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NONCOMMUTATIVE INTEGRATION

MASAMICHI TAKESAKI

To the Memory of Two Distinguished Operator Algebraists: *William B. Arveson and Gert K. Pedersen.*

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ABSTRACT. We will show that if $\mathcal{M}$ is a factor, then for any pair $\varphi, p \in \mathcal{M}_*^+$ of normal positive linear functionals on $\mathcal{M}$, the inequality:

$$\|\varphi\| \leq \|\psi\|$$

is equivalent to the fact that there exist a countable family $\{\varphi_i : i \in I\} \subset \mathcal{M}_*^+$ in $\mathcal{M}_*^+$ and a family $\{u_i : i \in I\} \subset \mathcal{M}$ of partial isometries in $\mathcal{M}$ such that

$$\varphi = \sum_{i \in I} \varphi_i, \quad \sum_{i \in I} u_i \varphi_i u_i^* \leq \psi, \quad \text{and} \quad u_i^* u_i = s(\varphi_i), i \in I,$$

where $s(\omega), \omega \in \mathcal{M}_*^+$, means the support projection of ω. Furthermore, if $\|\varphi\| = \|\psi\|$, then the equality replaces the inequality in the second statement. In the case that $\mathcal{M}$ is not of type III_1, the family of partial isometries can be replaced by a family of unitaries in $\mathcal{M}$. One cannot expect to have this result in the usual integration theory. To have a similar result, one needs to bring in some kind of non-commutativity. Let $\{X, \mu\}$ be a σ-finite semifinite measure space and G be an ergodic group of automorphisms of $L^\infty(X, \mu)$, then for a pair f and g of μ-integrable positive functions on X, the inequality:

$$\int_X f(x) \mathrm{d}\mu(x) \leq \int_X g(x) \mathrm{d}\mu(x)$$

is equivalent to the existence of a countable families $\{f_i : i \in I\} \subset L^1(X, \mu)$ of positive integrable functions and $\{\gamma_i : i \in I\}$ in G such that

$$f = \sum_{i \in I} f_i \quad \text{and} \quad \sum_{i \in I} \gamma_i(f_i) \leq g,$$

where the summation and inequality are all taken in the ordered Banach space $L^1(X, \mu)$ and the action of G on $L^1(X, \mu)$ is defined through the duality between $L^\infty(X, \mu)$ and $L^1(X, \mu)$, i.e.,

$$(\gamma(f))(x) = f\left(\gamma^{-1} x\right) \frac{\mathrm{d}\mu \circ \gamma^{-1}}{\mathrm{d}\mu}(x), \quad f \in L^1(X, \mu).$$

1. INTRODUCTION

Regardless of commutativity, the integration of a positive element is the numerical value indicating the size of the quantity represented by the element. The one faces the following basic question:

What does two positive elements to record
the same integration value mean?

Of course one cannot expect that two positive elements with the same integration value are isomorphic. In the classical integration theory, one cannot go further on this question. But in sharp contrast, in the non-commutative world, one can say that two positive elements with the same integration values are decomposed into the countable sum of two sequences of mutually isomorphic positive elements. This means that the non-commutative integration represents better the true meaning of integration than the classical commutative integration theory. Another important fact on this result is that the summation is taken over a countable set of objects. Otherwise, we are dealing with cardinality, which gives us very little room for analysis. This shows the distinguished position of the countability among infinities.

2. Preliminary, Noncommutative flow of weights

We will refer to either [4] or [7, Chapter XII Section 6] for the basic facts on noncommutative flow of weights. But unfortunately, [7, Exercise XII.6] contains a little imprecise statement, so we will present here the essence of that theory. We consider the translation flow $\{L^\infty(\mathbb{R}), \mathbb{R}, \rho\}$:

$$(\rho_t f)(s) = f(s+t), \quad f \in L^\infty(\mathbb{R}), s, t \in \mathbb{R}.$$

Lemma 2.1. *If μ is a normal weight on $\mathcal{A} = L^\infty(\mathbb{R})$ such that*

$$\mu{\circ}\rho_s(f) = e^{-s}\mu(f), \quad f \in \mathcal{A}_+;$$
$$0 < \mu(f_0) < +\infty \quad \textit{for some } f_0 \in \mathcal{A}_+,$$

then the weight μ is a faithful semi-finite normal weight on $\mathcal{A}$ such that

$$C = \mu((-\infty, 0]) < +\infty;$$
$$\mu(f) = C\int_{\mathbb{R}} f(s)e^s \mathrm{d}s, \quad f \in L^\infty(\mathbb{R})_+,$$

where we view the normal weight μ as a measure on $\mathbb{R}$ absolutely continuous relative to the Lebesgue measure, but not necessarily semi-finite.

