X$ such that for every nonempty open subset U of X, there exist a strictly increasing (n_k) of natural numbers and a constant $C > 0$ such that

$$T^{n_k}(x) \in U \text{ and } n_k \leq Ck, \ \forall \ k \in \mathbb{N}.$$

Analogous to the hypercyclicity criterion, we have the following criterion, proved in [3] and [11].

Theorem 2.2 (Frequent hypercyclicity criterion). *Let X be a separable F-space and $T \in L(X)$. If there exist a dense subset $D \subset X$ and a map $S : D \to D$ such that*

1. *$\sum T^n(x)$ and $\sum S^n(x)$ are unconditionally convergent for each $x \in D$; and*
2. *$TS = I$, the identity on D,*

then the operator T is frequently hypercyclic.

The operators of Birkhoff and MacLane satisfy the above criterion, cf. [3] and [5], and so they are frequently hypercyclic. In fact, any continuous operator, except a scalar multiple of the identity, that commutes with all translations on $H(\mathbb{C})$, has been shown to be frequently hypercyclic [10]. We also recall the hypercyclic comparison principle from [3] and [5], which says how to transfer the hypercyclicity via a linear quasi-conjugacy.

Proposition 2.3 (Hypercyclic comparison principle). *Let T and S be continuous linear operators on two topological vector spaces X and Y respectively and $A : X \to Y$ be a continuous linear map with dense range such that $SA = AT$. If T is hypercyclic (or frequently hypercyclic) on X, then S is hypercyclic (or frequently hypercyclic) on Y.*

3. q-Frequently hypercyclic operators

We first introduce the q-lower density of a subset of natural numbers, for $q \in \mathbb{N}$ and determine a useful characterization.

Definition 3.1. Let $A \subset \mathbb{N}$ and $q \in \mathbb{N}$. The q-lower density of A is defined as

$$q\text{-}\underline{\text{dens}}(A) = \liminf_{N \to \infty} \frac{\text{card}\{n \in A : n \leq N^q\}}{N}.$$

Let us note that the lower density of a set is always finite and lies in the interval $[0, 1]$, but the q-lower density can vary in $[0, \infty]$ for $q \geq 2$. As in the case of the lower density, we have the following.

Proposition 3.2. *Let (n_k) be a strictly increasing sequence of natural numbers. Then*

(1) $q\text{-}\underline{\text{dens}}(n_k) = \liminf\limits_{k \to \infty} \dfrac{k}{n_k^{1/q}}$.

(2) $q\text{-}\underline{\text{dens}}(n_k) > 0$ *if and only if there exists a constant $C > 0$ such that $n_k \leq Ck^q$ for all $k \in \mathbb{N}$.*

Proof. Fix a number $k \in \mathbb{N}$. For any $N \in \mathbb{N}$ with $n_k \leq N^q < n_{k+1}$, we have that

$$p_N = \frac{\text{card}\{k \in \mathbb{N} : n_k \leq N^q\}}{N} = k/N.$$

Thus the inequality

$$\frac{k}{n_{k+1}^{1/q}} < p_N \leq \frac{k}{n_k^{1/q}}$$

implies the first part in the theorem.

Part (2) follows immediately from the fact that $\liminf a_k > 0$ if and only if $\frac{1}{a_k} \leq C$ for some $C > 0$, where (a_k) is a sequence of positive numbers. $\square$

We now define the notion of q-frequent hypercyclicity of linear operators on topological vector spaces.

Definition 3.3. Let $q \in \mathbb{N}$. A continuous linear operator T on a separable topological vector space X is said to be q-*frequently hypercyclic* if there exists an $x \in X$ such that for any nonempty open subset U of X, the set $\mathbf{N}(x, U) = \{n \in \mathbb{N} : T^n(x) \in U\}$ has positive q-lower density. Such a vector is called a q-*frequently hypercyclic vector* for T.

Alternatively, a continuous linear operator T on a separable topological vector space X is q-*frequently hypercyclic* if there exists an $x \in X$ such that for any nonempty open subset U of X, we can find a strictly increasing sequence (n_k) of natural numbers and a constant $C > 0$ such that

$$T^{n_k}(x) \in U \text{ and } n_k \leq Ck^q, \text{ for all } k \in \mathbb{N}.$$

Such a vector is called a q-*frequently hypercyclic vector* for T.

Obviously, every frequently hypercyclic operator is q-frequently hypercyclic for any natural number q, and the two notions are the same for the case $q = 1$. Also this new property of linear operators is stronger than hypercyclicity; however, none of the converse implications is true, e.g. consider

Example 3.4. Here we show that there exists a hypercyclic operator on ℓ^1 that is not 2-frequently hypercyclic with respect to the weak topology of ℓ^1. Indeed, the weighted backward shift B_w with weights $w_n = \sqrt{\frac{n+1}{n}}$ is hypercyclic by the result of Salas, but was shown to be non-frequently hypercyclic on ℓ^2 in [3]. For showing the non-2-frequent hypercyclicity of B_w, choose the weakly open set $U = \{(y_n) \in \ell^1 : |y_1| > 1\}$. Let $x = (x_n)$ be a 2-frequently hypercyclic vector for

B_w. Enumerate the set $N(x, U) = \{n \in \mathbb{N} : B_w^n(x) \in U\}$ as (n_k). Thus we have a constant $c > 0$ such that $n_k \leq ck^2$, and hence

$$\sum_{k \geq 1} \frac{1}{\sqrt{n_k}} = \infty.$$

On the other hand, $B_w^{n_k}(x) \in U$ implies that

$$\sqrt{n_k + 1}\, |x_{n_k+1}| > 1, \text{ for all } k \geq 1$$

As $(x_n) \in \ell^1$, we get

$$\sum_{k \geq 1} \tfrac{1}{\sqrt{n_k+1}} < \infty,$$

which is a contradiction. Hence B_w is not 2-frequently hypercyclic on ℓ^1 for the norm topology.

Example 3.5. We show the existence of a hypercyclic operator that is not q-frequently hypercyclic with respect to the weak topology, for any $q \in \mathbb{N}$. Let us consider the unilateral shift B_w on ℓ^1 with weights $w_k = \dfrac{\ln(k+2)}{\ln(k+1)}$, $k \in \mathbb{N}$.

By the result of H.N.Salas, B_w is hypercyclic since $w_1 w_2 \cdots w_k = \frac{\ln(k+2)}{\ln 2} \to \infty$ as $k \to \infty$. Let $x = (x_n)$ be a q-frequently hypercyclic vector for B_w for some $q \in \mathbb{N}$. Enumerate the set $N(x, U) = \{n \in \mathbb{N} : B_w^n(x) \in U\}$ as (n_k), where $U = \{(y_n) \in \ell^1 : |y_1| > 1\}$. Thus we have a constant $c > 0$ such that $n_k \leq ck^q$. This implies that $\ln(n_k) \leq C \ln k$, for some constant $C > 0$ and for all $k \in \mathbb{N}$. Hence

$$\sum_{k \geq 1} \frac{1}{\ln n_k} = \infty.$$

On the other hand, $B_w^{n_k}(x) \in U$ implies that

$$\frac{\ln(n_k + 2)}{\ln 2} |x_{n_k+1}| > 1.$$

Consequently,

$$\sum_{k \geq 1} \frac{1}{\ln(n_k + 2)} < \infty,$$

which is a contradiction. Hence B_w is not q-frequently hypercyclic in the norm topology, for any $q \in \mathbb{N}$.