Proof. Let g be a continuous non-negative function with compact support on $\mathbb{R}$. Then we have

$$\begin{aligned}\mu(\rho_g(f_0)) &= \mu\left(\int_{\mathbb{R}} g(s)\rho_s(f_0)\mathrm{d}s\right) = \int_{\mathbb{R}} g(s)\mu(\rho_s(f_0))\mathrm{d}s \\ &= \left(\int_{\mathbb{R}} e^{-s}g(s)\mathrm{d}s\right)\mu(f_0),\end{aligned}$$

so that $0 < \mu(\rho_g(f_0)) < +\infty$ and

$$\left(\int_{\mathbb{R}} g(s)\rho_s(f_0)\mathrm{d}s\right)(t) = \int_{\mathbb{R}} g(s)f_0(s+t)\mathrm{d}s = (g * f_0)(t).$$

Hence $\rho_g(f_0)$ is continuous on $\mathbb{R}$ and takes a finite value on the normal weight μ. Thus we may and do take a continuous positive function as f_0 in the assumption of the lemma. So there are an interval $(a, b], a < b$, and a constant $C_1 > 0$ such that

$$C_1\chi_{(a,b]} \leq f_0 \quad \text{and} \quad 0 < \mu\big(\chi_{(a,b]}\big) < +\infty.$$

As $\rho_s\big(\chi_{(a,b]}\big) = \chi_{(a-s,b-s]}$, we have

$$\mu((a-s, b-s]) = e^{-s}\mu((a,b]) \quad \text{for every } s \in \mathbb{R}.$$

From this, it follows that the measure μ takes a positive finite value on every finite interval and also that

$$\begin{aligned} C = \mu((-\infty, 0]) &= \mu\left(\bigcup_{n=1}^{\infty}(-n, -n+1]\right) \\ &= \sum_{n=1}^{\infty} e^{-n}\mu((0,1]) = \frac{1}{e-1}\mu((0,1]) < +\infty, \\ \mu((-\infty, s]) &= \mu\big(\rho_{-s}\big(\chi_{(-\infty,0]}\big)\big) = e^{s}\mu((-\infty,0]) < +\infty, \\ \mathrm{d}\mu(s) &= \mu((-\infty,0])e^{s}\mathrm{d}s. \end{aligned}$$

This completes the proof. □

Fix a von Neumann algebra $\mathcal{M}$ and consider the associated noncommutative flow of weights $\left\{\widetilde{\mathcal{M}}, \mathbb{R}, \tau, \theta\right\}$ to have

$$\mathcal{M} = \widetilde{\mathcal{M}}^{\theta}, \quad \tau\circ\theta_s = e^{-s}\tau,$$
$$\mathcal{M}' \cap \widetilde{\mathcal{M}} = \mathcal{C} = \text{The Center of } \widetilde{\mathcal{M}},$$
$$\{\mathcal{C}, \mathbb{R}, \theta\} = \text{The flow of weights on } \mathcal{M}.$$

Let $\mathfrak{M}$ be the algebra of all τ-measurable densely defined closed operators affiliated to $\widetilde{\mathcal{M}}$. The following criteria for τ-measurability is very useful and easy to manage:

> A densely defined closed positive operator h affiliated to $\widetilde{\mathcal{M}}$ is **τ-measurable** if and only if there exists a positive number $\lambda_0 > 0$ such that
> $$\tau\big(\chi_{[\lambda_0,+\infty)}(h)\big) < +\infty.$$

We are going to write $E_\lambda = \chi_{[\lambda,+\infty)} \in L^\infty(\mathbb{R})$ for each $\lambda > 0$. The algebra $\mathfrak{M}$ is graded by the noncommutative flow $\{\theta_s : s \in \mathbb{R}\}$ as seen below.

Setting

$$\mathfrak{M}(\alpha) = \big\{x \in \mathfrak{M} : \theta_s(x) = e^{-\alpha s}x, \quad s \in \mathbb{R}\big\}, \quad \alpha \in \mathbb{C},$$

we obtain the following:

i) The original von Neumann algebra $\mathcal{M}$ is the fixed point algebra of θ, which is exactly the equality:
$$\mathcal{M} = \mathfrak{M}(0).$$

ii) For each $p > 0$ we write
$$L^p(\mathcal{M}) = \mathfrak{M}\left(\frac{1}{p}\right).$$

iii) The cases that $p = 1$ and $p = 2$ are of particular interest for us:
$$L^1(\mathcal{M}) = \mathfrak{M}(1) = \big\{x \in \mathfrak{M} \colon \theta_s(x) = e^{-s}x, s \in \mathbb{R}\big\},$$
$$L^2(\mathcal{M}) = \mathfrak{M}\left(\frac{1}{2}\right) = \big\{x \in \mathfrak{M} : \theta_s(x) = e^{-s/2}x, s \in \mathbb{R}\big\}.$$
as it will be identified with the predual $\mathcal{M}_*$ and the standard form of $\mathcal{M}$.