We now prove a criterion, similar to the frequent hypercyclicity criterion, which works even for operators defined on certain non-metrizable locally convex spaces. Using this, we obtain a 2-frequently hypercyclic operator that is not frequently hypercyclic. Before stating the result, let us recall that a series $\sum x_n$ in a topological vector space is *unconditionally convergent* if $\sum x_{\sigma(n)}$ is convergent for every permutation σ of $\mathbb{N}$. In any topological vector space, this mode of convergence is equivalent to the unordered convergence of $\sum x_n$, cf. [16], p.154. Thus a series $\sum x_n$ is unconditionally convergent if and only if for every non-empty open set U of 0, there corresponds an $N \in \mathbb{N}$ such that $\sum_{n \in F} x_n \in U$ for every finite set $F \subset [N, \infty)$. Also, for the proof of our criterion we need the following lemma from [5].

Lemma 3.6. *Let (N_k) be a strictly increasing sequence of natural numbers. Then there exists a pairwise disjoint sequence (J_k) of subsets of $\mathbb{N}$ such that*

(1) $\underline{\mathrm{dens}}(J_k) > 0$ *for each $k \geq 1$*
(2) $|n - m| \geq N_k + N_p$ *for $n \neq m$ and $(n, m) \in J_k \times J_p$.*
(3) $n \geq N_k$, *for each $n \in J_k$ and $k \geq 1$.*

We now state and prove the main result of the paper.

Theorem 3.7 (q-frequent hypercyclicity criterion). *Let (X, τ) be a separable locally convex space and $q \in \mathbb{N}$. Suppose that X is equipped with an F-norm $\|.\|$ such that the F-norm topology is stronger than τ and $(X, \|.\|)$ is complete. If T is an operator continuous with respect to the F-norm, D is a subset of X containing a countable τ-dense set and $S : D \to D$ is a map such that*

(1) $\sum T^{n^q}(x)$ *and* $\sum S^{n^q}(x)$ *are unconditionally convergent with respect to the F-norm, for each $x \in D$; and*
(2) $TS = I$, *the identity on D,*

then the operator T is q-frequently hypercyclic with respect to τ.

Proof. Our proof is inspired by that of the frequent hypercyclicity criterion given in [5]. However, we outline the proof for the sake of completeness. Let $Y = \{x_1, x_2, \cdots\} \subset D$ be a countable τ-dense set. Consider a summable sequence (ϵ_k) of positive real numbers, which are to be chosen later. By the hypothesis, corresponding to ϵ_k, we can find $N_k \in \mathbb{N}$ such that

$$\left\| \sum_{n \in F} T^{n^q}(x_i) \right\| + \left\| \sum_{n \in F} S^{n^q}(x_i) \right\| < \epsilon_k, 1 \leq i \leq k, \tag{3.1}$$

for any finite set $F \subset [N_k, \infty)$ of natural numbers. We may now assume that (N_k) is strictly increasing so that by Lemma 3.6, we get a sequence (J_k) of subsets of $\mathbb{N}$ with the properties mentioned therein. We now set

$$x = \sum_{k \geq 1} \sum_{n \in J_k} S^{n^q}(x_k).$$

Since unconditional convergence implies subseries convergence [16] p.154, the series $\sum_{n \in J_k} S^{n^q}(x_k)$ converges for each natural number k. It follows by (3.1) that

$$\sum_{k=1}^{\infty} \left\| \sum_{n \in J_k} S^{n^q}(x_k) \right\| \leq \sum_{k \geq 1} \epsilon_k < \infty$$

Thus $x \in X$.

Let us now fix $k \in \mathbb{N}$ and $m \in J_k$. Then

$$T^{m^q}(x) = \sum_{l \geq 1} \sum_{n \in J_l} T^{m^q} S^{n^q}(x_l).$$

and so

$$\left\|T^{m^q}(x) - x_k\right\| \le \sum_{l \ge 1}\left\|\sum_{n \in J_l, m > n} T^{m^q - n^q}(x_l)\right\| + \sum_{l \ge 1}\left\|\sum_{n \in J_l, m < n} S^{n^q - m^q}(x_l)\right\|. \quad (3.2)$$

Let us now consider the first sum on the right hand side of (3.2). Indeed, writing the first term as

$$\sum_{l=1}^{k}\left\|\sum_{n \in J_l, m > n} T^{m^q - n^q}(x_l)\right\| + \sum_{l \ge k+1}\left\|\sum_{n \in J_l, m > n} T^{m^q - n^q}(x_l)\right\|,$$

we have

$$\sum_{l \ge 1}\left\|\sum_{n \in J_l, m > n} T^{m^q - n^q}(x_l)\right\| \le k\epsilon_k + \sum_{j \ge k+1} \epsilon_j.$$

The last inequality arrives because whenever $m \in J_k$, we have $m^q - n^q > \max(N_k, N_l)$ for any $n \in J_l$ with $m > n$. Similarly, we evaluate the second term to get

$$\sum_{l \ge 1}\left\|\sum_{n \in J_l, m < n} S^{n^q - m^q}(x_l)\right\| \le k\epsilon_k + \sum_{j \ge k+1} \epsilon_j.$$

Set $\alpha_k = k\epsilon_k + \sum_{j \ge k+1} \epsilon_j$. So we arrive at the inequality,

$$\left\|T^{m^q}(x) - x_k\right\| \le 2\alpha_k < 3\alpha_k, \forall m \in J_k, k \ge 1. \quad (3.3)$$

Choose ϵ_k such that $\alpha_k \to 0$. We now show that x is a q-frequently hypercyclic vector for the operator T with respect to the topology τ. Let G be a nonempty τ-open set and $y + U \subset G$ for some τ-neighborhood U of the origin. Then we find a balanced neighborhood V of the origin such that $V + V \subset U$. Since Y is τ-dense in X, we find an increasing sequence of natural numbers (n_k) such that $x_{n_k} - y \in V$ for all $k \ge 1$. Since the $\|.\|$-topology is finer than τ, it follows from (3.3) that, for some $N \in \mathbb{N}$, $T^{m^q}(x) - x_k \in V$ for every $m \in J_k$ and $k \ge N$. Thus from the facts that $x_{n_N} - y \in V$ and $T^{m^q}(x) - x_{n_N} \in V$, we obtain that for all $m \in J_{n_N}$, $T^{m^q}(x) - y \in V + V \subset U$. Our conclusion now follows since $T^{m^q}(x) \in G$ for all $m \in J_{n_N}$, which has positive lower density. $\square$

Remark 3.8. It is evident from the proof of Theorem 3.7 that the sequence (T^{n^q}) is frequently hypercyclic and thus T is q-frequently hypercyclic. It would be interesting to know whether the converse is true or not, i.e., is (T^{n^q}) frequently hypercyclic whenever T is q-frequently hypercyclic?

Remark 3.9. Let us also note that in the above theorem, the countability assumption on a subset of D may be waived in case the topology τ is generated by an F-norm. Indeed, if (x_n) is a τ-dense sequence in X, choose a countable set $\{y_{n,m} : n, m \ge 1\}$ where $y_{n,m} \in D \cap B(x_n, \frac{1}{m})$, $B(x_n, \frac{1}{m})$ being the open ball of radius $\frac{1}{m}$ centered at x_n. It is now easy to verify that $\{y_{n,m} : n, m \ge 1\}$ is a τ-dense set in X.

4. Applications

In this section, we consider some applications of the q-frequent hypercyclicity criterion for obtaining q-frequently hypercyclic operators on spaces equipped with linear topologies which are not necessarily metrizable. Besides, we prove the frequent hypercyclicity of a non-convolution operator on $H(\mathbb{C})$, at the end of the section. Let us begin with the results on sequence spaces with metrizable topologies.

Proposition 4.1. *Let λ be a sequence space equipped with an F-norm and let $\{e_n\}$ be an unconditional basis in λ. If for some $q \in \mathbb{N}$,*

$$\sum_{n \in \mathbb{N}} \frac{1}{w_1 w_2 \cdots w_{n^q+j}} e_{n^q+j}$$

converges unconditionally for each $j \in \mathbb{N}$, then the backward shift B_w associated to the weight sequence (w_n), is q-frequently hypercyclic.