iv) If $\Re\alpha < 0$, then

$$\mathfrak{M}(\alpha) = \{0\}.$$

We now consider the operator valued weight I_θ from $\widetilde{\mathcal{M}}_+$ to the extended positive cone $\widehat{\mathcal{M}}_+$ of $\mathcal{M}$:

$$I_\theta(x) = \int_{\mathbb{R}} \theta_s(x)\mathrm{d}s, \quad x \in \widetilde{\mathcal{M}}_+.$$

As in the paper of Falcone–Takesaki, [4], we denote the element of $\mathfrak{M}$ corresponding to $\omega \in \mathcal{M}_*$ by $T(\omega) \in L^1(\mathcal{M})$ to avoid possible confusions and write

$$\omega(1) = \int \mathrm{d}T(\omega),$$

which is defined to be the following value:

$$\int \mathrm{d}T(\omega) = \tau\Big(a^{\frac{1}{2}}T(\omega)a^{\frac{1}{2}}\Big) = \omega(1)$$

for any $a \in \widetilde{\mathcal{M}}_+$ with $I_\theta(a) = 1$. The middle quantity $\tau\Big(a^{\frac{1}{2}}T(\omega)a^{\frac{1}{2}}\Big)$ does not depend on the choice of $a \in \widetilde{\mathcal{M}}_+$ with $I_\theta(a) = 1$ as shown in [4, Theorem 3.12].

3. Comparison of Integrals

Let $\mathcal{M}$ be a fixed von Neumann algebra. Fixing a pair $\varphi, \psi \in \mathcal{M}_*^+$ with $p = s(\varphi), q = s(\psi) \in \mathrm{Proj}(\mathcal{M})$, consider the one parameter group $\Big\{\sigma_t^{\varphi,\psi} : t \in \mathbb{R}\Big\}$ of isometries on $p\mathcal{M}q$ defined by the following:

$$\sigma_t^{\varphi,\psi}(x) = T(\varphi)^{\mathrm{i}t} x T(\psi)^{-\mathrm{i}t}, \quad x \in \mathcal{M},$$

which appears on the $(1,2)$-corner of $\mathrm{M}_2(\mathbb{C})\overline{\otimes}\mathcal{M}$ of the modular automorphism group σ^ρ of the balanced positive linear functional $\rho = \varphi \oplus \psi$:

$$\rho = \begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix} \in (\mathrm{M}_2(\mathbb{C})\overline{\otimes}\mathcal{M})_*^+.$$

We then consider the subspace $\mathcal{A}(\varphi, \psi)$ of entire elements in $p\mathcal{M}q$ relative to $\sigma^{\varphi,\psi}$, i.e., $\mathcal{A}(\varphi, \psi)$ is the set of all those elements $x \in p\mathcal{M}q$ such that the function: $t \in \mathbb{R} \mapsto \sigma_t^{\varphi,\psi}(x) \in \mathcal{M}$ has entire extension to $\mathbb{C}$. We denote its value at $\alpha \in \mathbb{C}$ by $\sigma_\alpha^{\varphi,\psi}(x) \in \mathcal{M}$. Of particular interest to us is the value at the half imaginary unit: $\pm\mathrm{i}/2$, which is $\sigma_{\pm\mathrm{i}/2}^{\varphi,\psi}(x) \in \mathcal{M}$.

Lemma 3.1. *If $x \in \mathcal{A}(\varphi, \psi), x \neq 0$, then the element $T(\varphi)^{\frac{1}{2}} x T(\psi)^{\frac{1}{2}} \in L^1(\mathcal{M})$ has the property*:

$$\Big|T(\varphi)^{\frac{1}{2}} x T(\psi)^{\frac{1}{2}}\Big| \leq \Big\|\sigma_{-\mathrm{i}/2}^{\varphi,\psi}(x)\Big\|\psi,$$

$$\Big|\Big(T(\varphi)^{\frac{1}{2}} x T(\psi)^{\frac{1}{2}}\Big)^*\Big| \leq \Big\|\sigma_{-\mathrm{i}/2}^{\psi,\varphi}(x^*)\Big\|\varphi,$$

$$T(\varphi)^{\frac{1}{2}} x T(\psi)^{\frac{1}{2}} \neq 0.$$

Proof. We consider the path:

$$t \in \mathbb{R} \mapsto \sigma_t^{\varphi,\psi}(x) = T(\varphi)^{\mathrm{i}t} x T(\psi)^{-\mathrm{i}t} \in \mathcal{A}(\varphi,\psi) \subset p\mathcal{M}q,$$

which admits entire extension:

$$\sigma_z^{\varphi,\psi} = T(\varphi)^{\mathrm{i}z} x T(\psi)^{-\mathrm{i}z} \in \mathcal{A}(\varphi,\psi), \quad z \in \mathbb{C}.$$

The evaluation at $-\mathrm{i}/2$ gives

$$\sigma_{-\mathrm{i}/2}^{\varphi,\psi}(x) = T(\varphi)^{\frac{1}{2}} x T(\psi)^{-\frac{1}{2}},$$

so that we get

$$T(\varphi)^{\frac{1}{2}} x T(\psi)^{\frac{1}{2}} = T(\varphi)^{\frac{1}{2}} x T(\psi)^{-\frac{1}{2}} T(\psi) = \sigma_{-\mathrm{i}/2}^{\varphi,\psi}(x) T(\psi) \in L^1(\mathcal{M}).$$