Proof. Since λ is an F-space, in view of Remark 3.9 choose D to be the dense set spanned by $\{e_n : n \geq 1\}$. Define S_w on D by $S_w(e_n) = \frac{1}{w_{n+1}} e_{n+1}$. To apply our criterion, we are only required to prove the unconditional convergence of $\sum S_w^{n^q}(x)$ for each $x \in D$. Indeed, for a given $k \in \mathbb{N}$, we have

$$S_w^n(e_k) = \frac{1}{w_{k+1} \cdots w_{k+n}} e_{k+n}.$$

Thus, by the hypothesis, the series

$$\sum_{n \geq 1} S_w^{n^q}(e_k) = w_1 w_2 \cdots w_k \sum_{n \in \mathbb{N}} \frac{1}{w_1 \cdots w_{k+n^q}} e_{k+n^q}.$$

converges unconditionally in λ. Hence B_w is q-frequently hypercyclic. $\square$

As a consequence of the above result, we obtain a q-frequently hypercyclic operator that is not frequently hypercyclic. This is the Bergman shift, considered in [3].

Corollary 4.2. *Let B_w be the unilateral shift on ℓ^2 given by the weights $w_n = \sqrt{\frac{n+1}{n}}$, $n \geq 1$. Then B_w is 2-frequently hypercyclic and is not frequently hypercyclic; a fortiori, B_w is q-frequently hypercyclic for any $q \geq 2$.*

Proof. Since $w_1 w_2 \cdots w_{n^2+j} = \sqrt{n^2 + j + 1}$, the result follows. $\square$

Remark 4.3. One may apply Proposition 4.1 to obtain the q-frequent hypercyclicity of shift operators defined on Fréchet spaces, for example the space $H(\mathbb{C})$ of entire functions equipped with the compact-open topology, the space of all sequences with the topology of co-ordinate convergence and the classical ℓ^p spaces.

Before we move on to another application, let us see an example.

Example 4.4. Let $p \in \mathbb{N}$. Then there exists an operator which is not p-frequently hypercyclic, but it is q-frequently hypercyclic for all $q \geq p + 1$. We consider the

unilateral shift B_w on ℓ^2 with weights $w_k = \left(\dfrac{k+2}{k+1}\right)^{1/2p}$, $k \in \mathbb{N}$. Then the proof similar to that of Example 3.5 shows that B_w is not p-frequently hypercyclic on ℓ^2. We can also conclude that B_w is $(p+1)$-frequently hypercyclic by applying Proposition 4.1 to the dense set of finite sequences.

Our next application of Theorem 3.7 is for the bilateral backward shift operators. Let X be an F-sequence space over the set $\mathbb{Z}$ of integers such that the unit sequences $(e_n)_{n\in\mathbb{Z}}$ form an unconditional basis in X. For $w = (w_n) \in \ell^\infty(\mathbb{Z})$, the operator $T_w(e_n) = w_n e_{n-1}$ is the bilateral backward shift. The content of the following proposition is the q-frequent hypercyclicity of T_w.

Proposition 4.5. *Let $(e_n)_{n\in\mathbb{Z}}$ form a unconditional basis in an F-sequence space X and let $q \in \mathbb{N}$. If*

$$\sum_{n\in\mathbb{N}} \frac{1}{w_1 w_2 \cdots w_{n^q+j}} e_{n^q+j} \ and \ \sum_{n\in\mathbb{N}} w_j w_{j-1} \cdots w_{-n^q+j+1} e_{-n^q+j}$$

converge unconditionally for each $j \in \mathbb{N}$, then T_w is q-frequently hypercyclic on X.