Thus we get the following easy conclusion:

$$\begin{aligned}\left|T(\varphi)^{\frac{1}{2}} x T(\psi)^{\frac{1}{2}}\right| &= \left[\left(\sigma_{-\mathrm{i}/2}^{\varphi,\psi}(x)T(\psi)\right)^*\left(\sigma_{-\mathrm{i}/2}^{\varphi,\psi}(x)T(\psi)\right)\right]^{\frac{1}{2}} \\ &\leq \left\|\sigma_{-\mathrm{i}/2}^{\varphi,\psi}(x)\right\| T(\psi).\end{aligned}$$

The other inequality follows similarly.

The non-triviality of the element $T(\varphi)^{\frac{1}{2}} x T(\psi)^{\frac{1}{2}}$ follows from the fact that $T(\psi)$ is non-singular on the range of the projection q and $T(\varphi)$ is also on the range of p. ☐

Definition 3.2. A pair $\varphi, \psi \in \mathcal{M}_*^+$ of normal positive linear functionals is said to be **equivalent** and written

$$\varphi \sim \psi$$

if there exists a partial isometry $u \in \mathcal{M}$ such that

$$u^*u \geq s(\varphi), \quad uu^* \geq s(\psi) \quad \text{and} \quad u\varphi u^* = \psi,$$

which automatically gives

$$\varphi = u^*\psi u.$$

If the above u can be chosen to be unitary, then we say that φ and ψ are **unitarily conjugate** and write

$$\varphi \equiv \psi \quad \mod \operatorname{Int}(\mathcal{M}).$$

Lemma 3.3. *If $\mathcal{M}$ is a factor, then every pair $\varphi, \psi \in \mathcal{M}_*^+$ of non-zero normal positive linear functionals on $\mathcal{M}$ admits a pair $\varphi_1, \psi_1 \in \mathcal{M}_*^+$ such that*

$$0 \neq \varphi_1 \leq \varphi, \quad 0 \neq \psi_1 \leq \psi \quad \text{and} \quad \varphi_1 \sim \psi_1.$$

In the case that if every non-zero normal positive linear functional $\omega \in \mathcal{M}_*^+, \omega \neq 0$, majorizes a non-zero non-faithful positive linear functional $\omega_1 \in \mathcal{M}_*^+, \omega_1 \neq 0$, then the above φ_1 and ψ_1 may be chosen to be unitarily conjugate, i.e.,

$$\varphi_1 \equiv \psi_1 \quad \mod \operatorname{Int}(\mathcal{M}).$$

Proof. Choose $x \in \mathcal{A}(\varphi, \psi), x \neq 0$ and set

$$\rho = \frac{1}{\left\| \sigma^{\varphi,\psi}_{-\mathrm{i}/2}(x) \right\|} \varphi^{\frac{1}{2}} x \psi^{\frac{1}{2}} \in L^1(\mathcal{M}) = \mathcal{M}_*.$$

Then with the polar decomposition:

$$\rho = v|\rho|$$

we have

$$0 \neq \psi_1 = |\rho| \leq \psi \quad \text{and} \quad 0 \neq \varphi_1 = |\rho^*| \leq \varphi,$$
$$v^*v = s(\psi_1), \quad vv^* = s(\varphi_1) \quad \text{and} \quad v\psi_1 v^* = \varphi_1, \quad v^*\varphi_1 v = \psi_1.$$

Setting $u_1 = v^*$, we get the desired triplet $\{\varphi_1, \psi_1, u_1\}$ of the lemma. If the partial isometry u_1 admits a unitary extension w in the sense that

$$w^*w = ww^* = 1, \quad ws(\varphi_1) = u_1,$$

then the triplet $\{\varphi_1, \psi_1, w\}$ is the required one in the latter claim. Thus if the projections $1 - s(\varphi_1)$ and $1 - s(\psi_1)$ are equivalent in the projection lattice $\mathrm{Proj}(\mathcal{M})$, then the above w exists and the last assertion on the unitary choice of u_1 follows. We split the proof according to the type of $\mathcal{M}$. The case that $\mathcal{M}$ is finite has been taken care of by the above arguments. So we assume that $\mathcal{M}$ is infinite.

The case that $\mathcal{M}$ is semi-finite: Let τ be a faithful semi-finite normal trace. Then φ_1 and ψ_1 are of the following form:

$$\varphi_1(x) = \tau(h_1 x) \quad \text{and} \quad \psi_1(x) = \tau(k_1 x), \quad x \in \mathcal{M},$$
$$u_1 h_1 u_1^* = k_1.$$

Choose a spectral projection e of h_1 such that $eh_1 \neq 0$ and $\tau(e) < +\infty$ and set $f = u_1 e u_1^*$. Replacing the triplet $\{\varphi_1, \psi_1, u_1\}$ by $\{\varphi_1 e, \psi_1 f, u_1 e\}$, we can extend $u_1 e$ to a unitary w, which makes the situation back to the already treated case.

The case that $\mathcal{M}$ is purely infinite: Suppose $\mathcal{M}$ is purely infinite. In this case, all non-zero σ-finite projections are equivalent and also the orthogonal complements of σ-finite projections are equivalent in the case that $\mathcal{M}$ is not σ-finite. So if $s(\varphi_1) \neq 1$ and $s(\psi_1) \neq 1$, then we have

$$1 - s(\varphi_1) \sim 1 - s(\psi_1).$$

Thus we are back to the already treated case. Therefore the only remaining case is that either $s(\varphi_1) = 1$ or $s(\psi_1) \neq 1$ by symmetry. In this last case, the assumption on $\mathcal{M}$ guarantees the existence of a non-faithful $\omega \in \mathcal{M}_*^+, \omega \neq 0$ bounded by φ_1, so that $e = s(\omega) \neq 1$. Replace φ_1 by ω and set $\psi_2 = u_1 \omega u_1^*$. Then we have $s(\psi_2) = u_1 e u_1^*$ and

$$1 - s(\varphi_1) \sim 1 - s(\psi_2)$$

which allows us to extend $u_1 s(\varphi_1)$ to a unitary $w \in \mathcal{U}(\mathcal{M})$ with $u_1 s(\varphi_1) = we$. This completes the proof of lemma. □

Lemma 3.4. *Let $\{\varphi_i : i \in I\}$ be a family of non-zero positive linear functionals on a von Neumann algebra $\mathcal{M}$ such that there exists $\varphi \in \mathcal{M}_*^+$ which dominates all finite sums of φ_i, i.e.,*

$$\sum_{i \in J} \varphi_i \leq \varphi \quad \textit{for all finite subset } J \Subset I,$$

then the family $\{\varphi_i : i \in I\}$ is countable.

Proof. For each $n \in \mathbb{N}$, set

$$I_n = \left\{ i \in I : \|\varphi_i\| \geq \frac{1}{n} \right\}.$$

Then we have

$$n\varphi(1) \geq \sum_{i \in I_n} n\varphi_i(1) \geq \mathrm{Card}(I_n),$$

so that $\mathrm{Card}(I_n)$ is finite. Since $I = \cup_{n \in \mathbb{N}} I_n$, we conclude that I is countable. $\square$

Theorem 3.5. (Comparison of Positive Linear Functionals) Let $\mathcal{M}$ be a factor. For a pair $\varphi, \psi \in \mathcal{M}_*^+$ of non-zero normal positive linear functionals on $\mathcal{M}$, the following statements are equivalent:

i)
$$\|\varphi\| = \varphi(1) \leq \psi(1) = \|\psi\|.$$

ii) There exist sequences $\{\varphi_i : i \in I\} \subset \mathcal{M}_*^+$, $\{\psi_i : i \in I\} \subset \mathcal{M}_*^+$, $I \subset \mathbb{N}$ such that

$$\varphi = \sum_{i \in I} \varphi_i, \qquad \sum_{i \in I} \psi_i \leq \psi,$$
$$\varphi_i \sim \psi_i, \quad i \in I.$$

In the above equivalence, the equality of (i) corresponds to that of (ii).

Proof. Suppose that (i) holds. Let $\mathcal{F}$ be the set of following three sequences:

$$\Phi = \{\varphi_i : i \in I\} \subset \mathcal{M}_*^+, \quad \Psi = \{\psi_i : i \in I\} \subset \mathcal{M}_*^+,$$
$$U = \{u_i : i \in I\} \subset \mathcal{M}$$

such that

$$\sum_{i \in} \varphi_i \leq \varphi, \qquad \sum_{i \in I} \psi_i \leq \psi,$$
$$0 \neq u_i^* u_i = s(\varphi_i), \quad 0 \neq u_i u_i^* = s(\psi_i),$$
$$u_i \varphi_i u_i^* = \psi_i \quad u_i^* \psi_i u_i = \varphi_i, \quad i \in I.$$

From Lemma 3.4 it follows that $\mathcal{F}$ is an inductive set relative to the inclusion ordering. Hence it admits a maximal element $\{\Phi, \Psi, U\} \in \mathcal{F}$. The maximality and Lemma 2.3 implies that either

$$\varphi = \sum_{i \in I} \varphi_i \quad \text{or} \quad \psi = \sum_{i \in I} \psi_i.$$

If $\varphi \neq \sum_{i \in I} \varphi_i$, then the equality $\psi = \sum_{i \in I} \psi_i$ implies that

$$\varphi(1) > \sum_{i \in I} \varphi_i(1) = \sum_{i \in I} \psi_i(1) = \psi(1),$$

which contradicts the assumption $\varphi(1) \leq \psi(1)$. Hence we have

$$\varphi = \sum_{i\in I}\varphi_i \quad \text{and} \quad \psi \geq \sum_{i\in I}\psi_i.$$