Proof. Let D be the set spanned by the sequence $(e_n)_{n\in\mathbb{Z}}$. Consider the map $S_w(e_k) = \frac{1}{w_{k+1}} e_{k+1}$ on the dense set D of X, so that $T_w S_w$ is the identity operator on D. Then,

$$T_w^{n^q}(e_j) = w_j w_{j-1}..w_{j-n^q+1} e_{j-n^q}$$

and

$$S_w^{n^q}(e_j) = \frac{1}{w_{j+1} w_{j+2} \cdots w_{j+n^q}} e_{j+n^q}$$

for each $j \in \mathbb{Z}$. From the hypothesis, we obtain that the series $\sum T_w^{n^q}(e_j)$ and $\sum S_w^{n^q}(e_j)$ converge unconditionally in X. The desired result now follows since T_w and S_w are linear and D is the span of $(e_n)_{n\in\mathbb{Z}}$. $\qquad\square$

A particular case of Proposition 4.5 is when T_w is the bilateral backward shift defined on the sequence space $\ell^p(\mathbb{Z})$ or $c_0(\mathbb{Z})$ for $1 \le p < \infty$. Recall that hypercyclicity of T_w was characterized by H. N. Salas in [23]. Also a series $\sum_{n\in\mathbb{Z}} a_n e_n$ converges unconditionally in $\ell^p(\mathbb{Z})$ if and only if the sequence $(a_n) \in \ell^p(\mathbb{Z})$. Thus we derive the following result from Proposition 4.5.

Corollary 4.6. *Let $q \in \mathbb{N}$ and $1 \le p < \infty$. Assume that for each $j \in \mathbb{Z}$,*

$$\sum_{n\in\mathbb{N}} \frac{1}{(w_1 w_2 \cdots w_{n^q+j})^p} < \infty \ and \ \sum_{n\in\mathbb{N}} (w_j w_{j-1} \cdots w_{-n^q+j+1})^p < \infty.$$

Then T_w is q-frequently hypercyclic on $\ell^p(\mathbb{Z})$. If $\lim_{n\to\infty}(w_1 w_2 \cdots w_{n^q+j}) = \infty$ and $\lim_{n\to\infty}(w_j w_{j-1} \cdots w_{-n^q+j+1}) = 0$ for each $j \in \mathbb{Z}$, then T_w is q-frequently hypercyclic on $c_0(\mathbb{Z})$.

We have yet another application of Theorem 3.7. Let X be a Banach space having a Schauder basis $\{x_n, f_n\}$. Then the dual X^* is weak*-separable and the weak*-topology is not metrizable, when X is infinite dimensional. We assume that $\{x_n, f_n\}$ is a *symmetric Schauder basis* for X, which means that

$\sum_{n \geq 1} f_{\mu(n)}(x) x_{\sigma(n)}$ converges for each $x \in X$ and every pair (μ, σ) of permutations of $\mathbb{N}$. A symmetric base in a Banach space is regular $(\inf \|x_n\| > 0)$ and bounded $(\sup \|x_n\| < \infty)$, see for example [17], p. 133. Corresponding to a weight sequence (w_n) and a symmetric Schauder basis $\{x_n, f_n\}$ (where the index starts from 1), we define the backward shift B_w by $B_w(f_n) = w_n f_{n-1}$, $n \geq 1$ with $f_0 = 0$. This operator is continuous with respect to the norm as well as the weak*-topology of X^*, cf. [21, p. 1434]. We now prove:

Theorem 4.7. *Let X be a Banach space with a symmetric Schauder basis $\{x_n, f_n\}$. Then for $q \in \mathbb{N}$, the backward shift operator on X^* is q-frequently hypercyclic with respect to the weak*-topology of X^* if $\sum_{n \in \mathbb{N}} \dfrac{1}{w_1 \cdots w_{j+n^q}}$ converges for each $j \in \mathbb{N}$.*

In particular, if $\sum_{n \in \mathbb{N}} \dfrac{1}{w_1 w_2 \cdots w_n}$ converges, then B_w is weak-frequently hypercyclic on X^*.*

Proof. In order to apply Theorem 3.7, consider D to be the span of $\{f_n : n \geq 1\}$. Then D contains a countable weak*-dense subset. The forward shift $S_w(f_n) = \frac{1}{w_{n+1}} f_{n+1}$, $n \geq 1$ maps D to itself. Thus

$$S_w^n(f_k) = \frac{1}{w_{k+1} \cdots w_{k+n}} f_{k+n}$$

and

$$\sum_{n \in \mathbb{N}} S_w^{n^q}(f_k) = w_1 w_2 \cdots w_k \sum_{n \in \mathbb{N}} \frac{1}{w_1 \cdots w_{k+n^q}} f_{k+n^q}.$$