Suppose that (ii) holds, i.e., there exist $\Phi = \{\varphi_i : i \in I\}, \Psi = \{\psi_i : i \in I\}$ and $U = \{u_i : i \in I\}$ which satisfy the requirements in (ii). Since u_i is an isometry from the range of $p_i = s(\varphi_i)$ to that of $q_i = s(\psi_i)$, we have $\|\varphi_i\| = \|\psi_i\|, i \in I$. Then we get

$$\begin{aligned}\varphi(1) &= \sum_{i\in I}\varphi_i(1) = \sum_{i\in I}\|\varphi_i\| = \sum_{i\in I}\|\psi_i\| = \sum_{i\in I}\psi_i(1) \\ &\leq \psi(1).\end{aligned}$$

This completes the proof. □

Definition 3.6. A positive linear functional φ on a von Neumann algebra $\mathcal{M}$ is said to be **super** faithful if every non-zero positive linear functional ψ dominated by φ is faithful.

Remark 3.7. If φ is a super faithful state on a von Neumann algebra $\mathcal{M}$, then φ is automatically a normal faithful positive linear functional and $\mathcal{M}_\varphi = \mathbb{C}$, consequently $\mathcal{M}$ is a factor of type III_1.

Corollary 3.8. i)*If $\mathcal{M}$ is a factor which does not admits a super faithful state, then the equivalence $\varphi_i \sim \psi_i$ in the condition* (ii) *can be replaced by the unitary conjugacy*: $\varphi_i \equiv \psi_i \mod \mathrm{Int}(\mathcal{M}), i \in I$.

ii) *If the pair $\varphi, \psi \in \mathcal{M}_*^+$ are both non-faithful instead, then the equivalence $\varphi_i \sim \psi_i$ can be replaced by $\varphi_i \equiv \psi_i \mod \mathrm{Int}(\mathcal{M})$.*

iii) *If the pair $\varphi, \psi \in \mathcal{M}_*^+$ are both super faithful, then the equivalence $\varphi_i \sim \psi_i$ is replaced by $\varphi_i \equiv \psi_i \mod \mathrm{Int}(\mathcal{M})$.*

Proof. This follows from the fact that the equivalence of non-faithful normal positive linear functionals can be implemented by a unitary element in $\mathcal{M}$. □

4. Commutative Case

Let $\mathcal{A}$ be an abelian von Neumann algebra. In this case, as it stands, one cannot compare a pair of normal positive linear functionals beyond the absolute continuity. We need a device to move around elements of $\mathcal{A}$. So let G be a group of automorphisms of $\mathcal{A}$, i.e., G is a subgroup of $\mathrm{Aut}(\mathcal{A})$. For each member $\gamma \in G$, we consider the action of γ on the predual $\mathcal{A}_*$ as follows:

$$\langle x, \gamma(\varphi)\rangle = \left\langle \gamma^{-1}(x), \varphi\right\rangle, \quad x \in \mathcal{A}, \varphi \in \mathcal{A}_*.$$

We write

$$\varphi \equiv \psi \mod G$$

if there exists $\gamma \in G$ such that $\psi = \gamma(\varphi)$.

Proposition 4.1. *If $\mathcal{A}$ is an abelian von Neumann algebra equipped with an ergodic group G of automorphisms, then for every pair $\varphi, \psi \in \mathcal{A}_*^+$ of normal positive linear functionals the following two statements are equivalent*:

i)

$$\|\varphi\| \leq \|\psi\|.$$

ii) *There exist families* $\{\varphi_i : i \in I\}$ *and* $\{\psi_i : i \in I\}$ *of normal positive linear functionals on* $\mathcal{A}$ *such that*

$$\varphi = \sum_{i \in I} \varphi_i, \quad \sum_{i \in I} \psi_i \leq \psi;$$
$$\varphi_i \equiv \psi_i \mod G, \quad i \in I.$$

Proof. First we remark that the commutativity of $\mathcal{A}$ entails the lattice property of both the self-adjoint part of $\mathcal{A}$ and of the self-adjoint part of its predual $\mathcal{A}_*$. From the discussion in the last section, to prove the theorem it is enough to show that for every pair $\varphi, \psi \in \mathcal{A}_*^+$ of non-zero normal positive linear functionals there exists a pair $\varphi_1, \psi_1 \in \mathcal{A}_*^+$ such that

$$0 \neq \varphi_1 \leq \varphi, \quad 0 \neq \psi_1 \leq \psi,$$
$$\varphi_1 \equiv \psi_1 \mod G.$$

To this end, set

$$p = s(\varphi), \quad q = s(\psi).$$

Since $\varphi \neq 0$ and $\psi \neq 0$, we have $p \neq 0$ and $q \neq 0$ as well. Hence the ergodicity of G implies the existence of $\gamma_1 \in G$ such that

$$\gamma_1(p)q \neq 0.$$

This means that $\gamma_1(\varphi)q \neq 0$, so that $\gamma_1(\varphi) \wedge \psi = \psi_1 \neq 0$. Setting

$$\varphi_1 = \gamma_1^{-1}(\psi_1),$$

we obtained a pair $\varphi_1, \psi_1 \in \mathcal{A}_*^+$ with

$$0 \neq \varphi_1 \leq \varphi, \quad 0 \neq \psi_1 \leq \psi, \quad \varphi_1 \equiv \psi_1 \mod G.$$

This completes the proof. □

Application of the proposition yields the following fact which can be stated more general form such as the integration over a locally compact group. We just state here a special case which should be taught in the class on the Lebesgue integration.

Corollary 4.2. *Let* $L^1(\mathbb{R}^n)$ *be the Banach space of all integrable functions on the vector space* $\mathbb{R}^n$ *relative to the Lebesgue measure. For a pair* $f, g \in L^1(\mathbb{R}^n)_+$ *of positive integrable functions, the following two conditions are equivalent*:

i)

$$\int_{\mathbb{R}^n} f(x)\mathrm{d}x \leq \int_{\mathbb{R}^n} g(x)\mathrm{d}x.$$

ii) *There exist countable families,* $\{f_i : i \in I\}, \{g_i : i \in I\} \subset L^1(\mathbb{R}^n)_+$ *and* $\{a_i : i \in I\} \subset \mathbb{R}^n$ *such that*

$$f = \sum_{i \in I} f_i, \quad \sum_{i \in I} g_i \leq g$$

and

$$g_i(x) = f_i(x + a_i) \text{ for almost every } x \in \mathbb{R}^n.$$

Here the summation is taken relative to the convergence in the Banach space $L^1(\mathbb{R}^n)$.

Here the equality of (i) *corresponds to that of* (ii).

5. Commutativity of Normal Positive Linear Functionals

Fix a factor $\mathcal{M}$ and a pair $\varphi, \psi \in \mathcal{M}_*^+$ with $\|\varphi\| \leq \|\psi\|$. Then we have decomposition:

$$\varphi = \sum_{i \in I} \varphi_i, \quad \sum_{i \in I} \psi_i \leq \psi, \quad \varphi_i \sim \psi_i, \quad i \in I.$$

We are going to discuss the commutativity of the families $\{\varphi_i : i \in I\}$ and $\{\psi_i, i \in I\}$. To this end, we remind ourselves the following fact: the commutativity of a pair $\varphi, \psi \in \mathcal{M}_*^+$ of normal positive linear functionals was first introduced in [6] in the following form:

Definition 5.1. A pair $\omega_1, \omega_2 \in \mathcal{M}_*^+$ is said to **commute** if

$$|\omega_1 + \mathrm{i}\omega_2| = |\omega_1 - \mathrm{i}\omega_2|.$$

In the case that both functionals are faithful, it is shown in [6] that their commutativity is equivalent to the invariance of one relative to the modular automorphism group of the other. For the general pair $\varphi, \psi \in \mathcal{M}_*^+$, we don't have any tool to attack the commutativity question. So we restrict ourselves to the special case that φ and ψ are both factoring through a maximal abelian subalgebra $\mathcal{A}$ of $\mathcal{M}$ in the sense that

$$\varphi = \varphi \circ \mathcal{E}_{\mathcal{A}} \quad \text{and} \quad \psi = \psi \circ \mathcal{E}_{\mathcal{A}},$$

where $\mathcal{E}_{\mathcal{A}}$ means the $\mathcal{A}$-valued normal conditional expectation. In general, $\mathcal{E}_{\mathcal{A}}$ does not exist. For example there is no normal conditional expectation from $\mathcal{L}(L^2(\mathbb{R}))$ to $L^\infty(\mathbb{R})$. But if it does exist, then it is unique.

Proposition 5.2. *Let* $\mathcal{M}$ *be a factor and* $\mathcal{A}$ *be a maximal abelian subalgebra of* $\mathcal{M}$*. If* $\mathcal{A}$ *is semi-regular and the range of normal conditional expectation* $\mathcal{E}_{\mathcal{A}}$*, then for a pair* $\varphi, \psi \in \mathcal{M}_*^+$ *such that*

$$\varphi = \varphi \circ \mathcal{E}_{\mathcal{A}}, \quad \psi = \psi \circ \mathcal{E}_{\mathcal{A}},$$

the inequality

$$\|\varphi\| \leq \|\psi\|$$

is equivalent to the existence of the decomposition:

$$\begin{gathered}\varphi = \sum_{i \in I} \varphi_i, \quad \sum_{i \in I} \psi_i \leq \psi, \\ \varphi_i \equiv \psi_i \bmod \mathcal{N}(\mathcal{A}), \\ \varphi_i = \varphi_i {\circ} \mathcal{E}_{\mathcal{A}}, \quad \psi_i = \psi_i {\circ} \mathcal{E}_{\mathcal{A}}, \quad i \in I,\end{gathered}$$

where $\mathcal{N}(\mathcal{A}) = \{u \in \mathcal{U}(\mathcal{M}) : u\mathcal{A}u^* = \mathcal{A}\}$ *is the normalizer of* $\mathcal{A}$ *in* $\mathcal{M}$.