Since a symmetric Schauder basis is regular, we have that $\|f_n\| < K$ for some constant $K > 0$ and for all $n \in \mathbb{N}$, cf. [17], p. 261 and [25], p. 25. Hence by our hypothesis, the series $\sum_{n \in \mathbb{N}} S_w^{n^q}(f_k)$ is absolutely convergent and so unconditionally convergent. Consequently, the shift B_w is q-frequently hypercyclic with respect to the weak*-topology on X^* by Theorem 3.7. □

As a consequence of the above result, we derive:

Corollary 4.8. *The backward shift B_w is weak*-q-frequently hypercyclic on ℓ^∞ if $\sum_{n \in \mathbb{N}} \dfrac{1}{w_1 \cdots w_{j+n^q}}$ converges for each $j \in \mathbb{N}$.*

Proof. Immediate since $\{e_n : n \geq 1\}$ is a symmetric Schauder basis for ℓ^1. □

Thus if $\sum_{n \in \mathbb{N}} \dfrac{1}{w_1 w_2 \cdots w_n}$ converges, B_w is weak*-frequently hypercyclic on ℓ^∞. In fact, the following stronger result holds.

Proposition 4.9. *(1)If $\lim_{n \to \infty} (w_1 w_2 \cdots w_n) = \infty$, then the unilateral backward shift B_w is weak*-frequently hypercyclic on ℓ^∞.*
(2)If $\lim_{n \to \infty} (w_1 w_2 \cdots w_n) = \infty$ and $\lim_{n \to \infty} (w_{-1} w_{-2} \cdots w_{-n}) = 0$, then the bilateral backward shift T_w is weak-frequently hypercyclic on $\ell^\infty(\mathbb{Z})$.*

Proof. Since $\lim_{n\to\infty}(w_1w_2\cdots w_n) = \infty$, the weighted shift B_w is frequently hyper-cyclic on c_0, cf. [3] or [5]. Also, it is easy to see that the identity operator from c_0 to ℓ^∞ is norm to weak* continuous and has weak*-dense range. Hence B_w is weak*-frequently hypercyclic on ℓ^∞, by Proposition 2.3. Similarly, the identity map takes $c_0(\mathbb{Z})$ into $\ell^\infty(\mathbb{Z})$ continuously and densely. Thus T_w is weak*-frequently hypercyclic on $\ell^\infty(\mathbb{Z})$.

$\square$

Remark 4.10. We would like to mention here that hypercyclicity on ℓ^∞ which is weak*-separable, was studied in [7] and [21]. It was proved that a backward shift B_w is weak*-hypercyclic on ℓ^∞ if and only if $\limsup_{n\to\infty}(w_1w_2\cdots w_n) = \infty$. However, the condition $\lim_{n\to\infty}(w_1w_2..w_n) = \infty$ is not necessary for B_w to be weak*-frequently hypercyclic on ℓ^∞. Indeed, there exists a frequently hypercyclic B_w on c_0 (and thus weak*-frequently hypercyclic on ℓ^∞) such that $w_1w_2..w_n \nrightarrow \infty$. cf. [4], p. 205.

Remark 4.11. In view of Theorem 3.7, a weakly q-frequently hypercyclic operator on a separable Banach space, which satisfies the q-frequent hypercyclicity criterion with respect to a weakly dense set is necessarily norm-q-frequently hypercyclic; for the closed convex sets are the same in the weak and norm topologies .