Before the proof, we observe that the invariance $\varphi = \varphi {\circ} \mathcal{E}_{\mathcal{A}}$ is equivalent to the inclusion:

$$\mathcal{A} \subset \mathcal{M}_{\varphi}.$$

Proof. First we observe

$$u\mathcal{E}_{\mathcal{A}}(x)u^* = \mathcal{E}_{\mathcal{A}}(uxu^*), \quad u \in \mathcal{N}(\mathcal{A}).$$

Then with $G = \{\mathrm{Ad}(u) : u \in \mathcal{N}(\mathcal{A})\}$, G acts on $\mathcal{A}$ ergodically by the semi-regularity assumption on $\mathcal{A}$. Hence Proposition 4.1 implies that there exist families $\{\bar{\varphi}_i : i \in I\} \subset \mathcal{A}_*^+$ and $\big\{\bar{\psi}_i : i \in I\big\}\imath\mathcal{A}_*^+$ such that

$$\varphi|_{\mathcal{A}} = \sum_{i \in I} \bar{\varphi}_i, \quad \sum_{i \in I} \bar{\psi}_i \leq \psi|_{\mathcal{A}}, \quad \bar{\varphi}_i \equiv \bar{\psi}_i \quad \bmod G.$$

Setting $\varphi_i = \bar{\varphi}_i {\circ} \mathcal{E}_{\mathcal{A}}$ and $\psi_i = \bar{\psi}_i {\circ} \mathcal{E}_{\mathcal{A}}$, we get for every $x \in \mathcal{M}_+$,

$$\begin{aligned}\sum_{i \in I} \varphi_i(x) &= \sum_{i \in I} \bar{\varphi}_i(\mathcal{E}_{\mathcal{A}}(x)) = \left(\sum_{i \in I} \bar{\varphi}_i\right)(\mathcal{E}_{\mathcal{A}}(x)) \\ &= \varphi(\mathcal{E}_{\mathcal{A}}(x)) = \varphi(x); \\ \sum_{i \in I} \psi_i(x) &= \sum_{i \in I} \bar{\psi}_i(\mathcal{E}_{\mathcal{A}}(x)) = \left(\sum_{i \in I} \bar{\psi}_i\right)(\mathcal{E}_{\mathcal{A}}(x)) \\ &\leq \psi(\mathcal{E}_{\mathcal{A}}(x)) = \psi(x).\end{aligned}$$

If $u_i \in \mathcal{N}(\mathcal{A})$ gives $\mathrm{Ad}(u_i)|_{\mathcal{A}}(\bar{\varphi}_i) = \bar{\psi}_i, i \in I$, then we have for each $x \in \mathcal{M}$

$$\begin{aligned}\varphi_i(u_i^* x u_i) &= \bar{\varphi}_i(\mathcal{E}_{\mathcal{A}}(u_i^* x u_i)) = \bar{\varphi}_i(u_i^* \mathcal{E}_{\mathcal{A}}(x) u_i) = \bar{\varphi}_i {\circ} \mathrm{Ad}(u_i)^{-1}(\mathcal{E}_{\mathcal{A}}(x)) \\ &= \bar{\psi}_i(\mathcal{E}_{\mathcal{A}}(x)) = \psi_i(x).\end{aligned}$$

Consequently we get

$$\varphi_i \equiv \psi_i \quad \bmod G, \quad i \in I.$$

This completes the proof. □

6. Concluding Remark

Throughout the paper, we only consider the factor case. The generalization to the non-factor case is very much the same as the comparison theory of projections in the general frame work of von Neumann algebras. However the point of this paper is that the non-commutative theory of integration gives the natural answer about the question concerning the meaning of the same values on integrations. In the case of semi-finite factors, the work of Kadison and Pedersen, [5], on the additivity of a trace gives the same result. However the motivations of their work and this work are quite different. They are very much concerned about the natural proof of the additivity property of the trace which comes from the comparison of projections. In other words, their theory can be viewed as the one about the measure theory, whilst our work is more concerned with the result of integration. Technically, their work is more demanding as they don't assume the existence of a semi-finite normal trace on the base von Neumann algebra. Indeed, the result in the case of factors of type III is unexpected. Also the seek of natural answer brought about the new question on the existence of a super faithful state which was never considered before. The author has been unable to exclude the existence of a super faithful state so far. The author would like to leave the existence question of a super faithful state as a challenge for operator algebraists.

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Department of Mathematics, UCLA, P.O.Box 951555, Los Angeles, California 90095-1555. Mail Address: 3-10-39 Nankohdai, Izumi-ku, Sendai, 981-8003 Japan.

E-mail address: mt@math.ucla.edu