Finally, we consider the frequent hypercyclicity of a non-convolution operator. It is known that any convolution operator on $H(\mathbb{C})$ (a continuous linear operator that commutes with all translations) that is not a multiple of the identity operator is frequently hypercyclic [10]. The operator $T_\mu(f)(z) = f'(\mu z)$ on the space $H(\mathbb{C})$ is a non-convolution for $\mu \neq 1$ and was shown to be hypercyclic for $|\mu| \geq 1$, cf. [12] and [2]. In fact, this operator is a weighted backward shift with weights $w_n = n\mu^{n-1}$. So, our result can be derived using the Proposition 4.1. We rather prove this in the following way.

Proposition 4.12. *Let $H(\mathbb{C})$ be equipped with the compact-open topology. Then the operator T_μ is frequently hypercyclic on $H(\mathbb{C})$ for $|\mu| \geq 1$.*

Proof. Let D be the set of all polynomials. Define the map S_μ by,

$$S_\mu(f)(z) = \mu \int_0^{\frac{1}{\mu}z} f(u)\,du.$$

It is easy to see that $\sum T_\mu^n(f)$ is absolutely convergent in $H(\mathbb{C})$ and that $T_\mu S_\mu = I$, the identity on the set D. We fix a $k \geq 0$ and consider the function z^k. Then

$$S_\mu^n(z^k) = \frac{k!z^{k+n}}{(k+n)!\mu^{nk+n(n-1)/2}}.$$

Since $|\mu| \geq 1$, the series $\sum S_\mu^n(f)$ is absolutely convergent for any polynomial f and we conclude that T_μ is frequently hypercyclic by Theorem 2.2. $\square$

We ask the following question. Can one say that the operators $T_{\mu,b}(f)(z) = f'(\mu z + b)$ on $H(\mathbb{C})$ for $|\mu| \geq 1$ and $b \in \mathbb{C}$, $b \neq 0$, are frequently hypercyclic? These have been shown to be hypercyclic in [2] and [12].

5. Rotation and Powers

We now remark that rotations and powers of a q-frequently hypercyclic operator on an arbitrary separable topological vector space remain q-frequently hypercyclic for any $q \in \mathbb{N}$. They also share the same set of q-frequently hypercyclic vectors. These results have been proved for the case $q = 1$ in [5]. The hypercyclicity of powers and rotations have been considered by S. I. Ansari [1] and F. Leon-Saavedra and V. Müller [19] respectively. For establishing the following theorem on powers and rotations of q-frequently hypercyclic operators, we need a lemma stated as

Lemma 5.1. *Let $A \subset \mathbb{N}$ have positive q-lower density and $\bigcup_{j=1}^{k} I_j = \mathbb{N}$. If $n_1, \cdots, n_k$ are finitely many natural numbers, then*

$$\bigcup_{j=1}^{k} (n_j + A \cap I_j)$$

has positive q-lower density.

Proof. Omitted as it follows on the same lines as given in [5], p. 148. □

Let us denote by $qFHC(T)$, the set of all q-frequently hypercyclic vectors for T and S^1, the unit circle in the complex plane. Then we have

Theorem 5.2. *Let T be a q-frequently hypercyclic operator on a complex topological vector space X. Then λT and T^p are q-frequently hypercyclic for each $\lambda \in S^1$ and $p \in \mathbb{N}$. Also $qFHC(T) = qFHC(\lambda T) = qFHC(T^p)$.*

Proof. To get this result, we proceed on similar lines as in the case of frequent hypercyclicity, [5], p. 148. □

Finally, we would like to mention that the notion of q-frequent hypercyclicity is a particular case of (m_k)-hypercyclicity studied in [6]. Indeed, for a strictly increasing sequence (m_k) of natural numbers, an element $x \in X$ is called (m_k)-hypercyclic for an operator T on X if for every non-empty open set $U \subset X$, there exists a strictly increasing $(n_k) = O(m_k)$ such that $T^{n_k}(x) \in U$ for all k. Thus the case $m_k = k^q$, for all k coincides with the notion of q-frequent hypercyclicity; however, the results in our paper have no overlap with the results of [6] except that the q-frequent hypercyclicity $(q \geq 2)$ of the Bergman shift has been proved in [6] using the notion of a hypercyclicity set; see Example 5.3, [6].

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